Algebraic Geometry Course

Section 2

Zariski topology

Natural closed subsets

A topology on a space is a collection of open and closed subsets

- $\emptyset$ and the total space are closed
- any intersection of closed subsets is closed
- finite union of closed subsets is closed
- open are complements of closed

- $\emptyset$ and the total space are open
- finite intersection of open subsets is open
- any union of open subsets is open
- closed are complements of open

We have a good candidate for closed subsets : vanishing loci of sys. of pol. eq.

Zariski topology $\begin{cases} \text{Closed subsets = Affine subvarieties} \\ \text{Open subsets = Total Space} \setminus \text{Affine subvarieties} \end{cases}$

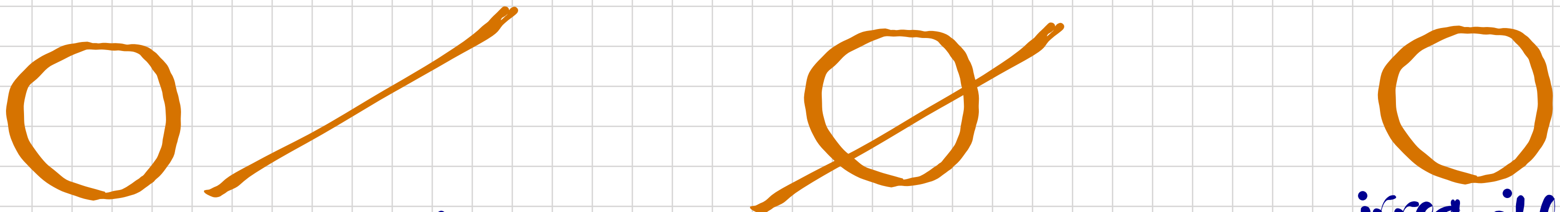
WEIRD! Most open subsets are dense

Irreducibility

X is disconnected $\Longleftrightarrow X = X_1 \cup X_2$ with X_i open; otherwise X is connected

X is reducible $\Longleftrightarrow X = X_1 \cup X_2$ with X_i closed; otherwise X is irreducible

Irreducibility is an important notion in the Zariski topology but not very interesting in the usual topology



disconnected & reducible

connected & reducible

irreducible (hence connected)

prop: X irreducible $\Longleftrightarrow$ $A(X)$ integral domain

$\nexists\, f_1 f_2 \in A(X)$ s.t. $f_1 \cdot f_2 = 0$

$$\left\{ \begin{array}{l} \text{irreducible} \\ \text{subvarieties of } X \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{prime} \\ \text{ideals in } A(X) \end{array} \right\}$$

Noetherian spaces

X is a Noetherian topological space $\Longleftrightarrow$ $\nexists$ infinite chain of closed subsets $X = X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$

Again, being Noetherian is an important notion in the Zariski topology but not very interesting in the usual topolog

When $Y \subset X$ and X noetherian $\Longrightarrow$ Y noetherian

Every $V(I) \subset \mathbb{A}^n_k$ noetherian $\Longleftrightarrow$ $\mathbb{A}^n_k$ noetherian $\Longleftrightarrow$ $\nexists$ infinite chain of ideals $I = I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \dots$

$\Updownarrow$

$K[x_1, \dots, x_n]$ is a noetherian ring

with respect to inclusion

irreducible components

prop: X noetherian $\Longrightarrow X = \bigcup_{\text{finite}} X_i$ with X_i irreducible

$X_i = V(I_i)$ $\quad I_i \trianglelefteq A(X)$ s.t. $J \ntrianglelefteq I_i$ is not prime

$$\left\{ \begin{array}{l} \text{irreducible} \\ \text{components of } X \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{minimal prime} \\ \text{ideals of } A(X) \end{array} \right\}$$

Dimension

One can define dimension using irreducibility

$$\dim(X) = \sup\{n \in \mathbb{N} : \emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n \subset X \ ; Y_i \text{ irred}\}$$

Given $Y \subset X$ we define the codimension

$$\operatorname{codim}(Y, X) = \sup\{n \in \mathbb{N} : Y \subset Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n \subset X \ ; Y_i \text{ irred}\}$$

Recall Krull dimension of a ring A and the Krull codimension of $I \subset A$

$$\dim(A) = \sup\{n \in \mathbb{N} : 0 \neq I_0 \subsetneq I_1 \subsetneq \dots \subsetneq I_n \subset A \ ; I_i \text{ prime}\} \quad (V(I_i) = Y_{n-i})$$

$$\operatorname{codim}(I, A) = \sup\{n \in \mathbb{N} : I \subset I_0 \subsetneq I_1 \subsetneq \dots \subsetneq I_n \subset A \ ; I_i \text{ prime}\}$$

Lemma X affine variety with coordinate ring A, consider $V(I) \subset X$ for $I \lhd A$

$$\dim(X) = \dim(A)$$

$$\operatorname{codim}(V(I), X) = \operatorname{codim}(I, A)$$

prop: Let X and Y be irreducible affine varieties

- $\dim(X \times Y) = \dim(X) + \dim(Y)$
- $Y \subset X \implies \dim(X) = \dim(Y) + \operatorname{codim}(Y, X)$
- $0 \neq f \in A(X) \implies$ Every irred. comp. of $V(f)$ has $\operatorname{codim} = 1$

A noetherian topological space is of pure dimension n if every irreducible component has $\dim = n$

(affine) curve = pure dimension 1 noetherian affine variety

(affine) surface = pure dimension 2 noetherian affine variety

(affine) hypersurface of Y irreducible = pure dimension $\dim Y - 1$ noetherian affine subvariety

For the future

A unique factorization domain $\iff$ $\forall f$ $\substack{\text{non-zero} \\ \text{non-unit}}$ is $f = f_1 \cdot \ldots \cdot f_j$ where f_i prime

prop: Let A noetherian integral domain

A is u.f.d. $\iff$ $\forall$ prime $I \trianglelefteq A$ with $\operatorname{codim}(I, A) = 1$ is principal

X locally factorial $\longleftrightarrow$ every codim $= 1$ irred. subvar gives a line bundle