INSTITUTO SUPERIOR TÉCNICO

Licenciatura em Matemática Aplicada e Computação

Ano Lectivo: 2010/2011 Semestre: 2º

MATEMÁTICA COMPUTACIONAL

Formulário - I

1. Representação de Números e Teoria de Erros

Erro, erro absoluto, erro relativo ($\tilde{x} \approx x$):

$$\text{(i)} \quad x \in \mathbb{R}: \quad e_{\tilde{x}} = x - \tilde{x}, \quad |e_{\tilde{x}}|, \quad \delta_{\tilde{x}} = \frac{e_{\tilde{x}}}{x}, \quad |\delta_{\tilde{x}}| \quad (x \neq 0)$$

$$\text{(ii)} \quad x \in \mathbb{R}^n: \quad e_{\tilde{x}} = x - \tilde{x}, \quad \|e_{\tilde{x}}\|, \quad \delta_{\tilde{x}} = \frac{e_{\tilde{x}}}{\|x\|}, \quad \|\delta_{\tilde{x}}\| \quad (x \neq 0)$$

Representação de números reais (notação científica):

$$x = \sigma\, m\, \beta^t \ \in \mathbb{R} \setminus \{0\}$$

$$\text{(base) } \beta \in \mathbb{N} \setminus \{1\}, \qquad \text{(sinal) } \sigma \in \{+,-\}, \qquad \text{(expoente) } t \in \mathbb{Z}$$

$$\text{(mantissa) } m = (0.a_1 a_2 \ldots)_\beta \in \ [\beta^{-1}, 1[, \qquad a_i \in \{0, 1, \ldots, \beta - 1\}, \qquad a_1 \neq 0$$

Sistema de ponto flutuante:

$$\text{FP}(\beta, n, t^-, t^+) = \{x \in \mathbb{Q}: \ x = \sigma\, m\, \beta^t\} \cup \{0\}$$

$$\beta \in \mathbb{N} \setminus \{1\}, \qquad \sigma \in \{+,-\}, \qquad t^- \leq t \leq t^+, \quad t, t^-, t^+ \in \mathbb{Z}$$

$$m = (0.a_1 a_2 \ldots a_n)_\beta \ \in [\beta^{-1}, 1 - \beta^{-n}], \qquad a_i \in \{0, 1, \ldots, \beta - 1\}, \qquad a_1 \neq 0$$

Arredondamentos:

$$x = \sigma(0.a_1 a_2 \ldots a_n a_{n+1} \ldots)_\beta \times \beta^t \ \in \mathbb{R}, \qquad \text{fl}(x) \in \text{FP}(\beta, n, t^-, t^+)$$

(i) arredondamento por corte:

$$\text{fl}_c(x) = \sigma(0.a_1 a_2 \ldots a_n)_\beta \times \beta^t$$

(ii) arredondamento simétrico (β par):

$$\text{fl}_s(x) = \begin{cases} \sigma(0.a_1 a_2 \ldots a_n)_\beta \times \beta^t, & 0 \leq a_{n+1} < \dfrac{\beta}{2} \\ \sigma\left[(0.a_1 a_2 \ldots a_n)_\beta + \beta^{-n}\right] \times \beta^t, & \dfrac{\beta}{2} \leq a_{n+1} < \beta \end{cases}$$

Erros de arredondamento ($x = \sigma m \beta^t \in \mathbb{R},\ \ \tilde{x} = \mathrm{fl}(x) \in \mathrm{FP}(\beta, n, t^-, t^+)$):

(i) arredondamento por corte:

$$|e_{\tilde{x}}| \leq \beta^{t-n}, \qquad |\delta_{\tilde{x}}| \leq \beta^{1-n} =: u_c$$

(ii) arredondamento simétrico:

$$|e_{\tilde{x}}| \leq \frac{1}{2}\,\beta^{t-n}, \qquad |\delta_{\tilde{x}}| \leq \frac{1}{2}\,\beta^{1-n} =: u_s$$

(u_c, u_s: unidade de arredondamento do sistema $\mathrm{FP}(\beta, n, t^-, t^+)$)

Operações aritméticas num sistema de ponto flutuante ($x, y \in \mathbb{R};\ \ \circ = +, -, \times, \div$):

$$x \boxed{\circ}\, y = \mathrm{fl}(\mathrm{fl}(x) \circ \mathrm{fl}(y))$$

Algarismo significativo:

$$x = \sigma m 10^t \in \mathbb{R}, \qquad \tilde{x} = \sigma(0.a_1 a_2 \ldots a_n)_{10} \times 10^t \in \mathrm{FP}(10, n, t^-, t^+),$$

$$a_i \quad \text{é algarismo significativo de } \tilde{x} \text{ se} \quad |e_{\tilde{x}}| \leq \frac{1}{2}\,10^{t-i}$$

Propagação de erros $\Big(x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ \ \phi : \mathbb{R}^n \to \mathbb{R},\ \ \tilde{x} \approx x,\ \ \tilde{\phi} = \mathrm{fl} \circ \phi\Big)$:

$$e_{\phi(\tilde{x})} = \phi(x) - \phi(\tilde{x}) \ \approx\ e^L_{\phi(\tilde{x})} = \sum_{k=1}^{n} \frac{\partial \phi}{\partial x_k}(x)\, e_{\tilde{x}_k}$$

$$\delta_{\phi(\tilde{x})} = \frac{e_{\phi(\tilde{x})}}{\phi(x)} \ \approx\ \delta^L_{\phi(\tilde{x})} = \sum_{k=1}^{n} p_{\phi,x_k}(x)\delta_{\tilde{x}_k}, \qquad p_{\phi,x_k}(x) = \frac{x_k \frac{\partial \phi}{\partial x_k}(x)}{\phi(x)}$$

$$\delta_{\tilde{\phi}(\tilde{x})} = \frac{\phi(x) - \tilde{\phi}(\tilde{x})}{\phi(x)} \ \approx\ \delta^L_{\phi(\tilde{x})} + \delta_{\mathrm{arr}}, \qquad \delta^L_{\phi(\tilde{x})} = \sum_{k=1}^{n} p_{\phi,x_k}\delta_{\tilde{x}_k}, \qquad \delta_{\mathrm{arr}} = \sum_{k=1}^{m} q_k \delta_{\mathrm{arr}_k}$$

2. Métodos Iterativos

Normas vectoriais ($x = (x_1, \ldots, x_n) \in \mathbb{R}^n$):

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \qquad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

(norma da soma) (norma Euclidiana) (norma do máximo)

Coeficiente assimptótico de convergência de ordem $r \geq 1$ de sucessão $\{x_m\},\ x_m \to x$:

$$K^{[r]}_\infty = \lim_{m \to \infty} \frac{\|x - x_{m+1}\|}{\|x - x_m\|^r}$$

3. Resolução de Equações Não-lineares ($f : \mathbb{R} \to \mathbb{R}$)

Método da bissecção ($f(z) = 0,\ \ f \in C([a,b]),\ \ f(a)f(b) < 0$):

$$x_{m+1} = x_m + \frac{b-a}{2^{m+2}}\, \mathrm{sgn}[f(a)f(x_m)], \qquad m = 0, 1, \ldots, \qquad x_0 = \frac{a+b}{2}$$

$$|z - x_m| \leq \frac{b-a}{2^{m+1}}, \qquad |z - x_{m+1}| \leq |x_{m+1} - x_m|$$

Método do ponto fixo ($f(z) = 0 \Leftrightarrow z = g(z)$):

$$\big(\ |g(x) - g(y)| \leq L|x - y|, \quad \forall x, y \in I \subset \mathbb{R}, \quad L < 1; \quad g(I) \subset I\ \big)$$

$$x_{m+1} = g(x_m), \qquad m = 0, 1, \ldots$$

$$|z - x_{m+1}| \leq L|z - x_m|, \qquad |z - x_m| \leq L^m |z - x_0|$$

$$|z - x_m| \leq \frac{1}{1-L}|x_{m+1} - x_m|, \qquad |z - x_{m+1}| \leq \frac{L}{1-L}|x_{m+1} - x_m|$$

$$|z - x_m| \leq \frac{L^m}{1-L}|x_1 - x_0|$$

∘ • $g'(z) \neq 0, \quad g \in C^1(I), \quad L = \max_{x \in I} |g'(x)| < 1$:

$$z - x_{m+1} = g'(\xi_m)(z - x_m), \qquad \xi_m \in]z; x_m[$$

$$|z - x_{m+1}| \leq L|z - x_m|, \qquad |z - x_m| \leq L^m |z - x_0|$$

$$\lim_{m \to \infty} \frac{z - x_{m+1}}{z - x_m} = g'(z), \qquad K_\infty^{[1]} = |g'(z)|$$

∘ • $g^{(r)}(z) = 0,\ r = 1, \ldots, p-1,\ \ g^{(p)}(z) \neq 0, \quad p = 2, 3, \ldots, \quad g \in C^p(I)$

$$z - x_{m+1} = \frac{1}{p!}(-1)^{p+1} g^{(p)}(\xi_m)(z - x_m)^p, \qquad \xi_m \in]z; x_m[$$

$$|z - x_{m+1}| \leq K_p |x - x_m|^p, \qquad |z - x_m| \leq K_p^{\frac{1}{1-p}} \left(K_p^{\frac{1}{p-1}} |z - x_0| \right)^{p^m}$$

$$K_p = \frac{1}{p!} \max_{x \in I} \left| g^{(p)}(x) \right|$$

$$\lim_{m \to \infty} \frac{z - x_{m+1}}{(z - x_m)^p} = \frac{(-1)^{p+1}}{p!}\, g^{(p)}(z), \qquad K_\infty^{[p]} = \frac{1}{p!} \left| g^{(p)}(z) \right|$$

Método de Newton ($f(z) = 0,\ \ f'(z) \neq 0,\ \ f \in C^2(I)$):

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)}, \qquad m = 0, 1, \ldots$$

$$z - x_{m+1} = -\frac{f''(\xi_m)}{2f'(x_m)}\, (z - x_m)^2\,, \qquad \xi_m \in]z, x_m[$$

$$|z - x_{m+1}| \leq K|z - x_m|^2, \qquad |z - x_m| \leq \frac{1}{K}\,(K|z - x_0|)^{2^m}$$

$$K = \frac{\max_{x\in I}|f''(x)|}{2\min_{x\in I}|f'(x)|}$$

$$\lim_{m\to\infty}\frac{z - x_{m+1}}{(z - x_m)^2} = -\frac{f''(z)}{2f'(z)}, \qquad K_\infty^{[2]} = \left|\frac{f''(z)}{2f'(z)}\right|$$

∘ • $f^{(r)}(z) = 0,\ r = 2, \ldots, p-1, \quad f^{(p)}(z) \neq 0, \quad p = 3, 4, \ldots, \quad f \in C^{p+1}(I)$

$$\lim_{m\to\infty}\frac{z - x_{m+1}}{(z - x_m)^p} = \frac{(-1)^{p+1}(p-1)}{p!}\,\frac{f^{(p)}(z)}{f'(z)}, \qquad K_\infty^{[p]} = \frac{p-1}{p!}\left|\frac{f^{(p)}(z)}{f'(z)}\right|$$

<u>Método da secante</u> ($f(z) = 0, \quad f'(z) \neq 0, \quad f \in C^2(I)$):

$$x_{m+1} = x_m - f(x_m)\,\frac{x_m - x_{m-1}}{f(x_m) - f(x_{m-1})}, \qquad m = 1, 2, \ldots$$

$$z - x_{m+1} = -\frac{f''(\eta_m)}{2f'(\xi_m)}\,(z - x_m)\,(z - x_{m-1})\,, \qquad \xi_m, \eta_m \in]x_{m-1}; z; x_m[$$

$$|z - x_{m+1}| \leq K\,|z - x_m|\,|z - x_{m-1}|\,, \qquad |z - x_m| \leq \frac{1}{K}\,\delta^{q_m}$$

$K = \dfrac{\max_{x\in I}|f''(x)|}{2\min_{x\in I}|f'(x)|}, \quad \delta = \max\{K|z - x_0|, K|z - x_1|\}, \quad q_m$: sucessão de Fibonnaci

$$\lim_{m\to\infty}\frac{|z - x_{m+1}|}{|z - x_m|^r} = \left|\frac{f''(z)}{2f'(z)}\right|^{r-1} =: K_\infty^{[r]}, \qquad r = \frac{\sqrt{5}+1}{2}$$