Algebraic Geometry Course

Morphisms

Ringed spaces

The functions on a geometric should make part of definition

Def Ringed space $(X, \mathcal{O}_X)$ X topological space
$\mathcal{O}_X$ structure sheaf

Example Affine varieties with sheaf of regular functions

Convention: The structure sheaf of a ringed space is a sheaf of functions
(useful to understand b))

Def Morphism (of ringed spaces) $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$

a) Continuous map of topological spaces

b) For all open subsets $U \subset Y$ we have a k-algebra homomorphism giving

$$f^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$
$$\varphi \longmapsto f^*\varphi := \varphi \circ f$$

Def Isomorphism $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ with inverse

Properties of morphisms

<u>Composition</u>

$$\begin{tikzcd}(X,\mathcal{O}_X) \arrow[r, "f"] \arrow[rr, bend left, "f\circ g"] & (Y,\mathcal{O}_Y) \arrow[r, "g"] & (Z,\mathcal{O}_Z)\end{tikzcd}$$

<u>Restriction</u>

$$\begin{tikzcd}(X,\mathcal{O}_X) \arrow[r, "f"] & (Y,\mathcal{O}_Y) \\ (\underset{\cup}{U},\mathcal{O}_X|_U) \arrow[r, "f|_U"] & (V,\mathcal{O}_Y|_V)\end{tikzcd}$$ where $f(U)\subset V$

<u>lemma</u> (<u>Gluing property</u>) $f: X \longrightarrow Y$ map of topological spaces

Let $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ ringed spaces, let $\{U_i\}_{i\in I}$ cover of X

s.t. $f|_{U_i}$ are morphisms (of ringed spaces), then f is a morphism

Morphism of affine varieties

prop: Let $U \subset X$ open of affine variety
$Y \subset \mathbb{A}^n$ affine variety

$$\left\{ \begin{array}{l} \text{morphisms} \\ f : X \longrightarrow Y \end{array} \right\} = \left\{ \begin{array}{l} \text{maps } f = (\varphi_1 \ldots \varphi_n) : X \longrightarrow Y \subset \mathbb{A}^n \\ \text{where } \varphi_i \in \mathcal{O}_X(U) \text{ regular functions} \end{array} \right\}$$

proof $\boxed{\subset}$ the i-th coordinate is a regular function on Y
by b) $\varphi_i = y_i \circ f$ is a regular function

$\boxed{\supset}$ $f = (\varphi_1 \ldots \varphi_n)$ satisfies a) and b)

a) (f is continuous) $Z = V(g_1 \ldots g_l) \subset Y$ closed,

$f^{-1}(Z) = V(f^* g_1 \ldots f^* g_l)$ closed

b) composition of regular functions is a regular function

Morphisms and coordinate rings

cor: Let X and Y affine varieties

$$\left\{\begin{array}{c}\text{morphisms}\\ f: X \to Y\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}k\text{-algebra homomorphisms}\\ f^*: A(Y) \to A(X)\end{array}\right\}$$

proof $\boxed{\Rightarrow}$ part of the definition

$\boxed{\Leftarrow}$ Let $y_i \in A(Y)$ be the coordinate of $Y \subset \mathbb{A}^n$

given $g: \underset{\{Y \to k\}}{\underset{\parallel}{A(Y)}} \longrightarrow \underset{\{X \to k\}}{\underset{\parallel}{A(X)}} \rightsquigarrow g_i = \underset{A(X)}{\underset{\cap}{g \circ y_i}} \rightsquigarrow \underset{f: X \longrightarrow \mathbb{A}^n}{f = (g_1 \dots g_n)}$

$Im(f) \subset Y$ since for $h \in I(Y)$

$f^* h = h(g_1 \dots g_n) = g(h) = 0$

g is a k-algebra homomorphism

Finitely generated K-algebras

R = finitely generated K-algebra without nilpotents (i.e. reduced)

let $a_1 \ldots a_n$ generators of R

$$g : K[x_1 \ldots x_n] \longrightarrow R \qquad f \longmapsto f(a_1 \ldots a_n)$$

then $R \cong K[x_1 \ldots x_n] / \ker g$

Construct $X = V(\ker g)$ affine variety with $A(X) \cong R$

Def Affine variety, ringed space associated to an affine variety old sense

$$\left\{ \begin{matrix} \text{affine} \\ \text{varieties} \end{matrix} \right\} / \cong \quad \xleftrightarrow{1:1} \quad \left\{ \begin{matrix} \text{finitely} \\ \text{generated} \\ K\text{-algebras} \end{matrix} \right\} / \cong$$

Distinguished open are affine

prop: X affine, $f \in A(X)$. $D(f)$ is affine with $A(D(f)) = A(X)_f$

proof consider $Y := \{(x,t) \in X \times \mathbb{A}^1$ with $t f(x) = 0\} \subset X \times \mathbb{A}^1$

with the maps $\begin{array}{l} Y \longrightarrow D(f) \\ (x,t) \longmapsto x \end{array}$ and $\begin{array}{l} D(f) \longrightarrow Y \\ x \longmapsto \left(x, \frac{1}{f(x)}\right) \end{array}$