# Relations in cohomology rings via mirror symmetry (a "preview")

based on j.w. with Hausel (topological)

Let $T$ be a torus

$H^*(T) \simeq \Lambda^* H^1(T)$

Let $T^\vee$ be the dual torus

$H^*(T^\vee) = \Lambda^* \underbrace{H^1(T)^*}_{\text{dual space}}$.

(1) $H^*(T^\vee) \cong H^{d-*}(T)$ $\quad d = \dim T$

When $T$ is abelian variety this is realized by a geometric operator called Fourier-Mukai transform.

Let $T$ be say principally polarized $\to$ ample line bundle $\to$ $w \in H^2(T)$.

induces symplectic form on $H^1(T)$.

$H^*(T^\vee) \cong H^*(T)$. and (1) is an operator on $H^*(T)$. $= S$-matrix

Explicit construction: $\exists$ $sl_2$ triple $E, F, H$ acting on $H^*(T)$ where $E = w$ $\quad H = \text{degree} - \frac{\dim}{2}$

because of Hard Lefschetz theorem:

$w^k \quad H^{d-k}(T) \to H^{d+k}(T)$ is an isomorphism.

$F$ is somewhat not explicit we call $F = L$ the Lefschetz operator.

$sl_2\text{-rep} \rightsquigarrow SL_2\text{-rep}$. $\quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2$ is the operator

giving iso $H^{d-k} \to H^{d+k}$

$S =$ F-M transform

$S = \exp(E) \exp(-F) \exp(E)$

finite sums.

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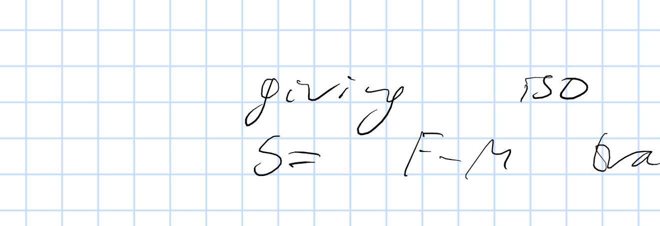
Character varieties $\quad \Sigma = \Sigma_{g,k}$ (genus, punctures)



base point

$X^{\text{Betti}} = \{\varphi : \pi_1(\Sigma) \to GL_n \mid \varphi(\gamma_i) \in C_i\}$ / conjugation

small loop; prescribed conj. classes.

To simplify: $k = 1$

1) "twisted case" $C_1 = e^{\frac{2\pi i}{n}} \cdot \mathbb{1}$

2) regular semisimple case $C_1 =$ fixed distinct e/v-s.

Non-abelian Hodge (a.k.a. Simpson's corresp.)

$X_{\text{Dol}} =$ 1) moduli space of stable $GL_n$-Higgs bundles $(\mathcal{E}, \theta)$ of degree $d$, rank $n$ $\quad (d, n) = 1$.

$x_0 \in \Sigma$ $\quad$ 2) Moduli space of stable triples $(\mathcal{E}, \theta, F)$ $\quad \mathcal{E}$ v.b. on $\Sigma$

$\theta : \mathcal{E} \to \mathcal{E} \otimes \omega_\Sigma$ (with a pole at $x_0$)

$F$ is a complete flag in $\mathcal{E}(x_0)$

s.t. $\operatorname{res} \theta$ is nilpotent and preserves $F$.

(need stability function).

stability functions $\Longleftrightarrow$ eigenvalues of $C_1$

$X_{\text{Dol}}$ is diffeo to $X_{\text{Betti}}$.

Du Pei: $\quad X_{\text{Dol}} \downarrow \mathbb{C}^N$ Hitchin fibration generic fibers are Abelian varieties

Since $GL_n = GL_n^\vee$ : $X$ is its own mirror dual.

F-M transform in generic fibers should be "integrated" and extended to special fibers.

$\Downarrow$

Cohomology should have $S$-matrix.

Suggestion: Use Lefschetz operator to construct $sl_2$ triple and obtain $S$-matrix as $\exp(E)\exp(-F)\exp(E)$.

Cohomology ring of $X$ (following Markman, Hausel, Thaddeus.)

$H^*(X)$ is generated by tautological classes:

$\mathcal{E} \downarrow X \times \Sigma$

($\mathcal{E}$ is not "canonical" only up to $\mathcal{E} \to \mathcal{E} \otimes \mathcal{L}$)

$\Downarrow$

$c_2(\mathcal{E}), c_3(\mathcal{E}) \ldots \in H^*(X \times \Sigma)$.

so $c_k(\mathcal{E})$ defines a map $H_*(\Sigma) \to H^*(X)$.

$H_*(\Sigma) \cong H^*(\Sigma) = H^0(\Sigma) \quad H^1(\Sigma) \quad H^2(\Sigma)$

$H_2$: $\alpha = [\Sigma]$; $\quad H_1$: basis $\psi_1, \ldots, \psi_g, \psi_{g+1}, \ldots, \psi_{2g}$; $\quad H_0$: $\beta = [pt]$

obtain classes

$\alpha_k \in H^{2k-2}(X) \quad \beta_k \in H^{2k}(X)$

$\psi_{ik} \in H^{2k-1}(X)$.

twisted case:

$H^*(X)$ is generated by $\alpha_k, \beta_k, \psi_{ik}$.

Relations known only for $n = 2$.

reg. semisimple case (not written down)

$H^*(X)$ is generated by $\alpha_k, \psi_{ik}$, $x_1, \ldots, x_n \in H^2(X)$.

$x_i = c_1(L_i)$ $\quad (\beta_k = e_k(x_1, \ldots, x_n))$

in fact

$H^*(\text{twisted case}) = H^*(\text{reg. semisimple})^{S_n}$.

take $\alpha_2 \in H^2(X)$. Satisfies "relative Lefschetz"

for $X_{\text{Dol}}$, Perverse filtration

"curious Lefschetz" for $X_{\text{Betti}}$, weight filtration

$\Downarrow$

$\alpha_2 = E \quad H =$ degree operator,

fit into $SL_2$-triple $\quad$ ($L = F$ is a not explicit Lefschetz operator)

<u>Main Question</u>: what if we apply the $S$-matrix to the generators? What do we obtain?

Answer for $x_i$: $\quad S x_i S^{-1} = y_i$.

<u>Idea</u> $y_i$ come from monodromy (move around the eigenvalues of $C_1$ ($\in \mathbb{C}^*$))

$\Downarrow$

unipotent endomorphisms of $H^*(X)$

$y_i = \log$ (of those).

<u>Proof</u> In Betti case $\frac{\partial}{\partial t_i} \alpha_2 = x_i$

Application "Voodoo operator".

Consider $H^*(X) = \mathbb{C}[\alpha_k, x_i, \psi_{ij}] / \mathcal{I}$

some ideal!

<u>Def</u> Operator $N : H^*(X) \to H^*(X)$ is called differential operator of order $\leq k$ if $[N, \alpha_k]$, $[N, x_i]$, $[N, \psi_{ij}]$ is a diff op of order $\leq k-1$.

$N =$ sum of partial derivatives.

<u>Conjecture</u> $(L, (\alpha_k), x_i)$ form a representation of the Lie algebra of Hamiltonian vector fields on $\mathbb{C}^2$.

$L \to \sum y_i \frac{\partial}{\partial x_i} \qquad \alpha_k = \sum x_i^k \frac{\partial}{\partial y_i}$

(based on Rozansky)

$\Downarrow$

$L$ is a diff op of order 2.

we need $[L, \alpha_k]$ ...

<u>Result</u> in rank 2 $\quad \alpha = \alpha_2 \quad \beta = \beta_2$

$$L = -2\beta \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \beta} + \alpha \frac{\partial}{\partial \alpha} - \alpha \frac{\partial^2}{\partial^2 \alpha} - 4\gamma \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \gamma} + \frac{\beta}{2} \gamma \frac{\partial^2}{\partial^2 \gamma} - \frac{\gamma}{2} \beta \frac{\partial}{\partial \gamma}.$$

"Voodoo operator".

$[L, \alpha] =$ degree

acts on $H^*(X)$ (rank $= 2$) $\Rightarrow L\mathcal{I} \subset \mathcal{I}$

<u>Conjecture</u> $L$ generates $\mathcal{I}$ from "trivial" relation $H^m(X) = 0$ if $m > \dim X$.

also in rank $= 3$.

<u>Expection</u>:

$y_i = \log(\text{monodromy})$

$\pi_1\left(\left((\mathbb{C}^*)^{n-1} \setminus \bigcup \Delta_{ij}\right) / S_n\right) = ABr_n$

$\downarrow$

$\mathbb{Z}^{n-1} \rtimes S_n$ acts on $H^*(X)$