A VARIATIONAL FORMULATION FOR THE NAVIER-STOKES EQUATION

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Contents

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1. Introduction

The Navier-Stokes equation describes the evolution of the velocity field of a viscous incompressible fluid. Beeing one of the most important equations of mathematical physics, it has been studied extensively. However, its theory is still incomplete, specially in three space dimensions where general existence and uniqueness results for smooth solutions are still partial. Several authors, see for instance [Rap01], [Rap00], [Bus99], [BFR] have studied representation formulas for solutions of the Navier-Stokes equation using probabilistic methods, and the idea of using random maps instead of deterministic ones can be traced back to Chorin [Cho73], Peskin [Pes85]. This paper is a contribuition in this direction, its main result (theorems 1 and 2) is a new variational formulation which asserts that the solutions of the Navier-Stokes equation are critical points of a stochastic control problem on the group of area-preserving diffeomorphisms. This problem is the stochastic analog to the result by Arnold [Arn66], further studied by Ebin and Marsden [EM70], concerning solutions to Euler equation. Our result is also related to the ones by Constantin in [Con01a], [Con01b], [Con03] as we discuss briefly in the final of the paper.

The plan of the paper is as follows: in the next section we review the magnetization formulation for the Navier-Stokes equation. Then we present a variational principle whose minimizers are solutions to the Navier-Stokes equation. Finally, we discuss the connection with the Lagrangian diffusive transformation theory developed by P. Constantin.

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2. Navier-Stokes equation in magnetization variables

The Navier-Stokes equation in $\mathbb{R}^n$ for the velocity field $u(x,t)$ of an incompressible fluid is

$$u_t + (u \cdot \nabla)u + \nabla p = \frac{1}{2}\Delta u \qquad \operatorname{div} u = 0, \tag{1}$$

with initial condition $u|_{t=0} = u^0$. The variable $p(x,t)$ is the pressure and is necessary to impose the incompressibility condition $\operatorname{div} u = 0$.

For our purposes in this paper, it is convenient to rewrite (1) in new variables, the magnetization variables. These have been used to study the Euler equation by several authors, namely Buttke [But93], Oseledets [Ose89], Russo and Smereka [RS99], among others. We will follow Chorin [Cho94] in the summary of results we present next.

The magnetization variable m is obtained by adding to the velocity field u a gradient

$$u = m + \nabla k.$$

The scalar function $k(x,t)$ is arbitrary at $t = 0$ and its evolution is chosen conveniently.

This transformation is a change of gauge, of which there are several possible choices, as discussed in [RS99]. Clearly, from m one can compute u by using the Leray projection on the divergence free vector fields:

$$u = \mathbb{P}m.$$

With an appropriate choice for k, the equation for the evolution of m is

$$\partial_t m_i + u_j D_j m_i + m_j D_i u_j = \frac{1}{2}\Delta m_i. \tag{2}$$

A main difference from (1) is that equation (2) does not involve pressure, nor $\operatorname{div} m = 0$. Furthermore, to any solution of (2) with $u = \mathbb{P}m$, corresponds a solution u to (1). In the other direction, to any solution of (1) and initial value of k there exists a solution of (2) such that $u = \mathbb{P}m$ for all times.

3. A variational principle

This is the core section of the paper in which we prove the main results (theorems 1 and 2). The first one asserts that any smooth solution to the Navier-Stokes equation is a critical point of variational problem. The second result states that the critical points of a closely related variational problem are solutions to the Navier-Stokes equation. Our approach is analogous to the one by Arnold [Arn66] and Ebin and Marsden [EM70]) for the Euler equation. However, the notation and methods were inspired in the paper [BCHM02]. The key idea is to replace the original variational problem on area preserving diffeomorphisms by random area preserving diffeomorphisms.

Before proceeding we would like to point out that our results are formal in the sense that in all the proofs we assume some smoothness and integrability of all functions. This is partially unavoidable as there is no global regularity result to the solutions to the Navier-Stokes equation.

Theorem 1. *Let u be a smooth solution to the Navier-Stokes equation with smooth initial condition u^0. Define $\phi^\omega : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ by* *solving the random equation*

$$\frac{\partial \phi^\omega}{\partial t} = u \circ B_t \circ \phi^\omega, \qquad \phi^\omega(x,0) = x, \tag{3}$$

in which B_t is a n-dimensional Brownian motion, identified, by convenience of notation, with the shift by B_t, that is,

$$u \circ B_t(x,t) = u(x + B_t, t), \qquad u \circ B_t \circ \phi^\omega = u(\phi^\omega(x,t) + B_t, t).$$

Similarly, let

$$\Xi^\omega : \mathbb{R}^{n\times n} \times [0,T] \to \mathbb{R}^{n\times n}$$

be a fundamental solution to the equation

$$\frac{\partial \Xi^\omega_{im}}{\partial t} + \Xi^\omega_{jm} D_i u_j \circ B_t \circ \phi^\omega = 0, \tag{4}$$

satisfying $\Xi(x,0) = I$. Then (u, ϕ^ω) is a critical point of the functional

$$S = \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} |u|^2 - E\int_{\mathbb{R}^n} \langle \Xi^\omega(x,T) u^0(x), \phi^\omega(x,T) - x\rangle,$$

under the constraints $\operatorname{div} u = 0$ *and (3).*

Proof. Let u be a smooth solution to the Navier-Stokes equation and ϕ^ω and Ξ^ω as in the statement. Define $\Pi^\omega : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ to be the solution to the linear equation

$$\frac{\partial \Pi^\omega_i}{\partial t} + \Pi^\omega_j D_i u_j \circ B_t \circ \phi^\omega = 0, \tag{5}$$

satisfying the initial condition

$$\Pi^\omega(x,0) = u^0(x).$$

Given the fundamental solution Ξ^ω, one can determine Π^ω by

$$\Pi^\omega(x,t) = \Xi^\omega(x,t) u^0(x).$$

Let $\tilde{S}[u,\phi^\omega]$ be the augmented Lagrangian:

$$\tilde{S}[u,\phi^\omega] = \int_0^T \int_{\mathbb{R}^n} \frac{|u|^2}{2} + E\langle \Pi^\omega, \frac{\partial \phi^\omega}{\partial t} - u \circ B_t \circ \phi^\omega\rangle$$
$$- E\int_{\mathbb{R}^n} \langle \Xi^\omega(x,T) u^0(x), \phi^\omega(x,T) - x\rangle.$$

Note that Π^ω is fixed once the solution u is given, and therefore the functional $\tilde{S}$ does not depend on u through Π^ω. Obviously, any critical point of $\tilde{S}$ which satisfies (3) is also a critical point of S under the constraint (3).

Let δu be a smooth compactly supported divergence free variation of u and $\delta\phi^\omega$ a C^2 in space and C^1 in time, progressively measurable variation of ϕ^ω. Then, using Einstein convention,

$$\delta\tilde{S} = \int_0^T \int_{\mathbb{R}^n} u_j \delta u_j - E\left[\Pi_j^\omega \delta u_j \circ B_t \circ \phi^\omega\right] + \int_0^T \int_{\mathbb{R}^n} E\left[\Pi_j^\omega \left(\frac{\partial \delta\phi_j^\omega}{\partial t} - D_i u_j \circ B_t \circ \phi^\omega \delta\phi_i^\omega\right)\right] - \int_{\mathbb{R}^n} E\langle \Xi^\omega(x,T) u^0(x), \delta\phi^\omega(x,T)\rangle.$$

Integrating by parts in time, one easily checks that

$$\int_0^T \int_{\mathbb{R}^n} E\left[\Pi_j^\omega \left(\frac{\partial \delta\phi_j^\omega}{\partial t} - D_i u_j \circ B_t \circ \phi^\omega \delta\phi_i^\omega\right)\right] - \int_{\mathbb{R}^n} E\langle \Xi^\omega(x,T) u^0, \delta\phi^\omega\rangle = 0.$$

Therefore, to show that $\delta\tilde{S} = 0$, we must prove that

$$\int_0^T \int_{\mathbb{R}^n} u_j \delta u_j - E\left[\Pi_j^\omega \delta u_j \circ B_t \circ \phi^\omega\right] = 0. \tag{6}$$

Since both ϕ^ω and B_t are measure preserving maps:

$$\int_{\mathbb{R}^n} E\left[\Pi_j^\omega \delta u_j \circ B_t \circ \phi^\omega\right] = E\left[\int_{\mathbb{R}^n} \Pi_j^\omega \circ (\phi^\omega)^{-1} \circ B_t^{-1} \delta u_j\right] = \int_{\mathbb{R}^n} E\left[\Pi_j^\omega \circ (\phi^\omega)^{-1} \circ B_t^{-1}\right] \delta u_j.$$

So we must show that

$$u = E\left[\Pi^\omega \circ (\phi^\omega)^{-1} \circ B_t^{-1}\right] + \nabla k, \tag{7}$$

since δu is divergence free.

Define

$$m = E\left[\Pi^\omega \circ (\phi^\omega)^{-1} \circ B_t^{-1}\right],$$

and choose k such that

$$\operatorname{div}(m + \nabla k) = 0.$$

To prove (7) we must show that m solves (2), and so is the magnetization variable.

Lemma 1. *Suppose Π^ω at $t=0$ is non-random. Then*
$$\frac{\partial m_i}{\partial t}+m_jD_iu_j+u_jD_jm_i=\frac{1}{2}\Delta m_i.$$

Proof. Since Π^ω is non-random at $t=0$, the process $\Pi^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}$ is progressively measurable. Furthermore, $\Pi^\omega\circ(\phi^\omega)^{-1}$ is C^1 in time and C^2 in space. Therefore we can apply Itô's formula and obtain:
$$\begin{aligned}\frac{\partial m_i}{\partial t}&=\frac{\partial}{\partial t}E\left[\Pi_i^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}\right]\\&=E\left[\frac{\partial \Pi_i^\omega}{\partial t}\circ(\phi^\omega)^{-1}\circ B_t^{-1}+D_j\Pi_i^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}\frac{\partial(\phi^\omega)_j^{-1}}{\partial t}\circ B_t^{-1}\right]\\&\quad+\frac{1}{2}\Delta E\left[\Pi_i^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}\right],\end{aligned}$$
since the martingale term
$$E\left[D_x\left[\Pi_i^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}\right]dB_t\right]$$
vanishes. We have
$$E\left[\frac{\partial \Pi_i^\omega}{\partial t}\circ(\phi^\omega)^{-1}\circ B_t^{-1}\right]=-E\left[\Pi_j^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}D_iu_j\right]=-m_jD_iu_j.$$
Since
$$\phi^\omega\circ(\phi^\omega)^{-1}=Id,$$
it follows, by differentiation,
$$\frac{\partial(\phi^\omega)^{-1}}{\partial t}=-\left[D_x\phi^\omega\circ(\phi^\omega)^{-1}\right]^{-1}u\circ B_t=-D_x\left[(\phi^\omega)^{-1}\right]u\circ B_t.$$
Therefore
$$\begin{aligned}&E\left[D_j\Pi_i^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}\frac{\partial(\phi^\omega)_j^{-1}}{\partial t}\circ B_t^{-1}\right]\\&\quad=-E\left[D_j\Pi_i^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}D_k\left((\phi^\omega)_j^{-1}\right)\circ B_t^{-1}\right]u_k\\&\quad=-D_kE\left[\Pi_i^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}\right]u_k=-u_jD_jm_i.\end{aligned}$$
Finally
$$\frac{1}{2}\Delta E\left[\Pi_i^\omega\circ(\phi^\omega)^{-1}\circ B_t^{-1}\right]=\frac{1}{2}\Delta m_i.$$

Therefore we conclude that

$$\frac{\partial m_i}{\partial t} + m_j D_i u_j + u_j D_j m_i = \frac{1}{2}\Delta m_i.$$

∎

Since $\Pi^\omega(x,0) = u^0(x)$, the lemma implies

$$u = m + \nabla k,$$

as required. ∎

The second theorem in this section asserts that the critical points of the augmented Lagrangian are solutions to the Navier-Stokes equation.

Theorem 2. *Suppose $(u, \phi^\omega, \Pi^\omega)$ is a smooth critical point with fixed endpoints of*

$$\hat{S} = \int_0^T \int_{\mathbb{R}^n} \frac{|u|^2}{2} + E\langle \Pi^\omega, \frac{\partial \phi^\omega}{\partial t} - u \circ B_t \circ \phi^\omega \rangle,$$

under the constraint $\operatorname{div} u = 0$. *Assume further that Π^ω at $t = 0$ is non-random. Then u is a solution to the Navier-Stokes equation.*

Proof. Assume that δu, $\delta\phi^\omega$ and $\delta\Pi^\omega$ are compactly supported variations with $\operatorname{div}\delta u = 0$, and $\delta\phi^\omega$, $\delta\Pi^\omega$ are C^2 in space, C^1 in time, and progressively measurable.

$$\begin{aligned}
\delta\hat{S} = &\int_0^T \int_{\mathbb{R}^n} u_j \delta u_j - E\left[\Pi_j^\omega \delta u_j \circ B_t \circ \phi^\omega\right] \\
&+ \int_0^T \int_{\mathbb{R}^n} E\left[\Pi_j^\omega \left(\frac{\partial \delta\phi_j^\omega}{\partial t} - D_i u_j \circ B_t \circ \phi^\omega \delta\phi_i^\omega\right)\right] \\
&+ \int_0^T \int_{\mathbb{R}^n} E\left[\delta\Pi_j^\omega \left(\frac{\partial \phi_j^\omega}{\partial t} - u_j \circ B_t \circ \phi^\omega\right)\right] = 0.
\end{aligned}$$

Thus

$$\int_0^T \int_{\mathbb{R}^n} E\left[\delta\Pi_j^\omega \left(\frac{\partial \phi_j^\omega}{\partial t} - u_j \circ B_t \circ \phi^\omega\right)\right] = 0,$$

which implies

$$\frac{\partial \phi_j^\omega}{\partial t} - u_j \circ B_t \circ \phi^\omega = 0.$$

Similarly,

$$\int_0^T \int_{\mathbb{R}^n} E\left[\Pi_j^\omega \left(\frac{\partial \delta\phi_j^\omega}{\partial t} - D_i u_j \circ B_t \circ \phi^\omega \delta\phi_i^\omega\right)\right] = 0,$$

integrating by parts, we have

$$\frac{\partial \Pi_i^\omega}{\partial t} + \Pi_j^\omega D_i u_j \circ B_t \circ \phi^\omega = 0.$$

Finally,

$$\int_0^T \int_{\mathbb{R}^n} u_j \delta u_j - E\left[\Pi_j^\omega \delta u_j \circ B_t \circ \phi^\omega\right] = 0.$$

Since both ϕ^ω and B_t are measure preserving maps, it follows

$$\begin{aligned}\int_{\mathbb{R}^n} E\left[\Pi_j^\omega \delta u_j \circ B_t \circ \phi^\omega\right] &= \int_{\mathbb{R}^n} E\left[\Pi_j^\omega \circ (\phi^\omega)^{-1} \circ B_t^{-1} \delta u_j\right] \\ &= \int_{\mathbb{R}^n} E\left[\Pi_j^\omega \circ (\phi^\omega)^{-1} \circ B_t^{-1}\right] \delta u_j,\end{aligned}$$

which implies

$$u = E\left[\Pi_j^\omega \circ (\phi^\omega)^{-1} \circ B_t^{-1}\right] + \nabla k,$$

since u is divergence free.

Define

$$m = E\left[\Pi^\omega \circ (\phi^\omega)^{-1} \circ B_t^{-1}\right],$$

and choose k such that

$$\operatorname{div}(m + \nabla k) = 0.$$

We need to prove that m is the magnetization variable and solves (2). Since Π^ω at $t = 0$ is non-random, lemma 1 implies this result.

■

4. Diffusive Lagrangian Transformations

In this section we use the formalism developed previously to give an interpretation to the theory developed by P. Constantin [Con01b] on near identity transformations to the Navier-Stokes equation.

Consider a minimizer (u, ϕ^ω) as before and define

$$A = E\left[(\phi^\omega)^{-1} \circ B_t^{-1}\right].$$

The next proposition shows that the vector A satisfies exactly the advection-diffusion equation as in [Con01b].

Proposition 1. *The vector A satisfies*

$$\frac{\partial A}{\partial t} + (u \cdot \nabla) A - \frac{1}{2}\Delta A = 0,$$

with $A(x, 0) = x$.

PROOF. Since $(\phi^\omega)^{-1} \circ B_t^{-1}$ is non-random at $t = 0$ and progressively measurable, we have, proceeding as in lemma 1,

$$\begin{aligned}\frac{\partial A}{\partial t} &= E\left[\frac{\partial (\phi^\omega)^{-1}}{\partial t} \circ B_t^{-1}\right] + \frac{1}{2}\Delta A \\ &= -E\left[u D_x (\phi^\omega)^{-1} \circ B_t^{-1}\right] + \frac{1}{2}\Delta A \\ &= -u \cdot \nabla A + \frac{1}{2}\Delta A.\end{aligned}$$

■

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