WIGNER MEASURES AND QUANTUM AUBRY-MATHER THEORY

DIOGO A. GOMES (*), CLAUDIA VALLS (**)

Abstract. In this paper we investigate the quantum action problem using Wigner measures on the torus. We prove existence, study its main properties, and prove convergence to Mather measures in the semiclassical limit. We also indicate how to extend these techniques to the study of stochastic Mather measures.

(*) Supported in part by FCT/POCTI/FEDER
(**) Supported in part by the Center for Mathematical Analysis, Geometry, and Dynamical Systems, through FCT/POCTI/FEDER and grant SFRH/BPD/14404/2003.

1. Introduction

Wigner measures [LP93] are a powerful tool to study the semiclassical limit in quantum mechanics. The objective of this paper is to present new applications of Wigner measures and related techniques to the study the Aubry-Mather theory [Mat91] and the related quantum action problem [Eva03].

In quantum mechanics, Wigner measures are measures on phase space which encode the semiclassical behaviour as h, the Plank constant, tends to zero. In particular, for stationary states of the Schrödinger equation, they yield measures which are invariant under the classical dynamics. In classical mechanics, the Mather measures are action minimizing measures which are invariant under the same dynamics. At the present, the connection between the semiclassical limit of stationary states of the Schrödinger equation and Mather measures is still unknown. In [Eva03] it was suggested a related quantum action problem which in the semiclassical limit yields Mather measures. In this paper, we investigate further the quantum action problem and we prove that the corresponding Wigner measures converge to Mather measures as $h \to 0$. We also indicate how these techniques extend to the study of

stochastic Mather measures [Gom02] and how to construct a quantum analog.

The plan of this paper is as follows: in section 2 we review some basic results concerning stationary states (eigenvectors) of the Schödinger equation. We focus our attention in variational principles and its relations with classical mechanics, namely the Aubry-Mather theory. Section 3 concerns the construction of Wigner measures in the periodic setting ($\mathbb{T}^n$). The construction of Wigner measures in $\mathbb{T}^n$ is somewhat similar to the one in $\mathbb{R}^n$, with the Fourier transform being replaced by Fourier series. In this section we also present some applications to the study of semiclassical limit of stationary states for the Schrödinger equation. Next, in section 4, we formulate in terms of Wigner measures the Evans' quantum action minimization problem. The existence of minimizers and the connection with Mather's theory is studied, respectively, in sections 5, and 6. Finally, the stochastic Mather problem and its quantum analog are discussed in section 7.

2. Variational principles in quantum and classical mechanics

In this section we review, without proof, some well known variational principles in quantum mechanics, as well as some (formal) relations with the Aubry-Mather theory in classical mechanics.

Proposition 1. *A state ψ is a critical point of the functional*

$$H[\psi] = \int_{\mathbb{T}^n} \frac{h^2}{2}|\nabla\psi|^2 + V(x)|\psi|^2,$$

under the constraint

$$M[\psi] \equiv \int_{\mathbb{T}^n} |\psi|^2 = 1$$

if and only if it is an eigenvector of the Schrödinger equation

$$\mathcal{L}\psi = -\frac{h^2}{2}\Delta\psi + V(x)\psi.$$

Since only the first eigenvalue is a minimum, it is natural to consider further variational principles which may give useful insight for high-frequency states. One such principle can be found in [GM83]:

Proposition 2. *Suppose ψ is an eigenvector of the Schrödinger equation with Bloch wave structure*

$$\psi = ae^{i\frac{Px+u(x)}{h}},$$

in which a and u are periodic real-valued functions and $a > 0$. Then

$$-\frac{h^2}{2}\Delta a + \left[\frac{|P + D_x u|^2}{2} + V(x)\right] a = Ea \tag{1}$$

and

$$\operatorname{div}(a^2(P + D_x u)) = 0. \tag{2}$$

Furthermore, ψ is a critical point of the action

$$S[\psi] = \int_{\mathbb{T}^n} -\frac{h^2}{2}|\nabla a|^2 + \left[\frac{|P + D_x u|^2}{2} - V(x)\right] a^2,$$

under the constraints:

$$\int_{\mathbb{T}^n} a^2 = 1,$$

stationary current

$$\operatorname{div}(a^2(P + D_x u)) = 0,$$

and total current intensity

$$\int_{\mathbb{T}^n} a^2(P + D_x u) = Q.$$

Proof. See [GM83], or also [Eva03]. ■

Unfortunately, the Guerra-Morato action S is not positive definite and in general critical points may not be minimizers.

Motivated by considerations in classical mechanics L. C. Evans [Eva03] introduced the quantum action, which, for a Bloch-Wave type state $\psi = ae^{i\frac{Px+u(x)}{h}}$ is

$$A_Q[\psi] = \int_{\mathbb{T}^n} \frac{h^2}{2}|\nabla a|^2 + \left[\frac{|P + D_x u|^2}{2} - V(x)\right] a^2.$$

Its critical points, under the same constraints as in proposition 2, are local minima of the quantum action, and are quasi-modes to the Schödinger operator.

The problem of minimizing the quantum action should be considered a quantum analog the Mather problem of finding a probability measure μ which minimizes the action

$$A[\mu] = \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, v) d\mu,$$

with

$$L = \frac{|v|^2}{2} - V(x),$$

under the constraints

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} v D_x \phi d\mu = 0, \quad \text{for all } \phi(x) \in C^1(\mathbb{T}^n),$$

which is the analog to (2), and the average rotation number, which is

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} v d\mu = Q.$$

3. Wigner measures and semiclassical limits

In this section we define and study Wigner measures for periodic problems, and present some applications to the semiclassical limit of stationary states of the Schrödinger operator as $h \to 0$. As one would expect, the results are parallel to the ones in the non-periodic setting, for instance see [LP93], with Fourier transform being replaced by Fourier series.

Given a state ψ, which is a function in $L^2(\mathbb{T}^n)$, one wants to compute the averages of observables, such as momentum or energy. Observables are linear operators a and its average is given by $\langle \psi, a\psi \rangle$.

Frequently, observables are pseudo-differential operators, associated with a smooth symbol $a(x, p)$. In the periodic setting the momenta p are quantized and instead of taking values in $\mathbb{R}^n$ they belong to the lattice $h\mathbb{Z}^n$. The main problem is that in pseudo-differential calculus there is not a unique way to associate an operator to a given symbol, as well as to define the average value of an observable a. A common choice is the Weyl quantization rule:

$$\langle \psi, a\psi \rangle = \sum_{p \in h\mathbb{Z}^n} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} a(x, 2\pi p) \bar{\psi}(x + \frac{y}{2}) \psi(x - \frac{y}{2}) e^{-2\pi i p y / h} dy dx.$$

If a is just a function of x then

$$\langle \psi, a\psi \rangle = \int_{\mathbb{T}^n} a(x) |\psi|^2 dx.$$

In the case in which a depends only on p the average value is

$$\langle \psi, a\psi \rangle = \sum_{p \in h\mathbb{Z}^n} a(2\pi p) |\hat{\psi}(p)|^2 dx,$$

in which, for $p \in h\mathbb{Z}^n$,

$$\hat{\psi}(p) = \int_{\mathbb{T}^n} \psi(x) e^{-2\pi i p\cdot x/h} dx$$

are the Fourier coefficients.

The Wigner measure on $\mathbb{T}^n \times \mathbb{R}^n$ is defined by its x-density for each $p \in h\mathbb{Z}^n$

$$W_h(x,p) = \int_{\mathbb{T}^n} \bar{\psi}(x + \frac{y}{2})\psi(x - \frac{y}{2}) e^{-2\pi i p y/h} dy.$$

In this way

$$\langle \psi, a\psi \rangle = \sum_{p \in h\mathbb{Z}^n} \int_{\mathbb{T}^n} a(x, 2\pi p) W_h(x,p).$$

It turns out that if ψ is an eigenvector of the Schrödinger equation then W_h has several remarkable properties which encode the semiclassical behaviour of ψ as $h \to 0$. These properties are well known for the non-periodic case and will record them here in the periodic case for the convenience of the reader.

Proposition 3 (Mass estimate)**.** *Suppose ψ is smooth and*

$$\int_{\mathbb{T}^n} |\psi|^2 = 1$$

then

$$\sum_{p \in h\mathbb{Z}^n} \int_{\mathbb{T}^n} W_h(x,p) = 1.$$

Proof. The Poisson summation formula asserts that, in $\mathbb{T}^n$,

$$\sum_{p \in h\mathbb{Z}^n} e^{-2\pi i p\cdot y/h} = \delta(y).$$

Therefore

$$\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \bar{\psi}(x + \frac{y}{2})\psi(x - \frac{y}{2}) \left[\sum_{p \in h\mathbb{Z}^n} e^{-2\pi i p y/h} \right] dy dx = \int_{\mathbb{T}^n} |\psi|^2 dx = 1.$$

■

Proposition 4 (Energy estimate)**.** *Suppose ψ is a normalized eigenvector of the Schrödinger equation*

$$\mathcal{L}\psi = E\psi.$$

Then,

$$\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\left[\frac{|2\pi p|^2}{2}+V(x)\right]W_h(x,p)=E.$$

PROOF. Observe that

(3)

$$\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\left[\frac{|2\pi p|^2}{2}+V(x)\right]W_h(x,p)$$
$$=\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\left[\frac{|2\pi p|^2}{2}+V(x)\right]\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})e^{-2\pi ipy/h}dydx.$$

Recall that

$$\frac{|2\pi p|^2}{2}e^{-2\pi ipy/h}=-\frac{h^2}{2}\Delta_y\left(e^{-2\pi ipy/h}\right),$$

therefore,

$$\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\frac{|2\pi p|^2}{2}\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})e^{-2\pi ipy/h}dydx$$
$$=-\frac{h^2}{2}\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})\Delta_y\left(e^{-2\pi ipy/h}\right)$$
$$=-\frac{h^2}{8}\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}e^{-2\pi ipy/h}\left[\Delta_x\left(\bar{\psi}(x+\frac{y}{2})\right)\psi(x-\frac{y}{2})\right.$$
$$\left.+\bar{\psi}(x+\frac{y}{2})\Delta_x\left(\psi(x-\frac{y}{2})\right)-2D_x\bar{\psi}(x+\frac{y}{2})D_x\psi(x-\frac{y}{2})\right].$$

By the Poisson summation formula, we conclude that this last term is

$$-\frac{h^2}{8}\int_{\mathbb{T}^n}(\Delta\bar{\psi})\psi+\bar{\psi}\Delta\psi-2|D_x\psi|^2=\frac{h^2}{2}\int_{\mathbb{T}^n}|D_x\psi|^2.$$

Using Poisson summation formula in (3),

$$\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\left[\frac{|2\pi p|^2}{2}+V(x)\right]W_h(x,p)=\frac{h^2}{2}\int_{\mathbb{T}^n}|D_x\psi|^2+V(x)|\psi|^2=E,$$

using that

$$-\frac{h^2}{2}\Delta\psi=(E-V(x))\psi,$$

and that $\int|\psi|^2=1$. ■

Proposition 5. *Suppose ψ_h is a family of (norm one) eigenvectors of the Schrödinger equation*

$$\mathcal{L}\psi = E_h\psi,$$

with E_h bounded. Then, through some subsequence, if necessary, W_h converges, weakly in an appropriate topology, to a positive measure W_0 as $h \to 0$.

Proof. As in [LP93], we consider the following space of sequences of continuous functions

$$\mathcal{A} = \left\{ (\phi_k)_{k\in h\mathbb{Z}^n} : \phi_k \in C_0(\mathbb{T}^n), \sum_{k\in h\mathbb{Z}^n} \phi_k(x)e^{2\pi iky/h} \in L^1(\mathbb{T}^n_y, C_0(\mathbb{T}^n_x)) \right\}.$$

With the norm

$$\|\phi_k\|_{\mathcal{A}} = \int_{\mathbb{T}^n} \sup_x \left| \sum_{k\in h\mathbb{Z}^n} \phi_k(x)e^{2\pi iky/h} \right| dy,$$

$\mathcal{A}$ is a separable Banach space.

Therefore, by weak convergence theory, it suffices to check that W_h is a bounded linear functional on $\mathcal{A}$. As, if this is the case, we can extract an weakly convergent subsequence. So, consider $\phi_k \in \mathcal{A}$ and write

$$\begin{aligned}
&\sum_{p\in h\mathbb{Z}^n} \int_{\mathbb{T}^n} \phi_k(x)W_h(x,p)dx \\
&= \sum_{p\in h\mathbb{Z}^n} \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \phi_k(x)\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})e^{-2\pi ipy/h}dydx \\
&= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left(\sum_{p\in h\mathbb{Z}^n} \phi_k(x)e^{-2\pi ipy/h} \right) \bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})dydx.
\end{aligned}$$

This implies

$$\begin{aligned}
&\left|\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\phi_k W_h(x,p)dx\right|\\
&\leq \int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\left|\sum_{p\in h\mathbb{Z}^n}\phi_k(x)e^{-2\pi i p y/h}\right||\bar\psi(x+\frac{y}{2})\psi(x-\frac{y}{2})|dydx\\
&\leq \int_{\mathbb{T}^n}\left[\sup_x\left|\sum_{p\in h\mathbb{Z}^n}\phi_k(x)e^{-2\pi i p y/h}\right|\int_{\mathbb{T}^n}|\bar\psi(x+\frac{y}{2})\psi(x-\frac{y}{2})|dx\right]dy\\
&\leq \left[\sup_y\int_{\mathbb{T}^n}|\bar\psi(x+\frac{y}{2})\psi(x-\frac{y}{2})|dx\right]\left[\int_{\mathbb{T}^n}\sup_x\left|\sum_{p\in h\mathbb{Z}^n}\phi_k(x)e^{-2\pi i p y/h}\right|dy\right]\\
&\leq \|\phi_k\|_{\mathcal{A}}\|\psi\|_{L^2}^2.
\end{aligned}$$

■

Proposition 6. *The limiting measure obtained in proposition 5 is invariant under the Hamiltonian dynamics, that is, as a measure in* $\mathbb{T}^n\times\mathbb{R}^n$*,*

$$p\cdot D_xW_0 - D_xV\cdot D_pW_0 = 0.$$

Proof. Observe that

$$\begin{aligned}
&pD_xW_h\\
&=\int_{\mathbb{T}^n} pe^{-2\pi i p y/h}\left[D_x\bar\psi(x+\frac{y}{2})\psi(x-\frac{y}{2})+\bar\psi(x+\frac{y}{2})D_x\psi(x-\frac{y}{2})\right]dy\\
&=\frac{h}{4\pi i}\int_{\mathbb{T}^n} e^{-2\pi i p y/h}\left[\Delta\bar\psi(x+\frac{y}{2})\psi(x-\frac{y}{2})-\bar\psi(x+\frac{y}{2})\Delta\psi(x-\frac{y}{2})\right]dy\\
&=\frac{1}{2\pi i h}\int_{\mathbb{T}^n} e^{-2\pi i p y/h}\left[V(x+\frac{y}{2})-V(x-\frac{y}{2})\right]\bar\psi(x+\frac{y}{2})\psi(x-\frac{y}{2})dy,
\end{aligned}$$

using in the last line the Schrödinger equation. Let $\phi(x,p)$ be a test function. Then

$$\begin{aligned}
&-\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}D_x\phi(x,2\pi p)pW_hdx=\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\phi(x,2\pi p)pD_xW_hdx\\
&=\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}e^{-2\pi i p y/h}\phi(x,p)\left[\frac{V(x+\frac{y}{2})-V(x-\frac{y}{2})}{2\pi i h}\right]\cdot\\
&\qquad\cdot\bar\psi(x+\frac{y}{2})\psi(x-\frac{y}{2})dydx.
\end{aligned}$$

Observe that, by expanding V is Fourier series

$$\frac{V(x+\frac{y}{2})-V(x-\frac{y}{2})}{2\pi ih}=\sum_{k\in\mathbb{Z}^n}\hat{V}_k e^{2\pi ikx}\frac{e^{\pi iky}-e^{-\pi iky}}{2\pi ih}.$$

Then

$$\begin{aligned}
&\sum_{p\in h\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}e^{-2\pi ipy/h}\phi(x,2\pi p)V_k e^{2\pi ikx}\frac{e^{\pi iky}-e^{-\pi iky}}{2\pi ih}\\
&=\sum_{p\in h\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\phi(x,2\pi p)V_k e^{2\pi ikx}\frac{e^{-2\pi i(p-kh/2)y/h}-e^{-2\pi i(p+kh/2)y/h}}{2\pi ih}\\
&=\sum_{p\in h\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}V_k e^{2\pi ikx}\frac{\phi(x,2\pi(p+\frac{kh}{2}))-\phi(x,2\pi(p-\frac{kh}{2}))}{2\pi ih}e^{-2\pi ipy/h}
\end{aligned}$$

using summation by parts. Therefore

$$\begin{aligned}
&-\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}D_x\phi(x,2\pi p)pW_h dx\\
&=\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\sum_{p\in h\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}V_k e^{2\pi ikx}\frac{\phi(x,2\pi(p+\frac{kh}{2}))-\phi(x,2\pi(p-\frac{kh}{2}))}{2\pi ih}\cdot\\
&\qquad\cdot e^{-2\pi ipy/h}\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2}).
\end{aligned}$$

Since V is smooth, then $V_k=O(|k|^{-N})$, for any N large enough. Also

$$\frac{\phi(x,2\pi(p+\frac{kh}{2}))-\phi(x,2\pi(p-\frac{kh}{2}))}{2\pi ih}=\frac{2\pi k}{i}D_p\phi(x,2\pi p)+O(h|k|^2).$$

Thus, as $h\to 0$,

$$\begin{aligned}
&-\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}D_x\phi(x,2\pi p)pW_h dx\\
&=\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\sum_{p\in h\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}2\pi ikV_k e^{2\pi ikx}D_p\phi(x,2\pi p)e^{-2\pi ipy/h}\cdot\\
&\qquad\cdot\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})dxdy+O(h)\\
&=\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\sum_{p\in h\mathbb{Z}^n}D_xV(x)D_p\phi(x,2\pi p)e^{-2\pi ipy/h}\cdot\\
&\qquad\cdot\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})dxdy+O(h),
\end{aligned}$$

and therefore

$$\lim_{h\to 0}\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n} D_x\phi(x,2\pi p)pW_h dx = -\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n} D_xV D_p\phi(x,2\pi p)W_0 dx.$$

■

4. Action minimizing Wigner measures

In Mather's theory, one is interested in probability measures μ on $\mathbb{T}^n\times\mathbb{R}^n$ that minimize the action $A[\mu]$ and satisfy the holonomy constraint

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} vD_x\phi(x)d\mu = 0,$$

for all $\phi\in C^1(\mathbb{T}^n)$, as well as fixed rotation vector

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} v d\mu = Q.$$

Inspired by the work [Eva03], which only uses classical tools, we suggest that in the semiclassical limit one should consider Wigner measures that minimize the quantum action. More precisely, given a state function ψ with L^2 norm

$$\int_{\mathbb{T}^n}|\psi|^2 = 1, \tag{4}$$

we define its action as

$$\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\left[\frac{|2\pi p|^2}{2}-V(x)\right]e^{-2\pi i py/h}\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})dydx. \tag{5}$$

The corresponding holonomy constraint is

$$\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} pD_x\phi e^{-2\pi i py/h}\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})dydx = 0, \tag{6}$$

for all $\phi\in C^1(\mathbb{T}^n)$. The rotation vector of the state ψ is

$$2\pi\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} pe^{-2\pi i py/h}\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})dydx = Q. \tag{7}$$

In terms of the function ψ this problem can be rewriten as minimizing

$$\int_{\mathbb{T}^n}\frac{h^2}{2}|D\psi|^2 - V(x)|\psi|^2 \tag{8}$$

under the constraints (4),

$$h\,\mathrm{div}\left[\bar{\psi}D_x\psi - \psi D_x\bar{\psi}\right] = 0, \tag{9}$$

and

$$\frac{h}{i}\int_{\mathbb{T}^n}\left[\bar{\psi}D_x\psi - \psi D_x\bar{\psi}\right] = Q. \tag{10}$$

In the case in which ψ has the Bloch wave form

$$\psi = ae^{iu/h},$$

with a and u real valued, a periodic and

$$u(x) = P\cdot x + v(x),$$

for some $P \in \mathbb{R}^n$ and a $\mathbb{Z}^n$-periodic function v, expressions (4), and (8) to (10) are equivalent to

$$\int_{\mathbb{T}^n}|a|^2 = 1, \tag{11}$$

the quantum action, to be minimized, is

$$\int_{\mathbb{T}^n}\left[\frac{h^2}{2}|Da|^2 + \frac{a^2}{2}|Du|^2 - Va^2\right]dx, \tag{12}$$

the holonomy constraint is

$$\mathrm{div}(a^2Du) = 0, \tag{13}$$

and the rotation vector is

$$\int_{\mathbb{T}^n} a^2 Du dx = Q. \tag{14}$$

Therefore this problem is equivalent, as long as we restrict our class of states to Bloch-wave type states to the one in [Eva03].

5. Existence of quantum-action minimizing measures

In this section we show the existence of a global minimizer of the quantum action. To do so, consider a sequence of functions ψ_n which satisfy the constraints (4) and (6) and are a minimizing sequence for (5). Then, since h is fixed, ψ_n is a bounded sequence in $W^{1,2}$ and therefore it converges, extracting a subsequence if necessary, strongly in L^2 and weakly in $W^{1,2}$ to some function ψ. This function satisfies

$$\int_{\mathbb{T}^n}|\psi|^2 = 1,$$

as well as, for all $\phi(x) \in C^1(\mathbb{T}^n)$,

$$h \int_{\mathbb{T}^n} \left(\bar{\psi} D_x \psi - \psi D_x \bar{\psi}\right) D_x \phi = 0,$$

since by (9)

$$h \int_{\mathbb{T}^n} \left(\bar{\psi}_n D_x \psi_n - \psi_n D_x \bar{\psi}_n\right) D_x \phi = 0,$$

and

$$\frac{h}{i} \int_{\mathbb{T}^n} \left(\bar{\psi} D_x \psi - \psi D_x \bar{\psi}\right) = Q.$$

Finally,

$$\int_{\mathbb{T}^n} V(x)|\psi_n|^2 \to \int_{\mathbb{T}^n} V(x)|\psi|^2,$$

and

$$\begin{aligned}
\liminf_{n\to\infty} \int_{\mathbb{T}^n} \frac{1}{2}|D_x\psi_n|^2 =& \liminf_{n\to\infty} \int_{\mathbb{T}^n} \frac{1}{2}|D_x\psi|^2 + D_x\psi D_x(\psi_n - \psi) \\
&+ \frac{1}{2}|D_x(\psi_n - \psi)|^2 \\
&\geq \int_{\mathbb{T}^n} \frac{1}{2}|D_x\psi|^2.
\end{aligned}$$

Therefore, the Wigner measure associated with ψ is a minimizer of the quantum action problem.

6. Connection with Mather theory

In general Wigner measures are not positive measures, which creates technical problems when we study the limit $h \to 0$. The standard idea to handle this problem [LP93] is to consider the convolution with Husimi functions, that is, properly normalized gaussians in p and x. More precisely,

Proposition 7. *Let*

$$\zeta_h(x,p) = Ch^n e^{-2\pi\left(|x|^2+|p|^2\right)/h},$$

in which the constant is chosen so that

$$\sum_{p\in h\mathbb{Z}^n} \int_{\mathbb{R}^n} \zeta_h(x,p)dx = 1.$$

Then

$$\tilde{W}_h = \zeta_h * W_h$$

satisfies

$$\tilde{W}_h + Ch \geq 0,$$

for some constant C and all h sufficiently small.

PROOF. The fact that $\tilde{W}_h$ has total mass one follows from Fubinni's theorem. To prove positivity, observe that

$$\begin{aligned}\tilde{W}_h(x,p) = &\sum_{j\in\mathbb{Z}^n}\sum_{k\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} Ch^n e^{-2\pi\left(|j+x-z|^2+|p-k|^2\right)/h} e^{-2\pi iky/h}\cdot \\ &\cdot \bar{\psi}(z+\frac{y}{2})\psi(z-\frac{y}{2})dzdy \\ = &\sum_{j\in\mathbb{Z}^n}\sum_{k\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} Ch^n e^{-2\pi\left(|j+x-(z_1+z_2)/2|^2+|p-k|^2\right)/h} e^{-2\pi ik(z_1-z_2)/h}\cdot \\ &\cdot \bar{\psi}(z_1)\psi(z_2)dz_1dz_2,\end{aligned}$$

using the change of coordinates

$$z_1 = z+\frac{y}{2} \qquad z_2 = z-\frac{y}{2}.$$

Set $u = p-k$, and then

$$\begin{aligned}&\sum_{k\in h\mathbb{Z}^n} e^{-2\pi|p-k|^2/h} e^{-2\pi ik(z_1-z_2)/h} \\ &\qquad = e^{-2\pi ip(z_1-z_2)/h}\sum_{u\in h\mathbb{Z}^n} e^{-\pi|u|^2/h} e^{2\pi iu(z_1-z_2)/h}.\end{aligned}$$

Let $w = \frac{z_1-z_2}{h}$. The sum

$$\sum h^{n/2} e^{-2\pi|u|^2/h} e^{2\pi iuw}$$

corresponds to the mid-point sum for the integral

$$\int_{\mathbb{R}^n} h^{-n/2} e^{-2\pi|u|^2/h} e^{2\pi iuw} du,$$

on a regular mesh of size h, or, by change of coordinates, to the integral

$$\int_{\mathbb{R}^n} e^{-2\pi|v|^2} e^{2\pi i\sqrt{h}vw} dv.$$

This integral is

$$e^{-\pi h|w|^2/2} = e^{-\pi|z_1-z_2|^2/(2h)}.$$

The error for the mid-point sum can be bounded uniformly by the absolute value of the second derivative multiplied by h^2, that is

$$h^2\int_{\mathbb{R}^n}(1+|v|^2+h|w|^2)e^{-2\pi|v|^2}dv = O(h),$$

for our choice of w.

Thus

$$\tilde{W}_h(x,p) = \sum_{j\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} C h^{n/2} e^{-2\pi|j+x-(z_1+z_2)/2|^2/h} \left[e^{-\pi|z_1-z_2|^2/(2h)} + O(h) \right] \cdot$$
$$\cdot\, e^{-2\pi i p(z_1-z_2)/h} \bar{\psi}(z_1)\psi(z_2)$$
$$= \sum_{j\in\mathbb{Z}^n} \left| \int_{\mathbb{T}^n} e^{-\pi|j+x-z|^2} e^{2\pi i p z} \psi(z) dz \right|^2 + O(h),$$

and the $O(h)$ term is bounded by $h\|\psi\|^2_{L^2}$. ■

Corollary 1. *W_h converges to a positive probability measure.*

Finally we would like to prove that this probability measure is indeed a Mather measure.

Proposition 8. *Let W_h be a quantum-action minimizing measure. Then, as $h \to 0$*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \frac{|p|^2}{2} - V(x) dW_h \to \int_{\mathbb{T}^n\times\mathbb{R}^n} \frac{|p|^2}{2} - V(x) dW_0.$$

Proof. It suffices to observe that both

$$\int_{\mathbb{T}^n} W_h(p,x)dx \geq 0 \qquad \sum_{p\in h\mathbb{Z}^n} W_h(p,x) \geq 0$$

and use standard weak convergence techniques. ■

We also have to check the holonomy constraint

Proposition 9. *The measure W_0 satisfies*

$$\int p D_x \phi dW_0 = 0.$$

Proof. This follows from the identity

$$h \int_{\mathbb{T}^n} \left(\bar{\psi} D_x \psi - \psi D_x \bar{\psi} \right) D_x \phi = 0,$$

and from the fact that

$$h \left(\bar{\psi} D_x \psi - \psi D_x \bar{\psi} \right)$$

is a bounded measure, uniformly in h. ■

Theorem 1. *The measure W_0 is a Mather measure.*

PROOF. The previous results show that the action of W_0 is the limit as $h \to 0$ of the quantum action of W_h and that W_0 satisfies the holonomy constraint. In [Eva03] it was proved that the limit of the quantum action as $h \to 0$ is the minimal Mather's action. Therefore W_0 is a Mather measure. ■

7. Stochastic Mather measures

In this section discuss briefly the extension of the previous results to construct quantum analogues of stochastic Mather measures [Gom02], [ISM04]. The techniques are essentially identical to the previous ones therefore we only highlight the main points.

In stochastic Mather's theory one is interested in probability measures μ on $\mathbb{T}^n \times \mathbb{R}^n$ that minimize the action $A[\mu]$ and satisfy the stochastic holonomy constraint

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} vD_x\phi(x) + \Delta\phi d\mu = 0,$$

for all $\phi \in C^2(\mathbb{T}^n)$.

As in section 4, we consider Wigner measures to study the semiclassical limit, we impose the constraints (4), (7), the action is defined as in (5), but the stochastic holonomy constraint is now defined by

$$\sum_{p\in h\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\left[pD_x\phi e^{-2\pi ipy/h} + \Delta\phi\right]\bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})dydx = 0, \tag{15}$$

for all $\phi \in C^2(\mathbb{T}^n)$.

In terms of the function ψ this problem can be rewriten as minimizing (8), under the constraints (4), (10) and

$$h\,\mathrm{div}\left[\bar{\psi}D_x\psi - \psi D_x\bar{\psi}\right] = \Delta|\psi|^2. \tag{16}$$

In the case in which ψ has the Bloch wave form, as in section 4, the stochastic holonomy constraint is

$$\Delta a^2 + \mathrm{div}(a^2 Du) = 0. \tag{17}$$

Proceeding as in section 5 one can easily obtain existence of a minimizing Wigner measure for the stochastic quantum action problem. Then, by sending $h \to 0$, and using the ideas of section 6 we prove that these measures converge indeed to a stochastic Mather measure.

References

[Eva03] L. C. Evans. Towards a quantum analog of weak kam theory. *preprint*, 2003.

[GM83] Francesco Guerra and Laura M. Morato. Quantization of dynamical systems and stochastic control theory. *Phys. Rev. D (3)*, 27(8):1774–1786, 1983.

[Gom02] Diogo Aguiar Gomes. A stochastic analogue of Aubry-Mather theory. *Nonlinearity*, 15(3):581–603, 2002.

[ISM04] R. Iturriaga and H. Snchez-Morgado. On the stochastic aubry-mather theory. *preprint*, 2004.

[LP93] Pierre-Louis Lions and Thierry Paul. Sur les mesures de Wigner. *Rev. Mat. Iberoamericana*, 9(3):553–618, 1993.

[Mat91] John N. Mather. Action minimizing invariant measures for positive definite Lagrangian systems. *Math. Z.*, 207(2):169–207, 1991.