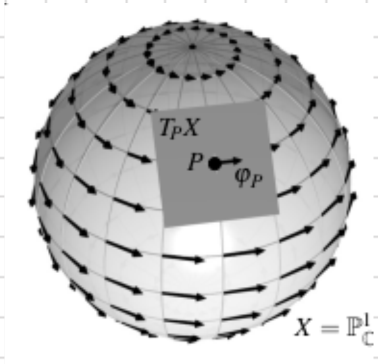
Algebraic Geometry Course

Sheaves of modules

First definitions

useful to describe analogs of functions taking values on <u>varying target</u>

<u>Example</u> Tangent sheaf



<u>Def</u> (X scheme) (pre)-sheaf of ($\mathcal{O}_X$-modules) = (pre) sheaf $\mathcal{F}$ s.t. $\forall U \subset X$ open $\mathcal{F}(U)$ has a $\mathcal{O}_X(U)$-module structure compatible with restrictions

<u>Def</u> Morphism $f : \mathcal{F} \to \mathcal{G}$ of (pre)sheaves of $\mathcal{O}_X$-modules = morphism $f|_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for every $U \subset X$ open compatible with restrictions

(induces morphisms on stalks $f_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$, $\overline{(U,\varphi)} \mapsto \overline{(U, f|_U(\varphi))}$

<u>Def</u>: (Push-forward of sheaves) Given $\mathcal{F}$ on X and $f : X \to Y$, set $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$

Important constructions

<u>Twisting sheaves on $\mathbb{P}^n$</u> $\mathcal{O}_{\mathbb{P}^n}(d)(U) = \left\{ \frac{g}{f} \;\middle|\; \begin{array}{l} f, g \text{ hom. pol } \deg(g) - \deg(f) = d \\ f(P) \neq 0 \text{ for all } P \in U \end{array} \right\}$

$\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological sheaf $K^{n+1} - \{0\} \to \mathbb{P}^n$

Def (Sheafification) $\mathcal{F}'$ presheaf, $\mathcal{F}(U) := \left\{ \varphi = (\varphi)_{P \in U} : \varphi_P \in \mathcal{F}'_P \ \forall P \in U \ \& \ \exists U_P \ni P \ \& \ \exists s \in \mathcal{F}'(U_P) \text{ s.t. } \varphi_Q = s_Q \ \forall Q \in U_P \right\}$

Given $f : \mathcal{F} \to \mathcal{G}$ <u>Kernel sheaf</u> $\ker(f)(U) := \ker\left(f_U : \mathcal{F}(U) \to \mathcal{G}(U)\right)$

<u>Image sheaf</u> $\operatorname{Im}(f)(U) := \operatorname{im}\left(f_U : \mathcal{F}(U) \to \mathcal{G}(U)\right)$

(f injective) <u>Quotient sheaf</u> $\mathcal{G}/\mathcal{F}(U) := \mathcal{G}(U)/\mathcal{F}(U)$

Given $\mathcal{F}, \mathcal{G}$ <u>tensor product</u> $\mathcal{F} \otimes \mathcal{G}(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$

<u>Dual sheaf</u> $\mathcal{F}^\vee(U) = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{O}_X(U))$

<u>Sky-scraper sheaf</u> $P \in X$ ($i : P \hookrightarrow X$) closed point $K_P(U) = i_* \mathcal{O}_P(U) = \begin{cases} K & \text{if } P \in U \\ 0 & \text{if } P \notin U \end{cases}$

Exact sequences

Def A sequence $\cdots \to \mathcal{F}_{i-1} \xrightarrow{f_{i-1}} \mathcal{F}_i \xrightarrow{f_i} \mathcal{F}_{i+1} \to \cdots$ is exact $\iff \operatorname{im}(f_{i-1}) = \ker(f_i)$

$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \iff f$ injective

$\mathcal{F} \to \mathcal{G} \xrightarrow{f} 0 \iff f$ surjective

Example Skyscraper sequence

$$0 \to I_p \to \mathcal{O}_X \to \kappa_p \to 0$$

sheaf of ideals of p

$I_p(U) = \{ \varphi \in \mathcal{O}_X(U) \text{ s.t. } \varphi(p) = 0 \}$

Lemma The sequence $\cdots \to \mathcal{F}_{i-1} \xrightarrow{f_{i-1}} \mathcal{F}_i \xrightarrow{f_i} \mathcal{F}_{i+1} \to \cdots$ is exact

$\Updownarrow$

the sequence $\cdots \to \mathcal{F}_{i-1}(U) \xrightarrow{f_{i-1}} \mathcal{F}_i(U) \xrightarrow{f_i} \mathcal{F}_{i+1}(U) \to \cdots$ is exact

$\Updownarrow$

the sequence $\cdots \to (\mathcal{F}_{i-1})_p \xrightarrow{f_{i-1}} (\mathcal{F}_i)_p \xrightarrow{f_i} (\mathcal{F}_{i+1})_p \to \cdots$ is exact