INSTITUTO SUPERIOR TÉCNICO
Licenciatura em Matemática Aplicada e Computação
Ano Lectivo: 2010/2011   Semestre: 2º

MATEMÁTICA COMPUTACIONAL

Resolução do Exame de 16 de Junho de 2011 – Parte II

[1]
(a)[15]

$$\frac{\|x_{\alpha,\beta}-x_{0,\beta}\|_\infty}{\|x_{0,\beta}\|_\infty} \le \frac{\mathrm{cond}_\infty(A_{0,\beta})}{1-\|\delta_{A_{\alpha,\beta}}\|_\infty \,\mathrm{cond}_\infty(A_{0,\beta})}\left(\left|\delta_{A_{\alpha,\beta}}\right|_\infty + \|\delta_{b_{\alpha,\beta}}\|_\infty\right)$$

$$\delta_{A_{\alpha,\beta}} = \frac{A_{\alpha,\beta}-A_{0,\beta}}{\|A_{0,\beta}\|_\infty}, \qquad \delta_{b_{\alpha,\beta}} = \frac{b_{\alpha,\beta}-b_{0,\beta}}{\|b_{0,\beta}\|_\infty}$$

$$\|\delta_{A_{\alpha,\beta}}\|_\infty \,\mathrm{cond}_\infty(A_{0,\beta}) < 1$$

$$x_{0,\beta} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad b_{0,\beta} = \begin{bmatrix} 1 \\ 1+\beta \\ 1-\beta \end{bmatrix}, \qquad A_{0,\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta \\ 0 & -\beta & 1 \end{bmatrix}$$

$$A_{0,\beta}^{-1} = \frac{1}{1+\beta^2}\begin{bmatrix} 1+\beta^2 & 0 & 0 \\ 0 & 1 & -\beta \\ 0 & \beta & 1 \end{bmatrix}$$

$$\|x_{0,\beta}\|_1 = 1, \qquad \|b_{0,\beta}\|_1 = 1+|\beta|, \qquad \|A_{0,\beta}\|_\infty = 1+|\beta|$$

$$\|A_{0,\beta}^{-1}\|_\infty = \max\left\{1, \frac{1+|\beta|}{1+\beta^2}\right\} = \begin{cases} \dfrac{1+|\beta|}{1+\beta^2}, & |\beta| \le 1 \\ 1, & 1 < |\beta| \end{cases}$$

$$K_\beta := \mathrm{cond}_\infty(A_{0,\beta}) = \|A_{0,\beta}\|_\infty \|A_{0,\beta}^{-1}\|_\infty = \begin{cases} \dfrac{(1+|\beta|)^2}{1+\beta^2}, & |\beta| \le 1 \\ 1+|\beta|, & 1 < |\beta| \end{cases}$$

$$A_{\alpha,\beta} - A_{0,\beta} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \|A_{\alpha,\beta}-A_{0,\beta}\|_\infty = |\alpha|$$

$$b_{\alpha,\beta} - b_{0,\beta} = [\alpha\beta \;\; 0 \;\; 0]^T, \qquad \|b_{\alpha,\beta}-b_{0,\beta}\|_\infty = |\alpha\beta|$$

$$\|\delta_{A_{\alpha,\beta}}\|_\infty = \frac{|\alpha|}{1+|\beta|}, \qquad \|\delta_{b_{\alpha,\beta}}\|_\infty = \frac{|\alpha\beta|}{1+|\beta|}$$

$$\|x_{\alpha,\beta}-x_{0,\beta}\|_\infty \le \frac{K_\beta}{1-\dfrac{K_\beta|\alpha|}{1+|\beta|}}\left(\frac{|\alpha|}{1+|\beta|}+\frac{|\alpha\beta|}{1+|\beta|}\right) = \frac{K_\beta|\alpha|}{1-\dfrac{K_\beta|\alpha|}{1+|\beta|}}$$

$$\text{para} \quad |\alpha| < \frac{1+\beta}{K_\beta}$$

(b)[15]

O método de Gauss-Seidel converge para a solução do sistema $A_{\alpha,\beta}x = b_{\alpha,\beta}$ se e só se o raio espectral da matriz iteradora do método for inferior à unidade, $r_\sigma(C_{GS}) < 1$.

$$C_{GS} = -M_{GS}^{-1}N_{GS}$$

$$A_{\alpha,\beta} = M_{GS} + N_{GS}, \qquad M_{GS} = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ 0 & -\beta & 1 \end{bmatrix}, \qquad N_{GS} = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}$$

A equação dos valores próprios

$$\det\left(C_{GS} - \lambda I\right) = 0$$

é equivalente à equação

$$\det\left(N_{GS} + \lambda M_{GS}\right) = 0$$

$$\det\left(N_{GS} + \lambda M_{GS}\right) = \begin{vmatrix} \lambda & \alpha & 0 \\ -\lambda\alpha & \lambda & \beta \\ 0 & -\lambda\beta & \lambda \end{vmatrix} = \lambda^2\left(\lambda + \alpha^2 + \beta^2\right)$$

$$\sigma(C_{GS}) = \{\ 0,\ 0,\ -\alpha^2 - \beta^2\}$$

$$r_\sigma\left(C_{GS}\right) = \alpha^2 + \beta^2$$

O método de Gauss-Seidel convergirá pois no conjunto

$$\left\{(\alpha,\beta) \in \mathbb{R}^2 : \quad \alpha^2 + \beta^2 < 1\right\}.$$

[2]
(a)[15]

Fórmula interpoladora de Lagrange:

$$p_3(x) = \sum_{j=0}^{3} f(x_j) l_{j,3}(x), \qquad l_{j,3}(x) = \prod_{i=0, i\neq j}^{3} \frac{x - x_i}{x_j - x_i}$$

$$p_3(x) = 4\, l_{0,3}(x) + 3\, l_{1,3}(x) + 2\, l_{2,3}(x) + 0\, l_{3,3}(x)$$

$$l_{0,3}(x) = \frac{(x-3)(x-4)(x-5)}{(2-3)(2-4)(2-5)} = -\frac{1}{6}\,(x-3)(x-4)(x-5)$$

$$l_{1,3}(x) = \frac{(x-2)(x-4)(x-5)}{(3-2)(3-4)(3-5)} = \frac{1}{2}\,(x-2)(x-4)(x-5)$$

$$l_{2,3}(x) = \frac{(x-2)(x-3)(x-5)}{(4-2)(4-3)(4-5)} = -\frac{1}{2}\,(x-2)(x-3)(x-5)$$

$$\begin{aligned} p_3(x) = \;& -\frac{2}{3}\,(x-3)(x-4)(x-5) + \frac{3}{2}\,(x-2)(x-4)(x-5) \\ & -(x-2)(x-3)(x-5) \end{aligned}$$

$$p_3(x) = -\frac{1}{6}\,(x-5)(x^2 - 4x + 12)$$

(b)[15]

Melhor aproximação mínimos quadrados:

$$q_2^*(x) = (a_0^* + a_1^* x)(x-5) = a_0^* \phi_0(x) + a_1^* \phi_1(x)$$

$$\phi_0(x) = x - 5, \qquad \phi_1(x) = x(x-5)$$

$$\begin{bmatrix} \langle \bar\phi_0, \bar\phi_0 \rangle & \langle \bar\phi_0, \bar\phi_1 \rangle \\ \langle \bar\phi_1, \bar\phi_0 \rangle & \langle \bar\phi_1, \bar\phi_1 \rangle \end{bmatrix} \begin{bmatrix} a_0^* \\ a_1^* \end{bmatrix} = \begin{bmatrix} \langle \bar f, \bar\phi_0 \rangle \\ \langle \bar f, \bar\phi_1 \rangle \end{bmatrix}, \qquad \langle \bar\phi, \bar\psi \rangle = \sum_{i=0}^{2} \bar\phi_i \bar\psi_i$$

$$\bar\phi_0 = [\phi_0(x_i)] = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}, \quad \bar\phi_1 = [\phi_1(x_i)] = \begin{bmatrix} -6 \\ -6 \\ -4 \end{bmatrix}, \quad \bar f = [f(x_i)] = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$\langle \bar\phi_0, \bar\phi_0 \rangle = 14$$

$$\langle \bar\phi_0, \bar\phi_1 \rangle = 34 = \langle \bar\phi_1, \bar\phi_0 \rangle$$

$$\langle \bar\phi_1, \bar\phi_1 \rangle = 88$$

$$\langle \bar f, \bar\phi_0 \rangle = -20$$

$$\langle \bar f, \bar\phi_1 \rangle = -50$$

$$\begin{bmatrix} 14 & 34 \\ 34 & 88 \end{bmatrix} \begin{bmatrix} a_0^* \\ a_1^* \end{bmatrix} = \begin{bmatrix} -20 \\ -50 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a_0^* \\ a_1^* \end{bmatrix} = -\frac{5}{19} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$q_2^* = -\frac{5}{19}\,(x+3)(x-5)$$

[3]

(a)[15]

$$Q(f) = w_0 f(0) + w_1 f(x_1)$$

$$Q(p) = I(p), \;\; \forall p \in \mathcal{P}_2 \qquad \Longleftrightarrow \qquad Q(x^m) = I(x^m), \;\; m = 0, 1, 2$$

$$\begin{cases} w_0 + w_1 = b \\ w_1 x_1 = \dfrac{b^2}{2} \\ w_1 x_1^2 = \dfrac{b^3}{3} \end{cases}$$

$$w_0 = \frac{b}{4}, \qquad w_1 = \frac{3b}{4}, \qquad x_1 = \frac{2b}{3}$$

$$Q(f) = \frac{b}{4}\left[f(0) + 3f\left(\frac{2b}{3}\right)\right]$$

(b)[15]

$$Q(f) = I(p_1), \quad p_1(x) = f[0] + f[0, x_1]x$$

$$E(f) = I(f) - Q(f) = I(f - p_1) = I(vW_2)$$

$$v(x) = f[0, x_1, x], \qquad W_2 = x(x - x_1), \qquad x_1 = \frac{2b}{3}$$

Introduzindo a função u definida por:

$$u'(x) = W_2(x), \qquad u(x) = \int_0^x W_2(t)dt = \frac{x^2}{3}\,(x - b)$$

$$E(f) = I(vu')$$

Integrando por partes:

$$E(f) = [u(x)v(x)]_0^b - I(v'u)$$

Sendo $u(0) = 0, \;\; u(b) = 0$ o primeiro termo é zero.

Sendo $u(x) \leq 0, \quad \forall x \in [0, b], \;$ o teorema do valor médio para integrais permite escrever

$$E(f) = -v'(\eta) I(u), \qquad \eta \in]0, b[$$

Usando a definição de derivada da diferença dividida e a relação desta com a derivada da função resulta:

$$v'(\eta) = f[0, c, \eta, \eta] = \frac{f'''(\xi)}{6}, \qquad \xi \in]0, b[$$

Atendendo finalmente a que:

$$I(u) = -\frac{b^4}{36}$$

obtém-se o resultado pretendido:

$$E(f) = \frac{b^4}{216} f'''(\xi)$$

[4]

$$\begin{cases} y'(x) = f(x, y(x)), & x \geq 1, \\ y(1) = 2, \end{cases} \qquad f(x, y) = x^2 + y^2$$

(a)[15]

Método de Euler modificado (passo h): $\; Y(1 + h) \approx y_1$

$$y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right), \qquad m \geq 0$$

$$x_0 = 1, \qquad y_0 = 2, \qquad f(x_0, y_0) = 5$$

$$y_1 = 2 + h\left[\left(1 + \frac{h}{2}\right)^2 + \left(2 + \frac{5h}{2}\right)^2\right]$$

$$y_1 = 2 + 5h + 11h^2 + \frac{13}{2} h^3$$

(b)[15]

Método de Taylor de ordem 3 (passo h): $\; Y(1 + h) \approx y_1$

$$y_1 = y_0 + hf(x_0, y_0) + \frac{h^2}{2}(d_f f)(x_0, y_0) + \frac{h^3}{6}(d_f^2 f)(x_0, y_0) \qquad m \geq 0$$

$$(d_f f)(x, y) = \left(\frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial y}\right) f(x, y)$$

$$(d_f^2 f)(x,y) = (d_f(d_f f))(x,y)$$

$$(d_f f)(x,y) = 2x + f(x,y)2y = 2(x + x^2 y + y^3)$$

$$(d_f^2 f)(x,y) = 2[1 + 2xy + f(x,y)(x^2 + 3y^2)]$$

$$x_0 = 1, \qquad y_0 = 2, \qquad f(x_0,y_0) = 5$$

$$(d_f f)(x_0,y_0) = 22, \qquad (d_f^2 f)(x_0,y_0) = 140$$

$$y_1 = 2 + 5h + 11h^2 + \frac{70}{3}\,h^3$$