Algebraic Geometry Course

Section 8

Grassmannians

Definition and motivation

Grassmannians are very important in classification problems in AG

1st examples of classifying spaces $\rightarrow$ $G_K(\ell,n) := \{\ell\text{-dim subspaces of } K^n\}$

We want to see that they are projective vars. (so we can use them in AG)

1st Plücker embedding $G_K(\ell,n) \hookrightarrow \mathbb{P}_K^N$

2nd Given by pol. eq.

Exterior algebra

Given $V \rightsquigarrow \Lambda^m V$ vector space with universal property through alternating m-linear maps

$$\Lambda^m V = \mathrm{Span}\,(v_1 \wedge v_2 \wedge \ldots \wedge v_m \,;\ \text{for all } v_i \in V)$$

$$\begin{tikzcd} V^m \arrow[r, "f"] \arrow[d, "\tau"'] & W \\ \Lambda^m V \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

$f(\ldots v_i, v_j \ldots) = -f(\ldots v_j, v_i \ldots)$

where

$$v_1 \wedge \ldots \wedge v_i \wedge v_j \wedge \ldots \wedge v_m = - v_1 \wedge \ldots \wedge v_j \wedge v_i \wedge \ldots \wedge v_m$$

$$(v_1 + v_1') \wedge \ldots \wedge v_i \wedge v_j \wedge \ldots \wedge v_m = (v_1 \wedge \ldots \wedge v_i \wedge v_j \wedge \ldots \wedge v_m) + (v_1' \wedge \ldots \wedge v_i \wedge v_j \wedge \ldots \wedge v_m)$$

$$(\lambda \cdot v_1) \wedge \ldots \wedge v_i \wedge v_j \wedge \ldots \wedge v_m = \lambda \cdot (v_1 \wedge \ldots \wedge v_i \wedge v_j \wedge \ldots \wedge v_m)$$

$$\Lambda^* V = \bigoplus_{m=0}^{\infty} \Lambda^m V$$

form an algebra

Properties:

$v_1 \wedge v_2 \wedge \ldots \wedge v_m = 0 \iff$ linearly dep.

$v_1 \wedge v_2 \wedge \ldots \wedge v_m = \underset{\uparrow\, \mathbb{C}^*}{\lambda} \cdot (w_1 \wedge w_2 \wedge \ldots \wedge w_s) \iff \mathrm{Span}(v_1, v_2, \ldots, v_m) = \mathrm{Span}(w_1, w_2, \ldots, w_m)$

Plücker embedding

$$G_K(\ell,n) \hookrightarrow \mathbb{P}_K(\Lambda^m K^n)$$

$$\mathrm{Span}(v_1,\dots,v_\ell) \longmapsto [v_1\wedge\dots\wedge v_\ell]_{K^*}$$

To see that $G_K(\ell,n) \subset \mathbb{P}_K^{\binom{n}{\ell}-1}$ is actually given by pol. eqs. we need a technical lemma

<u>lemma</u> for $w \in \Lambda^\ell K^n$ define $f_w : K^n \longrightarrow \Lambda^{\ell+1} K^n$

$$v \longmapsto v \wedge w$$

then $rk(f_w) \geq n-\ell$ with $rk(f_w) = n-\ell \iff w = v_1\wedge \_ \wedge v_\ell$ for some $v_1 \_ v_\ell$

projective variety (zeros of pol. eq. in $\mathbb{P}_K^{\binom{n}{\ell}-1}$) $\iff$ vanishing of all the $(n-k+1)\times(n-k+1)$ minors of the matrix f_w $\iff$ $w \in \mathrm{im}(G_K(\ell,n))$

Irreducibility

For that we first find an affine cover of $G_K(l,n)$

Consider $e_1 \_ e_n$ base of K^n

$\mathcal{U}_{(e_1\wedge\_\wedge e_l)} = G_K(l,n) \cap \{e_1 \wedge \_ \wedge e_l \neq 0\}$ open

Every vector space in $\mathcal{U}_{(e_1\wedge\_\wedge e_l)}$ is generated by

unique base of this form

$$\begin{cases} e_1 + c_{1,1} e_{l+1} + \dots + c_{1,n-l} e_n \\ \vdots \\ e_l + c_{l,1} e_{l+1} + \dots + c_{l,n-l} e_n \end{cases}$$

A subspace $V \in \mathcal{U}_{(e_1\wedge\_\wedge e_l)}$

$\Updownarrow$

$C = (c_{ij}) \in Mat(l, n-l)$

$\Longleftarrow$

$\mathcal{U}_{e_1\wedge..\wedge e_l} = K^{l\cdot(n-l)}$

affine & irreducible

$G_K(l,n) = \bigcup \mathcal{U}_{e_i \wedge \_ \wedge e_j}$ affine & irreducible cover

$\Rightarrow$ $G_K(l,n)$ irreducible