RELATIVE EQUILIBRIA IN LINEAR ELASTICITY

M.E. SOUSA DIAS
Dept. de Matemática, Instituto Superior Técnico,
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail: edias@math.ist.utl.pt

Relative equilibria for Hamiltonian dynamical systems modelling the motion of hyperelastic, homogeneous, frame indifferent and isotropic bodies, are studied. We find conditions on the potential energy function V for the existence of relative equilibria with certain prescribed symmetries, namely for those of type S.

1 Introduction

In the theory of elasticity different kinds of materials are characterized by the symmetry properties of a Piola -Kirchhoff stress tensor Π (see Marsden et al [4]). The tensor Π dependends through points of a reference body $\mathcal{B}$ and the deformation gradient, $d\phi$, of a configuration ϕ.

Here we will work in the context of linear elasticity and so the configuration space is the set of all invertible and orientation preserving linear maps $\phi : \mathcal{B} \subset \mathbb{R}^3 \to \mathbb{R}^3$, that is the linear group $GL^+(3)$ of all 3×3 matrices with positive determinant, and so the deformation gradient is constant, say Q.

The dynamics is considered to be given by a Hamiltonian system with Hamiltonian function H equal to the sum of the kinetic energy K and potential energy V. The Hamiltonian H is defined on the phase space $T^*GL^+(3)$, the cotangent bundle of the configuration space.

The type of elastic bodies we will consider are homogeneous, hyperelastic, frame indifferent and isotropic. A material is said hyperelastic if there exists a stored energy function W such that the tensor Π is its derivative with respect to the configuration. The homogeneity hypothesis implies that Π is independent of points of $\mathcal{B}$, that is the material exhibit the same mechanical behaviour on different portions of the body.

For hyperelastic materials the frame indifference and isotropy hypotheses are equivalent to requiring that the potential energy is respectively left and right invariant for the action of the orthogonal group $SO(3)$ on the configuration space. This means that V is an invariant function with respect to the following $SO(3) \times SO(3)$ action

$$(g, h) \cdot Q = g\, Q\, h^T \qquad Q \in GL^+(3), \qquad (g, h) \in SO(3) \times SO(3). \quad (1)$$

Frame indifference and isotropy imply that the potential energy depends on the configuration Q through the left and right Cauchy stress tensors, respectively QQ^T and Q^TQ.

Examples of materials that exhibit all the assumed properties are Mooney-Rivlin materials and perfect fluid masses. Another well known problem in this context is the Dirichlet problem for self-gravitating fluid masses (see Chandrasekhar [2]).

The main aim in this work is to find conditions on the potential energy which guarantee the existence of group orbits of relative equilibria with prescribed symmetry.

2 Geometric formulation

Consider a Hamiltonian system given by an invariant $SO(3)\times SO(3)$ Hamiltonian function H defined on $T^*GL^+(3)$ where the action on $T^*GL^+(3)$, is the lifted action of the action (1).

For $(Q,P)\in T^*_QGL^+(3)$ and $(Q,V)\in T_QGL^+(3)$ consider the pairing between $T^*_QGL^+(3)$ and $T_QGL^+(3)$ to be given by:

$$\langle P,V\rangle_E = \mathrm{tr}(P^TVE), \qquad \text{with} \qquad E=\int_{\mathcal{B}}\rho\,(X)X\otimes X dX.$$

E is the inertia dyadic of the reference body and ρ its density function.

By Noether's theorem the $SO(3)\times SO(3)$-invariance of H gives rise to conserved quantities expressed by the momentum map $J: T^*GL^+(3)\to so(3)^*\times so(3)^*$, where $so(3)^*$ denotes the dual of the Lie algebra of $SO(3)$. Consider the pairing between the Lie algebra of $SO(3)$ and its dual to be the restriction of the following pairing between elements of the set of 3×3 matrices, $gl(3)$

$$\langle \mu,\xi\rangle = \mathrm{tr}(\mu^T\xi\rangle \qquad \text{for} \qquad \mu\in gl^*(3) \qquad \text{and} \qquad \xi\in gl(3).$$

A relative equilibrium is a dynamic group orbit for the Hamiltonian vector field. This is a point $(Q_e,P_e)\in T^*GL^+(3)$ such that the trajectory through (Q_e,P_e) is given by $\exp(t\xi)\cdot(Q_e,P_e)\cdot\exp(-t\eta)$ for some $(\xi\,,\,\eta)\in so(3)\times so(3)$. Equivalently (Q_e,P_e) is a relative equilibrium for the Hamiltonian system defined by H if there is $(\xi\,,\,\eta)\in so(3)\times so(3)$ such that (Q_e,P_e) is a critical point of the modified Hamiltonian

$$H_{(\xi,\eta)}(Q,P) = H(Q,P) - \langle J(Q,P),(\xi,\eta)\rangle. \tag{2}$$

2.1 Relative equilibria

The value of the momentum map J, for the lifted action on cotangent bundles, at a point (Q,P) is the value of P at the infinitesimal generator for the action on the configuration space (see Abraham et al. [1]). For our problem, the momentum map has been computed in Roberts et al. [5] and Sousa Dias [6] and is

$$J(Q,P) = (J_L(Q,P), J_R(Q,P)) = \frac{1}{2}\left(PQ^T - QP^T\,, P^TQ - Q^TP\right). \tag{3}$$

The symmetry assumption implies that the potential energy depends on the configuration through the left and right Cauchy tensors. This is equivalent to saying that V depends on the configuration through three invariants, say I, II, II, with

$$I = \operatorname{tr} C \qquad II = \frac{1}{2}\left[(\operatorname{tr} C)^2 - \operatorname{tr} C^2\right] \qquad III = \det C,$$

where $\operatorname{tr} C$ and $\det C$ denote respectively the trace and the determinant of the matrix $C = Q^TQ$ (or $C = QQ^T$ since the invariants are the same).

Proposition 1 *The point (Q_e, P_e) is a relative equilibrium for the Hamiltonian system defined by the $SO(3)\times SO(3)$ invariant function $H(Q,P) = \frac{1}{2}||P||^2_{E^{-1}} + V(Q)$ if and only if there are skew-symmetric matrices $\hat{\xi}, \hat{\eta}$ verifying the following equation*

$$\begin{aligned} 2V_1 + 2V_2(IQ_e - Q_eQ_e^TQ_e) + 2V_3IIIQ_e^{-1} &= \\ &= -\hat{\xi}\left(\hat{\xi}\,Q_e - Q_e\,\hat{\eta}\right)E + \left(\hat{\xi}\,Q_e - Q_e\,\hat{\eta}\right)E\,\hat{\eta}, \end{aligned} \tag{4}$$

where $V_1 = \frac{\partial V}{\partial I}, V_2 = \frac{\partial V}{\partial II}, V_3 = \frac{\partial V}{\partial III}, I, II$ and III are all evaluated at (Q_e, P_e).

Proof: The first variation of (2) is:

$$DH_{(\hat{\xi},\hat{\eta})}(Q,P) = \langle\, \frac{\delta V}{\delta Q} + \hat{\xi}P - P\hat{\eta}\,,\, \delta Q\rangle + \langle\, PE^{-1} - \hat{\xi}Q + Q\hat{\eta}\,,\, \delta P\rangle$$

So (Q_e, P_e) is a relative equilibrium if and only if

$$\begin{aligned} P_eE^{-1} &= \hat{\xi}\,Q_e - Q_e\hat{\eta} \\ \tfrac{\delta V}{\delta Q} &= -\hat{\xi}P_e + P_e\,\hat{\eta}. \end{aligned} \tag{5}$$

That is if and only if

$$\frac{\delta V}{\delta Q} = -\hat{\xi}\left(\hat{\xi}\,Q_e - Q_e\,\hat{\eta}\right)E + \left(\hat{\xi}\,Q_e - Q_e\,\hat{\eta}\right)E\,\hat{\eta}.$$

Now $\frac{\delta V}{\delta Q} = V_1 \frac{\delta I}{\delta Q} + V_2 \frac{\delta II}{\delta Q} + \frac{\delta III}{\delta Q}$ and the result follows by the computation of the variations of the three invariants.

$$\langle \frac{\delta I}{\delta Q}\,,\, \delta Q \rangle = 2\,\mathrm{tr}(Q^T \delta Q) = 2\,\langle Q, \delta Q \rangle$$

$$\langle \frac{\delta II}{\delta Q}\,,\, \delta Q \rangle = 2\,\langle \mathrm{tr}(QQ^T)Q - QQ^TQ\,, \delta Q \rangle = 2\langle I - QQ^TQ\,,\ \delta Q \rangle$$

$$\langle \frac{\delta III}{\delta Q}\,,\, \delta Q \rangle = 2\mathrm{det}\,(Q)\langle \mathrm{det}\,(Q)\, Q^{-1}, \delta Q \rangle = 2\langle III\, Q^{-1}\,,\ \delta Q \rangle$$

Remark: For rigidly rotating relative equilibria (i.e $\hat{\eta} = 0$) the conditions (5) coincide with that of Lewis and Simo [3] where the existence and the nonlinear stability of some relative equilibria, namely the rigid rotating ones, is studied.

2.2 Symmetries of the relative equilibria

Any $SO(3) \times SO(3)$-invariant function Hamiltonian of the form $H = K(P) + V(Q)$ is also invariant under the action of the cyclic group Z_2 generated by an element τ that acts on the $GL^+(3)$ by transposition of matrices. This means that the full symmetry group is the semidirect product of Z_2 by $SO(3) \times SO(3)$. The $Z_2 \times_s (SO(3) \times SO(3))$ action on $GL^+(3)$ is given by

$$(1, g, h) \cdot Q = g\, Q\, h^T, \quad (\tau, g, h) \cdot Q = g\, Q^T\, h^T. \tag{6}$$

The induced Z_2 action on $so(3)^* \times so(3)^*$ is by interchanging the roles of J_L and J_R. Physically this means that if Q is an admissible configuration of our problem with conserved quantities $J = (J_L, J_R)$, then so is Q^T with conserved quantities (J_R, J_L). This result is known as Dedekind Theorem for the Dirichlet problem of self rotating fluid masses.

We are interested to know the conditions that must be satisfied by the potential energy for the existence of relative equilibria with prescribed symmetry, that is lying in a given fixed point set of an isotropy subgroup for the action (6). In next section we find these conditions for some classes of relative equilibria, namely for almost all S-type relative equilibria.

Following Riemann's classification of "ellipsoidal figures of equilibrium" for Dirichlet problem we divide the relative equilibria into two classes, those called of type S and those not of this type . A relative equilibrium configuration $\exp(t\hat{\xi})\, Q_0\, \exp(-t\hat{\eta})$ is said to be of type S if ξ and η are vectors aligned with one of the $\mathbb{R}^3$ basis axes ($\hat{\xi}$, $\hat{\eta}$ are skew symmetric matrices identifications of ξ and η respectively).

The $SO(3) \times SO(3)$ action on $so(3)^* \times so(3)^*$ is

$$(g,h) \cdot (\mu_L, \mu_R) \;=\; (g\,\mu_L\,g^T \;,\; h\,\mu_R\,h^T)$$

and so the isotropy lattice for the $Z_2 \times_s (SO(3) \times SO(3))$ on $so(3)^* \times so(3)^*$ is easily obtained from the isotropy lattice for the action of $SO(3)$ on the space of skew-symmetric matrices by conjugation. This latter action has for isotropy classes representatives $SO(3)$ and $SO(2)$, fixing respectively zero and a vector ξ aligned with a principal axis, say $\xi := (0,0,\xi) = \begin{pmatrix} 0 & -\xi & 0 \\ \xi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} := \hat{\xi}$.

We also distinguish relative equilibria with $J_L = J_R$ from those with $J_L = -J_R$. The first are said to be *self-adjoint* and the latter *skew-adjoint*. Note that the equivariance of the momentum map implies, for non-trivial values of J_L and J_R, that the vectors ξ and η are respectively parallel to J_L and J_R. Also by equivariance of the momentum map the isotropy subgroup of a relative equilibrium is contained in the isotropy subgroup of its momentum value.

Table 1. Some isotropy subgroups for the $Z_2 \times_s (SO(3) \times SO(3))$ action (6) and representative fixed points in $gl^*(3)$ and $so(3)^* \times so(3)^*$.

Isotropy Subgroup	$P \in gl^*(3)$	J_L, J_R
$Z_2 \times_s SO(3)$	diag (a,a,a)	$J_L = J_R = (0,0,0)$ self-adjoint
$Z_2 \times_s O(2)$	diag (a,a,c)	
$Z_2 \times_s D_2$	diag (a,b,c)	
$\widetilde{O(2)}$	$\begin{pmatrix} a & -d & 0 \\ d & a & 0 \\ 0 & 0 & c \end{pmatrix}$	$J_L = J_R = (0,0,\mu)$ S-type self-adjoint
$Z_2 \times_s Z_2$	$\begin{pmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & c \end{pmatrix}$	
$\widetilde{D_2}$	$\begin{pmatrix} a & -d & 0 \\ d & b & 0 \\ 0 & 0 & c \end{pmatrix}$	$J_L = (0,0,\mu) = -J_R$ S-type skew-adjoint

The isotropy lattice for the lifted action of (6) to $T^*GL^+(3)$ has been computed in Roberts et al. [5] and Sousa Dias [6]. There, $T^*GL^+(3)$ has been identified with $GL^+(3) \times gl^*(3)$ and the isotropy subgroups for the full action are showed to be the same as the isotropy subgroups of the following $Z_2 \times SO(3)$ on $gl^*(3)$:

$$(1,g)\cdot Q = g\,Q\,g^T, \quad (\tau,g)\cdot Q = g\,Q^T\,g^T.$$

In table 1 we list the isotropy subgroup classes which are the objects of study in next section. In this table, $Z_2 \times_s \Sigma$ is an isotropy subgroup that fixes a symmetric matrix, where Σ denotes a diagonally embedded isotropy subgroup for the $SO(3)\times SO(3)$ action on matrices by conjugation. $\widetilde{\Sigma}_1$ denotes an isotropy subgroup generated by an isotropy group $\Sigma \subset \Sigma_1$ (Σ is of the form defined above) and an element (τ,k) where k is a rotation by π about an axis perpendicular to the rotation axis of Σ.

Table 2. Conditions on the potential energy V for the existence of relative equilibria with the prescribed symmetry.

Isotropy Subgroup	$P_0 \in gl^*(3)$	$\nabla V = (V_1, V_2, V_3)$ velocity: $(0,0,\xi)$
$Z_2 \times_s SO(3)$	0	$a^2\,\left(2V_2 + a^2 V_3,\, V_2,\, V_3\right) = 0, \quad \xi = 0$
$Z_2 \times_s O(2)$	0	$V_1 = -a^2\,(c^2+a^2)$ $V_2 = a^2$ $V_3 = 1/(c^2-a^2)$
$Z_2 \times_s D_2$	0	Either $\nabla V = (0,0,0)$ or $c^2 = a^2 b^2 - a^2 - b^2$ and $\nabla V = -(c^2, c^2, 1)\,V_3$
$\widetilde{O(2)}$	0	Either $\nabla V = (0,0,0)$ or if $a^2 = d^2$ then $V_1 = -3a^2\,(a^2+3c^2)\,V_2$ $V_3 = \frac{c^2-a^2}{a^2\,(2c^2+a^2)}\,V_2$
$Z_2 \times_s Z_2$	$e_1 = e_2 = \epsilon$ $\epsilon\,\xi \begin{pmatrix} -2d & (a-b) & 0 \\ (a-b) & 2d & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Either, $c^2 + d^2 = ab$ and $V_1 = \epsilon\,\xi^2 = c^4\,V_3$, $V_2 = 0$, or $a^2 + b^2 = d^2 - ab$, $c^2 + d^2 \neq ab$ and $\epsilon\xi^2 = \frac{(d^2-ab)\,(c^2+ab)-c^4}{c^2+d^2-ab} V_2$ $V_1 = \epsilon\,\xi^2 - (c^2+2d^2)\,V_2$ $c^2\,(d^2-ab)\,V_3 = \epsilon\,\xi^2 + abV_2$
$\widetilde{D_2}$	$e_1 = e_2 = \epsilon$ $\epsilon\,\xi \begin{pmatrix} -2d & -(a+b) & 0 \\ (a+b) & 2d & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$(a+b)\,(a^2+b^2+ab-d^2)+$ $c^2(a^2+b^2-c^2) = ab$ $2\,\epsilon\xi^2 = (ab - c^2(a^2+b^2-c^2))\,V_2$ $V_1 = (a+b)\,\epsilon\,\xi^2 - (c^2+2d^2)\,V_2$ $c^2(d^2+ab)\,V_3 = (a+b)\,\epsilon\,\xi^2 - ab\,V_2$

3 Potential energy conditions for the existence of relative equilibria

We choose a basis in $\mathbb{R}^3$ in which the inertia dyadic E of the reference body is a diagonal matrix $E = \mathrm{diag}(e_1, e_2, e_3)$. We will look for relative equilibria $(Q_e, P_e) \in T^*GL^+(3)$ belonging to the fixed point sets of the isotropy subgroups listed in table 1.

Let V_1, V_2, V_3 be as in proposition 1, that is the derivatives of the potential energy with respect to the three principal invariants I, II, III calculated at the relative equilibrium configuration Q_e.

Theorem 1 *Table 2 gives the conditions on $\nabla V = (V_1, V_2, V_3)$ for the symmetric Hamiltonian system defined by $H(Q,P) = \frac{1}{2}||P||_{E^{-1}} + V(Q)$ to have a relative equilibrium with the listed respective symmetry.*

Note that for $Z_2 \times_s Z_2$ and $\widetilde{D_2}$ relative equilibria configurations the velocity $(0,0,\xi)$ is dependent on the configuration itself while this not true for the other symmetric relative equilibrium configurations.

Proof: Let $Qsym$ and $Qskew$ denote relative equilibria configurations of the form

$$Qsym = \begin{pmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & c \end{pmatrix} \qquad Qskew = \begin{pmatrix} a & -d & 0 \\ d & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

By proposition 1, equation (5), we have that $(Qsym, Psym)$ and $(Qskew, Pskew)$ are relative equilibria if and only if

$$Psym = \xi \begin{pmatrix} -2de_1 & (a-b)e_2 & 0 \\ (a-b)e_1 & 2de_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for self- adjoint}$$

$$Pskew = \xi \begin{pmatrix} -2de_1 & -(a+b)e_2 & 0 \\ (a+b)e_1 & 2de_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for skew- adjoint}$$

$$Pskew = \xi \begin{pmatrix} 0 & (a-b)e_2 & 0 \\ (a-b)e_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for self- adjoint}$$

By table 1 the first two matrices will belong to the fixed point sets of $Z_2 \times_s Z_2$ and $\widetilde{D_2}$ if and only if $e_1 = e_2$. The third belongs always to the fixed point set of $\widetilde{O(2)}$ since in this case $a = b$ which implies that $Pskew = 0$.

The nontrivial entries of the matrix of the left hand side of equation (4) are:

$$\begin{gathered}
\pm 2d\,\left(V_1 + (a^2+b^2+c^2+d^2)\,V_2 \,\pm\, c^2(d^2 \mp ab)\,V_3\right) \\
2d\,\left(V_1 + (a^2+b^2+c^2+d^2)\,V_2 \,\pm\, c^2(d^2 \mp ab)\,V_3\right) \\
2\,(a+b)\,\left(V_1 + (ab+c^2+2d^2)\,V_2 \,\mp\, c^2(d^2 \mp ab)\,V_3\right) \\
-2\,(a-b)\,\left(-V_1 - (-ab+c^2+2d^2)\,V_2 \,\mp\, c^2(d^2 \mp ab)\,V_3\right) \\
2c\,\left(V_1 + (a^2+b^2+2d^2)\,V_2 + (d^2 \mp ab)^2\,V_3.\right),
\end{gathered} \tag{7}$$

where the upper and lower signs stand for the symmetric and skew-symmetric configurations. Furthermore the first two expressions are the non trivial off-diagonal elements of the matrix, the second and the fourth are respectively the sum and the difference of the first two diagonal entries, and the last expression is the third diagonal entry.

The right hand side of equation (4), with $e_1 = e_2 = \epsilon$, gives for relative equilibria :

1. In $Fix\left(\widetilde{O(2)}\right)$ we have $P_0 = 0$ and so the right hand side of equation (4) is the matrix zero.

2. In $Fix\,(Z_2 \times_s Z_2)$: $\xi^2\,\epsilon \begin{pmatrix} 2(a-b) & 4d & 0 \\ 4d & -2(a-b) & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

3. In $Fix\left(\widetilde{D_2}\right)$: $\xi^2\,\epsilon \begin{pmatrix} 2(a+b) & -4d & 0 \\ 4d & 2(a+b) & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Equating the corresponding expressions for the left and right hand side of (4) we get:

1. In $Fix\left(\widetilde{O(2)}\right)$ where $P_0 = 0$ and $a = b$, the equation (4) is equivalent to

$$V_1 + (2a^2 + c^2 + d^2)V_2 - c^2(d^2 + a^2)V_3 = 0$$

$$V_1 + (a^2 + c^2 + 2d^2)V_2 - c^2(d^2 + a^2)V_3 = 0$$

$$V_1 + (2a^2 + 2d^2)V_2 - (d^2 + a^2)^2 V_3 = 0.$$

The solutions of this system are $(V_1, V_2, V_3) = (0, 0, 0)$ and

$$V_1 = -3a^2\left(a^2 + 3c^2\right)V_2, \qquad V_3 = \frac{c^2 - a^2}{a^2(2c^2 + a^2)}\, V_2 \qquad \text{if } a^2 = d^2.$$

2. In $Fix\left(\widetilde{D_2}\right)$:

$$V_1 + \left(a^2 + b^2 + c^2 + d^2\right)V_2 - c^2\left(d^2 + ab\right)V_3 = 2\epsilon\,\xi^2$$

$$V_1 + \left(ab + c^2 + 2d^2\right)V_2 + c^2\left(d^2 + ab\right)V_3 = 2\epsilon\,\xi^2$$

$$-V_1 - \left(-ab + c^2 + 2d^2\right)V_2 + c^2\left(d^2 + ab\right)V_3 = 0$$

$$V_1 + \left(a^2 + b^2 + 2d^2\right)V_2 + (d^2 + ab)^2 V_3 = 0.$$

Adding and subtracting the second and third equations we get

$$V_1 = (a+b)\epsilon\,\xi^2 - (c^2 + 2d^2)V_2, \qquad c^2(d^2 + ab)\,V_3 = (a+b)\epsilon\,\xi^2 - abV_2,$$

Plugging these expressions into the first and fourth equations we find nontrivial velocity ξ only when $(a+b)\left(a^2 + b^2 + ab - d^2\right) + c^2\left(a^2 + b^2 - c^2\right) - ab = 0$ and

$$2\,\epsilon\,\xi^2 = \left(ab - c^2(a^2 + b^2 - c^2)\right)\, V_2.$$

3. In $Fix\left(Z_2 \times_s Z_2\right)$:

$$V_1 + \left(a^2 + b^2 + c^2 + d^2\right)V_2 + c^2(d^2 - ab)V_3 = 2\,\epsilon\,\xi^2$$

$$V_1 + \left(ab + c^2 + 2d^2\right)V_2 - c^2(d^2 - ab)V_3 = 0$$

$$-V_1 - \left(-ab + c^2 + 2d^2\right)V_2 - c^2(d^2 - ab)V_3 = -\,2\,\epsilon\,\xi^2$$

$$V_1 + \left(a^2 + b^2 + 2d^2\right)V_2 + (d^2 - ab)^2 V_3 = 0.$$

Adding and subtracting the second and third equations we have:

$$V_1 = \epsilon\xi^2 - (c^2 + 2d^2)\, V_2 \qquad c^2\,(d^2 - ab)\, V_3 = \epsilon\xi^2 + ab\, V_2 \tag{8}$$

Plugging these into the first and fourth equations we get two cases:

a) $V_2 = 0$ and $c^2 + d^2 - ab = 0$ gives $V_1 = \epsilon\,\xi^2 \;=\; c^4\, V_3$.

b) If $a^2 + b^2 = d^2 - ab$ and $c^2 + d^2 \neq ab$ then there exists non trivial ξ with $\epsilon\,\xi^2 = \frac{(d^2-ab)\,(c^2+ab)-c^4)}{c^2+d^2-ab}\, V_2$ and relations (8) are verified.

4. In $Fix\,(Z_2 \times_s D_2)$, $Fix\,(Z_2 \times_s O(2))$ and $Fix\,(Z_2 \times_s SO(3))$ we have relative equilibria of the form (Q_0, P_0) (with $P_0 = 0$ and $Q_0 = \mathrm{diag}\,(a, b, c)$), where $a \neq b \neq c$, $a = b$ and $a = b = c$ respectively.

   From the expression of $Psym$ and table 1 we have that a relative equilibrium belongs to $Fix\,(Z_2 \times_s SO(3))$ only if its velocity is $\xi = 0$, i.e it must be an equilibrium.

   In all these cases, the expressions (7) simplify and the solutions of the homogeneous system obtained by setting (7) equal to zero are easy to compute and verify the conditions of table 2.

Acknowledgments

This work has been financially supported by FCT's Pluriannual Funding Program and projects PRAXIS XXI 2/2.1/MAT/199/94, PBIC/C/MAT/2139/95.

References

1. R. Abraham and J. Marsden, *Foundations of Mechanics*, (Addison-Wesley,1978).
2. S. Chandrasekhar, *Ellipsoidal figures of equilibrium*, (Dover Publ.,1987).
3. D. Lewis and Simo, *Nonlinear stability of pseudo rigid bodies* Proc. R. Soc. London **A-427**, 281-319 (1990).
4. J. Marsden and T. Hughes, *Mathematical Foundations of Elasticity*, (Prentice-hall, Inc.,1983).
5. R.M. Roberts and M.E.R. de Sousa-Dias, *Symmetries of Riemann ellipsoids* (Warwick preprints 30/1995).
6. M.E.R. de Sousa-Dias, *Local Dynamics of Symmetric Hamiltonian systems...*(PhD thesis, Univ. Warwick 1995).