INSTITUTO SUPERIOR TÉCNICO

Ano Lectivo: 2010/2011

MATEMÁTICA COMPUTACIONAL

Resolução do Exame de 12 de Julho de 2011

[1]

(a)[10]

$a = \pi \qquad \tilde{a} = 3.14159$

$b = \dfrac{2199}{700} \qquad \tilde{b} = 3.14143$

$\tilde{a} - \tilde{b} = 0.00016 \qquad \tilde{a} + \tilde{b} = 6.28302$

$\tilde{c} = 0.254655 \times 10^{-4}$

$\delta_{\tilde{c}} = \dfrac{c - \tilde{c}}{c} = 0.0249 \quad (2.49\%)$

(b)[10]

$c = \dfrac{u}{v}, \qquad u = a - b, \qquad v = a + b$

$\delta_{\tilde{u}} = p_{u,a}\,\delta_{\tilde{a}} + p_{u,b}\,\delta_{\tilde{b}} + \delta_S = \dfrac{a}{u}\,\delta_{\tilde{a}} - \dfrac{b}{u}\,\delta_{\tilde{b}} + \delta_S$

$\delta_{\tilde{v}} = p_{v,a}\,\delta_{\tilde{a}} + p_{v,b}\,\delta_{\tilde{b}} + \delta_A = \dfrac{a}{v}\,\delta_{\tilde{a}} + \dfrac{b}{v}\,\delta_{\tilde{b}} + \delta_A$

$$\begin{aligned}
\delta_{\tilde{c}} &= p_{c,u}\,\delta_{\tilde{u}} + p_{c,v}\,\delta_{\tilde{v}} + \delta_D \\
&= \delta_{\tilde{u}} - \delta_{\tilde{v}} + \delta_D \\
&= a\left(\frac{1}{u} - \frac{1}{v}\right)\delta_{\tilde{a}} - b\left(\frac{1}{u} + \frac{1}{v}\right)\delta_{\tilde{b}} + \delta_S - \delta_A + \delta_D \\
&= \frac{2ab}{a^2 - b^2}\,(\delta_{\tilde{a}} - \delta_{\tilde{b}}) + \delta_S - \delta_A + \delta_D
\end{aligned}$$

[2]

(a)[15] $f(x) = x - 3 - \dfrac{1}{2}\cos x - \dfrac{1}{3}\sin x$

Condições suficientes de convergência do método de Newton para z para qualquer $x_0 \in I$:

(0) $f \in C^2(I)$

(i) $\left.\begin{array}{l} f(2.6) = -0.143 \\ f(2.8) = 0.159 \end{array}\right\} \quad \Rightarrow \quad f(2.6)f(2.8) < 0$

(ii) $f'(x) = 1 + \dfrac{1}{2}\sin x - \dfrac{1}{3}\cos x, \qquad f'(x) > 0, \quad \forall x \in I$

(iii) $f''(x) = \dfrac{1}{2}\cos x + \dfrac{1}{3}\sin x, \qquad f''(x) < 0, \quad \forall x \in I$

(iv) $\left|\dfrac{f(2.6)}{f'(2.6)}\right| = 0.0929 < 0.2, \qquad \left|\dfrac{f(2.8)}{f'(2.8)}\right| = 0.108 < 0.2$

(b)[15] Método de Newton:

$$x_m = g(x_{m-1}), \quad m \geq 1, \qquad g(x) = x - \frac{f(x)}{f'(x)}$$

Critério de paragem: $|z - x_m| \leq B_m, \qquad B_m = KB_{m-1}^2, \qquad m \geq 0$

$$K = \frac{\max_{x\in I}|f''(x)|}{2\min_{x\in I}|f'(x)|} = \frac{|f''(2.8)|}{2|f'(2.8)|} = 0.1213$$

m	x_m	B_m
0	2.6	0.2
1	2.69291	0.485×10^{-2}
2	2.69368	0.286×10^{-5}

$\tilde{z} = 2.69368, \qquad |z - \tilde{z}| \leq B_2.$

[3][20] O método iterativo converge para a solução do sistema $Ax = b$ para qualquer $x^{(0)} \in \mathbb{R}^2$ para os valores de ω para os quais é satisfeita a condição necessária e suficiente de convergência:

$r_\sigma(C(\omega)) < 1, \qquad C(\omega)$ é a matriz iteradora do método.

$$C(\omega) = I - \frac{\omega}{2}A = \begin{bmatrix} 1-\omega & -\frac{\omega}{2} \\ \frac{\omega}{2} & 1-\omega \end{bmatrix}$$

$$\det[C(\omega) - \lambda I] = \begin{vmatrix} 1-\omega-\lambda & -\frac{\omega}{2} \\ \frac{\omega}{2} & 1-\omega-\lambda \end{vmatrix} = (1-\omega-\lambda)^2 + \left(\frac{\omega}{2}\right)^2$$

$$\det[C(\omega) - \lambda I] = 0 \quad \Leftrightarrow \quad \lambda \in \left\{1-\omega+i\frac{\omega}{2},\ 1-\omega-i\frac{\omega}{2}\right\}$$

$$r_\sigma(C(\omega)) = \sqrt{(1-\omega)^2 + \left(\frac{\omega}{2}\right)^2}$$

$$r_\sigma(C(\omega)) = 1 \quad \Leftrightarrow \quad \omega \in \left\{0,\ \frac{8}{5}\right\}$$

$$r_\sigma(C(\omega)) < 1 \quad \Leftrightarrow \quad \omega \in \left]0,\ \frac{8}{5}\right[$$

[4]

(a)[15]

A função $g : \mathbb{R}^2 \to \mathbb{R}^2$ tem um único ponto fixo em D pois satisfaz às hipóteses do teorema do ponto fixo. Com efeito:

(i) $g \in C^1(D)$

(ii) $J_g(x) = \begin{bmatrix} -\frac{1}{2}\sin\theta & -\frac{1}{2}\sin\theta \\ \frac{1}{3}\cos\theta & \frac{1}{3}\cos\theta \end{bmatrix}, \qquad \theta := x_1 + x_2$

$$\|J_g(x)\|_1 = \frac{1}{2}\,|\sin\theta| + \frac{1}{3}\,|\cos\theta|$$

$$\sup_{x\in D}\|J_g(x)\|_1 \le \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1$$

(iii) $g_1(x) \in \left[1 + \frac{1}{2}\cos\pi, 1 + \frac{1}{2}\cos\frac{13}{6}\right] = [0.5, 0.719385] \subset \left[\frac{1}{2}, \frac{3}{2}\right], \quad \forall x \in D$

$g_2(x) \in \left[2 + \frac{1}{3}\sin\frac{23}{6}, 2 + \frac{1}{3}\sin\frac{13}{6}\right] = [1.78737, 2.27589] \subset \left[\frac{5}{3}, \frac{7}{3}\right], \quad \forall x \in D$

$\Rightarrow \quad g(D) \subset D$

Sendo z o único ponto fixo de g em D então é também o único zero de f em D.

(b)[15]

$$\begin{cases} x^{(1)} = x^{(0)} + \Delta x^{(0)}, \\ J_f(x^{(0)})\Delta x^{(0)} = -f(x^{(0)}) \end{cases}$$

$$J_f(x) = \begin{bmatrix} 1 + \frac{1}{2}\sin\theta & \frac{1}{2}\sin\theta \\ -\frac{1}{3}\cos\theta & 1 - \frac{1}{3}\cos\theta \end{bmatrix}, \qquad \theta := x_1 + x_2$$

$$x^{(0)} = \begin{bmatrix} 1 \\ \pi - 1 \end{bmatrix}, \qquad \theta^{(0)} = \pi$$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{3} & \frac{4}{3} \end{bmatrix} \Delta x^{(0)} = \begin{bmatrix} -\frac{1}{2} \\ 3 - \pi \end{bmatrix} \quad \Leftrightarrow \quad \Delta x^{(0)} = \begin{bmatrix} -\frac{1}{2} \\ \frac{19-6\pi}{8} \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} \frac{1}{2} \\ \frac{11+2\pi}{8} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.1604 \end{bmatrix}$$

[5]
(a)[20]

Fórmula de Newton às diferenças divididas:

i	x_i	$f[x_i]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot]$
0	-2.0	-1.908				
			1.067			
1	-1.0	-0.841		0.387		
			1.841		-0.129	
2	0.0	1.000		0.0		0.0
			1.841		-0.129	
3	1.0	2.841		-0.387		
			1.067			
4	2.0	3.908				

$$\begin{aligned} p(x) = \ & f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) \\ & + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2) \\ & + f[x_0,x_1,x_2,x_3,x_4](x-x_0)(x-x_1)(x-x_2)(x-x_3) \end{aligned}$$

$$p(x) = -1.908 + 1.067(x+2) + 0.387(x+2)(x+1) - 0.129(x+2)(x+1)x$$

$$p(x) = 1.0 + 1.97x - 0.129x^3$$

(b)[20]

Melhor aproximação mínimos quadrados:

$$q(x) = a_0^*\phi_0(x) + a_1^*\phi_1(x) + a_2^*\phi_2(x)$$

$$\phi_0(x) = 1, \quad \phi_1(x) = x, \quad \phi_2(x) = x^3$$

$$\begin{bmatrix} \langle\bar\phi_0,\bar\phi_0\rangle & \langle\bar\phi_0,\bar\phi_1\rangle & \langle\bar\phi_0,\bar\phi_2\rangle \\ \langle\bar\phi_1,\bar\phi_0\rangle & \langle\bar\phi_1,\bar\phi_1\rangle & \langle\bar\phi_1,\bar\phi_2\rangle \\ \langle\bar\phi_2,\bar\phi_0\rangle & \langle\bar\phi_2,\bar\phi_1\rangle & \langle\bar\phi_2,\bar\phi_2\rangle \end{bmatrix} \begin{bmatrix} a_0^* \\ a_1^* \\ a_2^* \end{bmatrix} = \begin{bmatrix} \langle\bar f,\bar\phi_0\rangle \\ \langle\bar f,\bar\phi_1\rangle \\ \langle\bar f,\bar\phi_2\rangle \end{bmatrix}$$

$$\bar f = \begin{bmatrix} -1.908 \\ -0.841 \\ 1.000 \\ 2.841 \\ 3.908 \end{bmatrix}, \quad \bar\phi_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar\phi_1 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \bar\phi_2 = \begin{bmatrix} -8 \\ -1 \\ 0 \\ 1 \\ 8 \end{bmatrix}$$

$$\langle\bar g,\bar h\rangle = \sum_{i=0}^{4} \bar g_i, \bar h_i, \qquad \forall \bar g, \bar h \in \mathbb{R}^5$$

$$\langle \bar{\phi}_0, \bar{\phi}_0 \rangle = 5, \qquad \langle \bar{\phi}_0, \bar{\phi}_1 \rangle = 0 = \langle \bar{\phi}_1, \bar{\phi}_0 \rangle$$

$$\langle \bar{\phi}_0, \bar{\phi}_2 \rangle = 0 = \langle \bar{\phi}_2, \bar{\phi}_0 \rangle, \qquad \langle \bar{\phi}_1, \bar{\phi}_1 \rangle = 10$$

$$\langle \bar{\phi}_1, \bar{\phi}_2 \rangle = 34 = \langle \bar{\phi}_2, \bar{\phi}_1 \rangle, \qquad \langle \bar{\phi}_2, \bar{\phi}_2 \rangle = 130$$

$$\langle f, \bar{\phi}_0 \rangle = 5.0, \qquad \langle f, \bar{\phi}_1 \rangle = 15.314, \qquad \langle f, \bar{\phi}_2 \rangle = 50.21$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 34 \\ 0 & 34 & 130 \end{bmatrix} \begin{bmatrix} a_0^* \\ a_1^* \\ a_2^* \end{bmatrix} = \begin{bmatrix} 5.0 \\ 15.314 \\ 50.21 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} a_0^* \\ a_1^* \\ a_2^* \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.97 \\ -0.129 \end{bmatrix}$$

$q(x) = 1.0 + 1.97x - 0.129x^3$

[6]

(a)[20]

$$Q(f) = w_0 f(x_0) + w_1 f(x_1)$$

$$Q(p) = I(p), \quad \forall p \in \mathcal{P}_3 \qquad \Leftrightarrow \qquad Q(x^k) = I(x^k), \quad k = 0, 1, 2, 3$$

$$\begin{cases} w_0 + w_1 = 4 \\ w_0 x_0 + w_1 x_1 = 0 \\ w_0 x_0^2 + w_1 x_1^2 = 8 \\ w_0 x_0^3 + w_1 x_1^3 = 0 \end{cases} \qquad \begin{cases} w_0 = 2 \\ w_1 = 2 \\ x_0 = -\sqrt{2} \\ x_1 = \sqrt{2} \end{cases}$$

$$Q(f) = 2\left[f(-\sqrt{2}) + f(\sqrt{2})\right]$$

(b)[10]

Uma fórmula de quadratura $Q(f)$ que aproxima um integral $I(f)$ tem grau de precisão m se:

(i) $Q(q) = I(q), \quad \forall q \in \mathcal{P}_m$;

(ii) $Q(q) \neq I(q), \quad$ para algum $q \in \mathcal{P}_{m+1}$;

Por construção a fórmula obtida em (a) satisfaz a (i) com $m = 3$.

Uma vez que

$$Q(x^4) = 16 \quad \neq \quad I(x^4) = \frac{64}{3}$$

conclui-se que a fórmula obtida em (a) tem grau de precisão três.

[7]

$$\begin{cases} W'(t) = F(W(t)), \quad t \geq 0 \\ W(0) = W_0 \end{cases}$$

$$W = \begin{bmatrix} \phi \\ v \end{bmatrix} = \begin{bmatrix} \phi \\ \phi' \end{bmatrix}, \qquad F(W) = \begin{bmatrix} v \\ -\sin\phi \end{bmatrix}, \qquad W_0 = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$$

(a)[15]

Método de Euler modificado (um passo de comprimento h):

$$W_1 = W_0 + hF\left(\tilde{W}_1\right), \qquad \tilde{W}_1 = W_0 + \frac{h}{2}\,F(W_0)$$

$$\tilde{W}_1 = \begin{bmatrix} \frac{\pi}{2} \\ -\frac{h}{2} \end{bmatrix}$$

$$W_1 = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} + h \begin{bmatrix} -\frac{h}{2} \\ -1 \end{bmatrix}$$

$$\Phi(h) \approx \phi_1 = \frac{1}{2}\,(\pi - h^2), \qquad \Phi'(h) \approx v_1 = -h$$

(b)[15]

Método de Taylor de ordem 2 (um passo de comprimento h):

$$W_1 = W_0 + hF(W_0) + \frac{h^2}{2}\,(d_F F)(W_0)$$

$$(d_F F)(W) = (F \cdot \nabla_F)F(W) = \left(v\frac{\partial}{\partial\phi} - \sin\phi\frac{\partial}{\partial v}\right) F(W) = \begin{bmatrix} -\sin\phi \\ -v\cos\phi \end{bmatrix}$$

$$W_1 = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} + h \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{h^2}{2} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\Phi(h) \approx \phi_1 = \tfrac{1}{2}(\pi - h^2), \qquad \Phi'(h) \approx v_1 = -h$$