Algebraic Geometry Course

The sheaf of regular functions

Sheaves of functions

One of the most important features of a geometric object is the set of functions defined on it

In differential geometry $C^\infty(M)$ is huge and it is enough to consider $C^\infty(U) = C^\infty(M)|_U$

In algebraic geometry $A(X)$ is not that big, we need more than $A(X)|_U$

Given $U \subset X$ $\quad \mathcal{O}_X(U) = \left\{ \begin{array}{l} \text{functions that locally are } \frac{f}{g} \; (g \neq 0) \\ \text{quotients of polynomials} \end{array} \right\}$

<u>Recall</u>: Sheaf = the appropriate structure for the set of functions of a variety X

$\mathcal{F} : \left\{ \begin{array}{c} \text{open subsets} \\ \text{of } X \end{array} \right\} \longrightarrow$ Sets (... of functions) with restrictions $res_{U \subset V} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$

- gluing condition
- local equality is enough

Regular functions

Def: X affine variety, $A(X)$ coordinate ring, a regular function on $U \subset X$ is a map $\varphi: U \longrightarrow k$, such that for every $a \in U$ $\exists\, U_a \subset U$ open such that $\varphi|_{U_a} = \frac{f}{g}$ for $f, g \in A(X)$ with $g \neq 0$ on U_a

Lemma: The vanishing locus $V(\varphi)$ of a regular function is closed

Corollary (Identity theorem for regular functions) Let $\varphi_1, \varphi_2 \in \mathcal{O}_X(W)$, Let U open with $U \subset W$ s.t. $\varphi_1|_U = \varphi_2|_U$, then $\varphi_1 = \varphi_2$.

($U \subset V(\varphi_1 - \varphi_2)$ closed $\Rightarrow V(\varphi_1 - \varphi_2) = W$)

Def Distinguished open set of $f \in A(X)$

$D(f) := X - V(f) = \{x \in X \text{ s.t. } f(x) \neq 0\}$

("doesn't vanishing set")

- $D(f) \cap D(g) = D(fg)$
- Every open $U = \bigcup_{i=0}^{m<\infty} D(f_i)$

Localization

Given a ring R, given $f \in R$ we define the localization of R at f as

$$R_f = \left\{ \left[\frac{r}{f^n}\right]_\sim \text{ for some } n \in \mathbb{N}, \text{ where } \frac{r}{f^n} \sim \frac{r'}{f^m} \iff rm - r'n = 0 \right\}$$

prop $\mathcal{O}_X(D(f)) \cong A(X)_f$

proof $\supset$ is obvious

for $\subset$ we use the identity th for regular functions and relative Hilbert Nullstellensatz.

Stalk of the sheaf of regular functions

Def Given a (pre)sheaf $\mathcal{F}$ on X, define the stalk at $a \in X$

$$\mathcal{F}_a := \{(U,\varphi) : a \in U \subset X ;\ \varphi \in \mathcal{F}(U)\}/\sim$$

where $(U,\varphi) \sim (U',\varphi')$ iff $\exists V \subset U \cap U'$ s.t. $\varphi|_V = \varphi'|_V$

Def $\mathcal{O}_{X,a} := (\mathcal{O}_X)_a$ the local ring at $a \in X$

prop $\mathcal{O}_{X,a} = A(X)_{I(a)}$ (we localize at the entire $I(a)$)

prop $\mathcal{O}_{X,a} = \{\frac{g}{f}\ f,g \in A(X)$ with $f(a) \neq 0\}$ $\rightsquigarrow$ local ring

has maximal ideal $\mathfrak{m}_a = \{\frac{g}{f}$ with $g(a) = 0$ & $f(a) \neq 0\}$