Algebraic Geometry Course

Section 1

Affine Varieties

Affine Varieties

1st example of algebraic sp.

$\boxed{\begin{array}{l}\text{Affine space } \mathbb{A}^n_k \ (= k^n) \\ \text{equipped with polynomials } k[x_1, \dots x_n]\end{array}}$

(ALGEBRA)

(GEOMETRY)

Systems of polynomial equations $\longleftrightarrow$ Set of solutions

$$\begin{cases} f_1(x_1 - x_n) = 0 \\ \vdots \\ f_n(x_1 - x_n) = 0 \end{cases}$$

$$\left\{ x = (x_1 - x_n) \text{ s.t. } \begin{array}{c} f_1(x) = 0 \\ \vdots \\ f_n(x) = 0 \end{array} \right\} \subset \mathbb{A}^n_k$$

$$\|$$

$$V(f_1 - f_n) \subset \mathbb{A}^n_k$$

zero locus
& vanishing locus.

$$\|$$

adding eq. obtained
from lin. comb. of f_i

$I = \langle f_1 - f_n \rangle$ $\longleftrightarrow$ $\boxed{\begin{array}{l} V(I) \text{ for any ideal } I \triangleleft k[x_1 - x_n] \\ \text{definition of an affine variety} \end{array}}$

ideal of $k[x_1 - x_n]$
generated by $f_1 - f_n$

2nd example
of algebraic sp.

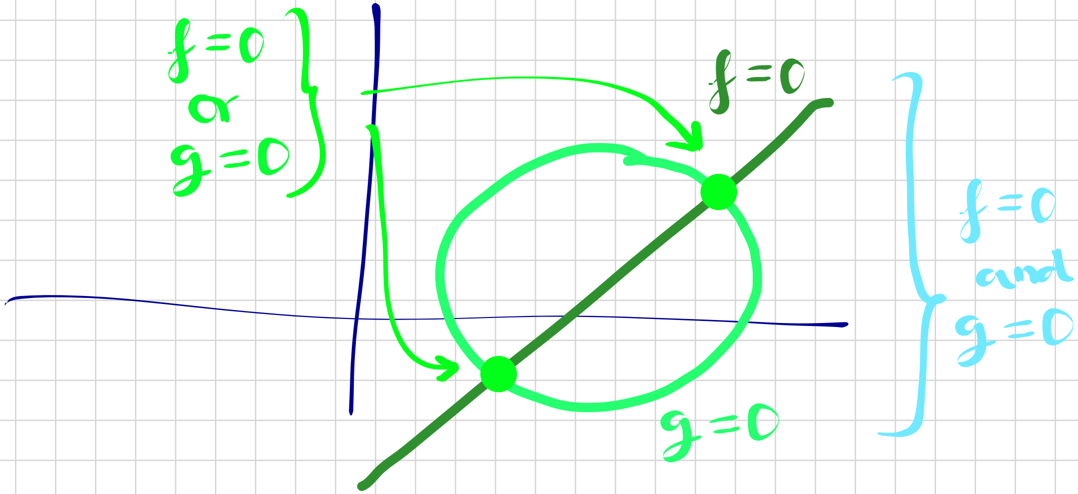
Properties of vanishing sets of ideals

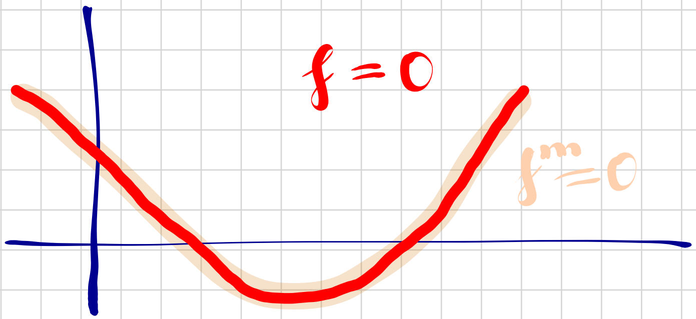
$\sqrt{J}$ = radical of the ideal J

$$= \{ f \subset R \ \text{ s.t. } \ f^m \in J \ \text{ for some } m \}$$

both have the same zeros

$\Leftrightarrow \quad V(J) = V(\sqrt{J})$





More properties

- $V(J_1) \cap V(J_2) = V(J_1 + J_2)$
- $V(J_1) \cup V(J_2) = V(J_1 \cap J_2)$

The ideal of a set

Define $I(Y) := \{ f \in k[x_1, \_ x_n] \text{ s.t. } f(y) = 0 \text{ for all } y \in Y \}$ $\left(I(X) = \sqrt{I(X)} \right)$

<u>Theorem</u> (Hilbert Nullstellensatz)

k alg. closed, $I \triangleleft k[x_1, \_ x_n]$

let f s.t. $f(x) = 0$ for all $x \in V(I)$

then $f \in \sqrt{I}$

Inmediate consequences:

- $V(I(Y)) = Y$
- $I(V(J)) = \sqrt{J}$

$$\left\{ \begin{matrix} \text{affine} \\ \text{varieties} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{radical} \\ \text{ideals} \end{matrix} \right\}$$

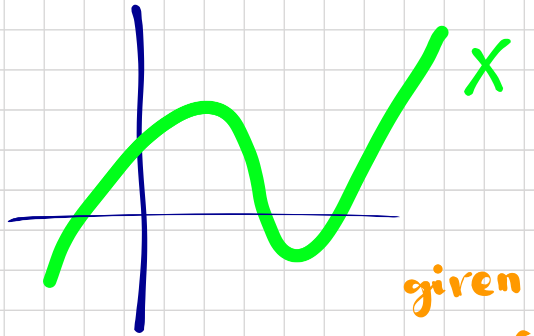
Then, more properties follow

- $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$
- $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$

Coordinate ring

Coordinate ring of X

$$A(X) := \frac{K[x_1 \_\_ x_n]}{I(X)} \approx \text{the polynomials defined over } X$$



given $f \in I(X)$

$$g|_X = (g+f)|_X$$

Relative vanishing locus

$$V_X(Y) := \{ x \in X \text{ s.t. } f(x) = 0 \text{ for all } f \in S \} \subset X$$

Relative ideal

$$I_X(Y) := \{ f \in A(X) \text{ s.t. } f(y) = 0 \text{ for all } y \in Y \} \trianglelefteq A(X)$$

Relative Nullstellensatz

$$\left\{ \begin{array}{l} \text{affine} \\ \text{subvars of } X \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{radical} \\ \text{ideals of } A(X) \end{array} \right\}$$