

# Algebraic Geometry Course

## Section 8

### Grassmannians

# Definition and motivation

Grassmannians are very important in classification problems in AG

1st examples of classifying spaces  $\rightarrow G_k(l, n) := \{l\text{-dim subspaces of } K^n\}$

We want to see that they are projective vars. (so we can use them in AG)

①st Plücker embedding  $G_k(l, n) \hookrightarrow \mathbb{P}_k^N$

②nd Given by pd. eq.

# Exterior algebra

Given  $V \rightsquigarrow \Lambda^m V$  vector space with universal property through alternating  $m$ -linear maps

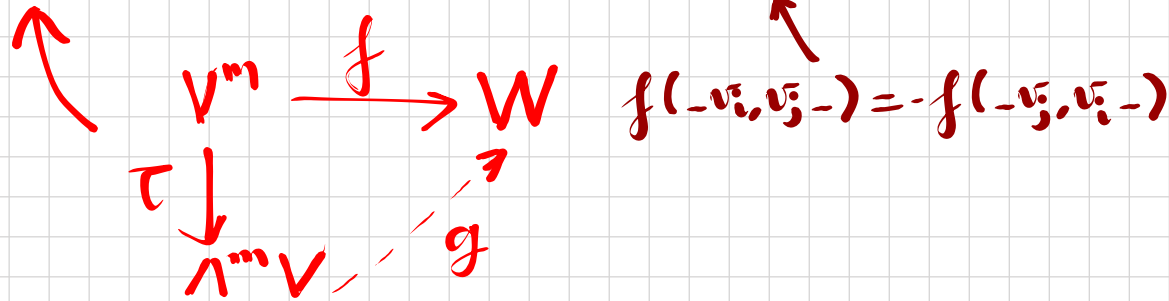
Span  $(v_1 \wedge v_2 \wedge \dots \wedge v_m; \text{ for all } v_i \in V)$

where

$$v_1 \wedge \dots \wedge v_i \wedge v_j \wedge \dots \wedge v_m = -v_1 \wedge \dots \wedge v_j \wedge v_i \wedge \dots \wedge v_m$$

$$(v_1 + v_1') \wedge \dots \wedge v_i \wedge v_j \wedge \dots \wedge v_m = (v_1 \wedge \dots \wedge v_i \wedge v_j \wedge \dots \wedge v_m) + (v_1' \wedge \dots \wedge v_i \wedge v_j \wedge \dots \wedge v_m)$$

$$(\lambda \cdot v_1) \wedge \dots \wedge v_i \wedge v_j \wedge \dots \wedge v_m = \lambda \cdot (v_1 \wedge \dots \wedge v_i \wedge v_j \wedge \dots \wedge v_m)$$



$$\Lambda^* V = \bigoplus_{n=0}^{\infty} \Lambda^n V$$

form an algebra

Properties:

$$v_1 \wedge v_2 \wedge \dots \wedge v_m = 0 \iff \text{linearly dep.}$$

$$v_1 \wedge v_2 \wedge \dots \wedge v_m = \lambda \cdot (w_1 \wedge w_2 \wedge \dots \wedge w_m) \iff \text{Span}(v_1, v_2, \dots, v_m) = \text{Span}(w_1, w_2, \dots, w_m)$$

# Plücker embedding

$$G_K(l, n) \hookrightarrow \mathbb{P}_K(\wedge^l K^n)$$

$$\text{Span}(v_1, \dots, v_l) \mapsto [v_1 \wedge \dots \wedge v_l]_K$$

To see that  $G_K(l, n) \subset \mathbb{P}_K^{\binom{n}{l}-1}$  is actually given by pt. eqs. we need a technical lemma

lemma for  $\omega \in \wedge^l K^n$  define  $f_\omega: K^n \rightarrow \wedge^{l+1} K^n$   
 $v \mapsto v \wedge \omega$

then  $\text{rk}(f_\omega) \geq n-l$  with  $\text{rk}(f_\omega) = n-l \iff \omega = v_1 \wedge \dots \wedge v_l$   
 for some  $v_1, \dots, v_l$

projective variety  
 (zeros of pt. eq. in  $\mathbb{P}_K^{\binom{n}{l}-1}$ )  $\iff$

vanishing of all the  
 $(n-l+1) \times (n-l+1)$  minors  $\iff \omega \in \text{im}(G_K(l, n))$

# Irreducibility

For that we first find an affine cover of  $G_K(l, n)$  Consider  $e_1, \dots, e_n$  base of  $K^n$

$$U_{(e_1, \dots, e_l)} = G_K(l, n) \cap \{e_1, \dots, e_l \neq 0\} \text{ open}$$

Every vector space in  $U_{(e_1, \dots, e_l)}$  is generated by  
unique base of this form  $\rightarrow \begin{cases} e_1 + c_{1,l+1}e_{l+1} + \dots + c_{1,n-l}e_n \\ \vdots \\ e_l + c_{l,l+1}e_{l+1} + \dots + c_{l,n-l}e_n \end{cases}$

$$U_{(e_1, \dots, e_l)} = K^{l \cdot (n-l)} \\ \text{affine \& irreducible}$$

$$\begin{array}{c} \text{A subspace } V \in U_{(e_1, \dots, e_l)} \\ \updownarrow \\ C = (c_{ij}) \in \text{Mat}(l, n-l) \end{array}$$

$$G_K(l, n) = \bigcup U_{(e_1, \dots, e_l)} \text{ affine \& irreducible cover}$$

$$\Downarrow \rightarrow G_K(l, n) \text{ irreducible}$$