Algebraic Geometry Course

Schemes

First definitions

Algebraic varieties over ground field k	$\xleftrightarrow{1:1}$	finitely generated k-algebras
$\cap$		$\cap$
Schemes	$\xleftrightarrow{1:1}$	Rings

No ground field required! $\rightsquigarrow$ many concepts redefined to avoid k

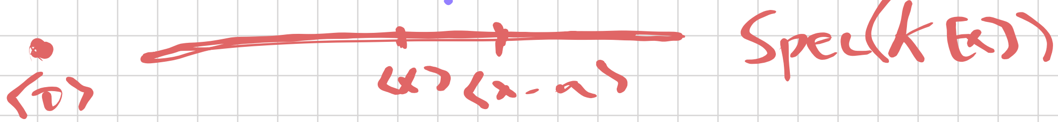
<u>Def</u> Spectrum of a ring

$$\mathrm{Spec}(R) = \{\text{all prime ideals of } R\}$$

<u>dictionary</u>

maximal prime ideal

= points of affine varieties



<u>Def</u> (Regular function) R ring $\quad P \in \mathrm{Spec}(R)$

for any $f \in R$, $f(P) :=$ class of f in the residue field (i.e. the quotient field of R/P)

<u>idea</u> evaluate $f(x=a)$ is the same as computing the class of $f/_{\langle x-a\rangle}$

Zariski topology

Note $f(P)=0 \iff P \ni f$

Def (Zero loci) $S \subset R$ subset $V(S) := \{P \in \mathrm{Spec}(R) \text{ s.t. } f(P)=0 \iff f \in P \ \forall f \in S\}$
$:= \{P \in \mathrm{Spec}(R) \text{ s.t. } P \supset S\}$

Def (Vanishing ideal) $X \subset \mathrm{Spec}(R)$ $I(X) := \{f \in R \text{ s.t } f(P)=0 \iff f \in P \text{ for all } P \in X\}$
$$:= \bigcap_{P \in X} P$$

Def (Zariski topology) $X \subset \mathrm{Spec}(R)$ closed $\iff \exists S \subset R$ s.t. $X = V(S)$

Note There are non-closed points of $\mathrm{Spec}(R)$

i.e. $\langle 0 \rangle \in \mathrm{Spec}(K[x])$
$\overline{\langle 0 \rangle} = V(0) = \{P \ni 0\} = \mathrm{Spec}(K[x])$
$\hookrightarrow \langle 0 \rangle$ generic (open) point

Open and closed subsets

Th (Scheme-theoretic Nullstellensatz) R ring

(a) $X \subset \mathrm{Spec}(R)$ closed $\quad V(I(X)) = X$

(b) For any ideal $J \trianglelefteq R$ $\quad I(V(J)) = \sqrt{J}$

$$\left\{ \begin{matrix} \text{closed subsets} \\ \text{of } \mathrm{Spec}(R) \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{radical ideals} \\ \text{in } R \end{matrix} \right\}$$

Properties of $V(\bullet)$ and $I(\bullet)$

(a) $J_1, J_2 \trianglelefteq R$ $\quad V(J_1) \cup V(J_2) = V(J_1 J_2)$ and $V(J_1) \cap V(J_2) = V(J_1 + J_2)$

(b) $X_1, X_2 \subset \mathrm{Spec}(R)$ closed $\quad I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$ and $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$

Def Distinguished open subsets $\quad f \in R \quad D(f) := \mathrm{Spec}(R) \setminus V(f) =$
$= \{ P \in \mathrm{Spec}(R) \text{ s.t. } f \notin P \}$

Basis of the Zariski topology. for every $\mathcal{U}$ open

$$\mathcal{U} = \mathrm{Spec}(R) \setminus V(S) = \mathrm{Spec}(R) \setminus \bigcap_{f \in S} V(f) = \bigcup_{f \in S} D(f)$$

Regular functions (without a ground field)

Def $\mathcal{U} \subset \operatorname{Spec}(R)$ open, a regular function φ on $\mathcal{U}$ collection of germs $\{\varphi_P\}_{P \in \mathcal{U}}$ s.t. for all $P \in \mathcal{U}$ $\exists\, \mathcal{U}_P \subset \mathcal{U}$ where on all $Q \in \mathcal{U}_P$

$\varphi_P \in K(P) :=$ field of fractions of R/P

$$\varphi_Q = \left[\frac{g}{f}\right] \in R_Q \quad \text{for } g, f \in R,\ f \notin Q \ (f(Q) \neq 0)$$

The structure sheaf $\mathcal{O}_{\operatorname{Spec}(R)}$ gives, for every open $\mathcal{U} \subset \operatorname{Spec}(R)$, the set of regular functions

Def Affine scheme $= (\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ for some ring R

lemma
- For every $P \in \operatorname{Spec}(R)$ $\quad \mathcal{O}_{\operatorname{Spec}(R),P} \cong R_P$ localization stalk
- For every $f \in R$ $\quad \mathcal{O}_{\operatorname{Spec}(R)}(D(f)) \cong R_f$

Problems with morphisms $f: X \to Y$ but $f^*: \mathcal{O}_Y(\mathcal{U}) \to \mathcal{O}_X(f^{-1}(\mathcal{U}))$?

the f^* must be compatible with the local morphisms

$$f_P^*: \mathcal{O}_{Y, f(P)} \to \mathcal{O}_{X,P}$$

Locally ringed spaces

Def A locally ringed space $(X, \mathcal{O}_X)$ = ringed space whose stalks $\mathcal{O}_{X,x}$ are local rings

Def A morphism of locally ringed spaces $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is

- $f : X \longrightarrow Y$ continuous
- for every $U \subset X$ open, a pull-back morphism $f_U^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$

such that f^* is compatible with restrictions and for all $p \in X$ we get $f_p : \mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{X,p}$ and $(f_p^*)^{-1}(I_p) = I_{f(p)}$

prop $\left\{ \begin{matrix} \text{morphisms} \\ Spec(R) \longrightarrow Spec(S) \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{ring homomorphism} \\ S \longrightarrow R \end{matrix} \right\}$

$f \longleftrightarrow f^*$

cor $\left\{ \begin{matrix} \text{affine} \\ \text{schemes} \end{matrix} \right\} / \cong \xleftrightarrow{1:1} \{\text{rings}\} / \cong$

Gluing affine schemes

For a proper notion of gluing schemes $\Longrightarrow$ notion of subschemes

Def Subscheme of $\mathrm{Spec}(R) = \mathrm{Spec}(S)$ with morphism $i: \mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(R)$

s.t. $i^*: R \to S$ surjective ($\Longleftrightarrow$ i injective)

Note: $\left\{ \begin{array}{c} \text{affine subschemes} \\ \text{of } \mathrm{Spec}(R) \end{array} \right\} \xleftrightarrow{1:1} \{ \text{ideals in } R \}$

Def $\mathrm{Spec}(R/J_1) \cap \mathrm{Spec}(R/J_2) := \mathrm{Spec}\left(R/_{J_1+J_2} \right)$

prop $D(f) \subset \mathrm{Spec}(R)$ are affine subschemes $D(f) = \mathrm{Spec}(R_f)$

Def Scheme = locally ringed space obtained from gluing affine schemes

Open subscheme $\quad \mathcal{U} \subset X$ open subset of a scheme $\Longrightarrow (\mathcal{U}, \mathcal{O}_X|_{\mathcal{U}})$ open subscheme

Closed subscheme $\quad Y \subset X$ closed + $i: Y \hookrightarrow X$ ($\Longleftrightarrow$ $i^*: \mathcal{O}_X \to \mathcal{O}_Y$ surjective)

Prevarieties are schemes

prop (pre)Varieties are schemes … (associated to a field)

Def Y scheme, a scheme X/Y (X over Y) is a scheme together with a (structural) morphism $f: X \longrightarrow Y$

A morphism of schemes over Y is
$$\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

__Note__ If $Y = \mathrm{Spec}(S)$ and $X = \mathrm{Spec}(R)$, then R is an S-algebra

__Def__ $f: X \to Y$ of finite type = __finite__ affine cover by __finitely generated__ S-algebra

__Cor__
$$\left\{\begin{matrix}\text{Prevarieties}\\ \text{over } K\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{schemes over } \mathrm{Spec}(K)\\ \text{of finite type}\end{matrix}\right\}$$

__Def__ Reduced scheme = for any open $U \subset X$ the rings $\mathcal{O}_X(U)$ have no nilpotent element

__prop__:
$$\left\{\begin{matrix}\text{Varieties}\\ \text{over } K\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{reduced schemes over}\\ \mathrm{Spec}(K) \text{ of finite type}\end{matrix}\right\}$$