Algebraic Geometry Course

Smooth varieties

The tangent space

The tangent cone $C_a(X)$ approximates the variety X at a by a cone

The tangent space $T_a(X)$ approximates the variety X at a by a linear space

Def X variety, $U \subset X$ affine with $a \in U$ being $a = 0$

$$T_aX := V(f_L \text{ where } f \in I(U)) \subset \mathbb{A}^n$$

$\left(\Rightarrow \not\exists\, f \in I(U) \text{ with constant term} \right)$

f_L linearization of f

$$\underbrace{x^3y + 2y^2x + \overbrace{3x - 2y}}_{f}$$

$$f_L = \sum x_i \frac{\partial f}{\partial x_i}\Big|_{x=0}$$

$C_a(X) = \{ \text{lines in } X \text{ passing through } a \} \Longrightarrow C_a(X) \subset T_a(X)$

$\Rightarrow \dim(T_a(X)) \geq \operatorname{codim}(a, X)$ $\quad (= \dim X \text{ if } X \text{ irreducible})$

Zariski tangent space

$X = V(I)$ affine variety containing the origin $a = 0$

$I(a) \trianglelefteq A(X)$ vanishing ideal of $a \in X$

<u>lemma</u> $\mathrm{Hom}_k(T_aX, k) \cong I(a)/I(a)^2$

<u>proof</u> a) $I(a)$ polynomials with no constant term

b) $W = \{ f_L \text{ for } f \in I(X) \}$ where $I(X) \trianglelefteq I(a)$

c) $T_aX \subset \mathbb{A}^m$ where W vanish

d) W = space of linear forms vanishing on T_aX

e) $\mathrm{Hom}_k(T_aX, k)$ linear op. on T_aX (contains the linear parts of $I(a)$)

f) $\psi : I(a) \longrightarrow \mathrm{Hom}_k(T_aX, k)$

$f \longmapsto f_L|_{T_aX}$

g) $I(a)^2 \subset \ker(\psi)$ $\quad (fg)_L = f(a) g_L + g(a) f_L = 0 \Rightarrow \psi(fg) = 0$ $\quad \begin{array}{c} I(a) \\ \cup\!\!\! \cup \end{array}$

h) $\ker(\psi) \subset I(a)^2 \rightsquigarrow f-g$ has no constant nor linear part) $\Rightarrow f-g$ (a) vanish twice

$\ker(\psi) = \{ f \in I(a) \text{ s.t. } f_L \in W \} \Rightarrow \exists\, g \in W$ ($g \in I(X)$) with $g_L|_{T_aX} = 0$ & $f_L = g_L$

Zariski tangent space

We choose a canonical open subset of $a \in X \rightsquigarrow \mathcal{O}_{X,a} = A(X)_S \left(= S^{-1}A(X)\right)$

$$S = A(X) \setminus I(a)$$

$$\cup$$

$$\mathfrak{m}_a = I(a)_S \left(= S^{-1}I(a)\right)$$

Every $f \in S$ has $f(a) \neq 0 \Rightarrow \overset{K}{\underset{c}{\cup}} = \frac{1}{f}$ in $A(X)/I(a) \cong K$

$S^{-1}\left(I(a)/I(a)^2\right) = I(a)/I(a)^2$ localizing does not affect

$$\| $$

$$S^{-1}I(a)/S^{-1}I(a)^2 =$$

$$\|$$

corollary: $\mathfrak{m}_a/\mathfrak{m}_a^2 = \mathrm{Hom}(T_a X, K)$ (canonical construction)

Smoothness

Two local descriptions $\left\{ \begin{array}{l} \text{Tangent cone } C_a X : \text{ same dim but not linear space} \\ \text{Tangent space } T_a X : \text{ linear space but not same dim} \end{array} \right.$

Def X is smooth at $a \iff C_a X = T_a X$

Lemma X smooth at $a \iff \dim T_a X = \operatorname{codim}_X(a)$ $(\leq)$

Remark Smoothness is a purely algebraic notion ($m_a \subset \mathcal{O}_{X,a}$)

$\dim\left(m_a / m_a^2 \right) =$ Krull dimension of $\mathcal{O}_{X,a} \rightsquigarrow$ regular local ring

proposition $\mathcal{O}_{X,a}$ regular local ring $\Rightarrow \mathcal{O}_{X,a}$ integral domain

idea: smooth varieties are locally irred.

Jacobi criterium

Let $a \in X = V(I) \subset \mathbb{A}^n$ with $I = \langle f_1 \ldots f_r \rangle$

fact: the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\Big|_a \right)$ has rank $\geq n - \operatorname{codim}_X(a)$

(i.e. n – local dim)

prop (Affine Jacobi criterium)

X smooth at a $\Longleftrightarrow$ $\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}\Big|_a \right) = n - \operatorname{codim}_X(a)$

cor (Projective Jacobi criterium) still valid for $X = V(I) \subset \mathbb{P}^n$

cor The set of smooth points is open (i.e. the set of singular points is closed)

Rk: when non-empty $\Longrightarrow$ smooth locus is dense

for $V(f)$ no smooth pts $\Longleftrightarrow \frac{\partial f}{\partial x}\Big|_a = 0 \xRightarrow{\text{Hilbert Nullstellensatz}} \frac{\partial f}{\partial x} \in \langle f \rangle$ & $\deg \frac{\partial f}{\partial x} < \deg f$