Algebraic Geometry Course

Proj construction

Graded rings

Def Graded ring: R ring with a group decomposition $R = \bigoplus_{d=0}^{\infty} R_d$ ($R_d \subset R$ subgps) such that $R_d \cdot R_e \subseteq R_{e+d}$

Def Graded A-algebra: A-algebra R with a $\mathbb{Z}$-grading

Def Homogeneous ideal = ideal in graded ring generated by homogeneous elements

Def Irrelevant ideal $R_+ = \bigoplus_{d=1} R_d$

Def M $\mathbb{Z}$-graded R-module: there exists M_i subgroups s.t. $M = \bigoplus_{i=-\infty}^{\infty} M_i$ and $R_d M_i \subseteq M_{d+i}$

Def f homogeneous: $M_f := \bigoplus_{i=-\infty}^{\infty} (M_f)_i$ where $(M_f)_i = \{ \frac{m}{f^k}$ where $i = \deg(m) + k \deg(f) \}$

Def $M(n)$ the graded R-module $M(n)_i := M_{n+i}$

Proj of a graded ring

Given $R=\bigoplus_{d=0}^{\infty} R_d$ **Def** $\mathrm{Proj}(R) = \{ \mathfrak{p}$ homogeneous prime ideal of R s.t. $R_+ \not\subseteq \mathfrak{p} \}$

$f \in R_d$ (homogeneous) **Def** $D(f) = \{ \mathfrak{p} \in \mathrm{Proj}(R)$ s.t. $f \notin \mathfrak{p} \}$

$V(f) = \{ \mathfrak{p} \in \mathrm{Proj}(R)$ s.t. $f \in \mathfrak{p} \}$

Def Zariski topology $:= \{$ closed $= V(I) \}$ **Th** $\{D(f)\}$ base of the Zariski topology

lemma $D(f) \cong \mathrm{Spec}((R_f)_0)$ $\rightsquigarrow$ Proj can be covered by affine schemes $\rightsquigarrow$ $\boxed{\text{cor: } \mathrm{Proj}(R) \text{ is a scheme}}$

$\mathfrak{p} \mapsto \mathfrak{p}_f \mapsfrom (\mathfrak{p}_f)_0$

idea $\mathfrak{p} \subseteq R$ homogeneous prime ideal with $f \notin \mathfrak{p}$ $\rightleftarrows$ $\mathfrak{p}_f \subseteq R_f$ homogeneous prime ideal

$(\mathfrak{p}_f)_0 :=$ summand of degree 0 $\quad \bigoplus \mathfrak{q}_i = \mathfrak{p} \mapsto \mathfrak{p}_0 \quad$ $\mathfrak{q}_i = \{ r \in R_d$ s.t. $\frac{r^{\deg(f)}}{f^d} \in \mathfrak{p}_0 \}$

Given $S \longrightarrow R$ morphism of graded rings $\rightsquigarrow$ $\mathrm{Proj}(R) \longrightarrow \mathrm{Proj}(S)$

lemma: $I \trianglelefteq R$ homogeneous ideal, $R \to R/I$ gives $\mathrm{Proj}(R/I) \hookrightarrow \mathrm{Proj}(R)$ closed

idea: locally closed on $D(f)$

Relative Proj

Def Y scheme, a scheme X/Y (X over Y) scheme with $f : X \longrightarrow Y$

A morphism of schemes over Y is
$$\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

Note If $Y = \mathrm{Spec}(S)$ and $X = \mathrm{Spec}(R)$, then R is an S-algebra

Given R a finitely generated graded S-algebra, $\mathrm{Proj}(R) \longrightarrow \mathrm{Spec}(S)$
$\left(\overset{\shortparallel}{\bigoplus_{d \geq 0}} R_d \text{ with } R_0 = S \right)$ $\quad \mathfrak{p} \longmapsto \mathfrak{p}_0$

X scheme $\mathcal{R}$ = quasicoherent sheaf of graded $\mathcal{O}_X$-algebras, $\mathcal{R} = \bigoplus_{d=0}^{\infty} \mathcal{R}_d$ $(\mathcal{R}_0 = \mathcal{O}_X)$

Def $\mathrm{Proj}(\mathcal{R}) = \bigcup_i \mathrm{Proj}(\mathcal{R}|_{\mathrm{Spec}(S_i)})$ gluing well defined by qcoh sheaf condition

Examples (a) $\mathrm{Sym}^\bullet(V^*) := k \oplus V^* \oplus \mathrm{Sym}^2(V^*) \oplus \ldots$ $(\cong k[v_1^* \ldots v_n^*]) \rightsquigarrow \mathrm{Proj}(\mathrm{Sym}^\bullet(V^*)) \cong \mathbb{P}(V)$

(b) $S[x] \rightsquigarrow \mathbb{P}^0_S := \mathrm{Proj}(S[x]) \cong \mathrm{Spec}(S)$

(c) $S[x_0 \_ x_n] \rightsquigarrow \mathbb{P}^n_S := \mathrm{Proj}(S[x_0 \_ x_n]) \cong \mathrm{Spec}(S) \times \mathbb{P}^n_k$

(d) $\mathcal{E}$ locally free on X $\quad \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{E}^*) \cong \mathbb{P}_X(\mathcal{E})$

Twisting sheaves

Given $R = \bigoplus_{d=0}^{\infty} R_d$, $X = \mathrm{Proj}(R)$, given $M = \bigoplus_{d=-\infty}^{\infty} M_i$ graded R-module

Def $\widetilde{M}$ quasicoherent sheaf on X obtained by gluing $\widetilde{M_f}$ on $D(f) \cong \mathrm{Spec}(R_{f,0})$

Def $\mathcal{O}_X(n) := \widetilde{R(n)}$

Observe $\mathcal{O}_{\mathbb{P}^N}(n) = \{ \text{locally } \frac{g}{f} \text{ where } \deg(g) - \deg(f) = n \}$

$\mathcal{O}_X(n) = i^* \mathcal{O}_{\mathbb{P}^N}(n)$ for $i : X \hookrightarrow \mathbb{P}^N$.

Blow-up

$$\underbrace{0 \to \mathcal{I} \to \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0}$$

X (Noetherian) scheme, $Y \xhookrightarrow{i} X$ subscheme, $\mathcal{I} \triangleleft \mathcal{O}_X$ coherent sheaf of ideals

Consider $\mathcal{O}_X$-algebra $\bigoplus_{i=0}^{\infty} \mathcal{I}^i$ where $\mathcal{I}^0 := \mathcal{O}_X$

__Def__ (Blow-up of X at Y) $Bl_Y(X) := \mathrm{Proj}\left(\bigoplus_{i=0} \mathcal{I}^i\right)$

__Example__: $\mathbb{A}^n = \mathrm{Spec}(K[x_1 \ldots x_n])$, $Y = \{0\}$, $\mathcal{I} = \langle x_1 \ldots x_n \rangle$, $S = \bigoplus_{i=0}^{\infty} \mathcal{I}^i$

$$K[x_1 \ldots x_n][y_1 \ldots y_n] \longrightarrow S = K[x_1 \ldots x_n, y_1 \ldots y_n] / \langle x_i y_j - x_j y_i \text{ forall } i,j \rangle$$

$$\mathrm{Proj}(K[x_1 \ldots x_n][y_1 \ldots y_n]) = \mathbb{P}^{n-1}_{\mathbb{A}^n} \cong \mathbb{A}^n \times \mathbb{P}^{n-1}$$

$$Bl_{\{0\}}(\mathbb{A}^n) := \mathrm{Proj}(S) = V_{\mathbb{A}^n \times \mathbb{P}^{n-1}}(\langle x_i y_j - x_j y_i \text{ forall } i,j \rangle)$$

__Example__: $Y \hookrightarrow \mathbb{A}^n$ given by $\mathcal{I} = \langle f_1 \ldots f_n \rangle$

$$K[x_1 \ldots x_n][y_1 \ldots y_n] \longrightarrow S = K[x_1 \ldots x_n, y_1 \ldots y_n] / \langle f_i y_j - f_j y_i \text{ forall } i,j \rangle$$

$$Bl_Y(\mathbb{A}^n) := \mathrm{Proj}(S) = V_{\mathbb{A}^n \times \mathbb{P}^{n-1}}(\langle f_i y_j - f_j y_i \text{ forall } i,j \rangle)$$

Universal property of blowing-up

$Y \overset{i}{\hookrightarrow} X$, $\mathcal{I}$ sheaf of ideals, $Z := Bl_Y(X) = \mathrm{Proj}\left(\bigoplus_{i=0}^{\infty} \mathcal{I}^i\right)$ with $\mathcal{I}^0 = \mathcal{O}_X$

$\mathcal{O}_X \hookrightarrow \bigoplus_{i=0}^{\infty} \mathcal{I}^i \rightsquigarrow \pi : Z \longrightarrow X$ structural morphism of blow-up

<u>Observe</u> $\pi^{-1}\mathcal{I} = \widetilde{\mathcal{I} \otimes_{\mathcal{O}_X} \bigoplus_{i=0} \mathcal{I}^i} = \widetilde{\bigoplus_{i=1}^{\infty} \mathcal{I}^i} = \widetilde{\bigoplus_{i=0}^{\infty} \mathcal{I}^i(1)} = \mathcal{O}_Z(1)$

<u>Theorem</u> Let $W \xrightarrow{g} X$ such that $g^{-1}\mathcal{I}$ is invertible, then there exists f such that

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[dr, dotted, "f"'] & Bl_Y(X) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

commutes