INSTITUTO SUPERIOR TÉCNICO
Licenciatura em Engenharia Física Tecnológica
Licenciatura em Engenharia e Gestão Industrial
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ANÁLISE NUMÉRICA

Formulário

1. Representação de Números e Teoria de Erros

Erro, erro absoluto, erro relativo ($\tilde{x} \approx x$):

$$\text{(i)} \quad x \in \mathbb{R}: \qquad e_{\tilde{x}} = x - \tilde{x}, \qquad |e_{\tilde{x}}|, \qquad \delta_{\tilde{x}} = \frac{e_{\tilde{x}}}{x}, \qquad |\delta_{\tilde{x}}| \qquad (x \neq 0)$$

$$\text{(ii)} \quad x \in \mathbb{R}^n: \qquad e_{\tilde{x}} = x - \tilde{x}, \qquad \|e_{\tilde{x}}\|, \qquad \delta_{\tilde{x}} = \frac{e_{\tilde{x}}}{\|x\|}, \qquad \|\delta_{\tilde{x}}\| \qquad (x \neq 0)$$

Representação de números reais (notação científica):

$$x = \sigma\, m\, \beta^t \ \in \mathbb{R} \setminus \{0\}$$

$$\text{(base) } \beta \in \mathbb{N} \setminus \{1\}, \qquad \text{(sinal) } \sigma \in \{+,-\}, \qquad \text{(expoente) } t \in \mathbb{Z}$$

$$\text{(mantissa) } m = (0.a_1 a_2 \ldots)_\beta \in \ [\beta^{-1}, 1[, \qquad a_i \in \{0, 1, \ldots, \beta - 1\}, \qquad a_1 \neq 0$$

Sistema de ponto flutuante:

$$\mathrm{FP}(\beta, n, t_1, t_2) = \{x \in \mathbb{Q}: \ x = \sigma\, m\, \beta^t\} \cup \{0\}$$

$$\beta \in \mathbb{N} \setminus \{1\}, \qquad \sigma \in \{+,-\}, \qquad t_1 \leq t \leq t_2, \quad t, t_1, t_2 \in \mathbb{Z}$$

$$m = (0.a_1 a_2 \ldots a_n)_\beta \ \in [\beta^{-1}, 1 - \beta^{-n}], \qquad a_i \in \{0, 1, \ldots, \beta - 1\}, \qquad a_1 \neq 0$$

Arredondamentos:

$$x = \sigma(0.a_1 a_2 \ldots a_n a_{n+1} \ldots)_\beta \times \beta^t \ \in \mathbb{R}, \qquad \mathrm{fl}(x) \in \mathrm{FP}(\beta, n, t_1, t_2)$$

(i) arredondamento por corte:

$$\mathrm{fl}(x) = \sigma(0.a_1 a_2 \ldots a_n)_\beta \times \beta^t$$

(ii) arredondamento simétrico (β par):

$$\mathrm{fl}(x) = \begin{cases} \sigma(0.a_1 a_2 \ldots a_n)_\beta \times \beta^t, & 0 \leq a_{n+1} < \dfrac{\beta}{2} \\ \sigma\left[(0.a_1 a_2 \ldots a_n)_\beta + \beta^{-n}\right] \times \beta^t, & \dfrac{\beta}{2} \leq a_{n+1} < \beta \end{cases}$$

Erros de arredondamento ($x = \sigma m \beta^t \in \mathbb{R}, \;\; \tilde{x} = \text{fl}(x) \in \text{FP}(\beta, n, t_1, t_2)$):

(i) arredondamento por corte:

$$|e_{\tilde{x}}| \leq \beta^{t-n}, \qquad |\delta_{\tilde{x}}| \leq \beta^{1-n} =: U$$

(ii) arredondamento simétrico:

$$|e_{\tilde{x}}| \leq \frac{1}{2}\,\beta^{t-n}, \qquad |\delta_{\tilde{x}}| \leq \frac{1}{2}\,\beta^{1-n} =: U$$

(U: unidade de arredondamento do sistema FP)

Algarismo significativo:

$$x = \sigma m 10^t \in \mathbb{R}, \qquad \tilde{x} = \sigma(0.a_1 a_2 \ldots a_n)_{10} \times 10^t \in \text{FP}(10, n, t_1, t_2),$$

$$a_i \quad \text{é algarismo significativo de } \tilde{x} \text{ se } \quad |e_{\tilde{x}}| \leq \frac{1}{2}\,\beta^{t-i}$$

Propagação de erros $\Big(x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \;\; \phi : \mathbb{R}^n \to \mathbb{R}, \;\; \tilde{x} \approx x, \;\; \tilde{\phi} = \text{fl} \circ \phi\Big)$:

$$e_{\phi(\tilde{x})} = \phi(x) - \phi(\tilde{x}) \quad \approx \quad \tilde{e}_\phi(x) = \sum_{k=1}^{n} \frac{\partial \phi}{\partial x_k}(x)\, e_{\tilde{x}_k}$$

$$\delta_{\phi(\tilde{x})} = \frac{e_{\phi(\tilde{x})}}{\phi(x)} \quad \approx \quad \tilde{\delta}_\phi(x) = \sum_{k=1}^{n} p_{\phi, x_k}(x) \delta_{\tilde{x}_k}, \qquad p_{\phi, x_k}(x) = \frac{x_k \frac{\partial \phi}{\partial x_k}(x)}{\phi(x)}$$

$$\delta_{\tilde{\phi}(\tilde{x})} = \frac{\phi(x) - \tilde{\phi}(\tilde{x})}{\phi(x)} \quad \approx \quad \tilde{\delta}_\phi + \delta_{\text{arr}}, \qquad \tilde{\delta}_\phi = \sum_{k=1}^{n} p_{\phi, x_k} \delta_{\tilde{x}_k}, \qquad \delta_{\text{arr}} = \sum_{k=1}^{m} q_k \delta_{\text{arr}_k}$$

2. Complementos de Álgebra Linear

Normas matriciais induzidas por normas vectoriais ($x \in \mathbb{R}^n, \;\; A \in L^n(\mathbb{R})$):

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} \qquad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \qquad \|A\|_2 = \sqrt{r_\sigma(A^* A)} \qquad \|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

(norma por coluna) (norma por linha)

$$r_\sigma(A) = \max_{1 \leq i \leq n} |\lambda_i|, \qquad \lambda_1, \ldots, \lambda_n : \text{ valores próprios de } A$$

Número de condição de uma matriz:

$$\text{cond}_p(A) = \|A\|_p \|A^{-1}\|_p, \quad p = 1, 2, \infty, \qquad \text{cond}_*(A) = r_\sigma(A) r_\sigma(A^{-1})$$

Condicionamento de sistemas lineares ($Ax = b,\ \tilde{A}\tilde{x} = \tilde{b}$):

$$\frac{\|\tilde{x} - x\|_p}{\|x\|_p} \le \frac{\mathrm{cond}_p(A)}{1 - \frac{\|\tilde{A}-A\|_p}{\|A\|_p}\,\mathrm{cond}_p(A)} \left(\frac{\|\tilde{A} - A\|_p}{\|A\|_p} + \frac{\|\tilde{b} - b\|_p}{\|b\|_p} \right)$$

$$\|\tilde{A} - A\|_p \|A^{-1}\|_p < 1, \qquad p = 1, 2, \infty$$

3. Resolução de Sistemas Lineares ($Ax = b,\ A \in L^n,\ b, x \in \mathbb{R}^n$)

Métodos directos:

$$Ax = b \;\Leftrightarrow\; (LU)x = b \;\Leftrightarrow\; \begin{cases} Lg = b \\ Ux = g \end{cases}$$

- Método de Doolittle ($A = LU,\ l_{11} = l_{22} = \cdots = l_{nn} = 1$):

$$k = 1, 2, \ldots, n: \begin{cases} u_{kj} = a_{kj} - \displaystyle\sum_{r=1}^{k-1} l_{kr} u_{rj}, & j = k, \ldots, n \\ l_{ik} = \dfrac{1}{u_{kk}} \left(a_{ik} - \displaystyle\sum_{r=1}^{k-1} l_{ir} u_{rk} \right), & i = k+1, \ldots, n \end{cases}$$

- Método de Crout ($A = LU,\ u_{11} = u_{22} = \cdots = u_{nn} = 1$):

$$k = 1, 2, \ldots, n: \begin{cases} l_{ik} = a_{ik} - \displaystyle\sum_{r=1}^{k-1} l_{ir} u_{rk}, & i = k, \ldots, n \\ u_{kj} = \dfrac{1}{l_{kk}} \left(a_{kj} - \displaystyle\sum_{r=1}^{k-1} l_{kr} u_{rj} \right), & j = k+1, \ldots, n \end{cases}$$

- Método de Cholesky ($A = LL^T,\ A$ simétrica definida positiva):

$$k = 1, 2, \ldots, n: \begin{cases} l_{kk} = \sqrt{a_{kk} - \displaystyle\sum_{r=1}^{k-1} l_{kr}^2}, & \\ l_{ik} = \dfrac{1}{l_{kk}} \left(a_{ik} - \displaystyle\sum_{r=1}^{k-1} l_{ir} l_{kr} \right), & i = k+1, \ldots, n \end{cases}$$

Métodos iterativos:

$$x^{(k+1)} = Cx^{(k)} + w, \qquad k = 0, 1, \ldots$$

$$C = -M^{-1}N, \qquad w = M^{-1}b, \qquad M + N = A = L + D + U$$

$$\|x - x^{(k+1)}\| \leq c\|x - x^{(k)}\|, \qquad \|x - x^{(k)}\| \leq c^k\|x - x^{(0)}\|$$

$$\|x - x^{(k)}\| \leq \frac{1}{1-c}\,\|x^{(k+1)} - x^{(k)}\|, \qquad \|x - x^{(k+1)}\| \leq \frac{c}{1-c}\,\|x^{(k+1)} - x^{(k)}\|$$

$$\|x - x^{(k)}\| \leq \frac{c^k}{1-c}\,\|x^{(1)} - x^{(0)}\|, \qquad (c = \|C\| < 1)$$

- Método de Jacobi ($M = D$):

$$x^{(k+1)} = D^{-1}\left[b - (L+U)x^{(k)}\right], \qquad k = 0, 1 \ldots$$

$$\|x - x^{(k)}\|_\infty \leq \mu^k \|x - x^{(0)}\|_\infty$$

$$\mu = \max_{1\leq i\leq n}(\alpha_i + \beta_i), \qquad \alpha_i = \sum_{j=1}^{i-1}\left|\frac{a_{ij}}{a_{ii}}\right|, \qquad \beta_i = \sum_{j=i+1}^{n}\left|\frac{a_{ij}}{a_{ii}}\right|$$

- Método de Gauss-Seidel ($M = D + L$):

$$x^{(k+1)} = D^{-1}\left(b - Lx^{(k+1)} - Ux^{(k)}\right), \qquad k = 0, 1 \ldots$$

$$\|x - x^{(k)}\|_\infty \leq \eta^k \|x - x^{(0)}\|_\infty$$

$$\eta = \max_{1\leq i\leq n}\frac{\beta_i}{1-\alpha_i}, \qquad \alpha_i = \sum_{j=1}^{i-1}\left|\frac{a_{ij}}{a_{ii}}\right|, \qquad \beta_i = \sum_{j=i+1}^{n}\left|\frac{a_{ij}}{a_{ii}}\right|$$

- Método da iteração simples ou da relaxação linear $\left(M = \dfrac{I}{\omega},\ \omega \in \mathbb{R}\setminus\{0\}\right)$:

$$x^{(k+1)} = x^{(k)} + \omega\left(b - Ax^{(k)}\right), \qquad k = 0, 1, \ldots$$

- Método de Jacobi modificado $\left(M = \dfrac{D}{\omega},\ \omega \in \mathbb{R}\setminus\{0\}\right)$:

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega D^{-1}\left[b - (L+U)x^{(k)}\right], \qquad k = 0, 1, \ldots$$

- Método de Gauss-Seidel modificado ou SOR $\left(M = \dfrac{D}{\omega} + L,\ \omega \in \mathbb{R}\setminus\{0\}\right)$:

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega D^{-1}\left(b - Lx^{(k+1)} - Ux^{(k)}\right), \qquad k = 0, 1, \ldots$$

4. Determinação de Valores e Vectores Próprios

Método das potências ($Au_1 = \lambda_1 u_1,\ \ \lambda_1 \in \mathbb{R},\ \ |\lambda_1| > |\lambda_i|,\ i = 2, \ldots, n$):

$$u^{(k+1)} = \sigma_k \frac{Au^{(k)}}{\|Au^{(k)}\|_\infty}, \qquad \lambda^{(k)} = \frac{[Au^{(k)}]_m}{u_m^{(k)}}, \qquad k = 0, 1, \ldots, \qquad \|u^{(0)}\|_\infty = 1$$

σ_k: sinal da componente do vector $Au^{(k)}$ de maior valor absoluto
m: índice da componente do vector $u^{(k)}$ de maior valor absoluto

$$\|u^{(k)} - u_1\|_\infty = O\left(\rho^k\right), \qquad |\lambda^{(k)} - \lambda_1| = O\left(\rho^k\right), \qquad \rho = \frac{\max_{2\le i\le n} |\lambda_i|}{|\lambda_1|}$$

Método das iterações inversas ($Au_j = \lambda_j u_j,\ \ \lambda_j \in \mathbb{R}$):

$$B = (A - \mu I)^{-1}, \qquad \mu \ \text{ t.q. } \frac{1}{|\lambda_j - \mu|} > \max_{i\ne j} \frac{1}{|\lambda_i - \mu|}$$

$$u^{(k+1)} = \sigma_k \frac{Bu^{(k)}}{\|Bu^{(k)}\|_\infty}, \qquad \lambda^{(k)} = \frac{[Bu^{(k)}]_m}{u_m^{(k)}}, \qquad k = 0, 1, \ldots \qquad \|u^{(0)}\|_\infty = 1$$

$$\|u^{(k)} - u_j\|_\infty = O\left(\rho^k\right), \qquad \left|\lambda^{(k)} - \frac{1}{\lambda_j - \mu}\right| = O\left(\rho^k\right), \qquad \rho = \frac{|\lambda_j - \mu|}{\min_{i\ne j} |\lambda_i - \mu|}$$

5. Resolução de Equações e Sistemas Não-lineares

Resolução de equações ($f : \mathbb{R} \to \mathbb{R}$):

- Método da bissecção ($f(z) = 0,\ f(a)f(b) < 0$):

$$x_{m+1} = x_m - \frac{b-a}{2^{m+1}} \operatorname{sgn}[f(b)] \operatorname{sgn}[f(x_m)], \qquad m = 0, 1, \ldots$$

$$|z - x_m| \le \frac{b-a}{2^m}, \qquad |z - x_{m+1}| \le |x_{m+1} - x_m|$$

- Método do ponto fixo ($f(z) = 0 \Leftrightarrow z = g(z)$):

$$x_{m+1} = g(x_m), \qquad m = 0, 1, \ldots$$

$$|z - x_{m+1}| \le L|z - x_m|, \qquad |z - x_m| \le L^m |z - x_0|$$

$$|z - x_m| \le \frac{1}{1-L}|x_{m+1} - x_m|, \qquad |z - x_{m+1}| \le \frac{L}{1-L}|x_{m+1} - x_m|$$

$$|z - x_m| \le \frac{L^m}{1-L}|x_1 - x_0| \qquad (L < 1,\ \ L : \text{constante de Lipschitz})$$

∘ • $g'(z) \ne 0$:

$$z - x_{m+1} = g'(\xi_m)(z - x_m), \qquad \xi_m \in \operatorname{int}(z, x_m)$$

$$|z - x_{m+1}| \le L|z - x_m|, \qquad L = \max_{x\in[a,b]} |g'(x)|$$

∘ • $g^{(r)}(z) = 0,\ r = 1, \ldots, p-1,\ \ g^{(p)}(z) \ne 0, \quad p = 2, 3, \ldots$

$$z - x_{m+1} = \frac{1}{p!}(-1)^{p+1} g^{(p)}(\xi_m)(z - x_m)^p, \qquad \xi_m \in \operatorname{int}(z, x_m)$$

$$|z - x_{m+1}| \le K_p |x - x_m|^p, \qquad |z - x_m| \le K_p^{\frac{1}{1-p}} \left(K_p^{\frac{1}{p-1}} |z - x_0|\right)^{p^m}$$

$$K_p = \frac{1}{p!} \max_{x\in[a,b]} \left| g^{(p)}(x) \right|$$

- Método de Newton ($f(z) = 0,\ f'(z) \neq 0$):

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)}, \qquad m = 0, 1, \ldots$$

$$z - x_{m+1} = -\frac{f''(\xi_m)}{2f'(x_m)} \left(z - x_m\right)^2, \qquad \xi_m \in \mathrm{int}(z, x_m)$$

$$|z - x_{m+1}| \leq K|z - x_m|^2, \qquad |z - x_m| \leq \frac{1}{K} \left(K|z - x_0|\right)^{2^m}$$

$$K = \frac{\max_{x\in[a,b]} |f''(x)|}{2 \min_{x\in[a,b]} |f'(x)|}$$

$$|z - x_{m+1}| \leq |x_{m+1} - x_m|, \quad \text{se} \quad K|z - x_m| \leq \frac{1}{2}$$

- Método de Newton modificado:

$$f^{(r)}(z) = 0, \quad r = 0, \ldots, p-1, \quad f^{(p)}(z) \neq 0, \quad p = 2, 3, \ldots$$

$$x_{m+1} = x_m - p\,\frac{f(x_m)}{f'(x_m)}, \qquad m = 0, 1, \ldots$$

- Método da secante ($f(z) = 0$):

$$x_{m+1} = x_m - f(x_m)\,\frac{x_m - x_{m-1}}{f(x_m) - f(x_{m-1})}, \qquad m = 1, 2, \ldots$$

$$z - x_{m+1} = -\frac{f''(\eta_m)}{2f'(\xi_m)} \left(z - x_m\right)\left(z - x_{m-1}\right), \qquad \xi_m, \eta_m \in \mathrm{int}(x_{m-1}, z, x_m)$$

$$|z - x_{m+1}| \leq K\,|z - x_m|\,|z - x_{m-1}|, \qquad K = \frac{\max_{x\in[a,b]} |f''(x)|}{2 \min_{x\in[a,b]} |f'(x)|}$$

$$|z - x_{m+1}| \leq |x_{m+1} - x_m|, \quad \text{se} \quad K|z - x_{m-1}| \leq \frac{1}{2}$$

$$\lim_{m\to\infty} \frac{|z - x_{m+1}|}{|z - x_m|^r} = \left| \frac{f''(z)}{2f'(z)} \right|^{r-1} =: K_\infty^{[r]}, \qquad r = \frac{\sqrt{5}+1}{2}$$

Resolução de sistemas de equações ($f : \mathbb{R}^n \to \mathbb{R}^n$):

- Método do ponto fixo ($f(z) = 0 \Leftrightarrow z = g(z)$):

$$x_{m+1} = g(x_m), \qquad m = 0, 1, \ldots$$

$$|z - x_{m+1}| \leq L|z - x_m|, \qquad |z - x_m| \leq L^m|z - x_0|$$

$$|z - x_m| \leq \frac{1}{1-L}|x_{m+1} - x_m|, \qquad |z - x_{m+1}| \leq \frac{L}{1-L}|x_{m+1} - x_m|$$

$$|z - x_m| \leq \frac{L^m}{1-L}|x_1 - x_0|, \qquad \left(L < 1,\ L = \sup_{x\in D} \|J_g(x)\|\right)$$

- Método de Newton generalizado ($f(z)=0,\ \det[J_f(z)]\neq 0$):

$$x_{m+1}=x_m+\Delta x_m, \qquad J_f(x_m)\Delta x_m=-f(x_m), \qquad m=0,1,\ldots$$

$$\|z-x_{m+1}\|\leq K\|z-x_m\|^2, \qquad \|z-x_m\|\leq \frac{1}{K}\left(K\|z-x_0\|\right)^{2^m}$$

$$K=\frac{M_2}{2M_1}\begin{cases}\dfrac{1}{M_1}=\sup\limits_{x\in D}\|[J_f(x)]^{-1}\|,\\ M_2=\max\limits_{1\leq i\leq n}\sup\limits_{x\in D}\|H_{f_i}(x)\|, \quad H_{f_i}\in L^n, \quad (H_{f_i})_{jk}=\dfrac{\partial^2 f_i}{\partial x_j\partial x_k}\end{cases}$$

6. Interpolação Polinomial

Fórmula interpoladora de Lagrange:

$$p_n(x)=\sum_{j=0}^{n} f_j\, l_j(x), \qquad l_j(x)=\prod_{i=0,i\neq j}^{n}\frac{x-x_i}{x_j-x_i}$$

Fórmula interpoladora de Newton:

$$p_n(x)=f[x_0]+\sum_{j=1}^{n} f[x_0,\ldots,x_j]\prod_{i=0}^{j-1}(x-x_i), \quad f[x_0,\ldots,x_j]=\sum_{l=0}^{j}\frac{f(x_l)}{\prod\limits_{i=0,i\neq l}^{j}(x_l-x_i)}$$

Fórmula do erro:

$$e_n(x)=f(x)-p_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}W_{n+1}(x)=f[x_0,\ldots,x_n,x]\,W_{n+1}(x)$$

$$W_{n+1}(x)=\prod_{i=0}^{n}(x-x_i), \qquad \xi\in \operatorname{int}(x_0,\ldots,x_n,x)$$

7. Teoria da Aproximação

Melhor aproximação uniforme ϕ^* de $f\in E$ em $F\subset E$, F subespaço de dimensão n, $E=C([a,b])$, espaço normado com a norma uniforme:

ϕ^* é a m.a. uniforme de f em F $\Leftrightarrow$ $\exists\ x_0<x_1<\cdots<x_n$ em $[a,b]$ tais que

(i) $f(x_i)-\phi^*(x_i)=(-1)^i\delta, \quad i=0,\ldots,n \qquad \delta\in\mathbb{R}$

(ii) $|\delta|=\|f-\phi^*\|_\infty=\inf\limits_{\phi\in F}\|f-\phi\|_\infty$

Melhor aproximação ϕ^* mínimos quadrados de $f \in E$ em $F \subset E$, F subespaço de dimensão n gerado por $\{\varphi_0, \ldots, \varphi_n\}$, E espaço pré-Hilbertiano:

$$\|f - \phi^*\|_2 = \inf_{\phi \in F} \|f - \phi\|_2 \quad \Leftrightarrow \quad \langle f - \phi^*, \phi \rangle, \quad \forall \phi \in F$$

$$\phi^* = \sum_{k=0}^{n} a_k^* \varphi_k, \quad a_k^* = \sum_{j=0}^{n} \left(M^{-1}\right)_{kj} \langle f, \varphi_j \rangle, \qquad M \in L^n, \quad M_{jk} = \langle \varphi_j, \varphi_k \rangle$$

$$\phi^* = \sum_{k=0}^{n} a_k^* \varphi_k, \quad a_k^* = \frac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle}, \qquad \text{se } \{\varphi_0, \ldots, \varphi_n\} \text{ é um sistema ortogonal}$$

Polinómios ortogonais com respeito ao produto interno

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx, \quad (f, g \in C([a,b]), \ w \in C(]a,b[), \ w(x) \geq 0)$$

- Polinómios de Legendre, P_n $\quad (x \in [a,b] = [-1,1], \ w(x) = 1)$:

$$\begin{cases} P_{n+1}(x) = \dfrac{2n+1}{n+1}\, x P_n(x) - \dfrac{n}{n+1}\, P_{n-1}(x), & n = 1, 2, \ldots \\ P_0(x) = 1, \qquad P_1(x) = x \end{cases}$$

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left[(1 - x^2)^n \right], \quad n = 1, \ldots$$

$$\langle P_n, P_m \rangle = 0, \quad \forall n \neq m, \qquad \langle P_n, P_n \rangle = \frac{2}{2n+1}, \quad n = 0, 1, \ldots$$

- Polinómios de Chebyshev, T_n $\quad \left(x \in [a,b] = [-1,1], \ w(x) = 1/\sqrt{1-x^2}\right)$:

$$\begin{cases} T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x), & n = 1, 2, \ldots \\ T_0(x) = 1, \qquad T_1(x) = x \end{cases}$$

$$T_n(x) = \cos(n \arccos x), \qquad n = 0, 1, \ldots$$

$$\langle T_n, T_m \rangle = 0, \quad \forall n \neq m, \qquad \langle T_0, T_0 \rangle = \pi, \qquad \langle T_n, T_n \rangle = \frac{\pi}{2}, \quad n = 1, 2, \ldots$$

$$T_n(x_i) = 0, \qquad x_i = \cos \frac{(2i+1)\pi}{2n}, \qquad i = 0, \ldots, n-1, \qquad n = 1, 2, \ldots$$

8. Integração

Fórmulas de Newton-Cotes fechadas de ordem n ($f \in C([a,b])$):

$$I(f) = \int_a^b f(x)\, dx \quad \approx \quad I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$$

$$I_n(x^k) = I(x^k), \qquad k = 0, 1, \ldots, n$$

$$x_{j,n} = a + jh, \quad j = 0, 1, \ldots, n, \qquad h = \frac{b-a}{n}$$

$$w_{j,n} = I(l_{j,n}) = \frac{h(-1)^{n-j}}{j!(n-j)!} \int_0^n \prod_{i=0, i\neq j}^{n} (t-i)\, dt, \qquad w_{j,n} = w_{n-j,n}$$

$$E_n(f) = I(f) - I_n(f) = \begin{cases} C_n h^{n+2} f^{(n+1)}(\xi), & n \text{ ímpar} \\ D_n h^{n+3} f^{(n+2)}(\xi), & n \text{ par} \end{cases} \qquad \xi \in (a,b)$$

$$C_n = \frac{1}{(n+1)!} \int_0^n \prod_{i=0}^{n} (t-i)\, dt, \qquad D_n = \frac{1}{(n+2)!} \int_0^n t \prod_{i=0}^{n} (t-i)\, dt$$

- Regra dos trapézios (Newton-Cotes fechada com $n = 1,\ h = b - a$):

$$I_1(f) = \frac{b-a}{2} [f(a) + f(b)], \qquad E_1(f) = -\frac{h^3}{12} f''(\xi), \quad \xi \in (a,b)$$

- Regra de Simpson $\left(\text{Newton-Cotes fechada com } n = 2,\ h = \frac{b-a}{2}\right)$:

$$I_2(f) = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right], \qquad E_2(f) = -\frac{h^5}{90} f^{(4)}(\xi)$$

- Regra dos três oitavos $\left((\text{Newton-Cotes fechada com } n = 3,\ h = \frac{b-a}{3}\right)$:

$$I_3(f) = \frac{b-a}{8} \left[f(a) + 3f(a+h) + 3f(b-h) + f(b) \right], \qquad E_3(f) = -\frac{3h^5}{80} f^{(4)}(\xi)$$

- Regra de Milne $\left((\text{Newton-Cotes fechada com } n = 4,\ h = \frac{b-a}{4}\right)$:

$$I_4(f) = \frac{b-a}{90} \left[7f(a) + 32f(a+h) + 12f\left(\frac{a+b}{2}\right) + 32f(b-h) + 7f(b) \right]$$

$$E_4(f) = -\frac{8h^7}{945} f^{(6)}(\xi), \qquad \xi \in (a,b)$$

Fórmulas de Newton-Cotes abertas de ordem n:

$$I(f) = \int_a^b f(x)\, dx \quad \approx \quad I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$$

$$I_n(x^k) = I(x^k), \qquad k = 0, 1, \ldots, n$$

$$x_{j,n} = a + (j+1)h, \quad j = 0, 1, \ldots, n, \qquad h = \frac{b-a}{n+2}$$

$$w_{j,n} = I(l_{j,n}) = \frac{h(-1)^{n-j}}{j!(n-j)!} \int_{-1}^{n+1} \prod_{i=0, i\neq j}^{n} (t-i)\, dt, \qquad w_{j,n} = w_{n-j,n}$$

- Regra do ponto médio $\left(\text{Newton-Cotes aberta com } n=0,\ h=\dfrac{b-a}{2}\right)$:

$$I_0(f) = (b-a)\, f\left(\frac{a+b}{2}\right), \qquad E_0(f) = \frac{h^3}{3} f''(\xi), \quad \xi \in (a,b)$$

Fórmulas de Newton-Cotes compostas:

$$x_j = a + j h_M, \quad j = 0,1,\ldots,M, \qquad h_M = \frac{b-a}{M}$$

- Regra dos trapézios composta :

$$I_1^{(M)}(f) = \frac{h_M}{2}\left[f(x_0) + f(x_M) + \sum_{j=1}^{M-1} f(x_j)\right]$$

$$E_1^{(M)}(f) = -\frac{(b-a)h_M^2}{12} f''(\xi), \qquad \xi \in (a,b)$$

- Regra de Simpson composta (M par):

$$I_2^{(M)}(f) = \frac{h_M}{3}\left[f(x_0) + f(x_M) + 4\sum_{j=1}^{M/2} f(x_{2j-1}) + 2\sum_{j=1}^{M/2-1} f(x_{2j})\right]$$

$$E_2^{(M)}(f) = -\frac{(b-a)h_M^4}{180} f^{(4)}(\xi), \qquad \xi \in (a,b)$$

- Regra do ponto médio composta (M par):

$$I_0^{(M)}(f) = 2h_M \sum_{j=1}^{M/2} f(x_{2j-1}), \qquad E_0^{(M)}(f) = \frac{(b-a)h_M^2}{6} f''(\xi), \quad \xi \in (a,b)$$

Fórmulas de Gauss:

$$I(f) = \int_a^b w(x) f(x)\, dx \quad \approx \quad I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$$

$$I_n(x^k) = I(x^k), \qquad k = 0,1,\ldots,2n+1$$

$x_{j,n}$, $j = 0,1,\ldots,n$: zeros do polinómio φ_{n+1} de grau $n+1$ pertencente ao sistema $\{\varphi_0, \varphi_1, \ldots\}$ de polinómios mónicos ortogonais com respeito ao produto interno $\langle f, g\rangle = I(fg)$.

$$w_{j,n} = I(l_{j,n}) = I(l_{j,n}^2) = -\frac{\langle \varphi_{n+1}, \varphi_{n+1}\rangle}{\varphi'_{n+1}(x_{j,n})\varphi_{n+2}(x_{j,n})}, \qquad j = 0,1,\ldots,n$$

$$E_n(f) = \frac{\langle \varphi_{n+1}, \varphi_{n+1}\rangle}{(2n+2)!} f^{(2n+2)}(\xi), \qquad \xi \in (a,b)$$

- Fórmulas de Gauss-Legendre ($[a,b] = [-1,1]$, $w(x) \equiv 1$):

$x_{j,n}$, $j = 0, 1, \ldots, n$: zeros do polinómio de Legendre P_{n+1}

$$w_{j,n} = -\frac{2}{(n+2)P'_{n+1}(x_{j,n})P_{n+2}(x_{j,n})}, \qquad j = 0, 1, \ldots, n$$

$$E_n(f) = \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3} f^{(2n+2)}(\xi), \qquad \xi \in (a, b)$$

- Fórmulas de Gauss-Chebyshev $\left([a, b] = [-1, 1],\ w(x) = 1/\sqrt{1 - x^2}\right)$:

$$x_{j,n} = \cos\left(\frac{2j+1}{2n+2}\pi\right), \qquad w_{j,n} = \frac{\pi}{n+1}, \qquad j = 0, 1, \ldots, n$$

$$E_n(f) = \frac{\pi 4^{-n}}{(2n+2)!} f^{(2n+2)}(\xi), \qquad \xi \in (a, b)$$

- Fórmulas de Gauss-Legendre compostas:

$$I(f) = \int_a^b f(x)\,dx \quad \approx \quad I_n^{(M)}(f) = \frac{h_M}{2} \sum_{j=0}^{n} w_{j,n} \sum_{m=1}^{M} f\left(x_{j,n}^{(m)}\right)$$

$$x_{j,n}^{(m)} = a + h_M(m-1) + \frac{h_M}{2}(x_{j,n} + 1), \qquad h_M = \frac{b-a}{M}$$

9. Resolução de Equações Diferenciais Ordinárias: Problemas de Valor Inicial

$$\begin{cases} Y'(x) = f(x, Y(x)), & x_0 \le x \le b \\ Y(x_0) = Y_0 \end{cases}$$

Métodos de passo simples:

$$y_{n+1} = y_n + h\,\varphi(x_n, y_n; h), \qquad n \ge 0$$

$$x_n = x_0 + nh, \quad n = 0, 1, \ldots, N, \qquad h = \frac{b - x_0}{N}$$

- Erro de discretização local:

$$\tau(x, y; h) = \frac{1}{h}[Y(x+h) - Y(x)] - \varphi(x, y; h), \qquad Y(x) = y$$

- Erro de discretização global:

$$|Y(x_n) - y_n(h)| \le \frac{e^{M(x_n - x_0)} - 1}{M} \max_{0 \le n \le N} \tau(x_n, Y(x_n); h), \qquad 0 \le n \le N$$

- Método de Euler:

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad \tau(x_n, y_n; h) = O(h)$$

- Métodos de Taylor de ordem p:

$$y_{n+1} = y_n + h\sum_{j=0}^{p-1} \frac{h^j}{(j+1)!}\left(D^j f\right)(x_n, y_n), \qquad \tau(x_n, y_n; h) = O(h^p)$$
$$(Df)(x,y) = \frac{\partial f}{\partial x}(x,y) + f(x,y)\frac{\partial f}{\partial y}(x,y)$$

- Métodos de Runge-Kutta de ordem 2 ($\tau(x_n, y_n; h) = O(h^2)$):

  - Método de Euler modificado ou do ponto médio:

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right)$$

  - Método de Heun:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_n + h, y_n + hf(x_n, y_n)\right)\right]$$

  - Método de Runge-Kutta de ordem 2 (propriamente dito):

$$y_{n+1} = y_n + \frac{h}{4}\left[f(x_n, y_n) + 3f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}f(x_n, y_n)\right)\right]$$

- Método de Runge-Kutta clássico de ordem 4 ($\tau(x_n, y_n; h) = O(h^4)$):

$$y_{n+1} = y_n + \frac{h}{6}\left[\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4\right],$$
$$\varphi_1 = f(x_n, y_n), \qquad \varphi_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\varphi_1\right)$$
$$\varphi_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\varphi_2\right), \qquad \varphi_4 = f(x_n + h, y_n + h\varphi_3)$$

Métodos multipasso lineares com $p+1$ passos, $p > 0$:

$$y_{n+1} = \sum_{k=0}^{p} a_k y_{n-k} + h\sum_{k=-1}^{p} b_k f(x_{n-k}, y_{n-k}), \qquad n \geq p$$
$$|a_p| + |b_p| \neq 0$$
$$x_n = x_0 + nh, \quad n = 0, 1, \ldots, N, \qquad h = \frac{b - x_0}{N}$$

- Erro de discretização local:

$$\tau(x, y; h) = \frac{1}{h}\left[Y(x+h) - \sum_{k=0}^{p} a_k Y(x - kh)\right] - \sum_{k=-1}^{p} b_k f(x - kh, Y(x - kh)), \qquad Y(x) = y$$

- Condições de consistência ($q \geq 1$):

$$\tau(x,y;h) = O(h^q) : \quad \begin{cases} \displaystyle\sum_{k=0}^{p} a_k = 1, \\ \displaystyle\sum_{k=0}^{p} (-k)^j a_k + j \sum_{k=-1}^{p} (-k)^{j-1} b_k = 1, \qquad j = 1, \ldots, q \end{cases}$$

- Condição da raiz:

$$\rho(r) = r^{p+1} - \sum_{k=0}^{p} a_k r^{p-k} = \prod_{j=0}^{p} (r - r_j)$$

$$\text{(i)} \ \ |r_j| \leq 1, \ \ j = 0, 1, \ldots, p; \qquad \text{(ii)} \ \ |r_j| = 1 \Rightarrow \rho'(r_j) \neq 0$$

- Métodos de Adams-Bashforth ($f_m := f(x_m, y_m)$):

$$y_{n+1} = y_n + h \sum_{k=0}^{p} b_k f_{n-k}, \qquad n \geq p$$

  - $p = 1, \ \ q = 2$:

$$y_{n+1} = y_n + \frac{h}{2} [3f_n - f_{n-1}]$$

  - $p = 2, \ \ q = 3$:

$$y_{n+1} = y_n + \frac{h}{12} [23f_n - 16f_{n-1} + 5f_{n-2}]$$

  - $p = 3, \ \ q = 4$:

$$y_{n+1} = y_n + \frac{h}{24} [55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$$

- Métodos de Adams-Moulton ($f_m := f(x_m, y_m)$):

$$y_{n+1} = y_n + h \sum_{k=-1}^{p} b_k f_{n-k}, \qquad n \geq p$$

  - $p = 0, \ \ q = 2$:

$$y_{n+1} = y_n + \frac{h}{2} [f_{n+1} + f_n]$$

  - $p = 1, \ \ q = 3$:

$$y_{n+1} = y_n + \frac{h}{12} [5f_{n+1} + 8f_n - f_{n-1}]$$

  - $p = 2, \ \ q = 4$:

$$y_{n+1} = y_n + \frac{h}{24} [9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]$$