Algebraic Geometry Course

Quasicoherent sheaves

Sheaves from modules

<u>Def</u> (Sheaf associated to a module) Let $X = \mathrm{Spec}(R)$, M R-module

$$\tilde{M}(U) := \left\{ \varphi = (\varphi_P)_{P \in U} \text{ such that } \varphi \in M_P \text{ for all } P \in U,\ \exists U_P \ni P \atop \exists g \in M,\ f \in R \text{ with } \varphi_Q = \frac{g}{f} \text{ for all } Q \in U_P \right\}$$

<u>prop</u> $X = \mathrm{Spec}(R)$, M an R-module

(a) for every $P \in X$, $\tilde{M}_P \cong M_P$ stalk localization

(b) for every $f \in R$, $\tilde{M}(D(f)) = M_f$

<u>Def</u> (Quasicoherent sheaf) $\mathcal{F}$ on X is quasicoherent $\iff$ exists cover of X by (locally associated to R-modules) $U_i = \mathrm{Spec}(R_i)$ and $\mathcal{F}|_{U_i} = \tilde{M}_i$ $M_i = R_i$-mod

<u>Def</u> (Coherent sheaf) $\mathcal{F}$ on X is coherent $\iff$ M_i are finitely generated

<u>lemma</u> (a) $\left\{ \begin{matrix} \text{morphism of sheaves} \\ \text{associated to modules} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \{ R\text{-module homomorphisms} \}$

(b) exact $\iff$ exact ; $\tilde{M} \oplus \tilde{N} = (M \oplus N)^\sim$; $\tilde{M} \otimes \tilde{N} = (M \otimes N)^\sim$; $\tilde{M}^\vee = (M^\vee)^\sim$

Sheaf of ideals

Let $i: Y \hookrightarrow X$ embedding of a closed subvariety

<u>Lemma</u> $\mathcal{F}$ quasicoherent on $Y \Rightarrow i_* \mathcal{F}$ quasicoherent on X

Consider the (surjective) morphism $\mathcal{O}_X \to i_* \mathcal{O}_Y$ $\left(R \longrightarrow R/J\right)$

$$I_{Y/X} := \ker(\mathcal{O}_X \to i_* \mathcal{O}_Y)$$ ideal sheaf of Y

$$0 \longrightarrow I_{Y/X} \to \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y \longrightarrow 0 \qquad (0 \to J \to R \to R/J \to 0)$$

<u>Def</u> Ideal sheaf $\Leftrightarrow$ subsheaf of $\mathcal{O}_X$

Pull-back & push-forward

Def (a) (affine) $f: \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(S)$

$$f^* \widetilde{M} := (M \otimes_S R)^{\sim}$$

(b) (arbitrary) $f: X \longrightarrow Y$

$$f^* \mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

Lemma Let $P \in \mathrm{Spec}(R) \subset X$; $f(P) \in \mathrm{Spec}(S) \subset Y$

$$(f^*\mathcal{F})_P = (M \otimes_S R)_P$$

prop $i: Y \to X$ closed embedding, $\mathcal{F}, \mathcal{G}$ q.coh. sheaves

(a) $i^* i_* \mathcal{F} \cong \mathcal{F}$

(b) (Projection formula) $i_*(\mathcal{F} \otimes i^*\mathcal{G}) \cong (i_*\mathcal{F}) \otimes \mathcal{G}$

$\boxed{\text{for more general morphisms}}$

Locally-free sheaves

Def (locally free sheaves) $\mathcal{F}$ on X loc. free $\Longleftrightarrow$ exists an affine cover $\{U_i\}$ of X and $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$ for some $n \in \mathbb{N}$

(VECTOR BUNDLES)

Lemma Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ exact sequence of quasicoherent sh.

(a) for $\mathcal{E}$ locally free

$$0 \to \mathcal{F}_1 \otimes \mathcal{E} \to \mathcal{F}_2 \otimes \mathcal{E} \to \mathcal{F}_3 \otimes \mathcal{E} \to 0$$

(b) if $\mathcal{F}_i$ locally-free $\Longrightarrow$ $0 \to f^*\mathcal{F}_1 \to f^*\mathcal{F}_2 \to f^*\mathcal{F}_3 \to 0$ exact

Lemma Consider a short exact sequence of locally-free sheaves

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

then $\det(\mathcal{E}_2) \cong \det(\mathcal{E}_1) \otimes \det(\mathcal{E}_3)$