Algebraic Geometry Course

Birational maps & blow-ups

Birrational maps

Def Rational map $f : X \dashrightarrow Y$

$$\begin{array}{l} X \dashrightarrow Y \\ \uparrow \nearrow \\ U \end{array}$$

open subset U (dense)

[- Good definition $[f : X \dashrightarrow Y]$ equivalence class

$f_i : U_i \to Y$, $U_i \cap X$

$f_i \sim f_j \iff f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$]

Def $f : X \dashrightarrow Y$ Dominant rational map $\iff$ $\mathrm{im}(f)$ open in Y [$g \circ f$ is rat.]

Def $f : X \dashrightarrow Y$ Birrational map $\iff$ f dominant & $\exists\, g : Y \dashrightarrow X$ s.t. $g \circ f \sim \mathrm{id}_X$ & $f \circ g \sim \mathrm{id}_Y$

Def X and Y are birationally equivalent $\iff$ $\exists\, f : X \to Y$ birrat.

Obs X birrat. eq. to Y $\iff$ they have isomorphic dense open subsets

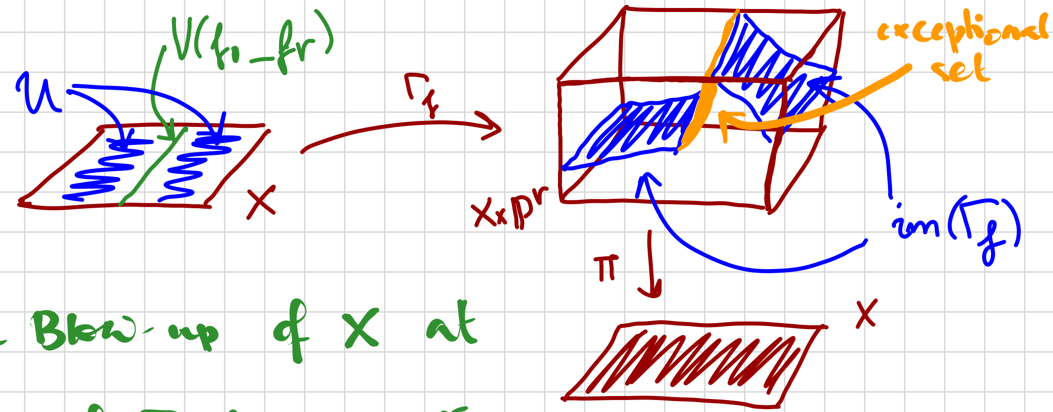
Important notion in Alg. Geometry, as open subsets are dense

Blow-up

(An example of birational map)

Given $f_0 - f_{r+1} \in A(X)$ $\quad V(f_0 - f_{r+1})$ $\quad U := X - V(f_0 - f_{r+1}) = \left\{ \begin{array}{l} \text{where } f_0 - f_{r+1} \text{ do not} \\ \text{vanish simultaneously} \end{array} \right.$

Def $\Gamma_f := \operatorname{im}\left(\begin{array}{ccc} U & \longrightarrow & X \times \mathbb{P}^r \\ x & \longmapsto & (x, [f_0 : \dots : f_r]) \end{array} \right)$



Def Given X affine and $f_0 - f_r \in A(X)$, the Blow-up of X at $V(f_1 - f_r) \subset X$ is $\tilde{X} = Bl_{(f_0 - f_r)}(X) =$ closure of Γ_f in $X \times \mathbb{P}^r$

Since $\tilde{X} \hookrightarrow X \times \mathbb{P}^r \longrightarrow X$

$\pi : \tilde{X} \longrightarrow X$ structure morphism of the blow-up

$\pi^{-1}(V(f_1 - f_r)) \subset \tilde{X}$ exceptional set of the blow-up.

Def Given $Y \subset X$ closed. consider $f_0|_Y - f_r|_Y$

strict transform $\tilde{Y} := Bl_{(f_0 - f_r)}(Y) \subset \tilde{X} = Bl_{(f_0 - f_r)}(X)$

Description of blow-ups

If $r=0$ (only 1 function) lemma $Bl_f(X) \cong X$ $\quad X - V(f) \cong im(\Gamma_f) \subset X \times \mathbb{P}^0 \cong X$

$$\tilde{X} := \overline{X - V(f)} \cong X$$

prop Given X affine, $f_0 - f_r \in A(X)$

$$\tilde{X} = V_{X \times \mathbb{P}^r}\left(y_1 f_2 - y_2 f_1, y_1 f_3 - y_3 f_1, \dots, y_i f_j - y_j f_i \dots\right)$$

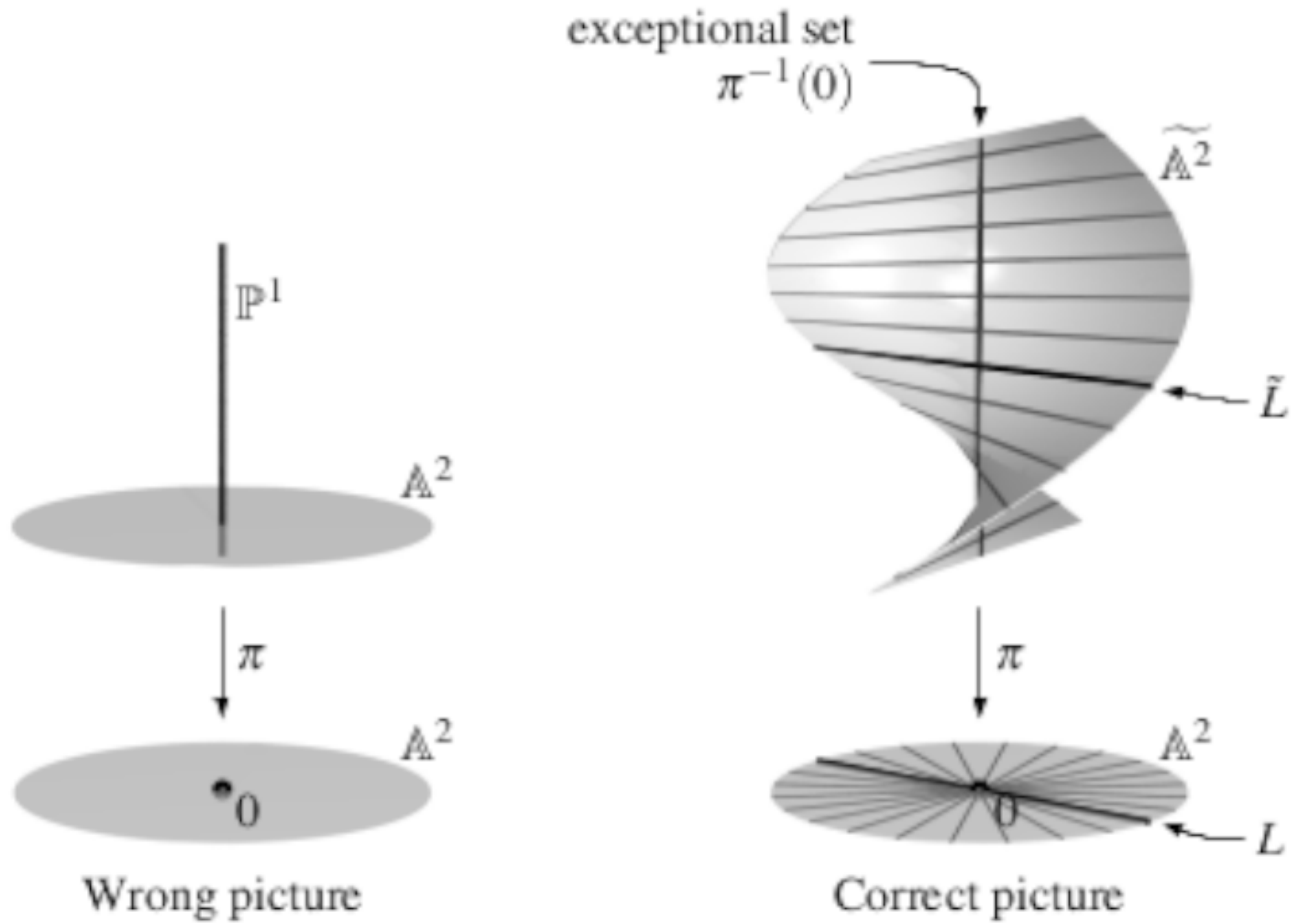
(all combinations)

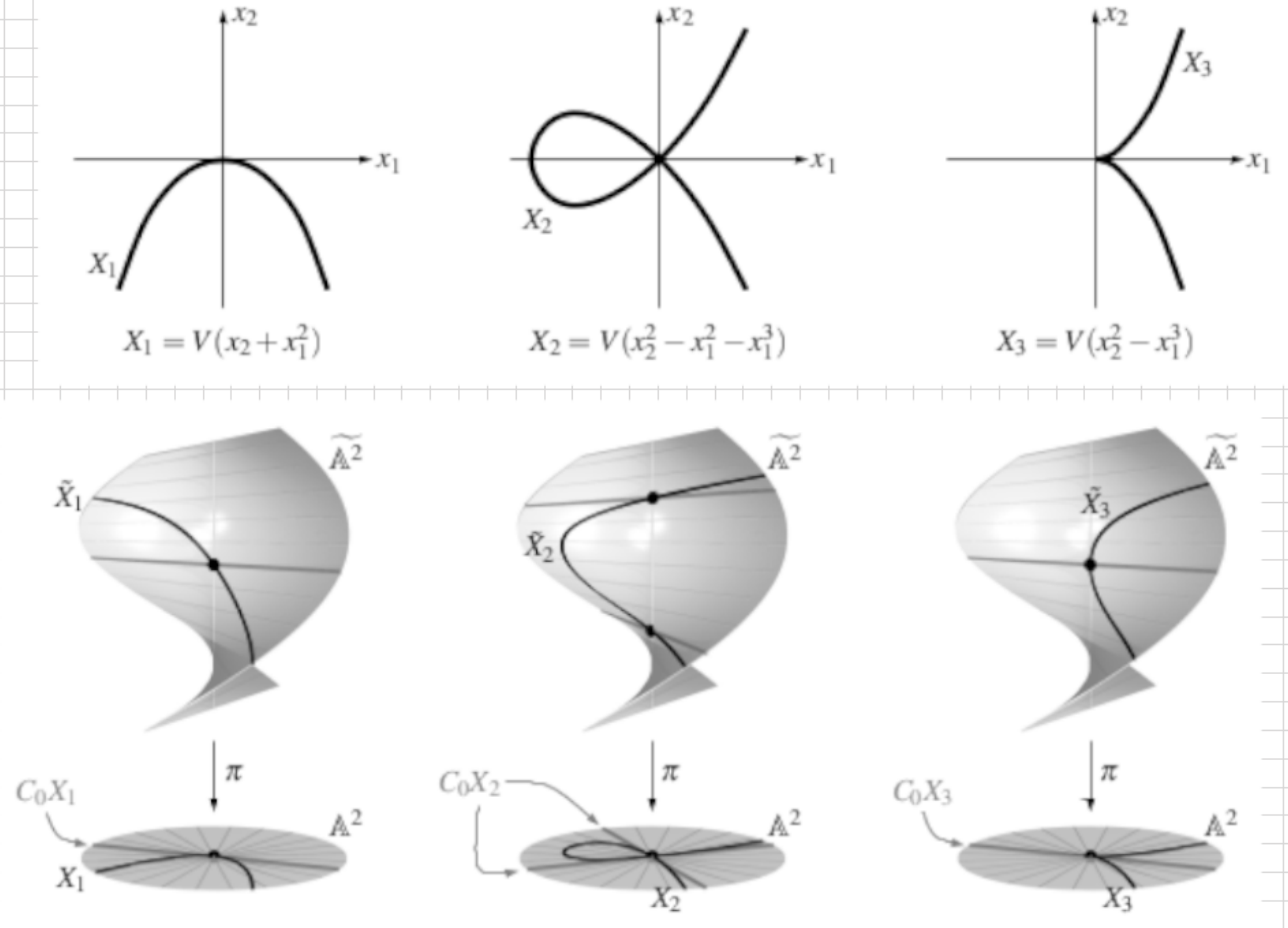
idea: on $X - V(f_0 - f_r)$ $\{(x, y_0 - y_r) = (x, f_0 - f_r)\}$

Description of $\tilde{\mathbb{A}}_n := Bl_{(x_0 - x_n)}(\mathbb{A}_n)$ $\quad V(x_0 - x_n) = 0$, $\pi : \tilde{\mathbb{A}}_n \longrightarrow \mathbb{A}_n$

exceptional locus $\pi^{-1}(0) = \mathbb{P}^{n-1}$ $\quad \tilde{\mathbb{A}}_n - \pi^{-1}(0) \cong \mathbb{A}_n - \{0\}$

a line $\subset \mathbb{A}_n$ passing through 0 defines $p \in \mathbb{P}^{n-1}$ $\Longrightarrow$ $\tilde{\mathbb{A}}_n$ = union of lines of $\mathbb{A}_n$ passing through 0


exceptional set
$\pi^{-1}(0)$
$\widetilde{\mathbb{A}^2}$
$\tilde{L}$
$\mathbb{P}^1$
$\mathbb{A}^2$
π
π
$\mathbb{A}^2$
$\mathbb{A}^2$
0
0
L
Wrong picture
Correct picture


x_2
x_1
X_1
X_2
X_3
$X_1 = V(x_2 + x_1^2)$
$X_2 = V(x_2^2 - x_1^2 - x_1^3)$
$X_3 = V(x_2^2 - x_1^3)$
$\tilde{X}_1$
$\tilde{X}_2$
$\tilde{X}_3$
$\widetilde{\mathbb{A}^2}$
π
C_0X_1
C_0X_2
C_0X_3
$\mathbb{A}^2$

Blowing-up subvarieties

prop: The blow-up depends on the ideal $I = \langle f_0 - f_r \rangle$ (or on $V(I) \subset X$)

$f_0' - f_r'$ generators of $I \Rightarrow f_i' = \sum h_{ij} f_j \Rightarrow X \times \mathbb{P}^r \xrightarrow{(h)} X \times \mathbb{P}^r$

$\tilde{X}' := Bl_{(f_0' - f_r')}(X)$ $\qquad \tilde{X} \longmapsto \tilde{X}'$

$\Rightarrow Bl_{V(I)}(X)$ = blow-up at the affine subvariety $V(I)$

For projective varieties $\rightsquigarrow$ $Bl_Y(X) = \bigcup_i Bl_{Y_i}(X_i)$ where we identify

$$\underset{\substack{\uparrow \\ X_i \cap Y}}{\bigcup_i Y_i} = Y \subset X = \underset{\substack{\uparrow \\ \text{affine}}}{\bigcup_i X_i}$$

$$\begin{array}{ccc} & B_{Y_i \cap Y_j}(X_i \cap X_j) & \\ \swarrow\!\!\!\supset & & \subset \\ Bl_{Y_i}(X_i) & & Bl_{Y_j}(X_j) \end{array}$$

prop: Given $Bl_{(f_0 - f_r)}(X)$. one has $\operatorname{cod}(\pi^{-1}(a), X) = 1$

Let $U_1 \subset X$ affine given by $x_1 = 1$, $f_1 = 0 \Leftrightarrow$ all $f_i = 0$ in $\tilde{X}$ as $x_1 f_i = x_i f_1$

principal $\qquad$ eq of $\pi^{-1}(a)$

Tangent cones

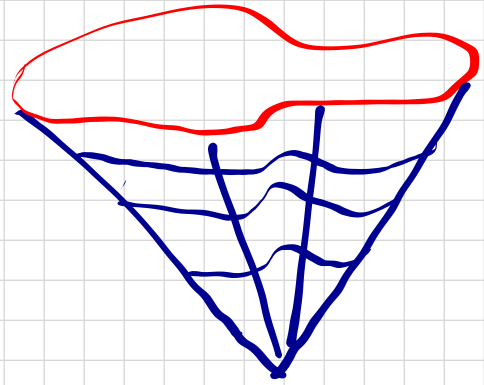
Given a projective variety $X \hookrightarrow \mathbb{P}^\ell$

$$\uparrow p$$

(The cone of X) $C(X) := p^{-1}(X) \cup \{0\} \subset \mathbb{A}^{\ell+1} - \{0\}$

$$\parallel$$

$v \in C \Leftrightarrow \lambda v \in C$



Def Tangent cone of $a \in X$ $\quad C_a(X) := C(\pi^{-1}(a))$

where we consider $Bl_a(X) \xrightarrow{\pi} X$

$\pi^{-1}(a)$ classifies the lines from where X approaches a $\rightsquigarrow$ $C_a(X) = C(\pi^{-1}(a))$ union of all these lines $\rightsquigarrow$ $C_a(X)$ gives a local description of X at a

If $\dim(X) = n$ (locally) $\Rightarrow \dim(C_a(X)) = n$

Blow-up to extend morphism

Given X and $f_0 - f_n \in A(X)$

$$\begin{tikzcd} {[f_0 : \_ : f_n] : X} \arrow[r, dashed] & \mathbb{P}^n \\ {[y_0 : \_ : y_n] : \tilde{X}} \arrow[u] \arrow[ur] & \end{tikzcd}$$

by going to $\tilde{X}$ we extend the morphism to $\mathbb{P}^n$

Theorem Universal property of blow-ups, for $\tilde{X} = Bl_Y(X)$, given f. if $\pi^{-1}(Y) \subset Z$ smooth, then $\exists\, g$ such that

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

(wrong statement! more general)