Algebraic Geometry Course

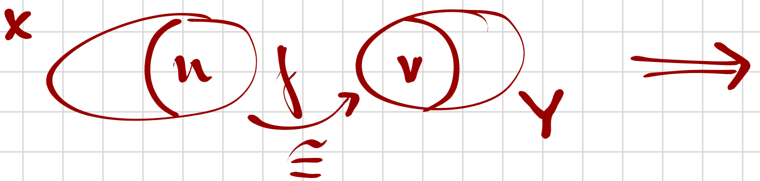
Varieties

Gluing prevarieties

Def Prevariety = Ringed space with a finite cover by affine varieties

locally affine varieties

Gluing 2 prevarieties $\quad X, Y$ prevarieties $\quad U \subset X$, $V \subset Y$ open subsets $\quad f: U \xrightarrow{\cong} V$ iso of ring. sp.



$X \cup_f Y$

a) $X \sqcup Y /_{\sim_f}$ $\quad a \sim_f f(a)$ $a \in U$ $\quad \left(\begin{array}{l} i_X : X \hookrightarrow X \cup_f Y \text{ naturally} \\ i_Y : Y \hookrightarrow \end{array} \right)$

b) (quotient topology) $W \subset X \cup_f Y$ open $\iff i_X^{-1}(W) \,\&\, i_Y^{-1}(W)$ open

c) $\mathcal{O}_{X \cup_f Y}(W) = \{ \varphi : W \to K \;\; i_X^* \varphi \in \mathcal{O}_X(i_X^{-1}(W)) \,\&\, i_Y^* \varphi \in \mathcal{O}_Y(i_Y^{-1}(W)) \}$

Gluing prevarieties

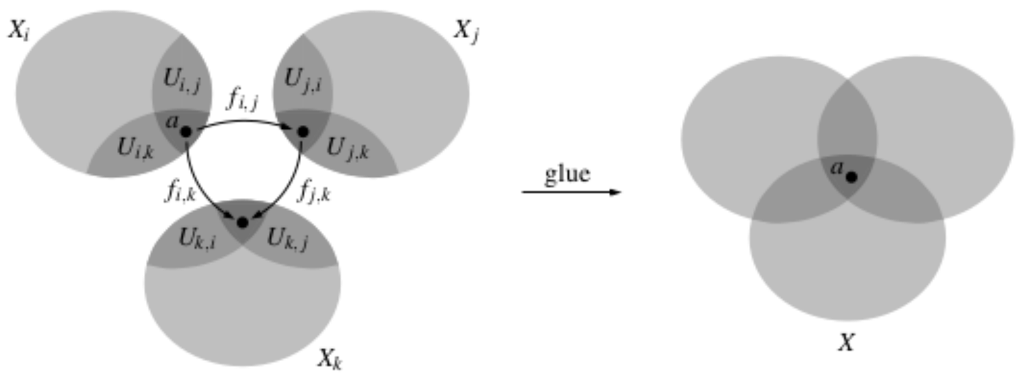
<u>General gluing</u> $\{X_i\}_{i\in I}$ with I finite.

let $U_{i,j} \subset X_i$ and $f_{i,j} : U_{i,j} \xrightarrow{\cong} U_{j,i}$

satisfying a) $f_{ji} = f_{ij}^{-1}$

b) $U_{ij} \cap f_{ij}^{-1}(U_{jk}) \subset U_{ik}$

c) $f_{jk} \circ f_{ij} = f_{ik}$ on $U_{ij} \cap f_{ij}^{-1}(U_{jk})$



Open and closed subprevarieties

Let $(X, \mathcal{O}_X)$ be a prevariety

- <u>Open subprevariety</u> Take $U \subset X$ open, $\mathcal{O}_U = \mathcal{O}_X|_U$

  X covered by affine $\Rightarrow$ U covered by affine

- <u>Closed subprevariety</u> Take $Y \subset X$ closed, $V \subset Y$ open in Y

  $\mathcal{O}_Y(V) := \{ \varphi : V \to K \text{ s.t. } \forall a \in V \ \exists U \ni a$ and $f \in \mathcal{O}_X(U)$ with $\varphi = f$ on $U \cap V \}$

  Y intersected with an affine cover gives closed subsets on the affine sets

The image of an open or closed prevariety might be nor open or closed

By continuity the preimage of a closed (open) is closed (open).

Product of prevarieties

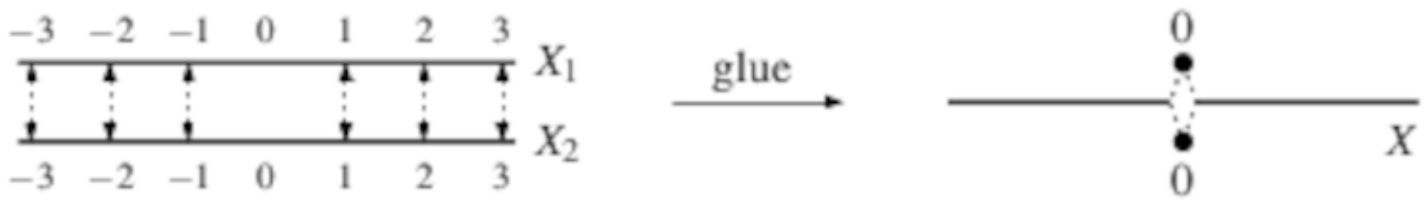
product of prevarieties independendtly of affine cover $\rightsquigarrow$ through universal property

<u>Def</u> X, Y prevarieties. the product $X \times Y$, naturally equipped with projections $\pi_X : X \times Y \to X$; $\pi_Y : X \times Y \to Y$ such that for any $f_X : Z \to X$ and $f_Y : Z \to Y$ $\exists\ f : Z \to X \times Y$ with $f_X = \pi_X \circ f$
$f_Y = \pi_Y \circ f$

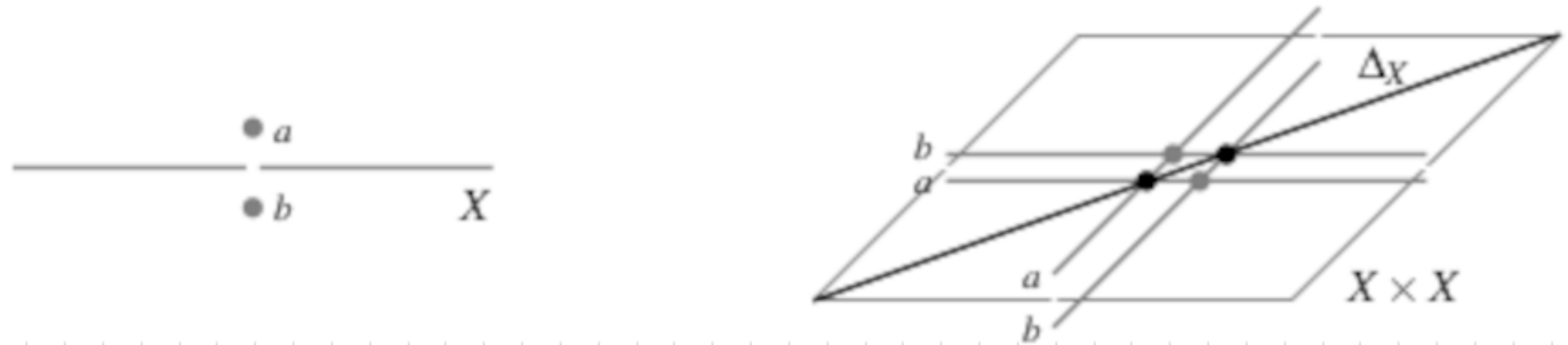
<u>Prop</u> The product of prevarieties exists and is another prevariety

Non-Hausdorff gluing

__Example__ $X := \mathbb{A}^1 \cup_f \mathbb{A}^1$ where $f = \mathrm{id}_{\mathbb{A}^1 - \{0\}} : \mathbb{A}^1 - \{0\} \to \mathbb{A}^1 - \{0\}$



considering the diagonal $\Delta_X \subset X \times X$ we see that we leave points in the closure of Δ_X out of it



Varieties

Def A prevariety X is called a variety (or separated) if $\Delta_X \subset X \times X$ is closed in the Zariski topology

lemma
- affine varieties are varieties
- open or closed subprevarieties of varieties are varieties

Def Curve = variety of dim 1
Surface = variety of dim 1
Hypersurface = variety of dim $n-1$

prop Y variety. $f : X \to Y$ morphism of prevarieties
$\Gamma_f = \{ (x, f(x)) \ x \in X \}$ closed in Y