# Algebraic Geometry Course

## Differentials

# The sheaf of differentials

Differenciating was important to study the tangent space (and smoothness)

Very easy for polynomials, determined by

(a) $d(f+g) = df + dg$

(b) $d(fg) = f\,dg + g\,df$  $f,g \in R$  $k$-algebra

(c) $df = 0$ for $f \in k$

def **Differentials of R** : $\Omega_R$ = free $R$-module generated by $df$ $(f \in R)$ modulo (a).(b) and (c)

$$\begin{aligned} d : R &\longrightarrow \Omega_R \\ f &\longmapsto df \end{aligned}$$

(not a homomorphism, only $\mathbb{C}$-linear)

**Example**

(a) $\mathbb{A}^n = \operatorname{Spec}(R = k[x_1,\ldots,x_n])$  $\Omega_R = R\,dx_1 \oplus \ldots \oplus R\,dx_n$

(b) $\operatorname{Spec}\left(R = {}^{k[x_1,\ldots,x_n]}/_{\langle f_1,\ldots,f_m\rangle}\right)$  $\Omega_R = R\,dx_1 \oplus \ldots \oplus R\,dx_n \Big/ \left\langle \sum_j \frac{\partial f_i}{\partial x_j} dx_j \;\; \begin{matrix} i \in \{1,\ldots,m\} \\ j \in \{1,\ldots,n\} \end{matrix} \right\rangle$

for $P \in X$ closed $(R/_P = k)$

linear forms that vanish on the tangent

$$T_pX = V\left(\sum_j \frac{\partial f_i}{\partial x_j}(P)\, x_j\right)$$

$$\Omega_R \otimes_R R/_P = k\,dx_1 \oplus \ldots \oplus k\,dx_n \Big/ \left\langle \sum_j \frac{\partial f_i}{\partial x_j}(P)\, dx_j \;\; \begin{matrix} i \in \{1,\ldots,m\} \\ j \in \{1,\ldots,n\} \end{matrix} \right\rangle$$

$\cong$

$T_pX^\vee$ dual of the tangent space at $P \in X$

$k\,dx_1 \oplus \ldots \oplus k\,dx_n$ ← linear forms

# The cotangent sheaf

**Observe**

(a) $(f+g)\otimes_k 1 - 1\otimes_k (f+g) = (f\otimes_k 1 - 1\otimes_k f) + (g\otimes_k 1 - 1\otimes_k g)$

(b) $(fg)\otimes_k 1 - 1\otimes_k (fg) = f(g\otimes_k 1 - 1\otimes_k g) + (f\otimes_k 1 - 1\otimes_k f)g$

(c) $\lambda\otimes_k 1 - 1\otimes_k \lambda = 0$

**lemma** $R$ a $k$-algebra, $\delta : R\otimes_k R \longrightarrow R$, $f\otimes g \longmapsto fg$, then $\Omega_R = \ker(\delta)/\ker(\delta)^2$

**idea of proof:**

1st) $R\otimes_k R$ has 2 $R$-mod struc. but $\ker(\delta)/\ker(\delta)^2$ has 1 $R$-mod str.

$$h(\underset{\underset{\ker(\delta)}{\cap}}{\textstyle\sum f_i\otimes g_i}) - (\underset{\underset{\ker(\delta)}{\cap}}{\textstyle\sum f_i\otimes g_i})h = \textstyle\sum hf_i\otimes g_i - \sum f_i\otimes hg_i = \underbrace{(\textstyle\sum f_i\otimes g_i)(\underset{\underset{\ker(\delta)}{\cap}}{1\otimes h - h\otimes 1})}_{\ker(\delta)^2}$$

2nd)
$$\begin{tikzcd} \ker(\delta)/\ker(\delta)^2 \arrow[r] & \Omega_R \arrow[r] & \ker(\delta)/\ker(\delta)^2 \\ f\otimes 1 - 1\otimes f \arrow[r, mapsto] & df \arrow[r, mapsto] & f\otimes 1 - 1\otimes f \end{tikzcd}$$

**Def** $X$ separated $\Delta_X \overset{i}{\hookrightarrow} X\times X$ closed. $\mathcal{J}$ ideal sheaf of $\Delta_X$. Cotangent sheaf $\Omega_X := i^*(\mathcal{J}/\mathcal{J}^2)$

**lemma** $X$ smooth $\Longleftrightarrow$ $\Omega_X$ locally free

$\Uparrow$

**idea:** $\dim(T_pX^\vee) = n$ $\quad \operatorname{rk}\left(\frac{\partial f}{\partial x}\right) = \overset{\#\text{eqs}}{r} - \overset{\dim X}{n} \Rightarrow \left(dx_1 \ldots dx_n \,\middle|\, \overset{\det\neq 0}{dx_{n+1} \ldots dx_{n+r}}\right) \Rightarrow \Omega_U = \mathcal{O}_U dx_1 \oplus \ldots \oplus \mathcal{O}_U dx_n$

**Def** if further $X$ smooth Tangent bdle $T_X := \Omega_X^\vee$ Canonical bundle $\omega_X := \Lambda^n \Omega_X$

# The projective space

**prop:** (Euler sequence) $0 \to \Omega_{\mathbb{P}^n} \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \xrightarrow{g} \mathcal{O}_{\mathbb{P}^n} \to 0$

**idea** on $U_0 = \{x_0 \neq 0\}$ $\frac{x_i}{x_0}$ generate $\mathcal{O}_{U_0}$ $d\left(\frac{x_i}{x_0}\right) = \frac{1}{x_0} dx_i - \frac{x_i}{x_0^2} dx_0$ generate $\Omega_{U_0}$

Formal

$$\Omega_{\mathbb{P}^n} \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n}(-1)dx_0 \oplus \text{—} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)\, dx_n$$

$$\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \xrightarrow{g} \mathcal{O}_{\mathbb{P}^n}$$

$$(\varphi_0 \text{—} \varphi_n) \longmapsto x_0\varphi_0 + \text{—} + x_n\varphi_n$$

setting $x_0 = 1$

$dx_i = dx_i - x_i$

$$f \sim \underbrace{\left.\begin{pmatrix} -x_1 & \text{—} & -x_n \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\right\}}_{n} n+1$$

$$g \sim (1, x_1 \text{—} x_n)$$

**cor** $\omega_{\mathbb{P}^n} = \Lambda^{n+1}\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$

**prop** (Conormal sequence) $X = V_p(f) \xhookrightarrow{i} \mathbb{P}^n$ hypersurface of deg $= d$

$$0 \to \mathcal{O}_X(-d) := i^*\mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow i^*\Omega_{\mathbb{P}^n} \longrightarrow \Omega_X \longrightarrow 0$$

**idea** above maps given by $\varphi \longmapsto d(f\varphi)$

$d\psi \longmapsto d\psi|_X$

$U_0 = \{x_0 \neq 0\}$ $X|_{U_0} = V(f^{dh})$

$R = k[x_1 \text{—} x_n]$ $S = R/f^{dh}$ $0 \longrightarrow S \longrightarrow S dx_1 \oplus \text{—} \oplus S dx_n \longrightarrow S dx_1 \oplus \text{—} \oplus S dx_n / \langle df^{dh} \rangle \longrightarrow 0$

$\varphi \longmapsto \cancel{f^{dh} d\varphi} + \varphi\, df^{dh}$