28/6/21 Lisboa

Friday, September 11, 2020 5:41 PM

The Gamma and Strominger-Yau-Zaslow conjectures

(joint with Abouzaid, Ganatra, Iritani)

Outline: If X, Y are mirror Calabi-Yau varieties, then:

Homological mirror symmetry: $\underline{DFuk(X) \simeq DCoh(Y)}$ $\overset{HC_\bullet}{\Longrightarrow}$ Hodge-theoretic mirror symmetry: $\underline{\mathcal{V}^A(X) \underset{\mathbb{C}\text{-VHS}}{\simeq} \mathcal{V}^B(Y)}$ (2875 lines on quintic ...) 2

$\Uparrow \otimes_{\mathbb{Z}} \mathbb{C}$

SYZ conjecture:

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

$B \leftarrow$ "tropical"

4 $\overset{?}{\Longrightarrow}$ Gamma conjecture: $\mathcal{V}^A(X) \underset{\mathbb{Z}\text{-VHS}}{\simeq} \mathcal{V}^B(Y)$ (study periods $\int_C \Omega$ via MS, e.g. $\zeta(k)$) 3

Plan: ① Local tropical period calculations ③ Gamma conjecture

② Hodge MS ④ SYZ $\Rightarrow \Gamma$

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① Local computations.

$$\lim_{T\to\infty} \log_T \left(T^x + T^y\right) = \max(x,y) + O\left(T^{-\varepsilon}\right) \quad \text{if } |x-y| > \varepsilon$$

alg. geom. $\longleftrightarrow$ trop. geom.

$$\int_{-B}^{B} \text{LHS}\, dx = \int_{-B}^{B} \text{RHS}\, dx + \frac{2\zeta(2)}{\log T} + O(T^{-\varepsilon}), \quad \varepsilon > 0$$

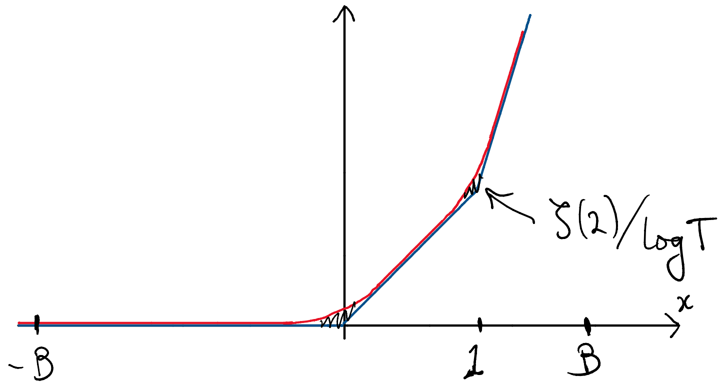
where $\zeta(k) := \sum_{n=1}^{\infty} \frac{1}{n^k}$, $\zeta(2) = \frac{\pi^2}{6}$.

(We use the integral:

alg. geom. $\longleftrightarrow$ trop. geom. ... where $\zeta(k) = \sum_{n=1} \frac{1}{n^k}$, ... 6.

E.g. $p_T(a) = 1 + a + T^{-1}a^2$

$$\lim_{T\to\infty} \log_T\left(p_T(T^x)\right) = \max(0, x, 2x-1)$$



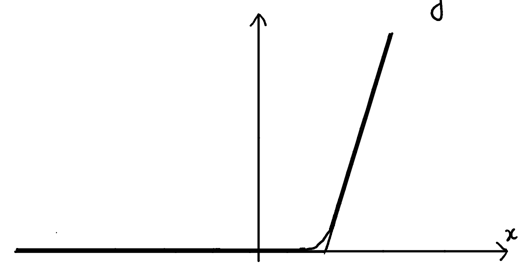
(We use the integral:

$$\int_{-\infty}^{\infty} \log(1+e^x) - \max(0,x)\, dx = \zeta(2).)$$

E.g. $p_T(a) = 1 + T^{-2}a + T^{-1}a^2$

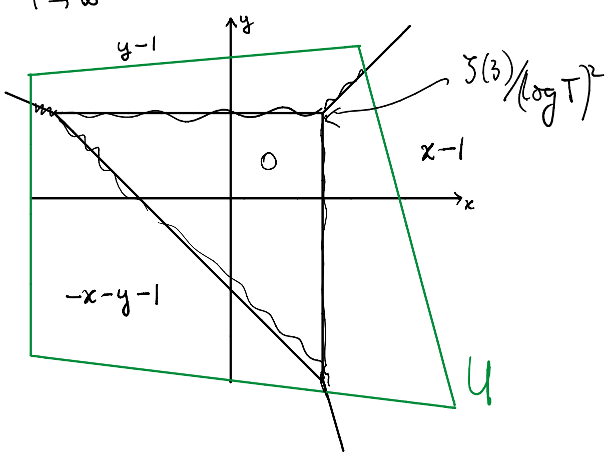
$$\lim_{T\to\infty} \log_T\left(p_T(T^x)\right) = \max(0, x-2, 2x-1)$$

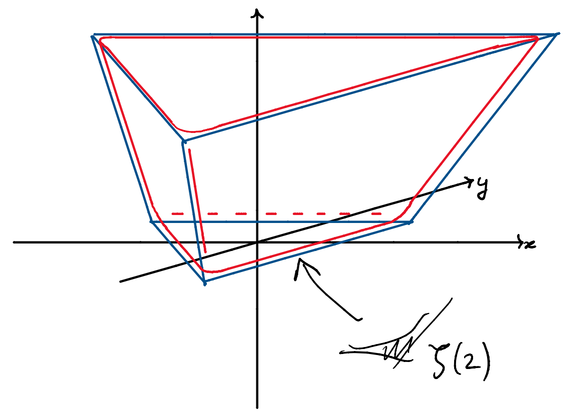
$$\int_{-B}^{B} \text{LHS}\, dx = \int_{-B}^{B} \text{RHS}\, dx + \frac{\frac{1}{2}\zeta(2)}{\log T} + O(T^{-\varepsilon})$$



E.g. $p_T(a,b) = 1 + T^{-1}\left(a + b + \frac{1}{ab}\right)$

$$\lim_{T\to\infty} \log_T\left(p_T(T^x, T^y)\right) = \max(0, x-1, y-1, -x-y-1)$$





$$\iint_U \text{LHS}\, dA = \iint_U \text{RHS}\, dA + \overset{\text{length inside } U}{L} \cdot \frac{\zeta(2)}{\log T} + 3\,\frac{\zeta(3)}{(\log T)^2} + O(T^{-\varepsilon})$$

② **Defn:** A $\mathbb{C}$-VHS$/\mathcal{M}$ consists of

- holomorphic vec. bun. $\mathcal{V} \to \mathcal{M}$
- filtration $F^{\geq *}\mathcal{V}$
- flat connection $\nabla : \mathcal{V} \to \Omega^1_{\mathcal{M}} \otimes \mathcal{V}$

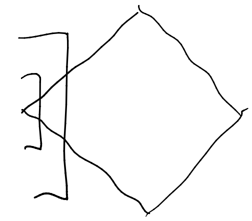
satisfying Griffiths transversality:

$$\nabla(F^{\geq p}\mathcal{V}) \subset \Omega^1_{\mathcal{M}} \otimes F^{\geq p-1}\mathcal{V}.$$

E.g. $\mathcal{Y} \to \mathcal{M}$ fam. of smooth compact CY Kähler manifolds $\rightsquigarrow$ $\mathbb{C}$-VHS$/\mathcal{M}$ $\mathcal{V}^B(\mathcal{Y})$, with:

$$V_m = H^\bullet(Y_m;\mathbb{C})$$

$$F^{\geq k}V_m = \bigoplus_{p-q \geq 2k} H^{p,q}$$



$$\nabla = \text{"Gauss–Manin connection"}$$

$$(\alpha,\beta) = i^{|\alpha|}\int \alpha \wedge \beta$$

A polarization of a $\mathbb{C}$-VHS is a pairing

$$(\cdot,\cdot) : \mathcal{V}^{\otimes 2} \to \mathcal{O}_{\mathcal{M}}$$

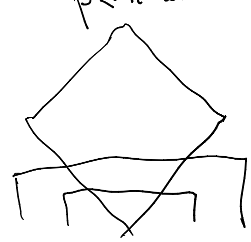
satisfying: (1) it is covariantly constant; (2) $(F^{\geq p}\mathcal{V}, F^{\geq q}\mathcal{V}) = 0$ if $p+q>0$; (3) $(\cdot,\cdot) : Gr_F^p \mathcal{V} \otimes Gr_F^{-p}\mathcal{V} \to \mathcal{O}$ non-degenerate; (4) $(u,v) = (-1)^n (v,u)$ for some $n \in \mathbb{Z}/2$.

E.g. X = sm. cpct. CY, ω = Kähler $\rightsquigarrow$ $\mathbb{C}$-VHS$/\Delta^*$, $\mathcal{V}^A(X,\omega)$:

$$\Delta^* = \{t \in \mathbb{C}^* : |t| < \delta\} \qquad \underline{T = t^{-1}}$$

$$V_T = H^\bullet(X;\mathbb{C})$$

$$F^{\geq k}V_T = \bigoplus_{p \leq n-2k} H^p(X;\mathbb{C})$$



$$\nabla_{\frac{\partial}{\partial T}}(\alpha) = \frac{\partial \alpha}{\partial T} - T^{-1}[\omega] \star_T \alpha$$

quantum cup product

(2875, ...)

$$(\alpha,\beta) = i^{n(n+2)-|\alpha|}\int \alpha \wedge \beta$$

$\mathcal{Y}$ $\xleftarrow{\text{mirror}}\!\!\rightarrow$ (X,ω)

fam. of CYs over Δ^*_t; $t \to 0$ "large complex structure limit"

fam. of symp. CYs over Δ^*_t (think: $\omega_t = -\log t \cdot \omega$); $t \to 0$ "large volume limit"

③ Gamma conjecture: refine this to an iso. of $\mathbb{Z}$-VHS

Def: $\mathbb{Z}$-VHS = $\mathbb{C}$-VHS + lattice of flat sections $V_{\mathbb{Z}} \subset \mathcal{V}_{flat}$.

(s.t. $F^{\geq *}\mathcal{V}$ is complementary to $\overline{F^{\geq -*}\mathcal{V}}$ ...)

complex structure limit" $\qquad$ $t \to 0$ "large volume limit"

$$\Downarrow$$

$$\begin{tikzcd}
\mathcal{V}^B(\mathcal{Y}) \arrow[r, phantom, "\underset{\mathbb{C}\text{-VHS}}{\simeq}"] \arrow[d] & \mathcal{V}^A(X,\omega) \arrow[d] \\
\Delta^* \arrow[r, leftrightarrow, "\sim"', "\text{``mirror map''}"] & \Delta^*
\end{tikzcd}$$

flat sections $V_{\mathbb{Z}} \subset V_{flat}$.

(s.t. $F^{\geq *}\mathcal{V}$ is complementary to $\overline{F^{\geq -*}\mathcal{V}}$ ...)

E.g. $\bigoplus_p (2\pi i)^{\frac{p}{2}} H^p(Y_m;\mathbb{Z}) \subset H^*(Y_m;\mathbb{C}) = \mathcal{V}^B(\mathcal{Y})$

$\rightsquigarrow$ info. about periods:

$$\int_c \Omega = \langle \underset{V_{\mathbb{Z}}}{\underset{\cap}{PD(c)}}, \underset{F^{\geq n}\mathcal{V}}{\underset{\cap}{\Omega}} \rangle$$

E.g.

$$\left\{ \begin{array}{l} \text{flat sections of } \mathcal{V}^A(X) \text{ asymptotic, as } T \to \infty, \text{ to} \\ \underbrace{\hat{\Gamma}_X \wedge \underbrace{T^{\omega}}_{\exp(\log T \cdot [\omega])} \wedge \bigoplus_p (2\pi i)^{\frac{p}{2}} H^p(X;\mathbb{Z})} \end{array} \right\} \subset \mathcal{V}^A(X)$$

fibres are $H^*(X;\mathbb{C})$

where

$$\hat{\Gamma}_X := \prod_i \Gamma(1+\delta_i) = \exp\Big(\sum_{k\geq 2} (-1)^k \zeta(k)\cdot(k-1)!\, ch_k(TX)\Big) \in H^*(X;\mathbb{C})$$

$$= \hat{\Gamma}_0 + \hat{\Gamma}_2 + \hat{\Gamma}_3 + \dots$$

$$= 1 + \zeta(2)\, ch_2(TX) - 2\zeta(3)\, ch_3(TX) + \dots$$

Chern roots of TX

Gamma Conjectures (Horja, Libgober, Gelfand-Kapranov-Zelevinsky, Golyshev, Lian, Yau ...)

(A) (Iritani, Katzarkov-Kontsevich-Pantev)

$$\mathcal{V}^B(Y) \underset{\mathbb{Q}\text{-VHS}}{\simeq} \mathcal{V}^A(X)$$

(for $\mathbb{Z}$-VHS, replace $\bigoplus_p (2\pi i)^{p/2} H^p(X;\mathbb{Z})$ with $\mathrm{im}\big(ch : K^{top}(X) \to H^*(X;\mathbb{C})\big)$.)

(B) (Hosono)

If $L_T \subset Y_T$ is a Lagrangian mirror to $E \in D^b Coh(X)$, then

$PD(L_T) \longleftrightarrow$ flat section asymp. to $\hat{\Gamma}_X \wedge T^{\omega} \wedge (2\pi i)^{\frac{\deg}{2}} ch(E)$.

$$PD(L_T) \longleftrightarrow \text{flat section asymp. to } \hat{\Gamma}_X \wedge T^{\omega} \wedge (2\pi i)^{\frac{\deg}{2}} ch(E).$$

$$\mathcal{V}^B(Y)_{\mathbb{Z}} \cap \simeq \mathcal{V}^A(X)_{\mathbb{Z}} \cap$$

$$\rightsquigarrow \int_{L_T} \Omega_T = \int_X \hat{\Gamma}_X \wedge T^{\omega} \wedge (2\pi i)^{\deg/2} ch(E) + O(T^{-\varepsilon})$$

$\varepsilon > 0$

Ω_T: holomorphic volume form

<u>Thm</u> (Barannikov-Kontsevich, Costello, Ganatra-Perutz-S.):

$$DFuk(X) \simeq DCoh(Y) \quad \Rightarrow \quad \mathcal{V}^A(X) \underset{\mathbb{C}\text{-VHS}}{\simeq} \mathcal{V}^B(Y)$$

Hope: upgrade to $\mathbb{Z}$- or $\mathbb{Q}$-VHS.

Note: suffices to match the lattices asymptotically as $T \to \infty$; hence our formulation of (B).

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(4) <u>Thm</u> (Abouzaid-Ganatra-Iritani-S.):

Γ Conj (B) holds if (X, Y) are Batyrev mirrors, E = 'ambient' line bundle.

(due to Iritani; give new proof based on SYZ).

Focus on $E = \mathcal{O}_X$:

$$\boxed{\begin{aligned} \int_{L_T} \Omega_T + O(T^{-\varepsilon}) &= \int_X \hat{\Gamma}_X \cup T^{\omega} \\ &= \sum_i \int_X \hat{\Gamma}_i \cup \frac{(\log T \cdot \omega)^{n-i}}{(n-i)!} \end{aligned}}$$

$i = 0$

SYZ:

$$\begin{array}{ccc} X & & Y_T \\ & \searrow \quad \swarrow & \\ & B \supset B^{sing} & \end{array}$$

"codim-2"

Local models:

$X \longrightarrow B \setminus B^{sing}$ looks like moment map for T^n-action.

$Y_T \longrightarrow B \setminus B^{sing}$ looks like

$$(\mathbb{C}^*)^n \xrightarrow{\log_T} \mathbb{R}^n;$$

"correct" by holom. discs emanating from B^{sing}, weighted by $T^{-\text{area}}$.

$i > 0$ terms come from 'corrections';

Ω_T looks like $d\log z_1 \wedge \dots \wedge d\log z_n$

L_T looks like $\mathbb{R}^n_+$

$$\rightsquigarrow \int_{L_T} \Omega_T = (\log T)^n \cdot \mathrm{vol}(B) + \text{l.o.t.}$$

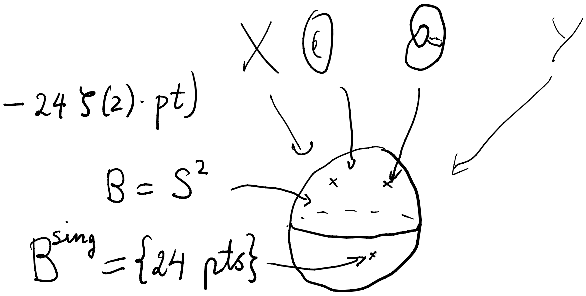
$$= (\log T)^n \underbrace{\int_X \frac{\omega^n}{n!}}_{i=0 \text{ term}} + \text{l.o.t.} \quad \text{(Duistermaat-Heckman)}$$

$i > 0$ terms come from 'corrections'; modulo $O(T^{-\varepsilon})$, contributions come from constant discs on B^{sing}

$\Longrightarrow$ local in B.

$\hat{\Gamma}_i$ arises from codim-i part.

E.g. $X = K3$, $B = S^2$, $B^{sing} = \{24 \text{ pts}\}$



$$\int_X T^\omega \cdot \hat{\Gamma}_X = \int_X \left(1 + \log T \cdot \omega + \tfrac{1}{2}(\log T)^2 \omega^2\right) \cdot \left(1 - 24\,\zeta(2) \cdot pt\right)$$

X: $$= (\log T)^2 \int_X \frac{\omega^2}{2} - 24\,\zeta(2)$$

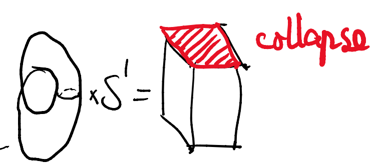
B: $$= (\log T)^2 \cdot \mathrm{vol}(B) - 24\,\zeta(2)$$

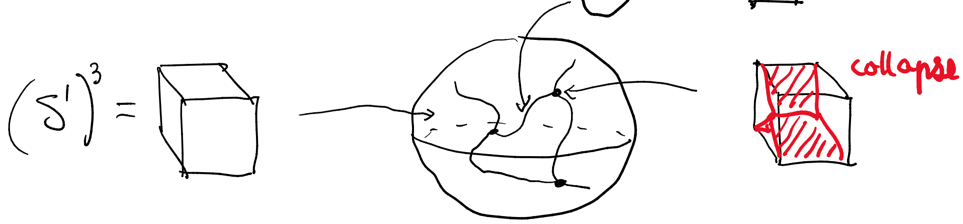
(DH) $\quad$ ↑ contribution of B^{sing} to asymptotics of $\int_{L_T} \Omega_T$.

Y: $$\boxed{= \int_{L_T} \Omega_T + O(T^{-\varepsilon})}$$

local computations

E.g. $X = CY3$, $B = S^3$, B^{sing} = trivalent graph





$$\int_X T^\omega \cdot \hat{\Gamma}_X = (\log T)^3 \int_X \frac{\omega^3}{3!} - (\log T) \cdot \zeta(2) \cdot \int_X c_2(TX) \cdot \omega - \zeta(3) \cdot \chi(X)$$

$$= \underbrace{(\log T)^3 \mathrm{vol}(B)}_{\text{codim-}0} - \underbrace{(\log T) \cdot \zeta(2) \cdot \mathrm{length}(B^{sing})}_{\text{codim-}2} - \underbrace{\zeta(3) \cdot \#\text{ verts of } B^{sing}}_{\text{codim-}3}$$

$$= \underline{\int_{L_T} \Omega_T + O(T^{-\varepsilon})}$$

E.g. Focus-focus singularity $\{xy = 1+z\} \subset \mathbb{C}^2 \times \mathbb{C}^*$, $\Omega_T = d\log z \wedge d\log y$
$= -d\log z \wedge d\log x$

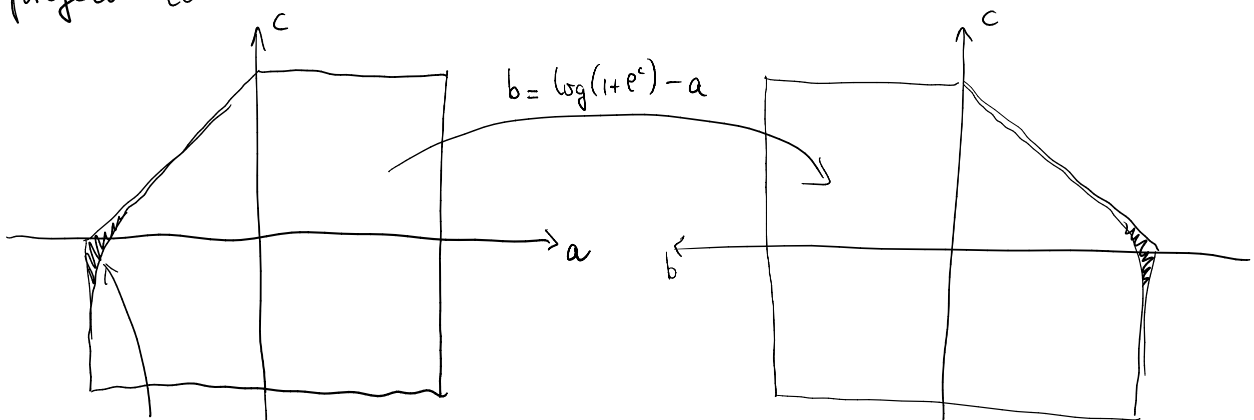
$a = \log|x| \quad b = \log|y| \quad c = \log|z|$

L_T = pos. real. loc.

region $\{c \in [-K, K],\ a \le K,\ b \le K\} \subset L_T$

$\Omega_T|_{L_T} = dc \wedge db$
$= -dc \wedge da$

projects to



$\zeta(2)$ "missing" from linear approx.