Algebraic Geometry Course

Projective varieties I

Projective spaces

Def Projective space $\mathbb{P}^n = \{ \ell \subset K^{n+1} \text{ linear subspace } 0 \in \ell \}$

$$\mathbb{P}^n = (K^{n+1} - \{0\}) / K^* \qquad (x_0, x_1, \_, x_n) \sim (\lambda x_0, \_, \lambda x_n)$$

Homogeneous coordinates $[x_0 : x_1 : \_ : x_n] \sim [\lambda x_0 : \_ : \lambda x_n] \in \mathbb{P}^n$

Rmk one of the x_i must be non-zero!

Affine cover of $\mathbb{P}^n$

$$f_i : \mathbb{A}^n \longrightarrow \mathbb{P}^n \qquad \mathbb{A}^n \cong U_i = \mathrm{im}(f_i) \longleftrightarrow x_i \text{ non-zero}$$

$$(x_1, \_, x_n) \mapsto [x_1 : \_ : \underset{\uparrow \, i\text{-th position}}{1} : \_ : x_n]$$

$$f_i^{-1} : U_i \longrightarrow \mathbb{A}^n$$

$$[x_1 : \_ : x_i : \_ : x_n] \mapsto \left(\frac{x_1}{x_0}, \_, \frac{x_n}{x_0} \right) \qquad \underset{\wr}{\{[\overset{\neq 0}{x_0} : x_1 : \_ : x_n]\}} \sqcup \underset{/\!/}{\{[0 : x_1 : \_ : x_n]\}}$$

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$$

$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$ one of the x_i must be non-zero!

Graded rings and homogeneous ideals

Def Graded ring R ring with a group decomposition $R = \bigoplus_{d=0}^{\infty} R_d$ ($R_d \subset R$ subgps) such that for $f \in R_d$, $g \in R_e \Rightarrow fg \in R_{ed}$

Def Graded k-algebra If R k-algebra and $\forall f \in R_d \ \forall \lambda \in k \Rightarrow \lambda f \in R_d$

Every $f \in R$ decomposes $f = \sum_d f_d$ with $f_d \in R_d - \{0\} = \{$homogeneous elements of degree d$\}$

Def Homogeneous ideal = ideal in graded ring generated by homogeneous elements

lemma

a) J homogeneous ideal $\Leftrightarrow \forall f \in J$ $f = \sum f_d$ satisfies $f_d \in J$

b) J_1, J_2 homogeneous $\Rightarrow J_1 + J_2$, $J_1 \cap J_2$, $J_1 J_2$, and $\sqrt{J_1}$ homogeneous

c) J homogeneous $\Rightarrow R/J$ is graded ring with $R/J = \bigoplus_{d=0}^{\infty} R_d/(J \cap R_d)$

Projective varieties

Def Projective vanishing locus

a) $S \subset K[x_0 . \_ x_n]$ set of homogeneous polynomials

$\mathbb{P}^n \supset V(S) := \{ x \in \mathbb{P}^n \; f(x) = 0 \text{ for all } f \in S \}$

b) let $J \trianglelefteq R$ homogeneous ideal

$\mathbb{P}^n \supset V(S) := \{ x \in \mathbb{P}^n \; f(x) = 0 \text{ for all } f \in J \text{ homogeneous} \}$

f homogeneous

$\Updownarrow$

$f(\lambda x) = \lambda^d f(x)$

$\Updownarrow$

$f(\lambda x) = 0 \Longleftrightarrow f(x) = 0$

Def Projective vanishing ideal $X \subset \mathbb{P}^n$

$$I(X) := \langle f \in K[x_0; \_ ; x_n] \text{ homogeneous s.t. } f(x) = 0 \text{ for all } x \in X \rangle$$

Notation: $V_p(J)$ & $I_p(X)$ projective vanishing locus and ideal

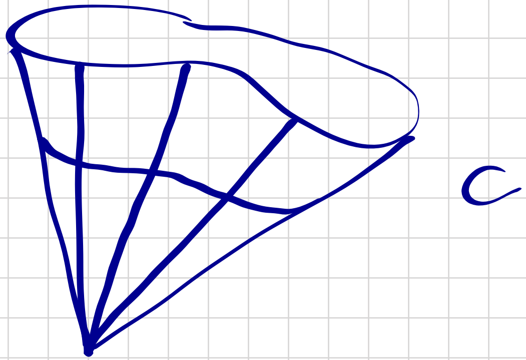
$V_a(J)$ & $I_a(X)$ affine vanishing locus and ideal

Cones and projective varieties

Def Cone = affine variety C s.t.

a) $0 \in C$

b) $x \in C \iff \lambda x \in C$ for all $\lambda \in k^*$



Consider the projection $\pi : \mathbb{A}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$

Def Projectivization of a cone C

$$\mathbb{P}(C) := \pi(C - \{0\}) \subset \mathbb{P}^n$$

Def Cone of a projective variety X

$$C(X) := \pi^{-1}(X) \cup \{0\}$$

Lemma There exists a bijection

$$\begin{tikzcd}\{\text{cones in } \mathbb{A}^{n+1}\} \arrow[r, shift left, "\mathbb{P}(-)"] & \{\text{projective varieties in } \mathbb{P}^n\} \arrow[l, shift left, "C(-)"]\end{tikzcd}$$

Projective Nullstellensatz

Def Irrelevant ideal $I_0 := \langle x_0, x_1, \_, x_n \rangle$ (radical & homogeneous)

Every homogeneous ideal is contained in I_0

Theorem (Projective Nullstellensatz)

a) X projective variety $\Longrightarrow V_p(I_p(X)) = X$

b) $J \lhd k[x_0 \_ x_n]$ homogeneous ideal with $\sqrt{J} \neq I_0 \Rightarrow I_p(V_p(J)) = \sqrt{J}$

Hence, there is a bijection

$$\left\{\begin{array}{c}\text{projective}\\ \text{varieties in } \mathbb{P}^n\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{homogeneous radical ideals}\\ \text{in } k[x_0 \_ x_n] \text{, not equal to } I_0\end{array}\right\}$$

properties: a) $V_p(J_1) \cup V_p(J_2) = V_p(J_1 \cap J_2)$; $V_p(J_1) \cap V_p(J_2) = V_p(J_1 + J_2)$

b) $I_p(X_1 \cap X_2) = \sqrt{I_p(X_1) + I_p(X_2)}$; $I_p(X_1 \cup X_2) = I_p(X_1) \cap I_p(X_2)$

Zariski topology

Def Homogeneous coordinate ring of $Y \subset \mathbb{P}^n$ projective variety

$$S(Y) := K[x_0 \_ x_n] / I(Y)$$ (graded ring)

Def Projective subvariety associated to homogeneous ideal $J \lhd S(Y)$

$$V(J) := \{ x \in Y : f(x) = 0 \text{ for all homogeneous elements of } J \}$$

Def Vanishing ideal of a projective subvariety $Z \subset Y$

$$I(J) := \langle f \in S(Y) \text{ homogeneous s.t. } f(x) = 0 \text{ for all } x \in X \}$$

Theorem Relative projective Nullstellensatz ($I_0(Y)$ irrelevant ideal of Y)

$$\left\{ \begin{matrix} \text{projective subvarieties} \\ \text{of } Y \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{homogeneous ideals of} \\ S(Y) \text{ different from } I_0(Y) \end{matrix} \right\}$$

Zariski topology ⟶

$$\boxed{\begin{matrix} \text{closed subset of } Y = \text{projective subvariety of } Y \\ \text{open subset of } Y = \text{complement of projective subvariety} \end{matrix}}$$

Homogenization and dehomogenization

Idea Dehomogenization: Restrict a projective variety to the affine cover
Homogenization: Find a projective variety restricting to our affine one

Def Let $f(x_0 x_1 \text{—} x_n)$ homogeneous polynomial
its dehomogenization $f^{dh}(x_1 \text{—} x_n) := f(x_0 = 1, x_1 \text{—} x_n)$

$$(fg)^{dh} = f^{dh} g^{dh} \qquad (f+g)^{dh} = f^{dh} + g^{dh}$$

Def Let $g(x_1 \text{—} x_n)$ arbitrary polynomial with $g = \sum_{l=m}^{n} g^l$ g^l hom of $\deg = l$
its homogenization $g^h(x_0 x_1 \text{—} x_n) := \sum_{l=m}^{n} x_0^{n-l} g^l(x_1 \text{—} x_n) = x_0^n g\left(\frac{x_1}{x_0}, \text{—}, \frac{x_n}{x_0}\right)$

homogeneous of degree n

$$(fg)^h = f^h g^h \qquad (f+g)^h \neq f^h + g^h$$