Algebraic Geometry Course

Cohomology of sheaves

Quasicoherent sheaves

Def (Sheaf associated to a module) Let $X = \mathrm{Spec}(R)$, M R-module

$$\widetilde{M}(U) := \left\{ \varphi = (\varphi_P)_{P \in U} \text{ such that } \varphi \in M_P \text{ for all } P \in U,\ \exists\, U_P \ni P \atop \exists\, g \in M,\ f \in R \text{ with } \varphi_Q = \frac{g}{f} \text{ for all } Q \in U_P \right\}$$

Def (Quasicoherent sheaf) $\mathcal{F}$ on X is quasicoherent $\Leftrightarrow$ exists cover of X by $U_i = \mathrm{Spec}(R_i)$ and $\mathcal{F}|_{U_i} = \widetilde{M}_i$ $M_i = R_i$-mod
(locally associated to R-modules)

lemma $X = \mathrm{Spec}(R)$

(a) $\left\{ \begin{matrix} \text{morphism of sheaves} \\ \text{associated to modules} \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ R\text{-module homomorphisms} \right\}$

(b) $0 \to \widetilde{M}_1 \to \widetilde{M}_2 \to \widetilde{M}_3 \to 0$ exact $\Leftrightarrow$ $0 \to M_1 \to M_2 \to M_3 \to 0$ exact

What happens on the case of a general scheme $X = \bigcup \mathrm{Spec}(R_i)$?

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \Rightarrow 0 \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X)$$

Motivation

Objective :

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \implies 0 \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X) \to H^1(\mathcal{F}_1) \to H^1(\mathcal{F}_2)$$

$$\to H^1(\mathcal{F}_3) \to H^2(\mathcal{F}_1) \to H^2(\mathcal{F}_2) \to H^2(\mathcal{F}_3) \to \dots$$

long exact sequence

$H^i(\mathcal{F})$ finite dimensional vector spaces

$H^i(\mathcal{F}) \neq 0$ only for a finite set of i

$\implies$ numerical invariants

On affine schemes exactness is preserved

construction of $H^i(\mathcal{F})$ $\rightsquigarrow$ failure of exactness gluing affine subschemes

Definition

Given $X = \bigcup_{i=0}^{r} \mathcal{U}_i = \mathrm{Spec}(R_i)$ and given $\mathcal{F}$ quasicoherent

Def $C^p(\mathcal{F}) = \bigoplus_{i_0 < \ldots < i_p} \mathcal{F}(\mathcal{U}_{i_0} \cap \ldots \cap \mathcal{F}(\mathcal{U}_{i_p}))$

$$d^p : C^p(\mathcal{F}) \longrightarrow C^{p+1}(\mathcal{F})$$

$$\{\varphi_{i_0 \ldots i_p}\} \longmapsto \{(d^p\varphi)_{i_0 \ldots i_{p+1}} = \sum_{\ell=0}^{r} (-1)^{\ell} \varphi_{i_0 \ldots \hat{i}_\ell \ldots i_{p+1}}|_{\mathcal{U}_{i_0} \cap \ldots \cap \mathcal{U}_{p+1}}\}$$

Lemma $d^{p+1} \circ d^p = 0 \Rightarrow 0 \to C^0(\mathcal{F}) \to C^1(\mathcal{F}) \to C^2(\mathcal{F}) \to \ldots$ is a complex

Def (Cohomology of $\mathcal{F}$) $H^i(\mathcal{F}) := \mathrm{ker}\, d^i / \mathrm{im}(d^{i-1})$

Well defineness of $H^i(\mathcal{F})$ thanks to :

Th Given $\{\mathcal{U}_i\}$ and $\{V_i\}$ affine covers of $X \Rightarrow H^i(\{\mathcal{U}_i\}, \mathcal{F}) = H^i(\{V_i\}, \mathcal{F}) =: H^i(X, \mathcal{F})$

Properties of cohomology

Given $\mathcal{F}$ quasicoherent on a variety X, then

(a) $H^0(X,\mathcal{F}) = \mathcal{F}(X)$ $\left(C^0(\mathcal{F}) = \bigoplus_{i=0}^{r} \mathcal{F}(U_i),\ \ker(d^0) = \{ \varphi_i - \varphi_j |_{U_i \cap U_j} = 0 \} \right)$

(b) X = affine $H^i(X,\mathcal{F}) = 0$ for $i > 0$ (exactness preserved)

(c) X = projective $H^i(X,\mathcal{F}) = 0$ for $i > \dim(X)$ $\left(\dim(X) = n \Leftrightarrow \Leftrightarrow X - V(f_0 \ldots f_n) = \emptyset \Leftrightarrow \Leftrightarrow X \text{ covered by } n+1 \text{ affine} \right)$

(d) $X \text{ closed } \overset{i}{\hookrightarrow} Y$ $H^i(Y, i_*\mathcal{F}) = H^i(X,\mathcal{F})$ $\left(C^r(i_*\mathcal{F}) = \bigoplus_{i_0 < \ldots < i_r} \mathcal{F}(X \cap \bigcap_{i=0}^{r} U_i) = C^r(\mathcal{F}) \right)$

(e) given $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ exact, then there is

$$0 \to H^0(X,\mathcal{F}_1) \to H^0(X,\mathcal{F}_2) \to H^0(X,\mathcal{F}_3) \to H^1(X,\mathcal{F}_1) \to$$
$$\to H^1(X,\mathcal{F}_2) \to H^1(X,\mathcal{F}_3) \to H^2(X,\mathcal{F}_1) \to H^2(X,\mathcal{F}_2) \to \ldots$$

long exact sequence (thanks to the snake lemma)

Cohomology of twisting sheaves

prop (a) $H^0(\mathbb{P}^n, \mathcal{O}_P(d)) = K[x_0 \_ x_n]_d$ $\quad \underline{d \geq 0}$

$$h^0(\mathbb{P}^n, \mathcal{O}_P(d)) = \binom{n+d}{n}$$

(b) $H^n(\mathbb{P}^n, \mathcal{O}_P(d)) = \dfrac{1}{x_0 x_1 \_ x_n} K[x_0^{-1} \_ x_n^{-1}]_d$ $\quad \underline{-n-1 \geq d}$

$$h^n(\mathbb{P}^n, \mathcal{O}_P(d)) = \binom{-d-1}{n}$$

(c) $H^i(\mathbb{P}^n, \mathcal{O}_P(d)) = 0$ any other case

<u>idea</u> (a) $\mathcal{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_P(d)$ global sections $= K[x_0 \_ x_n]$

(b) $U_i = \{x_i \neq 0\}$ $\quad \mathcal{F}(U_{i_0} \cap \_ \cap U_{i_l}) = K[x_0 x_1 \_ x_n]_{x_{i_0} \_ x_{i_l}}$ $\quad C^n(\mathcal{F}) = \bigoplus_{i=0}^{n} K[x_0 \_ x_n]_{x_0 \_ \hat{x}_i \_ x_n}$

$$H^n(\mathbb{P}^n, \mathcal{F}) = C^{n+1}(\mathcal{F}) / \mathrm{im}(d^n) = \frac{\mathrm{Lin}(x_0^{k_0} \cdot \_ \cdot x_n^{k_n} \quad k_i \in \mathbb{Z})}{\mathrm{Lin}(x_0^{k_0} \cdot \_ \cdot x_n^{k_n} \text{ one } k_i \geq 0)}$$

$$= \mathrm{Lin}(x_0^{k_0} \cdot \_ \cdot x_n^{k_n} \text{ all } k_i < 0)$$

$C^{n+1}(\mathcal{F}) = K[x_0 \_ x_n]_{x_0 \_ x_n}$

$C^{n+2}(\mathcal{F}) = 0$

(c) By induction on n $\quad \mathbb{P}^1$ ✓ $\quad i: \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$

$$0 \to \mathcal{F}_{\mathbb{P}^n} \xrightarrow{x_0} \mathcal{F}_{\mathbb{P}^n} \to i_* \mathcal{F}_{\mathbb{P}^{n-1}} \to 0 \xRightarrow{\text{long exact seq.}} x_0 : H^p(\mathcal{F}) \xrightarrow{\cong} H^p(\mathcal{F}) \Rightarrow H^p(\mathcal{F}) = 0$$

Euler characteristic

Def (Euler characteristic) $\chi(X,\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i h^i(X,\mathcal{F})$

A numerical invariant well behaved under short exact sequences

lemma Given $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ exact $\Rightarrow \chi(X,\mathcal{F}_2) = \chi(X,\mathcal{F}_1) + \chi(X,\mathcal{F}_3)$

idea The alternating sum of dims of elements in a long exact seq. is 0

lemma (a) $\chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{n}$

(b) $X \subset \mathbb{P}^n$ hypersurface of deg $= d \Rightarrow \chi(X, \mathcal{O}_X(\ell)) = \binom{n+\ell}{n} - \binom{n+\ell-d}{n}$

$$\chi(X, \Omega_X) = \binom{n-2d}{n} - (n+1)\binom{n-d-1}{n} - 1$$

(idea: use $0 \to \mathcal{O}_P(-d) \to \mathcal{O}_P \to i_* \mathcal{O}_X \to 0$

$i^*(0 \to \Omega_P \to \mathcal{O}_P(-1)^{n+1} \to \mathcal{O}_P \to 0)$

$0 \to \mathcal{O}_X(-d) \to i^*\Omega_P \to \Omega_X \to 0$ conormal seq.)