Algebraic Geometry Course

Projective varieties II

Sheaf of regular functions

X projective variety with homogeneous coordinate ring $S(X)$

Note: $f \in S_d(X)$ and $g \in S_{d'}(X)$ with $d \neq d' \Rightarrow \frac{f(\lambda x)}{g(\lambda x)} = \frac{\lambda^d f(x)}{\lambda^{d'} g(x)}$ not well defined

Def Regular function $\varphi \in \mathcal{O}_X(U)$, $U \subset X$ open, is a map $\varphi : U \to K$ s.t.

for all $a \in U$ $\exists U_a \ni a$, $d \in \mathbb{N}$, $f, g \in S_d(X)$ with $f(x) \neq 0$ for all $x \in U_a$ and $\varphi|_{U_a} = \frac{g}{f}$

$\hookrightarrow$ $\mathcal{O}_X :=$ sheaf of regular functions

prop: $X \subset \mathbb{P}^n$ projective var, then $U_i = \{(x_0 : x_1 : \_ : x_n) \in X : x_i \neq 0\}$ is affine var

$\boxed{\text{Hence } X \text{ is a prevariety.}}$

idea: $X = V_p(J)$ and $U_i \cong V_a(J^{dh})$ sending $\mathcal{O}_{U_i}$ to $\mathcal{O}_{V_a(J^{dh})}$

prop projective varieties are varieties (prevarietie + separated $\Delta_X \subset X \times X$ closed)

idea: enough to prove it for $\mathbb{P}^n$ (subprevarieties of a variety are varieties).

$\Delta_{\mathbb{P}^n} = V_{\mathbb{P}^n \times \mathbb{P}^n}(x_i y_j - x_j y_i \text{ forall } i,j)$ as $(x_0 \_ x_n) = \lambda(y_0 \_ y_n)$ is $\operatorname{rk}\begin{pmatrix} x_0 & \_ & x_n \\ y_0 & \_ & y_n \end{pmatrix} = 1$

Segre embedding

<u>lemma</u> $X \subset \mathbb{P}^n$ projective variety, $f_0 \ldots f_m \in S_d(X)$. then

$f: X \setminus V_X(f_0 \ldots f_m) \longrightarrow \mathbb{P}^m$ is a morphism (of prevarieties)

Example <u>Segre embedding</u> n, m and $N = (n+1)(m+1) - 1$

$[x_0 : \ldots : x_n] \in \mathbb{P}^n$; $\in \mathbb{P}^m$; $[z_{00} : \ldots : z_{0m} : z_{10} : \ldots : z_{1m} : \ldots : z_{n0} : \ldots : z_{nm}] \in \mathbb{P}^N$

$$f: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$$
$$[x_0 : \ldots : x_n] \times [y_0 : \ldots : y_m] \longmapsto [x_0 y_0 : \ldots : x_n y_m]$$
$$(z_{ij} = x_i y_j)$$

- f is an embedding as so it is restricted to the open $\{x_i = 1\}$
- $f(\mathbb{P}^n \times \mathbb{P}^m) = V_p(z_{ij} z_{kl} - z_{il} z_{kj})$ ($\subset$ is trivial
  $\supset$ $z_{00} = 1 \Rightarrow x_0 = 1 \quad y_0 = 1$
  $z_{0j} = y_j \quad z_{i0} = x_i \quad z_{ij} = z_{i0} z_{0j} = x_i y_j$

<u>Corollary</u> If X, Y projective varieties, $X \times Y$ is projective as well

Closed morphisms

def A map of varieties $f: X \to Y$ is closed iff
for all $Z \subset X$ closed, $f(Z)$ is closed in Y

prop $\pi: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^m$ is closed

Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ closed, $Z = V_{\mathbb{P}^N}(f_1 \dots f_r)$ in Segre coordinates $x_i y_j$

$\boxed{\text{idea } \mathbb{P}^m - \pi(Z) = \text{open}}$

$\mathbb{P}^m \ni a \notin \pi(Z) \iff V_p(f_1(x,a) \dots f_r(x,a)) = \emptyset$

$\iff \sqrt{\langle f_1(x,a) \dots f_r(x,a) \rangle} = \langle x_1 \dots x_n \rangle \iff$ for some N $\langle f_1(x,a) \dots f_r(x,a) \rangle$ contains all hom. pol. of deg $= N$

$\iff K[x_0 \dots x_n]^r_{N-d} \longrightarrow K[x_1 \dots x_n]_N$ is surjective $\iff$ non-vanishing of certain minors

$(h_1 \dots h_r) \longmapsto h_1 f_1(x,a) + \dots + h_r f_r(x,a)$

th $\pi: \mathbb{P}^n \times Y \longrightarrow Y$ is closed for any variety Y

1st (Y affine) Z closed in $\mathbb{P}^n \times Y$. $\bar{Z} \subset \mathbb{P}^n \times \mathbb{P}^m \Rightarrow \pi(\bar{Z})$ closed; $\pi(Z) = \pi(\bar{Z}) \cap Y$ closed

2nd (Y arbitrary) cover Y with affine subvarieties

Complete varieties

Compact makes no sense in Zariski topology as every variety is compact

Need for (an algebraic) property that encodes the idea of being compact in the usual topology

Idea: Continuous maps send compact sets (closed) to compact sets (closed)

Def X complete $\iff$ $\pi : X \times Y \longrightarrow Y$ is closed for all Y

Cor: $\mathbb{P}^n$ complete, every projective var. complete, affine var not complete

lemma $f : X \to Y$ morphism, X complete $\Rightarrow$ $f(X)$ is closed $\left(f(X) = \pi(\Gamma_f) \right)$

cor X connected and complete $\Rightarrow$ $\mathcal{O}_X(X) = k$ (only constant global regular functions)

idea $\varphi \in \mathcal{O}_X(X)$ defines $\tilde{\varphi} : X \longrightarrow \mathbb{P}^1$ not being all $\mathbb{P}^1$ $\Longrightarrow$ im f finite

Veronese embedding

Let $N = \binom{n+d+1}{n} - 1$, the Veronese embedding of degree d is

$$v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$$

$$[x_0 : \_ : x_n] \mapsto [x_0^d : x_0^{d-1}x_1 : \_ : x_0^{d-l_1-\dots-l_n} x_1^{l_1} \dots x_n^{l_n} : \_ : x_n^d]$$

(all degree d monomials)

It is an embedding as $v_d|_{\{x_0=1\}}$ is an embedding $(x_i \longmapsto x_0^{d-1} x_i = x_i)$

Note: Every degree d polynomial becomes linear in Veronese coordinates

prop for $f \in S(X)$, X projective, $X - V(f)$ is affine

idea for $f = x_0$ $\quad X - V(x_0) = \{x_0 \neq 0\}$ affine

for f hom. deg d use Veronese coordinates to traslate it into the first case