Duality Principles for Fully Nonlinear Elliptic Equations

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Abstract. In this paper we use duality theory to associate certain measures to fully-nonlinear elliptic equations. These measures are the natural extension of the Mather measures to controlled stochastic processes and associated second-order elliptic equations. We apply these ideas to prove new a-priori estimates for smooth solutions of fully nonlinear elliptic equations.

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1. Introduction

This paper builds upon the connections between variational principles in classical mechanics, such as Aubry-Mather theory, and viscosity solutions of Hamilton-Jacobi equations and tries to illustrate how similar techniques can be used to study fully-nonlinear elliptic equations and associated controlled Markov processes.

The variational principle in Classical Mechanics asserts that the trajectories $x(\cdot)$ of a mechanical system are critical points of the *action*:

$$\int_0^T L(x(t), \dot{x}(t))dt.$$

Of particular interest are the minimizers of the action. In Mather's theory, the problem of determining minimizers is relaxed and, instead, one looks for probability measures $\mu(x, v)$ which are generalized curves, that is,

$$\int v D_x \phi(x) d\mu = 0,$$

for all C^1 functions $\phi(x)$ and minimize the action, which is:

$$\int L(x, v)d\mu. \tag{1}$$

The support of such minimizing measures, the *Mather set*, is invariant under the *Euler-Lagrange equations*

$$\frac{d}{dt} D_v L(x, \dot{x}) - D_x L(x, \dot{x}) = 0.$$

Since the work of J. Mather [Mat91] this area of research has been extremely active. Several authors [E99], [Fat97a, Fat97b, Fat98a, Fat98b], [EG01a, EG01b], among others, have studied the connection between Mather's theory and Hamilton-Jacobi partial differential equations. The minimization problem (1) is a infinite dimensional linear programming problem. The dual problem is related with the stationary Hamilton-Jacobi equation

$$H(D_x u, x) = \overline{H}. \tag{2}$$

As in finite dimensional linear programming, the dual problem yields important information about the primal and vice-versa. For instance, if μ is a minimizing measure and u a solution of (2) then μ is supported on a graph $(x, v(x))$, with $v(x)$ is defined through

$$v(x) = -D_p H(D_x u, x).$$

In the other direction, one can use the measure to prove partial regularity [EG01a, Gom03] for the solutions to (2), and, in fact, one has

$$\int |D^2_{xx}u|^2 d\mu \leq C,$$

which is a weaker version of the Lipschitz graph theorem for Mather sets [Mat91].

In this paper we focus our attention at the class of nonlinear elliptic operators which have the form:

$$H(D^2_{xx}u, D_x u, x) = \sup_{\omega \in U} \left[-A_\omega u - L(\omega, x) \right].$$

The set $U \subset \mathbb{R}^m$ is the control space. We assume U to be a closed and convex set. The linear operator $A_\omega u$ is, for each ω, a (possibly degenerate) second-order elliptic operator whose zeroth-order coefficient vanishes, that is,

$$A_\omega u = a_\omega(x) : D^2_{xx}u + b_\omega(x) \cdot D_x u,$$

in which $a : b = \mathrm{Tr}(a^T b)$. We assume that the Lagrangian $L(\omega, x)$ is a convex and supperlinear (if U is unbounded) function in ω. Observe that $H(M, p, x)$ is jointly convex in (M, p) and monotone in M, that is,

$$H(M + B, p, x) \geq H(M, p, x)$$

for all non-negative matrices B. This class of operators arises in controlled Stochastic Dynamics and has been studied extensively, see, for instance, [CC95], [FS93] and the references therein for an introduction to fully nonlinear elliptic equations and stochastic optimal control theory.

We are particularly interested in periodic solutions to the stationary Hamilton-Jacobi equation

$$H(D^2_{xx}u, D_x u, x) = \overline{H}.$$

We associate to this equation a variational problem in a space of measures. The dual of this variational problem is closely related with the Hamilton-Jacobi equation. We apply these methods to study regularity of second-order Hamilton-Jacobi equations, extending some of the

results from [Gom02b] concerning generalizations of Aubry-Mather theory to a stochastic setting.

The outline of the paper is as follows: in section 2 we prove a representation formula for $\overline{H}$, and study its connections with generalized Mather measures. In section 3 we study some applications to prove a-priori regularity results for smooth solutions.

2. Duality

Proposition 1. *There is at most one value $\overline{H}$ for which*

$$H(D^2_{xx}u, D_x u, x) = \overline{H}. \tag{3}$$

has a periodic viscosity solution.

Proof. Suppose, by contradiction, $\overline{H}_1 > \overline{H}_2$ are such that (3) admits a viscosity solutions u_1 and u_2 for $\overline{H} = \overline{H}_1, \overline{H}_2$. We may assume $v_1 \equiv u_1 + C > u_2$, for a sufficiently large positive constant C. For ϵ sufficiently small

$$\epsilon v_1 + H(D^2_{xx}v_1, D_x v_1, x) \geq \epsilon u_2 + H(D^2_{xx}u_2, D_x u_2, x).$$

in the viscosity sense. The comparison principle for viscosity solutions implies $v_1 \leq u_2$, which is a contradiction. ■

Proposition 2. *Suppose that there exists a viscosity solution u of (3) then*

$$\overline{H} = \inf_{\phi \in C^2_{per}} \sup_x H(D^2_{xx}\phi, D_x\phi, x). \tag{4}$$

in which the infimum is taken over all C^2 periodic functions.

Remark. In the case of first order Hamiltonians this was proved in [CIPP98]. A proof in for a special class of 2nd order equations can be found in [Gom02b]

Proof. Let

$$\overline{H}^* = \inf_{\phi \in C^2_{per}} \sup_x H(D^2_{xx}\phi, D_x\phi, x).$$

At some point x_0, $u-\phi$ has a local minimum. By the viscosity property

$$H(D^2_{xx}\phi, D_x\phi, x_0) \geq \overline{H}.$$

which implies $\overline{H}^* \geq \overline{H}$.

To prove the reverse inequality we need to recall a few facts concerning the sup convolution whose proof can be found in [FS93].

Lemma 1. *Suppose u is a viscosity of (3). Define*

$$u_\epsilon(x) = \sup_y \left[u(y) - \frac{|x-y|^2}{\epsilon} \right].$$

Then

1. $u_\epsilon \to u$ *uniformly as* $\epsilon \to 0$,
2. u_ϵ *is semiconvex,*
3. u_ϵ *satisfies*

$$H(D^2_{xx}u_\epsilon, D_x u_\epsilon, x) \leq \overline{H} + O(\epsilon),$$

in the viscosity sense and almost everywhere.

Set $v_\epsilon = u_\epsilon * \eta_\epsilon$. Then

$$H(D^2_{xx}v_\epsilon, D_x v_\epsilon, x) \leq \overline{H} + O(\epsilon),$$

thus $\overline{H}^* \leq \overline{H}$. ■

This representation formula can be best understood in light of a dual problem that involves generalized Mather measures. Choose a function $\gamma : \mathbb{T}^n \times U \to \mathbb{R}$, $\gamma \geq 1$, that satisfies

$$\lim_{|\omega|\to\infty} \frac{L(x,\omega)}{\gamma(\omega)} = +\infty \qquad \lim_{|\omega|\to\infty} \frac{|\omega|}{\gamma(\omega)} = 0,$$

we use the convention that if U is bounded then the previous identities are trivially satisfied.

Let $\mathcal{M}$ be the set of Radon measures on $\mathbb{T}^n \times U$ that satisfy

$$\int_{\mathbb{T}^n \times U} \gamma d\mu \leq \infty.$$

Note that $\mathcal{M}$ can be identified with the dual space of $C^0_\gamma(\mathbb{T}^n \times U)$, that is, the set of continuous functions ϕ such that

$$\|\phi\|_\gamma = \sup_{\mathbb{T}^n\times U} \left|\frac{\phi}{\gamma}\right| \qquad \lim_{|\omega|\to\infty} \frac{\phi}{\gamma} = 0.$$

Define $\mathcal{M}_0$ to be the set of all measures in $\mathcal{M}$ that satisfy the constraint

$$\int_{\mathbb{T}^n\times U} A_\omega \phi d\mu = 0$$

for all $\phi \in C^2(\mathbb{T}^n)$. Let $\mathcal{M}_1 \subset \mathcal{M}$ be the set of all positive probability measures that belong to $\mathcal{M}$.

We look for measures in $\mathcal{M}_0 \cap \mathcal{M}_1$ that minimize the action

$$\int_{\mathbb{T}^n\times U} L(\omega, x) d\mu.$$

If L is strictly convex in ω and A_ω is linear in ω then $\omega = \omega(x)$ almost everywhere in the support of μ. However, we do not make this assumption and will work in a more general framework.

This variational problem is, in fact, a linear programming problem, in an infinite dimensional space, and by Fenchel-Rockafellar duality theory [Roc66] it admits a dual problem. This is close in spirit to the papers [VL78a], [VL78b], [LV80], [FV89], [FV88] and [Fle89], in which Fenchel-Rockafellar duality theory is used to analyze optimal control problems. In the first order case this dual problem has been identified and studied [CIPP98] and involves a Hamilton-Jacobi equation.

Before proceeding we need to recall some facts concerning convex duality. Let E be a Banach space with dual E'. The pairing between E and E' is denoted by $(\cdot, \cdot)$. Suppose $h_1 : E \to (-\infty, +\infty]$ is a convex, lower semicontinuous function. The Legendre-Fenchel transform $h_1^* : E' \to [-\infty, +\infty]$ of h_1 is defined by

$$h_1^*(y) = \sup_{x\in E} \left(-(x, y) - h_1(x)\right),$$

for $y \in E'$. Similarly, for concave, upper semicontinuous functions $h_2 : E \to (-\infty, +\infty]$ we define

$$h_2^*(y) = \inf_{x \in E} \left(-(x, y) - h_2(x)\right).$$

Theorem 1 (Rockafellar [Roc66]). *Let E be a locally convex Hausdorff topological vector space over $\mathbb{R}$ with dual E^*. Suppose $h_1 : E \to (-\infty, +\infty]$ is convex and lower semicontinuous, $h_2 : E \to [-\infty, +\infty)$ is concave and upper semicontinuous. Then*

$$\sup_x h_2(x) - h_1(x) = \inf_y h_1^*(y) - h_2^*(y), \tag{5}$$

provided that either h_1 or h_2 is continuous at some point where both functions are finite.

To apply this theorem we define two functions h_1 and h_2 on $C^0_\gamma(\mathbb{T}^n \times U)$ and consider the dual problem of

$$\sup_{\phi \in C^0_\gamma(\mathbb{T}^n \times U)} h_2(\phi) - h_1(\phi).$$

The first function is defined by

$$h_1(\phi) = \sup_{(x,\omega) \in \mathbb{T}^n \times U} \left[-\phi(x, \omega) - L(x, \omega)\right].$$

Let

$$\mathcal{C} = \operatorname{cl}\left\{\phi : \phi = A_\omega \varphi, \varphi \in C^2(\mathbb{T}^n)\right\},$$

and set

$$h_2(\phi) = \begin{cases} 0 & \text{if } \phi \in \mathcal{C} \\ -\infty & \text{otherwise.} \end{cases}$$

Proposition 3. *We have*

$$h_1^*(\mu) = \begin{cases} \int L d\mu & \textit{if } \mu \in \mathcal{M}_1 \\ +\infty & \textit{otherwise,} \end{cases}$$

and

$$h_2^*(\mu) = \begin{cases} 0 & \textit{if } \mu \in \mathcal{M}_0 \\ -\infty & \textit{otherwise.} \end{cases}$$

Proof. The Legendre-Fenchel transform h_1^* of h_1 is

$$h_1^*(\mu) = \sup_{\phi \in C^0_\gamma(\mathbb{T}^n \times U)} \left(-\int \phi d\mu - h_1(\phi) \right).$$

We claim that for all non-positive measures μ, $h_1^*(\mu) = \infty$.

Lemma 2. *If $\mu \not\geq 0$ then $h_1^*(\mu) = +\infty$.*

Proof. If $\mu \not\geq 0$ there is a sequence of non-negative functions $\phi_n \in C^0_\gamma(\mathbb{T}^n \times U)$ such that

$$\int -\phi_n d\mu \to +\infty.$$

Thus, since $L \geq 0$,

$$\sup_{\mathbb{T}^n \times U} -\phi_n - L \leq 0.$$

Thus, if $\mu \not\geq 0$, then $h_1^*(\mu) = +\infty$. ∎

Lemma 3. *If $\mu \geq 0$ then*

$$h_1^*(\mu) \geq \int L d\mu + \sup_{\psi \in C^0_\gamma(\mathbb{T}^n \times U)} \left(\int \psi d\mu - \sup \psi \right).$$

Proof. Let L_n be a sequence of functions in $C^0_\gamma(\mathbb{T}^n \times U)$ increasing pointwise to L. Any function ϕ in $C^0_\gamma(\mathbb{T}^n \times U)$ can be written as $\phi = -L_n - \psi$, for some ψ also in $C^0_\gamma(\mathbb{T}^n \times U)$. Therefore

$$\sup_{\phi \in C^0_\gamma(\mathbb{T}^n \times U)} \left(-\int \phi d\mu - h_1(\phi) \right) =$$

$$= \sup_{\psi \in C^0_\gamma(\mathbb{T}^n \times U)} \left(\int L_n d\mu + \int \psi d\mu - \sup(L_n + \psi - L) \right).$$

Note that $L_n - L \leq 0$ implies

$$\sup_{\mathbb{T}^n \times U} L_n - L \leq 0,$$

thus

$$\sup_{\mathbb{T}^n \times U} (L_n + \psi - L) \leq \sup_{\mathbb{T}^n \times U} \psi.$$

Thus

$$\sup_{\phi\in C^0_\gamma(\mathbb{T}^n\times U)}\left(-\int \phi d\mu - h_1(\phi)\right)\geq$$
$$\geq \sup_{\psi\in C^0_\gamma(\mathbb{T}^n\times U)}\left(\int L_n d\mu+\int \psi d\mu - \sup(\psi)\right).$$

By the monotone convergence theorem $\int L_n d\mu \to \int L d\mu$. Therefore

$$\sup_{\phi\in C^0_\gamma(\mathbb{T}^n\times U)}\left(-\int \phi d\mu - h_1(\phi)\right)\geq$$
$$\geq \int L d\mu + \sup_{\psi\in C^0_\gamma(\mathbb{T}^n\times U)}\left(\int \psi d\mu - \sup(\psi)\right),$$

as required. ■

If $\int L d\mu = +\infty$ then $h_1^*(\mu)=+\infty$. Also if $\int d\mu \neq 1$ then

$$\sup_{\psi\in C^0_\gamma(\mathcal{D})}\left(\int \psi d\mu - \sup\psi\right)\geq \sup_{\alpha\in\mathbb{R}} \alpha(\int d\mu - 1) = +\infty,$$

by taking $\psi\equiv\alpha$, constant. So, $h_1^*(\mu)=+\infty$, and therefore a finite value of h_1^* is only possible if $\int d\mu = 1$.

If $\int d\mu = 1$ we have, from the previous lemma,

$$h_1^*(\mu)\geq \int L d\mu,$$

by taking $\psi\equiv 0$.

Also, for any function ϕ

$$\int(-\phi - L)d\mu \leq \sup_{\mathbb{T}^n\times U}(-\phi - L),$$

if $\int d\mu = 1$. Hence

$$\sup_{\phi\in C^0_\gamma(\mathbb{T}^n\times U)}\left(-\int\phi d\mu - h_1(\phi)\right)\leq \int L d\mu.$$

Thus

$$h_1^*(\mu)=\begin{cases}\int L d\mu & \text{if} \quad \mu\in\mathcal{M}_1\\ +\infty & \text{otherwise.}\end{cases}$$

Now we compute h_2^*. Note that if $\mu \notin \mathcal{M}_0$ there exists $\hat{\phi} \in \mathcal{C}$ such that

$$\int \hat{\phi} d\mu \neq 0.$$

and so

$$\inf_{\phi \in \mathcal{C}} - \int \phi d\mu \leq \inf_{\alpha \in \mathbb{R}} \alpha \int \hat{\phi} d\mu = -\infty.$$

If $\mu \in \mathcal{M}_0$ then $\int \phi d\mu = 0$, for all $\phi \in \mathcal{C}$. Therefore

$$h_2^*(\mu) = \inf_{\phi \in \mathcal{C}} - \int \phi d\mu = \begin{cases} 0 & \text{if } \mu \in \mathcal{M}_0 \\ -\infty & \text{otherwise.} \end{cases}$$

■

Theorem 1 yields then

$$\sup_{\phi \in C_\gamma^0(\mathbb{T}^n \times U)} h_2(\phi) - h_1(\phi) = \inf_{\mu \in \mathcal{M}} h_1^*(\mu) - h_2^*(\mu),$$

provided we prove that h_1 is continuous on the set $h_2 > -\infty$. This is the content of the next lemma.

Lemma 4. h_1 *is continuous.*

Proof. Suppose $\phi_n \to \phi$ in C_γ^0. Then $\|\phi_n\|_\gamma$ and $\|\phi\|_\gamma$ are bounded uniformly by some constant C. The growth condition on L implies that there exists $R > 0$ such that

$$\sup_{\mathbb{T}^n \times U} -\hat{\phi} - L = \sup_{T^n \times (B_R \cap U)} -\hat{\phi} - L,$$

for all $\hat{\phi}$ in $C_\gamma^0(\mathbb{T}^n \times U)$ with $\|\hat{\phi}\|_\gamma < C$. On $B_R \cap U$, $\phi_n \to \phi$ uniformly and so

$$\sup_{\mathbb{T}^n \times U} -\phi_n - L \to \sup_{\mathbb{T}^n \times U} -\phi - L.$$

■

The next theorem summarizes the main result of this section

Theorem 2.

$$\overline{H} = - \inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int L d\mu. \tag{6}$$

Proof. This is a corollary to proposition 2 and the duality result proved above. ■

3. A-priori estimates

In this section we apply the ideas from the previous section to prove a-priori bounds for smooth solutions of second-order nonlinear equations such as the maximal eigenvalue operator, streamline diffusion controlled dynamics, and mean curvature flow.

Proposition 4. *Let u be a smooth periodic solution to*

$$H(D^2_{xx}u, D_xu, x) = \overline{H},$$

and μ a corresponding Mather measure. Then

$$\omega \in \operatorname{argmin}\left[A_\omega u + L(x,\omega)\right]$$

μ almost everywhere.

Proof. Since

$$-\overline{H} = -H(D^2_{xx}u, D_xu, x) \leq A_\omega u + L(x,\omega),$$

with equality if and only if $\omega \in \operatorname{argmin}\left[A_\omega u + L(x,\omega)\right]$, integrating with respect to μ yields

$$-\overline{H} \leq \int L(x,\omega)d\mu,$$

and, unless

$$\omega \in \operatorname{argmin}\left[A_\omega u + L(x,\omega)\right],$$

μ-a.e. this would yield a contradiction. ■

Proposition 5. *Suppose u is a smooth periodic solution to*

$$H(D^2_{xx}u, D_xu, x) = \overline{H}.$$

Then , μ almost everywhere,

$$A_\omega\varphi = a_\omega : D^2_{xx}\varphi + b_\omega \cdot D_x\varphi,$$

for any $\phi \in C^2(\mathbb{T}^n)$, with

$$a_\omega = -D_M H(D^2_{xx}u, D_xu, x),$$

and

$$b_\omega = -D_p H(D^2_{xx}u, D_xu, x).$$

Proof. It suffices to observe that almost everywhere in the support of μ one has

$$A_\omega u + L(x,\omega) = -H(D^2_{xx}u, D_x u, x),$$

and, for any $\varphi \in C^2(\mathbb{T}^n)$

$$A_\omega(u+\epsilon\varphi) + L(x,\omega) \geq -H(D^2_{xx}(u+\epsilon\varphi), D_x(u+\epsilon\varphi), x).$$

Thus, by subtracting the last two equations and sending $\epsilon \to 0$ one gets the theorem. ■

Theorem 3. *Suppose u is a smooth periodic solution to*

$$H(D^2_{xx}u, D_x u, x) = \overline{H}.$$

Then u satisfies the following a-priori identity: let $\xi \in \mathbb{R}^n$ be arbitrary, then

$$\begin{aligned}\int \Big[&H_{M_{ij}M_{lm}}\xi_k D^3_{x_ix_jx_k}u\xi_{k'}D^3_{x_ix_jx_{k'}}u + 2H_{M_{ij}p_m}\xi_k D^3_{x_ix_jx_k}u\xi_{k'}D^2_{x_mx_{k'}}u \\ &+H_{p_ip_j}\xi_k D^2_{x_ix_k}u\xi_{k'}D^2_{x_jx_{k'}}u\Big]\,d\mu \\ = -\int &\Big[2H_{M_{ij}x_m}\xi_k D^3_{x_ix_jx_k}u + \xi_{k'}2H_{p_ix_{k'}}\xi_k D^2_{x_ix_k}u + \xi_{k'}H_{x_kx_{k'}}\Big]\,d\mu.\end{aligned}$$

Remark. Note that the left-hand side of this estimate is a non-negative quadratic form on $D^2(D_\xi u)$ and $D(D_\xi u)$ since $H(M,p,x)$ is jointly convex in M and p and the right-hand side depends on lower-order terms. Therefore this identity yields an a-priori estimate for second and third derivatives.

Proof. By differentiating the equation

$$H(D^2_{xx}u, D_x u, x) = \overline{H}$$

with respect to x_k and multiplying by ξ_k we obtain

$$H_{M_{ij}}\xi_k D^3_{x_ix_jx_k}u + H_{p_i}\xi_k D^2_{x_ix_k}u + \xi_k H_{x_k} = 0.$$

Differentiating this last expression with respect to $x_{k'}$ and multiplying by $\xi_{k'}$ we obtain

$$\begin{aligned}
&H_{M_{ij}M_{lm}}\xi_k D^3_{x_ix_jx_k}u\xi_{k'}D^3_{x_ix_jx_{k'}}u + 2H_{M_{ij}p_m}\xi_k D^3_{x_ix_jx_k}u\xi_{k'}D^2_{x_mx_{k'}}u \\
&\quad + H_{p_ip_j}\xi_k D^2_{x_ix_k}u\xi_{k'}D^2_{x_jx_{k'}}u + 2H_{M_{ij}x_m}\xi_k D^3_{x_ix_jx_k}u \\
&\quad + 2\xi_{k'}H_{p_ix_{k'}}\xi_k D^2_{x_ix_k}u + \xi_k\xi_{k'}H_{x_kx_{k'}} + H_{M_{ij}}\xi_k\xi_{k'}D^4_{x_ix_jx_kx_{k'}}u \\
&\quad + \xi_k\xi_{k'}H_{p_i}D^3_{x_ix_kx_{k'}}u = 0.
\end{aligned}$$

Integrating with respect to μ, and observing that the last two terms integrate to 0 since

$$\begin{aligned}
&H_{M_{ij}}\xi_k\xi_{k'}D^4_{x_ix_jx_kx_{k'}}u + \xi_k\xi_{k'}H_{p_i}D^3_{x_ix_kx_{k'}}u \\
&\qquad = H_{M_{ij}}D^2_{x_ix_j}(D^2_{\xi\xi}u) + H_{p_i}D_{x_i}(D^2_{\xi\xi}u),
\end{aligned}$$

we obtain the result. ■

Next, we briefly illustrate the estimates discussed above for a fully-nonlinear second-order equation. Let u be a periodic solution to the one-dimensional equation

$$e^{u_{xx}} + \frac{u_x^2}{2} + V(x) = \overline{H}.$$

The projection $\theta(x)$ of a minimizing measure in the x-axis satisfies

$$(\theta(x)e^{u_{xx}})_{xx} - (\theta u_x)_x = 0,$$

weakly as a measure. Furthermore

$$\int \left[e^{u_{xx}}u^2_{xxx} + u^2_{xx}\right]\theta dx \leq C.$$

There are several important examples for which these estimates apply, two of them, the Stochastic Mather problem [Gom02b]

$$A_\omega u + L(x,\omega) = \Delta u + \omega \cdot D_x u + \frac{|\omega|^2}{2} - V(x),$$

and the vakonomic mechanics operator [Gom02a]

$$A_\omega u + L(x,\omega) = \omega f(x) \cdot D_x u + \frac{|\omega|^2}{2} - V(x),$$

have been studied in detail. However, important cases such as the maximal eigenvalue operator

$$A_\omega u = \frac{\omega \otimes \omega}{|\omega|^2} : D^2_{xx} u,$$

the related streamline-diffusion controlled dynamics problem

$$A_\omega u + L(x, \omega) = \frac{\omega \otimes \omega}{|\omega|^2} : D^2_{xx} u + \omega \cdot D_x u + \frac{|\omega|^2}{2} + V(x),$$

and the mean curvature flow (see [ST02] for a control theory formulation of the mean curvature flow) with drift

$$A_\omega u + L(x, \omega) = (I - \frac{\omega \otimes \omega}{|\omega|^2}) : D^2_{xx} u + b(x) D_x u + V(x),$$

have not been studied using these techniques. We believe that our estimates and ideas may give important insight on these problems.

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