On the top weight rational cohomology of $\mathcal{A}_g$

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Idea

$\mathcal{M} \subset \overline{\mathcal{M}}$ $\rightsquigarrow$ $\underbrace{\mathcal{M}^{trop}}$

compactified moduli space

tropical modular interpretation for comb. data used to compactify

- study $\mathcal{M}^{trop}$
- translate results back to $\mathcal{M}$

(eg. $H_\bullet(\mathcal{M}^{trop}, \mathbb{Q}) \rightsquigarrow H^\bullet(\mathcal{M}, \mathbb{Q})$)

Today

Def Let $\mathcal{A}_g$ be the moduli space of abelian varieties of dim g

$\mathcal{A}_g$ is a smooth and separated Deligne-Mumford stack of dim $d = \binom{g+1}{2}$, endowed with quasiprojective moduli space A_g.

- $\mathcal{A}_g$ is not proper, but admits several compactifications

[- We will consider toroidal compactifications $\overline{\mathcal{A}_g}^{\Sigma}$,

Σ comb. data

 - $\forall \Sigma$, can construct $\overline{\mathcal{A}_g}^{trop, \Sigma}$ (joint w. Brannetti, Viviani)

We will compute $H_\bullet(L\overline{\mathcal{A}_g}^{trop, \Sigma}, \mathbb{Q})$, $g \leq 7$

$\rightsquigarrow$

$\hookrightarrow$ translate results $\underline{H^\bullet(\mathcal{A}_g, \mathbb{Q})}$.

What is known about $\underline{H^\bullet(\mathcal{A}_g, \mathbb{Q})}$?

- $\underline{g=2}$ [Igusa' 62] $H^k(\mathcal{A}_2, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & k=0,2 \\ 0, & \text{else} \end{cases}$

- $\underline{g=3}$ [Hain' 02] $H^k(\mathcal{A}_3, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & k=0,2,4 \\ \mathbb{Q}^2, & k=6 \\ 0, & \text{else} \end{cases}$

- $\underline{g=4}$ $H^\bullet(\mathcal{A}_4, \mathbb{Q})$ is not completely described but a lot is known (e.g. Hulek-Tommasi)

<u>Question</u> [Grushevsky' 09] Are there odd degree classes in $H^\bullet(\mathcal{A}_g, \mathbb{Q})$?

$\rightarrow$ Answer should be yes, but these are hard to produce, answer is yes for M_g

<u>Recall</u> Deligne's theory of weights

$H^\bullet(\mathcal{A}_g, \mathbb{Q})$ admits a weight filtration

$$\cdots \supset W_j H^k(\mathcal{A}_g, \mathbb{Q}) \supset W_{j-1} H^k(\mathcal{A}_g, \mathbb{Q}) \supset \cdots$$

with graded pieces

$$\mathrm{Gr}^W_j H^k(\mathcal{A}_g, \mathbb{Q}) := W_j H^k(\mathcal{A}_g, \mathbb{Q}) / W_{j-1} H^k(\mathcal{A}_g, \mathbb{Q})$$

with weights $k \leq j \leq 2k$.

<u>Def</u> The top weight rational cohomology of $\mathcal{A}_g$ is

$$\boxed{\mathrm{Gr}^W_{\underbrace{2d}_{= g(g+1)}} H^\bullet(\mathcal{A}_g, \mathbb{Q})}$$

Theorem [BBCMMW]

i) $\mathrm{Gr}^W_6 H^\bullet(\mathcal{A}_2, \mathbb{Q}) = 0$

ii) $\mathrm{Gr}^W_{12} H^k(\mathcal{A}_3, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k = 6 \\ 0 & \text{else} \end{cases}$ [Hain]

iii) $\mathrm{Gr}^W_{20} H^\bullet(\mathcal{A}_4, \mathbb{Q}) = 0$ [Hulek-Tommasi]

iv) $\mathrm{Gr}^W_{30} H^\bullet(\mathcal{A}_5, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 15, 20 \\ 0 & \text{else} \end{cases}$

v) $\mathrm{Gr}^W_{42} H^k(\mathcal{A}_6, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 30 \\ 0 & \text{else} \end{cases}$

vi) $\mathrm{Gr}^W_{56} H^k(\mathcal{A}_7, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 28, 33, 37, 42 \\ 0 & \text{else} \end{cases}$

Rmk
- Answer to Grushevsky's question is yes
- Speculate on non-vanishing in general, e.g. $\mathrm{Gr}^W H^d(\mathcal{A}_g, \mathbb{Q})$ if g odd?

Toroidal compactifications of $\mathcal{A}_g$ and $\mathcal{A}_g^{trop}$

Let $\Omega_g = \{$ positive definite quadratic forms in $\mathbb{R}^g \} \subset \mathbb{R}^{\binom{g+1}{2}}$

$\cap$

$\Omega_g^{rat} = \{$ positive semi-definite quadratic forms in $\mathbb{R}^g$ with rational kernel $\}$

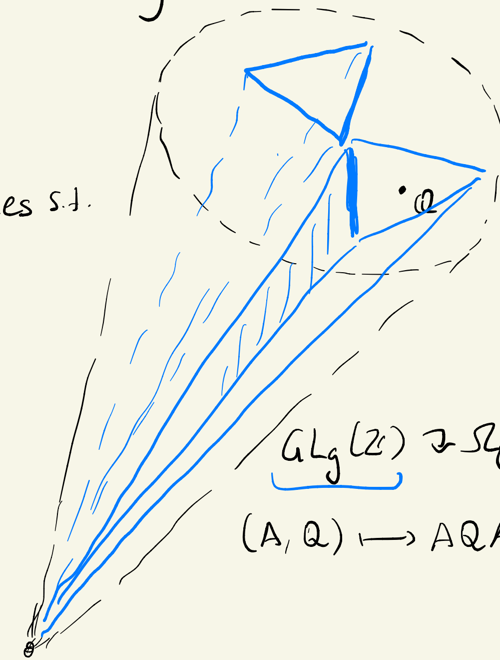
Def An admissible decomposition of Ω_g^{Rat} is a family $\{\sigma\}_{\sigma\in\Sigma}$ of rat'l poly. cones s.t.

1) $\Omega_g^{rat} = \bigcup_{\sigma\in\Sigma} \sigma$

2) Σ is closed under taking faces and intersections

3) Σ is stable under action of $GL_g(\mathbb{Z})$

4) $\underbrace{\# \Sigma / GL_g(\mathbb{Z})}$ $< \infty$



$\underbrace{GL_g(\mathbb{Z})} \curvearrowright \Omega_g^{rat}$

$(A, Q) \longmapsto AQA^T$

Facts Given Σ can construct $\overline{\mathcal{A}}_g^{\Sigma}$ and $\mathcal{A}_g^{\Sigma,\mathrm{trop}}$ s.t.

- $\mathcal{A}_g \subset \overline{\mathcal{A}}_g^{\Sigma}$ is a toroidal compactification
- $\overline{\mathcal{A}}_g^{\Sigma} \setminus \mathcal{A}_g$ is stratified according to poset structure on Σ
- $\exists\, \Sigma : \overline{\mathcal{A}}_g^{\Sigma} \setminus \mathcal{A}_g$ is simple normal crossings (SNC)
- $\mathcal{A}_g^{\Sigma,\mathrm{trop}} := \varinjlim \{\sigma\}_{\sigma \in \Sigma}$, arrows given by inclusions of faces composed with action of $GL_g(\mathbb{Z})$

  is a generalized cone complex [BMV]

Remarks

- $\overline{\mathcal{A}}_g^{\Sigma}$ has different regularity properties according to Σ, (not always simplest Σ will give better behaved $\overline{\mathcal{A}}_g^{\Sigma}$)
- $\forall\, \Sigma$, $\overline{\mathcal{A}}_g^{\Sigma} \longrightarrow \overline{\mathcal{A}}_g^{SAT} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0$

<u>Thm</u> [Comparison theorem]

Let Σ be as above. then

$$Gr^W_{2d} H^{2d-i}(\mathcal{A}_g, \mathbb{Q}) \cong \widetilde{H}_{i-1}(L\mathcal{A}_g^{trop, \Sigma}, \mathbb{Q})$$

<u>Proof</u> (Sketch)

1) Can assume that all cones are smooth and that $\overline{\mathcal{A}_g}^{\Sigma} \setminus \mathcal{A}_g$ is SNC

- start with Σ, can consider a refinement of Σ, Σ' with these properties [FC]
- $L\mathcal{A}_g^{trop, \Sigma} \cong L\mathcal{A}_g^{trop, \Sigma'}$

2) $\mathcal{A}_g$ is smooth DM stack

$\mathcal{A}_g \subset \overline{\mathcal{A}_g}^{\Sigma}$ is <u>SNC</u>

$\Delta(\mathcal{A}_g \subset \overline{\mathcal{A}_g}^{\Sigma}) \cong L\mathcal{A}_g^{trop, \Sigma}$ ([FC] + const of $\mathcal{A}_g^{trop, \Sigma}$)

3) Deligne's mixed Hodge theory

+ [CGP] gen'l to smooth DM stacks

$$\mathrm{Gr}^W_{2d} H^{2d-i}(\mathcal{A}_g, \mathbb{Q}) \cong \tilde{H}_{i-1}(\Delta(\mathcal{A}_g \subset \mathcal{A}_g^{\Sigma}), \mathbb{Q})$$

$$\cong$$

$$\tilde{H}_{i-1}(L\mathcal{A}_g^{\mathrm{trop},\Sigma}, \mathbb{Q}).$$

$\blacksquare$

<u>Rmk</u> This implies that

$\mathrm{Gr}^W_{2d} H^{\cdot}(\mathcal{A}_g, \mathbb{Q})$ can be computed by considering any Σ, even though $\overline{\mathcal{A}_g}^{\Sigma}$ not very regular.

Perfect cone decomposition Σ_g^P

Let $Q \in \Omega_g$

- $\Pi(Q) := \{\bar{x} \in \mathbb{Z}^g \setminus \{\bar{0}\} : \bar{x} \text{ minimizes } Q\}$
- $\sigma(Q) := \mathbb{R}_{\geq 0}\langle \bar{x}\bar{x}^T, \bar{x} \in \Pi(Q)\rangle$
- $\Sigma_g^P = \{\sigma(Q) : Q \in \Omega_g\}$

Thm [Voronoi 1908] Σ_g^P is an admissible decomposition of Ω_g^{rat}, called perfect cone decomposition.

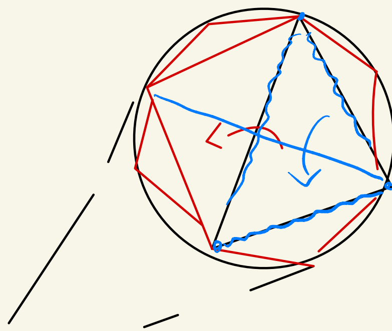
__Example__ $g=2$

$$Q = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad Q(x,y) = x^2 + xy + y^2$$

$$\mathcal{M}(Q) = \{ \pm(1,0),\ \pm(0,1),\ \pm(1,-1) \}$$

$$\sigma(Q) = \mathbb{R}_{\geq 0} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}$$

$$= \sigma^2_{\mathrm{prin}}$$



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

σ^2_{prin} not or.-preserving

Def: A cone is <u>alternating</u> if given $A \in GL_g(\mathbb{Z})$ s.t. $A\sigma A^T = \sigma$, the action of A is orientation preserving.

Def [Perfect complex $P_\bullet^g$]

$$P_k^g = \mathbb{Q}\left\langle \begin{matrix} GL_g(\mathbb{Z})\text{-orbits of alternating} \\ \text{cones in } \Sigma_g^P \text{ of dim } k+1 \end{matrix} \right\rangle$$

$$\partial_k([\sigma]) = \sum_{\substack{\tau \subseteq \sigma \\ \text{facet}}} (-1)^{\omega} [\tau]$$

Prop $\forall i \geq 0$, $H_{i-1}(P_\bullet^g) \cong \tilde{H}_{i-1}(L\mathcal{A}_g^{trop, \Sigma}, \mathbb{Q})$

$$\cong Gr^W_{2d} H^{2d-i}(\mathcal{A}_g, \mathbb{Q})$$

$\underline{g=2}$ $P^2_\bullet$: $0 \to P^2_2 \to P^2_1 \to P^2_0 \to P^2_{-1} \to 0$

$$0 \to 0 \to \mathbb{Q} \xrightarrow{=} \mathbb{Q} \to 0$$

$$H_k(P^2_\bullet) = 0, \ \forall k$$

$$\Rightarrow \mathrm{Gr}^W_6 H^\cdot(\mathcal{A}_2, \mathbb{Q}) = 0$$