INSTITUTO SUPERIOR TÉCNICO

Licenciatura em Matemática Aplicada e Computação

Ano Lectivo: 2010/2011 Semestre: 2º

MATEMÁTICA COMPUTACIONAL

Formulário – II

4. Resolução de Sistemas Lineares ($Ax = b,\ \ A \in \mathbb{M}^n(\mathbb{R}),\ \ b, x \in \mathbb{R}^n$)

Normas matriciais induzidas por normas vectoriais ($A = [a_{ij}] \in \mathbb{M}^n(\mathbb{R})$):

$$\|A\|_p = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|_p}{\|x\|_p}, \qquad p = 1, 2, \infty$$

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|, \qquad \|A\|_2 = \sqrt{r_\sigma(A^*A)}, \qquad \|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

(norma por colunas) (norma Euclidiana) (norma por linhas)

$$r_\sigma(A) = \max_{\lambda\in\sigma(A)} |\lambda|, \qquad \sigma(A): \text{ espectro de } A$$

Número de condição de uma matriz:

$$\mathrm{cond}_p(A) = \|A\|_p\,\|A^{-1}\|_p, \quad p = 1, 2, \infty, \qquad \mathrm{cond}_*(A) = r_\sigma(A)r_\sigma(A^{-1})$$

Condicionamento de sistemas lineares ($Ax = b,\ \tilde{A}\tilde{x} = \tilde{b}$):

$$\frac{\|x-\tilde{x}\|_p}{\|x\|_p} \le \frac{\mathrm{cond}_p(A)}{1 - \frac{\|A-\tilde{A}\|_p}{\|A\|_p}\mathrm{cond}_p(A)} \left(\frac{\|A-\tilde{A}\|_p}{\|A\|_p} + \frac{\|b-\tilde{b}\|_p}{\|b\|_p} \right)$$

$$\frac{\|A-\tilde{A}\|_p}{\|A\|_p}\mathrm{cond}_p(A) = \|A-\tilde{A}\|_p\,\|A^{-1}\|_p < 1, \qquad p = 1, 2, \infty$$

Métodos iterativos:

$$Mx^{(k+1)} = -Nx^{(k)} + b, \qquad k = 0, 1, \ldots$$

$$M + N = A = L + D + U$$

$$x^{(k+1)} = Cx^{(k)} + w, \qquad k = 0, 1, \ldots$$

$$C = -M^{-1}N = I - M^{-1}A, \qquad w = M^{-1}b$$

$$\|x - x^{(k+1)}\| \le c\|x - x^{(k)}\|, \qquad \|x - x^{(k)}\| \le c^k\|x - x^{(0)}\|$$

$$\|x - x^{(k)}\| \le \frac{1}{1-c}\|x^{(k+1)} - x^{(k)}\|, \qquad \|x - x^{(k+1)}\| \le \frac{c}{1-c}\|x^{(k+1)} - x^{(k)}\|$$

$$\|x - x^{(k)}\| \le \frac{c^k}{1-c}\|x^{(1)} - x^{(0)}\|, \qquad (c = \|C\| < 1)$$

- Método de Jacobi ($M = D$):

$$x^{(k+1)} = D^{-1}\left[b - (L+U)x^{(k)}\right], \qquad k = 0, 1 \ldots$$

- Método de Gauss-Seidel ($M = D + L$):

$$x^{(k+1)} = D^{-1}\left(b - Lx^{(k+1)} - Ux^{(k)}\right), \qquad k = 0, 1 \ldots$$

- Método de Jacobi modificado $\left(M = \dfrac{D}{\omega},\ \ \omega \in \mathbb{R} \setminus \{0\}\right)$:

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega D^{-1}\left[b - (L+U)x^{(k)}\right], \qquad k = 0, 1, \ldots$$

- Método de Gauss-Seidel modificado ou SOR $\left(M = \dfrac{D}{\omega} + L,\ \ \omega \in \mathbb{R} \setminus \{0\}\right)$:

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega D^{-1}\left(b - Lx^{(k+1)} - Ux^{(k)}\right), \qquad k = 0, 1, \ldots$$

5. Resolução de Sistemas Não-lineares ($f : \mathbb{R}^n \to \mathbb{R}^n$)

Método do ponto fixo ($f(z) = 0 \Leftrightarrow z = g(z)$):

$$\left(\ \|g(x) - g(y)\| \leq L\|x - y\|,\ \ \forall x, y \in D \subset \mathbb{R}^n, \quad L < 1; \quad g(D) \subset D\ \right)$$

$$\left(\ g \in C^1(D), \quad L = \sup_{x \in D} \|J_g(x)\|\ \right)$$

$$x^{(m+1)} = g(x^{(m)}), \qquad m = 0, 1, \ldots$$

$$\|z - x^{(m+1)}\| \leq L\|z - x^{(m)}\|, \qquad \|z - x^{(m)}\| \leq L^m\|z - x^{(0)}\|$$

$$\|z - x^{(m)}\| \leq \frac{1}{1-L}\|x^{(m+1)} - x^{(m)}\|, \qquad \|z - x^{(m+1)}\| \leq \frac{L}{1-L}\|x^{(m+1)} - x^{(m)}\|$$

$$\|z - x^{(m)}\| \leq \frac{L^m}{1-L}\|x^{(1)} - x^{(0)}\|$$

Método de Newton generalizado ($f(z) = 0,\ \ f \in C^2(D),\ \ \det[J_f(z)] \neq 0$):

$$\begin{cases} x^{(m+1)} = x^{(m)} + \Delta x^{(m)}, \\ J_f(x^{(m)})\Delta x^{(m)} = -f(x^{(m)}), \end{cases} \qquad m = 0, 1, \ldots$$

$$\|z - x^{(m+1)}\| \leq K\|z - x^{(m)}\|^2, \qquad \|z - x^{(m)}\| \leq \frac{1}{K}\left(K\|z - x^{(0)}\|\right)^{2^m}$$

$$K = \frac{M_2}{2M_1} \begin{cases} \dfrac{1}{M_1} = \sup\limits_{x \in D} \|[J_f(x)]^{-1}\|, \\ M_2 = \max\limits_{1 \leq i \leq n} \sup\limits_{x \in D} \|H_{f_i}(x)\|, \quad H_{f_i} \in L^n, \quad (H_{f_i})_{jk} = \dfrac{\partial^2 f_i}{\partial x_j \partial x_k} \end{cases}$$

6. Interpolação Polinomial

Fórmula interpoladora de Lagrange:

$$p_n(x) = \sum_{j=0}^{n} f_j \, l_j(x), \qquad l_j(x) = \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}$$

Fórmula interpoladora de Newton:

$$p_n(x) = f[x_0] + \sum_{j=1}^{n} f[x_0, x_1, \ldots, x_j] \, W_j(x)$$

$$W_j(x) = \prod_{i=0}^{j-1} (x - x_i), \qquad j = 1, 2, \ldots, n$$

$$f[x_0] = f(x_0), \qquad f[x_0, x_1, \ldots, x_j] = \sum_{l=0}^{j} \frac{f(x_l)}{\prod_{i=0, i\neq l}^{j} (x_l - x_i)}, \quad j = 1, 2, \ldots, n$$

$$f[x_0, x_1, \ldots, x_j] = \frac{f[x_1, x_2, \ldots, x_j] - f[x_0, x_1, \ldots, x_{j-1}]}{x_j - x_0}, \qquad j = 1, 2, \ldots, n$$

Fórmula do erro:

$$e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} W_{n+1}(x) = f[x_0, x_1, \ldots, x_n, x] \, W_{n+1}(x)$$

$$W_{n+1}(x) = \prod_{i=0}^{n} (x - x_i), \qquad \xi \in]x_0; x_1; \ldots; x_n; x[$$

Relação entre as diferenças divididas e as derivadas de uma função:

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}, \qquad \xi \in]x_0; x_1; \ldots; x_n[$$

7. Aproximação Mínimos Quadrados

Melhor aproximação mínimos quadrados ϕ^* de $f \in E$ em $F \subset E$, F subespaço de dimensão n gerado por $\{\varphi_0, \ldots, \varphi_n\}$, E espaço pré-Hilbertiano:

$$\|f - \phi^*\|_2 = \min_{\phi \in F} \|f - \phi\|_2 \quad \Leftrightarrow \quad \langle f - \phi^*, \phi \rangle = 0, \quad \forall \phi \in F$$

$$\phi^* = \sum_{k=0}^{n} a_k^* \varphi_k, \qquad \sum_{k=0}^{n} \langle \varphi_j, \varphi_k \rangle a_k^* = \langle f, \varphi_j \rangle, \quad j = 0, 1, \ldots, n$$

$$\phi^* = \sum_{k=0}^{n} a_k^* \varphi_k, \quad a_k^* = \frac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle}, \qquad \text{se } \{\varphi_0, \ldots, \varphi_n\} \text{ é um sistema ortogonal}$$

<u>Polinómios ortogonais com respeito ao produto interno</u>

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx, \quad (f, g \in C([a,b]), \ \ w \in C(]a,b[), \ \ w(x) \geq 0)$$

- Fórmula de recorrência:

$$\begin{cases} \varphi_{n+1}(x) = (x - B_{n+1})\varphi_n(x) - C_{n+1}\varphi_{n-1}(x), & n = 1, 2, \ldots \\ \varphi_0(x) = 1, \qquad \varphi_1(x) = x - B_1 & \end{cases}$$

$$B_{n+1} = \frac{\langle x\varphi_n, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}, \quad n = 0, 1, \ldots, \qquad C_{n+1} = \frac{\langle x\varphi_n, \varphi_{n-1} \rangle}{\langle \varphi_{n-1}, \varphi_{n-1} \rangle}, \quad n = 1, 2, \ldots$$

- Polinómios de Legendre, P_n $\quad (x \in [a,b] = [-1,1], \ \ w(x) = 1)$:

$$\begin{cases} P_{n+1}(x) = \dfrac{2n+1}{n+1}\, x P_n(x) - \dfrac{n}{n+1}\, P_{n-1}(x), & n = 1, 2, \ldots \\ P_0(x) = 1, \qquad P_1(x) = x & \end{cases}$$

$$\langle P_n, P_m \rangle = 0, \quad \forall n \neq m, \qquad \langle P_n, P_n \rangle = \frac{2}{2n+1}, \quad n = 0, 1, \ldots$$

$$A_n = \lim_{x \to \infty} x^{-n} P_n(x) = \frac{(2n)!}{2^n (n!)^2}, \qquad n = 1, 2, \ldots$$

- Polinómios de Chebyshev, T_n $\quad \big(x \in [a,b] = [-1,1], \ \ w(x) = 1/\sqrt{1-x^2}\big)$:

$$\begin{cases} T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), & n = 1, 2, \ldots \\ T_0(x) = 1, \qquad T_1(x) = x & \end{cases}$$

$$\langle T_n, T_m \rangle = 0, \quad \forall n \neq m, \qquad \langle T_0, T_0 \rangle = \pi, \qquad \langle T_n, T_n \rangle = \frac{\pi}{2}, \quad n = 1, 2, \ldots$$

$$A_n = \lim_{x \to \infty} x^{-n} T_n(x) = 2^{n-1}, \qquad n = 1, 2, \ldots$$

$$T_n(x) = \cos(n \arccos x), \qquad n = 0, 1, \ldots$$

$$T_n(x_i) = 0, \qquad x_i = -\cos\frac{(2i+1)\pi}{2n}, \qquad i = 0, \ldots, n-1, \qquad n = 1, 2, \ldots$$

8. Integração Numérica

<u>Fórmulas de Newton-Cotes fechadas de ordem n</u> $(f \in C([a,b]))$:

$$I(f) = \int_a^b f(x)\, dx \quad \approx \quad I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$$

$$I_n(x^k) = I(x^k), \qquad k = 0, 1, \ldots, n$$

$$x_{j,n} = a + jh, \quad j = 0, 1, \ldots, n, \qquad h = \frac{b-a}{n}$$

$$w_{j,n} = I(l_{j,n}) = \frac{h(-1)^{n-j}}{j!(n-j)!} \int_0^n \prod_{i=0, i\neq j}^{n} (t-i)\, dt, \qquad w_{j,n} = w_{n-j,n}$$

$$E_n(f) = I(f) - I_n(f) = C_n h^{n+1+\nu_n} f^{(n+\nu_n)}(\xi)$$

$$C_n = \frac{1}{(n+\nu_n)!} \int_0^n t^{\nu_n - 1} \prod_{i=0}^{n} (t-i)\, dt, \qquad \nu_n = 1 + \frac{1}{2}\left[1 + (-1)^n\right], \qquad \xi \in]a, b[$$

- $n = 1,\ h = b - a$ (Regra dos trapézios):

$$I_1(f) = \frac{b-a}{2}\,[f(a) + f(b)], \qquad E_1(f) = -\frac{h^3}{12} f''(\xi), \quad \xi \in]a, b[$$

- $n = 2,\ h = \dfrac{b-a}{2}$ (Regra de Simpson):

$$I_2(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right], \qquad E_2(f) = -\frac{h^5}{90} f^{(4)}(\xi)$$

- $n = 3,\ h = \dfrac{b-a}{3}$ (Regra dos três oitavos):

$$I_3(f) = \frac{b-a}{8}\left[f(a) + 3f(a+h) + 3f(b-h) + f(b)\right], \quad E_3(f) = -\frac{3h^5}{80} f^{(4)}(\xi)$$

- $n = 4,\ h = \dfrac{b-a}{4}$ (Regra de Milne):

$$I_4(f) = \frac{b-a}{90}\left[7f(a) + 32f(a+h) + 12f\left(\frac{a+b}{2}\right) + 32f(b-h) + 7f(b)\right]$$

$$E_4(f) = -\frac{8h^7}{945} f^{(6)}(\xi), \qquad \xi \in]a, b[$$

<u>Fórmulas de Newton-Cotes abertas de ordem n</u>:

$$I(f) = \int_a^b f(x)\, dx \quad \approx \quad I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$$

$$I_n(x^k) = I(x^k), \qquad k = 0, 1, \ldots, n$$

$$x_{j,n} = a + (j+1)h, \quad j = 0, 1, \ldots, n, \qquad h = \frac{b-a}{n+2}$$

$$w_{j,n} = I(l_{j,n}) = \frac{h(-1)^{n-j}}{j!(n-j)!} \int_{-1}^{n+1} \prod_{i=0, i\neq j}^{n} (t-i)\, dt, \qquad w_{j,n} = w_{n-j,n}$$

$$E_n(f) = I(f) - I_n(f) = C_n h^{n+1+\nu_n} f^{(n+\nu_n)}(\xi)$$

$$C_n = \frac{1}{(n+\nu_n)!} \int_{-1}^{n+1} t^{\nu_n - 1} \prod_{i=0}^{n} (t-i)\, dt, \qquad \nu_n = 1 + \frac{1}{2}\left[1 + (-1)^n\right], \qquad \xi \in]a,b[$$

- $n = 0,\ h = \dfrac{b-a}{2}$ (Regra do ponto médio):

$$I_0(f) = (b-a)\, f\left(\frac{a+b}{2}\right), \qquad E_0(f) = \frac{h^3}{3} f''(\xi), \quad \xi \in]a,b[$$

- $n = 1,\ h = \dfrac{b-a}{3}$:

$$I_1(f) = \frac{b-a}{2}\left[f(a+h) + f(b-h)\right], \qquad E_1(f) = \frac{3h^3}{4} f''(\xi), \quad \xi \in]a,b[$$

- $n = 2,\ h = \dfrac{b-a}{4}$:

$$I_2(f) = \frac{b-a}{3}\left[2f(a+h) - f\left(\frac{a+b}{2}\right) + 2f(b-h)\right]$$

$$E_2(f) = \frac{14h^5}{45} f^{(4)}(\xi), \quad \xi \in]a,b[$$

Fórmulas de Newton-Cotes fechadas compostas:

$$x_j = a + jh_M, \quad j = 0, 1, \ldots, M, \qquad h_M = \frac{b-a}{M}, \qquad f_j := f(x_j)$$

- $n = 1$:

$$I_1^{(M)}(f) = \frac{h_M}{2}\left[f_0 + f_M + 2\sum_{j=1}^{M-1} f_j\right]$$

$$E_1^{(M)}(f) = -\frac{b-a}{12}\, h_M^2 f''(\xi), \qquad \xi \in]a,b[$$

- $n = 2$ (M par):

$$I_2^{(M)}(f) = \frac{h_M}{3}\left[f_0 + f_M + 4\sum_{j=1}^{M/2} f_{2j-1} + 2\sum_{j=1}^{M/2-1} f_{2j}\right]$$

$$E_2^{(M)}(f) = -\frac{b-a}{180}\, h_M^4 f^{(4)}(\xi), \qquad \xi \in]a,b[$$

- $n = 3$ (M múltiplo de 3):

$$I_3^{(M)}(f) = \frac{3h_M}{8}\left[f_0 + f_M + 2\sum_{j=1}^{M/3-1} f_{3j} + 3\sum_{j=1}^{M/3} (f_{3j-1} + f_{3j-2})\right]$$

$$E_3^{(M)}(f) = -\frac{b-a}{80}\, h_M^4 f^{(4)}(\xi), \qquad \xi \in]a,b[$$

- $n=4$ (M múltiplo de 4):

$$I_4^{(M)}(f) = \frac{4h_M}{90}\left[7\,(f_0+f_M) + 14\sum_{j=1}^{M/4-1} f_{4j} + 32\sum_{j=1}^{M/4}(f_{4j-1}+f_{4j-3}) + 12\sum_{j=1}^{M/4} f_{4j-2}\right]$$

$$E_4^{(M)}(f) = -\frac{2(b-a)}{945}\,h_M^6 f^{(6)}(\xi), \qquad \xi \in]a,b[$$

Fórmulas de Newton-Cotes abertas compostas:

$$x_j = a + jh_M, \quad j=0,1,\ldots,M, \qquad h_M = \frac{b-a}{M}, \qquad f_j := f(x_j)$$

- Regra do ponto médio composta (M par):

$$I_0^{(M)}(f) = 2h_M\sum_{j=1}^{M/2} f_{2j-1}, \qquad E_0^{(M)}(f) = \frac{(b-a)h_M^2}{6} f''(\xi), \quad \xi \in]a,b[$$

Fórmulas de Gauss:

$$I(f) = \int_a^b w(x)f(x)\,dx \quad \approx \quad I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$$

$$I_n(x^k) = I(x^k), \qquad k=0,1,\ldots,2n+1$$

$x_{j,n}$, $j=0,1,\ldots,n$: zeros do polinómio Φ_{n+1} de grau $n+1$ pertencente ao sistema $\{\Phi_0,\Phi_1,\ldots\}$ de polinómios mónicos ortogonais com respeito ao produto interno $\langle f,g\rangle = I(fg)$.

$$w_{j,n} = I(l_{j,n}) = I(l_{j,n}^2) = -\frac{\langle \Phi_{n+1},\Phi_{n+1}\rangle}{\Phi'_{n+1}(x_{j,n})\Phi_{n+2}(x_{j,n})}, \qquad j=0,1,\ldots,n$$

$$E_n(f) = \frac{\langle \Phi_{n+1},\Phi_{n+1}\rangle}{(2n+2)!} f^{(2n+2)}(\xi), \qquad \xi \in]a,b[$$

- Fórmulas de Gauss-Legendre ($[a,b]=[-1,1]$, $w(x)\equiv 1$):

$x_{j,n}$, $j=0,1,\ldots,n$: zeros do polinómio de Legendre P_{n+1}

$$w_{j,n} = -\frac{2}{(n+2)P'_{n+1}(x_{j,n})P_{n+2}(x_{j,n})}, \qquad j=0,1,\ldots,n$$

$$E_n(f) = \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3}\, f^{(2n+2)}(\xi), \qquad \xi \in]a,b[$$

- ∘• $I_0(f) = 2f(0)$
- ∘• $I_1(f) = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$
- ∘• $I_2(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}\, f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$
- ∘• $I_3(f) = w_{0,3} f(x_{0,3}) + w_{1,3} f(x_{1,3}) + w_{2,3} f(x_{2,3}) + w_{3,3} f(x_{3,3})$

$$x_{0,3} = -\sqrt{\frac{1}{7}\left(3 + 2\sqrt{\frac{6}{5}}\right)} = -x_{3,3}, \qquad x_{1,3} = -\sqrt{\frac{1}{7}\left(3 - 2\sqrt{\frac{6}{5}}\right)} = -x_{2,3}$$

$$w_{0,3} = \frac{1}{6}\left(3 - \sqrt{\frac{5}{6}}\right) = w_{3,3}, \qquad w_{1,3} = \frac{1}{6}\left(3 + \sqrt{\frac{5}{6}}\right) = w_{2,3}$$

- Fórmulas de Gauss-Chebyshev $\big([a,b] = [-1,1],\ \ w(x) = 1/\sqrt{1-x^2}\big)$:

$$x_{j,n} = \cos\left(\frac{2j+1}{2n+2}\,\pi\right), \qquad w_{j,n} = \frac{\pi}{n+1}, \qquad j = 0, 1, \ldots, n$$

$$E_n(f) = \frac{\pi}{2^{2n+1}(2n+2)!}\, f^{(2n+2)}(\xi), \qquad \xi \in]a,b[$$

- Fórmulas de Gauss-Legendre compostas:

$$I(f) = \int_a^b f(x)\,dx \quad \approx \quad I_n^{(M)}(f) = \frac{h_M}{2} \sum_{j=0}^{n} w_{j,n} \sum_{m=1}^{M} f\left(x_{j,n}^{(m)}\right)$$

$$x_{j,n}^{(m)} = a + h_M(m-1) + \frac{h_M}{2}\left(x_{j,n} + 1\right), \qquad h_M = \frac{b-a}{M}$$

($x_{j,n}$ e $w_{j,n}$ são os nós e os pesos das fórmulas de Gauss-Legendre)

$$E_n^{(M)}(f) = \frac{b-a}{2}\left(\frac{h_M}{2}\right)^{2n+2} \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3}\, f^{(2n+2)}(\xi), \quad \xi \in]a,b[$$

10. Resolução de Equações Diferenciais Ordinárias: Problemas de Valor Inicial

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = Y_0 \end{cases}$$

$$f : D \subset \mathbb{R}^{1+M} \to \mathbb{R}^M, \qquad D \text{ aberto}, \qquad M \in \mathbb{Z}^+$$

$$f \in C(D), \qquad \|f(x,y) - f(x,z)\| \le L\|y - z\|, \quad \forall (x,y),(x,z) \in D$$

$$(x_0, Y_0) \in D$$

Métodos de passo simples explícitos:

$$y_{n+1} = y_n + h\,\varphi(x_n, y_n; h)$$

$$x_n = x_0 + nh, \qquad n = 0, 1, \ldots, N, \qquad N \in \mathbb{Z}^+$$

$$\varphi : D\times]0,\infty[\to \mathbb{R}^M, \qquad \varphi \in C(D\times]0,\infty[)$$

$$\|\varphi(x,y;h) - \varphi(x,z;h)\| \le K\|y - z\|, \quad \forall (x,y;h),(x,z;h) \in D\times]0,\infty[$$

- Erro de discretização local:

$$\tau(x,y;h) = \frac{1}{h}\left[Z(x+h) - Z(x)\right] - \varphi(x,y;h)$$

$$Z'(t) = f(t, Z(t)), \qquad Z(x) = y$$

- Erro de discretização global:

$$\|Y(x_n) - y_n(h)\| \le e^{K(x_n - x_0)}\|Y(x_0) - y_0(h)\| + \frac{\tau(h)}{K}\left[e^{K(x_n-x_0)} - 1\right]$$

$$\tau(h) = \max_{0\le n\le N} \|\tau(x_n, Y(x_n); h)\|$$

- Método de Euler:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$\tau(x,y;h) = \frac{h}{2}(d_f f)(x,y) + \mathcal{O}(h^2)$$

$$(d_f g)(x,y) = \left(\frac{\partial}{\partial x} + f(x,y)\cdot\nabla_y\right) g(x,y), \qquad \forall g \in C^1(D)$$

- Métodos de Taylor de ordem q:

$$y_{n+1} = y_n + h\sum_{j=0}^{q-1} \frac{h^j}{(j+1)!}\left(d_f^j f\right)(x_n, y_n)$$

$$\tau(x,y;h) = \frac{h^q}{(q+1)!}\,(d_f^q f)(x,y) + \mathcal{O}(h^{q+1})$$

- Métodos de Runge-Kutta de ordem 2:

$$\varphi(x,y;h) = (1-\gamma)f(x,y) + \gamma f\left(x + \frac{h}{2\gamma}, y + \frac{h}{2\gamma}f(x,y)\right)$$

$$\tau(x,y;h) = \frac{h^2}{6}\left[d_f^2 f(x,y) - 3\,\frac{\partial^2\varphi}{\partial h^2}(x,y;0)\right] + \mathcal{O}(h^3)$$

  - ∘• Método de Euler modificado ou do ponto médio ($\gamma = 1$):

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\,f(x_n, y_n)\right)$$

○• Método de Runge-Kutta clássico de ordem 2 $\left(\gamma = \dfrac{3}{4}\right)$:

$$y_{n+1} = y_n + \frac{h}{4}\left[f(x_n, y_n) + 3f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}\, f(x_n, y_n)\right)\right]$$

○• Método de Heun $\left(\gamma = \dfrac{1}{2}\right)$:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_n + h, y_n + hf(x_n, y_n)\right)\right]$$

• Método de Runge-Kutta clássico de ordem 4:

$$y_{n+1} = y_n + \frac{h}{6}\left[\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4\right]$$

$$\varphi_1 = f(x_n, y_n), \qquad \varphi_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\,\varphi_1\right)$$

$$\varphi_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\,\varphi_2\right), \qquad \varphi_4 = f(x_n + h, y_n + h\varphi_3)$$

$$\tau(x, y; h) = \frac{h^4}{120}\left[d_f^4 f(x, y) - 5\,\frac{\partial^4 \varphi}{\partial h^4}(x, y; 0)\right] + \mathcal{O}(h^5)$$

<u>Métodos de passo simples implícitos:</u>

$$y_{n+1} = y_n + h\left[b_{-1} f\left(x_{n+1}, y_{n+1}\right) + b_0 f(x_n, y_n)\right], \qquad n \geq 0$$

$$b_{-1} + b_0 = 1$$

• Erro de discretização local:

$$\tau(x, y; h) = \frac{1}{h}\,\left[Z(x + h) - Z(x)\right] - b_{-1} f(x + h, Z(x + h)) - b_0 f(x, Z(x))$$

$$Z'(t) = f(t, Z(t)), \qquad Z(x) = y$$

• Erro de discretização global:

$$\|Y(x_n) - y_n(h)\| \leq e^{\tilde{K}(x_n - x_0)}\|Y(x_0) - y_0(h)\| + \frac{\tau(h)}{(|b_{-1}| + |b_0|)L}\left[e^{\tilde{K}(x_n - x_0)} - 1\right]$$

$$\tilde{K} = \frac{(|b_{-1}| + |b_0|)L}{1 - h|b_{-1}|L}, \qquad \tau(h) = \max_{0 \leq n \leq N} \|\tau(x_n, Y(x_n); h)\|$$

• Método de Euler regressivo ($b_{-1} = 1, b_0 = 0$):

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

$$\tau(x, y; h) = -\frac{h}{2}\,(d_f f)(x, y) + \mathcal{O}(h^2)$$

• Método trapezoidal ($b_{-1} = b_0 = \frac{1}{2}$):

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_{n+1}, y_{n+1}) + f(x_n, y_n)\right]$$

$$\tau(x, y; h) = -\frac{h^2}{12}\,(d_f^2 f)(x, y) + \mathcal{O}(h^3)$$