

VISCOSITY SOLUTION METHODS AND THE DISCRETE AUBRY-MATHER PROBLEM

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Abstract. In this paper we study a discrete multi-dimensional version of Aubry-Mather theory using mostly tools from the theory of viscosity solutions. We set this problem as an infinite dimensional linear programming problem. The dual problem turns out to be a discrete analog of the Hamilton-Jacobi equations. We present some applications to discretizations of Lagrangian systems.

1. Introduction. Certain variational problems which arise in applications such as Lagrangian mechanics, Mather's problem [Mn92, Mn96], Monge-Kantorovich optimal transport problems [Eva99], or stationary stochastic optimal control [Gom02a], can be written as infinite dimensional linear programming problems in spaces of measures. The standard strategy to study such problems is to compute the dual problem. In general, the dual yields important information about the original (primal) problem. This dual may be of interest itself, and therefore the primal problem may provide useful insights about the dual.

In the continuous case, the dual of Mather's problem is related to viscosity solutions of Hamilton-Jacobi equations [Fat97a, Fat97b, Fat98a, Fat98b]. For stationary stochastic optimal control, it turns out to be related with second order nonlinear elliptic equations.

For discrete problems, such as optimal transport and the problem discussed in this paper, the dual is not a partial differential equation but a difference equation. These equations can be seen as discretizations of the corresponding partial differential equations in the appropriate limit.

The objective of this paper is twofold, one is to show that viscosity solutions methods can be adapted and used to study certain discrete dynamical system, and can be used to prove many well known facts about Mather's theory such as the existence of invariant sets and measures, the graph theorem, and asymptotic behaviour. The other one is to show that in the continuous limit one can recover the corresponding objects such as Mather's measures and viscosity solutions. We would like to point out that our results concerning Mather sets and measures are not new, the main novelty consists in the formulation and the methods used to obtain them,

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as well as, in pointing out the connection between certain difference equations and the corresponding Hamilton-Jacobi equations in the continuous setting.

Before stating the main problem, we need some notation. To that effect, let \mathbb{T}^n be the n dimensional torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, $\Omega = \mathbb{T}^n \times \mathbb{R}^n$, and consider a Lagrangian $L(x, v) : \Omega \rightarrow \mathbb{R}$, smooth, periodic in x (which is implicit by assuming $x \in \mathbb{T}^n$), coercive, and strictly convex in v . We look for positive probability measures $\mu(x, v)$ in Ω ,

$$\int_{\Omega} d\mu = 1, \quad (1.1)$$

that are invariant under the map $x \rightarrow x + v$, that is, for all continuous functions $w : \mathbb{T}^n \rightarrow \mathbb{R}$

$$\int_{\Omega} w(x + v) - w(x) d\mu = 0, \quad (1.2)$$

and that minimize the average action

$$\int_{\Omega} L(x, v) d\mu. \quad (1.3)$$

This is a linear programming problem on a space of measures, and so it admits a dual problem, which, as we will see in section 2, is given by:

$$\inf_{w \in C(\mathbb{T}^n)} \sup_{(x, v) \in \Omega} [w(x) - w(x + v) - L(x, v)]. \quad (1.4)$$

In analogy with the continuous case we will study the discrete stationary Hamilton-Jacobi equation associated with (1.4):

$$H(u(\cdot), x) = \overline{H}, \quad (1.5)$$

in which

$$H(w(\cdot), x) = \sup_v [w(x) - w(x + v) - L(x, v)].$$

The solution to (1.5) contains information about properties of minimizing measures such as the support. The analog of the Euler-Lagrange equations that arise in continuous time, are the discrete maps that we study in this paper.

It is instructive to observe several facts about the invariant measure that differ from the continuous case. In the continuous case, we look for probability measures that minimize (1.3) with the constraint

$$\int_{\Omega} v D_x w d\mu = 0 \quad \forall w \in C^1(\mathbb{T}^n). \quad (1.6)$$

Since L is strictly convex in v , and the constraints (1.1) and (1.6) are linear in v , it is immediate that the minimizing measure is supported on a graph $v = v(x)$. This is, however, an issue for the discrete version since (1.2) is non-linear in v . Nevertheless, if this measure were supported on a graph $v = v(x)$, and its projection on \mathbb{T}^n had a density $\theta(x)$ then it would satisfy the Monge-Ampere equation:

$$\theta(x) \det [I + D_x v(x)] = \theta(x + v(x)). \quad (1.7)$$

We will prove that, indeed, μ is supported on a graph and therefore if this graph is smooth the minimizing measure satisfies (1.7) as consequence of the change of coordinates formula. Furthermore, as we will see in the last section the constraint (1.7) should be regarded as the discrete version of

$$\nabla \cdot (\theta(x) v(x)) = 0,$$

which, together with the minimality condition, implies the invariance of the Mather measure in the continuous case.

2. Duality. As it was mentioned in the introduction, the minimization problem (1.3) with the constraints (1.1) and (1.2) is a linear programming problem in a space of Radon measures. The Fenchel-Rockafellar duality theorem [Roc66] is the natural tool to study such problems. In particular, it identifies the dual problem that, as in the continuous Aubry-Mather theory, contains important information concerning the original minimization.

The setting is the following: let E be a Banach space with dual E' and dual pairing denoted by (\cdot, \cdot) . Let $h_1(x)$ be a convex lower semicontinuous function in E assuming values in $(-\infty, +\infty]$. Define $h_1^*(y)$ for $y \in E'$ by

$$h_1^*(y) = \sup_{x \in E} [-(x, y) - h_1(x)].$$

Similarly, if h_2 is a concave upper semicontinuous function in E assuming values in $[-\infty, +\infty)$, let

$$h_2^*(y) = \inf_{x \in E} [-(x, y) - h_2(x)].$$

Theorem 2.1 (Rockafellar). *Let h_1 and h_2 be as above. Then*

$$\sup_{x \in E} [h_2(x) - h_1(x)] = \inf_{y \in E'} [h_1^*(y) - h_2^*(y)],$$

provided either h_1 or h_2 is continuous at some point where both functions are finite.

Set

$$C_*^0 = \left\{ \phi \in C^0(\Omega) : \lim_{|v| \rightarrow \infty} \frac{\phi(x, v)}{|v|} = 0 \right\}.$$

and observe that the dual of C_*^0 is the space \mathcal{M} of Radon measures μ in Ω with

$$\int_{\Omega} |v| d|\mu| < \infty.$$

Define

$$h_1(\phi) = \sup_{(x, v) \in \Omega} [-\phi(x, v) - L(x, v)].$$

Let

$$\mathcal{C} = \{ \phi \in C(\Omega) : \phi(x, v) = \psi(x + v) - \psi(x) \text{ for some } \psi \in C(\mathbb{T}^n) \},$$

and set

$$h_2(\phi) = \begin{cases} 0 & \text{if } \phi \in \mathcal{C} \\ -\infty & \text{otherwise.} \end{cases}$$

We should observe that the functions $\phi \in \mathcal{C}$ are discrete versions of exact one forms.

In fact, if (x_i, v_i) , with $1 \leq i \leq n$ satisfies

$$x_{i+1} = x_i + v_i \quad (1 \leq i \leq n-1) \quad x_1 = x_n,$$

then

$$\sum_{i=1}^n \phi(x_i, v_i) = 0,$$

as in the continuous case the integral along a closed curve of an exact one-form is zero.

Define

$$\mathcal{M}_0 = \left\{ \mu \in \mathcal{M} : \int_{\Omega} \psi(x + v) - \psi(x) d\mu = 0, \forall \psi \in C(\mathbb{T}^n) \right\},$$

and

$$\mathcal{M}_1 = \left\{ \mu \in \mathcal{M} : \mu \geq 0 \text{ and } \int_{\Omega} d\mu = 1 \right\}.$$

Proposition 2.2.

$$h_1^*(\mu) = \begin{cases} \int L d\mu & \text{if } \mu \in \mathcal{M}_1 \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$h_2^*(\mu) = \begin{cases} 0 & \text{if } \mu \in \mathcal{M}_0 \\ -\infty & \text{otherwise.} \end{cases}$$

PROOF. Recall that

$$h_1^*(\mu) = \sup_{\phi \in C_*^0} \left[- \int_{\Omega} \phi d\mu - h_1(\phi) \right].$$

If μ is non-positive then we can choose a sequence of non-negative functions $\phi_n \in C_*^0$ such that

$$- \int_{\Omega} \phi_n d\mu \rightarrow \infty.$$

Since $L \geq 0$ we have

$$h_1(\phi_n) = \sup [-\phi_n - L] \leq 0.$$

Therefore $h_1^*(\mu) = +\infty$.

Lemma 2.3. *If $\mu \geq 0$ then*

$$h_1^*(\mu) \geq \int_{\Omega} L d\mu + \sup_{\psi \in C_*^0} \left[\int_{\Omega} \psi d\mu - \sup_{\Omega} \psi \right].$$

PROOF. Let L_n be a sequence in C_*^0 that increases pointwise to L , $0 \leq L_n \uparrow L$. Any $\phi \in C_*^0$ can be written as $\phi = -L_n - \psi$ for some $\psi \in C_*^0$. Therefore

$$\sup_{\phi \in C_*^0} \left[- \int_{\Omega} \phi d\mu - h_1(\phi) \right] = \sup_{\psi \in C_*^0} \left[\int_{\Omega} L_n + \psi d\mu - h_1(L_n - \psi) \right].$$

Since $L_n - L \leq 0$ we have

$$h_1(-L_n - \psi) \leq \sup_{\Omega} \psi,$$

and so

$$\sup_{\phi \in C_*^0} \left[- \int_{\Omega} \phi d\mu - h_1(\phi) \right] \geq \int_{\Omega} L_n d\mu + \sup_{\psi \in C_*^0} \left[\int_{\Omega} \psi d\mu - \sup_{\Omega} \psi \right].$$

Letting $n \rightarrow \infty$, and using monotone convergence theorem proves the lemma. \blacksquare

Now suppose $\int_{\Omega} d\mu \neq 1$. Then by choosing $\psi = \alpha \in \mathbb{R}$ we get

$$\sup_{\psi \in C_*^0} \left[\int_{\Omega} \psi d\mu - \sup_{\Omega} \psi \right] \geq \sup_{\alpha \in \mathbb{R}} \alpha \left(\int_{\Omega} d\mu - 1 \right) = +\infty.$$

On the contrary, if $\int_{\Omega} d\mu = 1$ then

$$\int_{\Omega} (-\phi - L) d\mu \leq \sup_{\Omega} (-\phi - L) = h_1(\phi).$$

Therefore, for any ϕ

$$- \int_{\Omega} \phi d\mu - h_1(\phi) \leq \int_{\Omega} L d\mu,$$

and so

$$h_1^*(\mu) \leq \int_{\Omega} L d\mu.$$

To compute h_2^* observe that if $\mu \notin \mathcal{M}_0$ then there exists $\hat{\phi} \in \mathcal{C}$ such that

$$\int_{\Omega} \hat{\phi} d\mu \neq 0.$$

Thus

$$\inf_{\phi \in \mathcal{C}} - \int_{\Omega} \phi d\mu \leq \inf_{\alpha \in \mathbb{R}} \int_{\Omega} \alpha \hat{\phi} d\mu = -\infty.$$

So for $\mu \notin \mathcal{M}_0$ $h_2^*(\mu) = -\infty$. If $\mu \in \mathcal{M}_1$ then

$$\int_{\Omega} \phi d\mu = 0 \quad \forall \phi \in \mathcal{C}.$$

Therefore $h_2^*(\mu) = 0$. ■

Since h_1 is continuous, the previous result together with the Fenchel-Rockafellar theorem yields:

Theorem 2.4.

$$- \inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int_{\Omega} L d\mu = \inf_{\psi} \sup_{(x,v) \in \Omega} [\psi(x) - \psi(x+v) - L(x,v)].$$

3. Solution to the discrete Hamilton-Jacobi equation. The previous section motivates the study of the equation:

$$\sup_v [u(x) - u(x+v) - L(x,v)] = \overline{H}, \quad (3.8)$$

which should be seen as a discrete analog of the Hamilton-Jacobi equation:

$$\sup_v [-v D_x u - L(x,v)] = H(D_x u, x) = \overline{H}.$$

In this section we discuss the existence of solutions to (3.8), and, as we will see, the techniques used for the continuous case extend easily to this problem.

To construct a solution to (3.8) we are going to consider an approximate problem and then pass to the limit. To that effect, we define an operator T^α acting on bounded periodic functions u by

$$T^\alpha u(x) = e^{-\alpha} \inf_v [u(x+v) + L(x,v)],$$

for $\alpha > 0$. We look for fixed points u^α of T^α .

First observe that

$$\|T^\alpha u - T^\alpha w\|_\infty \leq e^{-\alpha} \|u - w\|_\infty,$$

for any u and w bounded and periodic. Therefore it is a strict contraction, and so there is a unique fixed point. Without loss of generality we may assume $L \geq 0$, by adding a constant to L . Therefore this fixed point is non-negative since T^α maps non-negative functions into non-negative functions. Furthermore, we have some a-priori bounds for the fixed point:

Lemma 3.1. *$T^\alpha u$ is uniformly (with respect to α) semiconcave and therefore, since it is periodic, Lipschitz.*

PROOF. Let v^* such that

$$u(x) = e^{-\alpha} [u(x + v^*) + L(x, v^*)].$$

Then

$$u(x \pm y) \leq e^{-\alpha} [u(x + v^*) + L(x, v^* \mp y)].$$

Then, since

$$L(x, v^* + y) + L(x, v^* - y) - 2L(x, v^*) \leq C|y|^2,$$

we have:

$$u(x + y) - 2u(x) + u(x - y) \leq C|y|^2.$$

The last remark in the statement follows from the fact that for periodic functions, semiconcavity implies Lipschitz continuity. \blacksquare

Note also that we have an a-priori bound on any fixed point:

Lemma 3.2. *Assume that $0 \leq L(x, 0) \leq C$. Then*

$$\|u^\alpha\|_\infty \leq \frac{C}{1 - e^{-\alpha}}.$$

REMARK. The assumption $L(x, 0) \geq 0$ is not critical (and of course implied by the assumption $L \geq 0$) but simplifies the bound. The other bound on the Lagrangian L is just a consequence of the compactness of \mathbb{T}^n .

PROOF. It suffices to observe that

$$0 \leq u^\alpha(x) \leq e^{-\alpha} [u^\alpha(x) + L(x, 0)],$$

by choosing $v = 0$. \blacksquare

Finally, the main result is

Theorem 3.3. *There exists a semiconcave periodic function u such that, through some subsequence, $u^\alpha - \min u^\alpha \rightarrow u$. There exists a number \overline{H} such that $(1 - e^{-\alpha}) \min u^\alpha \rightarrow \overline{H}$. Furthermore*

$$u(x) = \inf_v [u(x + v) + L(x, v) + \overline{H}],$$

that is, u solves (3.8).

PROOF. Since $u^\alpha - \min u^\alpha$ is uniformly Lipschitz, with respect to α , and consequently bounded by periodicity, it converges uniformly, possibly through some subsequence, to a limit u .

The uniform bounds on the fixed point imply that $(1 - e^{-\alpha}) \min u^\alpha$ is bounded and its Lipschitz constant tends to zero. Therefore, through a subsequence, it converges to a constant which define to be \overline{H} .

To check that (3.8) holds, it suffices to observe that

$$\begin{aligned} u^\alpha(x) - \min u^\alpha &= \\ &= -(1 - e^{-\alpha}) \min u^\alpha + \inf_v e^{-\alpha} [u^\alpha(x + v) - \min u^\alpha + L(x, v)], \end{aligned}$$

therefore in the limit we have

$$u(x) = \overline{H} + \inf_v [u(x + v) + L(x, v)].$$

\blacksquare

Proposition 3.4. *The number \overline{H} is unique.*

PROOF. By contradiction, assume that there is $\overline{H}_1 > \overline{H}_2$ and corresponding functions u_1, u_2 that satisfy

$$u_i(x) = \inf_v [u_i(x+v) + L(x, v) + \overline{H}_i].$$

Then it is possible to find a sequence of points (x_j, v_j) with $x_{j+1} = x_j + v_j$ and $1 \leq j \leq n$ such that

$$u_1(x_1) = u_1(x_n + v_n) + \sum_{j=1}^n L(x_j, v_j) + \overline{H}_1 n,$$

and

$$u_2(x_1) \leq u_2(x_n + v_n) + \sum_{j=1}^n L(x_j, v_j) + \overline{H}_2 n.$$

Then, since u_1 and u_2 are bounded we obtain

$$C \leq (\overline{H}_2 - \overline{H}_1)n \rightarrow -\infty,$$

as $n \rightarrow \infty$ if $\overline{H}_1 > \overline{H}_2$, which is a contradiction. \blacksquare

The last result in this section shows that the value \overline{H} obtained in (3.8) is the value of the infimum in Theorem 2.4.

Theorem 3.5. *We have*

$$- \inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int_{\Omega} L d\mu = \overline{H}.$$

PROOF. We have

$$\begin{aligned} \inf_{\phi} \sup_{(x,v) \in \Omega} [\phi(x) - \phi(x+v) - L(x, v)] &\leq \\ &\leq \sup_{(x,v) \in \Omega} [u(x) - u(x+v) - L(x, v)] = \overline{H}. \end{aligned}$$

To prove the reverse inequality, suppose that ψ is such that

$$\sup_{(x,v) \in \Omega} [\psi(x) - \psi(x+v) - L(x, v)] < \overline{H}.$$

Then we would have for all x

$$\psi(x) < \inf_v [\psi(x+v) + L(x, v) + \overline{H}] - \epsilon$$

for all x and some $\epsilon > 0$ sufficiently small. The difference $\psi - u$ has an absolute maximum at some point x_0 . At this point we have

$$u(x_0) = u(x_0 + v_0) + L(x_0, v_0) + \overline{H},$$

and

$$\psi(x_0) \leq \psi(x_0 + v_0) + L(x_0, v_0) + \overline{H} - \epsilon.$$

By subtracting the two expressions we have

$$\psi(x_0) - u(x_0) \leq \psi(x_0 + v_0) - u(x_0 + v_0) - \epsilon,$$

which is a contradiction. \blacksquare

REMARK. This last proof resembles the ones that arise in the theory of viscosity solutions - in fact, one should look at the function u as a viscosity solution of a discrete Hamilton-Jacobi equation.

4. Discrete Dynamic Programming principle. In this section we discuss the dynamic programming principle and discrete Euler equations for the discrete Hamilton-Jacobi equation.

Theorem 4.1. *Let u be a continuous solution of*

$$u(x) = \inf_v [u(x + v) + L(x, v) + \overline{H}]. \quad (4.9)$$

Then

1. *For each x there exists an optimal $v(x)$ (possibly not unique) such that*

$$u(x) = u(x + v(x)) + L(x, v(x)) + \overline{H}.$$

2. *u is differentiable at $x + v(x)$, and*

$$D_x u(x + v(x)) + D_v L(x, v(x)) = 0. \quad (4.10)$$

3. *If u is differentiable at x then*

$$D_x u(x) = D_x L(x, v(x)) - D_v L(x, v(x)). \quad (4.11)$$

PROOF. The first statement is a consequence that u is bounded and $L(x, v) \rightarrow \infty$ as $v \rightarrow \infty$.

To prove the second part, observe that for any x

$$u(x + y) \leq u(x + v(x)) + L(x + y, v(x) - y) + \overline{H}, \quad (4.12)$$

with equality for $y = 0$. Since the right hand side is differentiable in y , u has non-empty super-differential at any point (this also follows from the semiconcavity of u , but this proof is instructive).

At the point $x + v(x)$ we have

$$u(x) \leq u(x + v(x) + y) + L(x, v(x) + y) + \overline{H}, \quad (4.13)$$

which implies that at $x + v(x)$ u has non-empty sub-differential. When a function has non-empty sub and super differentials it is differentiable. From (4.13), by differentiating in y we obtain

$$D_x u(x + v(x)) + D_v L(x, v(x)) = 0. \quad (4.14)$$

Finally observe that from (4.12), by differentiating in y , we get

$$D_x u(x) = D_x L(x, v(x)) - D_v L(x, v(x)),$$

provided u is differentiable at x . ■

An easy consequence of this theorem is recorded in the next corollary. Let H , the Hamiltonian be defined by

$$H(x, p) = \sup_v [-v \cdot p - L(x, v)].$$

Corollary 4.2 (Discrete Hamilton equations). *Set*

$$p_n = D_x u(x_n) \quad x_{n+1} = x_n + v(x_n).$$

Then

$$\begin{cases} p_{n+1} - p_n = D_x H(x_n, p_{n+1}) \\ x_{n+1} - x_n = -D_p H(x_n, p_{n+1}). \end{cases} \quad (4.15)$$

PROOF. Observe that from (4.10) we have

$$p_{n+1} = -D_v L(x_n, v(x_n))$$

and (4.11) reads

$$p_n = D_x L(x_n, v(x_n)) - D_v L(x_n, v(x_n)).$$

Subtracting these two equations we obtain the first identity in (4.15), since

$$D_x L(x_n, v(x_n)) = -D_x H(x_n, p_{n+1}).$$

The second line follows from the fact that $v(x_n) = -D_p H(x_n, p_{n+1})$. ■

5. Regularity and the graph property. In this section, we study the regularity of the solution of the discrete Hamilton-Jacobi equation, and prove the graph property, that is, that a minimizing measure lies on a Lipschitz graph.

The first two propositions deal with the semiconcavity of the solution of the discrete Hamilton-Jacobi equation (4.9) (proposition 5.1), as well as the local semiconcavity along a minimizing trajectory (proposition 5.2). Then, following the ideas of [EG01], we prove that in the minimizing trajectory the solution of (4.9) is Lipschitz. The graph theorem, which characterizes the support of the minimizing measure is proven in theorem 5.5.

Proposition 5.1. *Let u be any solution of (4.9). Then u is semiconcave, that is*

$$u(x+y) - 2u(x) + u(x-y) \leq C|y|^2.$$

PROOF. Let v^* be the optimal velocity for x , that is

$$u(x) = u(x+v^*) + L(x, v^*) + \overline{H}$$

and

$$u(x, \pm y) \leq u(x+v^*) + L(x, v^* \mp y) + \overline{H},$$

thus, since L is C^2 ,

$$u(x+y) - 2u(x) + u(x-y) \leq C|y|^2. ■$$

Proposition 5.2. *Let u be a solution of (4.9), x_0 an arbitrary point in \mathbb{T}^n , and $v_0 = v(x_0)$ the optimal solution to (4.9). Then, at $x = x_0 + v_0$, u is locally semiconvex, that is,*

$$u(x+y) - 2u(x) + u(x-y) \geq -C|y|^2.$$

PROOF. Note that

$$u(x_0) = u(x) + L(x_0, v_0) + \overline{H}$$

and

$$u(x_0) \leq u(x \pm y) + L(x_0, v_0 \mp y) + \overline{H},$$

thus, since L is C^2

$$u(x+y) - 2u(x) + u(x-y) \geq -C|y|^2. ■$$

Note that this proposition does not hold at all points x , otherwise u would be both semiconcave and semiconvex and thus differentiable everywhere.

Now we borrow the ideas from [EG01], to show that along a minimizing trajectory, Du is Lipschitz.

Theorem 5.3. *Let x_0 be any point in \mathbb{T}^n and $v_0 = v(x_0)$ the optimal solution to (4.9). Let $x = x_0 + v_0$. There exists a constant C such that*

$$|u(x) - u(y) - Du(x)(y - x)| \leq C|x - y|^2,$$

furthermore

$$|Du(x) - Du(y)| \leq C|x - y|.$$

PROOF. For any h semiconcavity yields:

$$u(x + h) - u(x) - Du(x)h \leq C|h|^2, \quad (5.16)$$

and

$$u(x - h) - u(x) + Du(x)h \leq C|h|^2. \quad (5.17)$$

Also, by local semiconvexity at x ,

$$u(x + h) - 2u(x) + u(x - h) \geq -C|h|^2$$

Combining this with (5.17) we get

$$u(x + h) - u(x) - Du(x)h \geq -C|h|^2,$$

which, together with (5.16), proves the first estimate. Let now z be such that $|z| \leq 2|h|$. We have,

$$u(x + z) = u(x) + Du(x)z + O(|z|^2),$$

and

$$u(x + h) = u(x) + Du(x)h + O(|h|^2).$$

Furthermore, the semiconcavity of u implies

$$u(x + z) \leq u(x + h) + Du(x + h)(z - h) + C|z - h|^2.$$

Therefore by combining these estimates we have

$$(Du(x) - Du(x + h))(z - h) \leq C|h|^2$$

Choose

$$z = h + |h| \frac{Du(x) - Du(x + h)}{|Du(x) - Du(x + h)|},$$

which implies the second part. ■

Now we should observe that to a solution of (4.9) we can associate a dynamics by (4.15), and the initial conditions are determined by $D_x u(x_0)$, provided that at the initial point x_0 u is differentiable. To this dynamics corresponds a measure on $\mathbb{T}^n \times \mathbb{R}^n$ which is supported in the graph $(x, v(x))$. Furthermore u is differentiable in the support of this measure. Observe that

$$D_x u(x) = D_x L(x, v(x)) - D_v L(x, v(x))$$

has a unique solution if L is strictly convex in v and $D_x L$ has a small Lipschitz constant in v , these hypothesis are quite natural as we will see in the last section.

Theorem 5.4. *Let μ be any minimizing measure and u any solution of (4.9). Then μ -almost everywhere*

$$u(x + v) - u(x) + L(x, v) = -\overline{H},$$

in particular, if u is differentiable $D_x u = D_x L(x, v) - D_v L(x, v)$.

PROOF. By (4.9) we have

$$u(x+v) - u(x) + L(x, v) \geq -\overline{H},$$

for all (x, v) , namely those in $\text{supp } \mu$. Therefore it suffices to prove the reverse inequality. By contradiction, assume that there is $\epsilon > 0$ such that

$$u(x+v) - u(x) + L(x, v) > -\overline{H} + \epsilon.$$

Recall that

$$\int u(x+v) - u(x) d\mu = 0,$$

and so

$$\int L(x, v) d\mu > -\overline{H},$$

which contradicts Theorem 3.5. The last part of the statement is simply the optimality condition from the previous section. \blacksquare

Theorem 5.5. *If the equation*

$$p = D_x L(x, v) - D_v L(x, v)$$

has a unique differentiable solution $v(x, p)$ for every p and x then any minimizing measure is supported on a Lipschitz graph.

PROOF. By the previous theorem almost every v in the support of μ is optimal, and thus u is differentiable at $x+v$ with

$$D_x u(x+v) = -D_v L(x, v).$$

Furthermore, we have

Lemma 5.6. *Almost every point (x, v) in the support of μ is of the form $(x, v) = (x_0 + v_0, v)$ with (x_0, v_0) in the support of μ .*

PROOF. By contradiction, there would be a function $\phi(x) \geq 0$ such that

$$\int \phi(x) d\mu = 1$$

and

$$\int \phi(x+v) d\mu < \epsilon,$$

and this contradicts $\int \phi(x) - \phi(x+v) d\mu = 0$. \blacksquare

But then,

$$D_x u(x) = D_x L(x, v) - D_v L(x, v)$$

defines v uniquely, and since $D_x u$ is Lipschitz at x then v is Lipschitz. \blacksquare

6. Asymptotic behavior. By replacing the Lagrangian L with

$$L(x, v) + P \cdot v$$

for $P \in \mathbb{R}^n$ we obtain a family of solutions $u(x, P)$ and a function $\overline{H}(P)$ that satisfy

$$u(x) = u(x+v(x)) + L(x, v(x)) + P \cdot v(x) + \overline{H}(P).$$

Proposition 6.1. *$\overline{H}(P)$ is convex in P .*

PROOF. It suffices to observe that $\overline{H}(P)$ can be written as the supremum of a family of convex functions:

$$\overline{H}(P) = \sup_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} - \int_{\Omega} L(x, v) + P \cdot v d\mu.$$

■

The next theorem should be compared with the corresponding result for Hamilton-Jacobi equations [Gom02b].

Theorem 6.2. *Let x_n be an optimal trajectory for the problem corresponding to P . Then*

$$\lim_{n \rightarrow \infty} \frac{x_n}{n} = -D_P \overline{H}(P),$$

provided \overline{H} is differentiable at P .

PROOF. We have

$$u(x_0, P) = u(x_n, P) + \sum_{i=0}^{n-1} L(x_i, v_i) + P(x_{i+1} - x_i) + \overline{H}(P)$$

and, for any Δ

$$u(x_0, P + \Delta) \leq u(x_n, P + \Delta) + \sum_{i=0}^{n-1} L(x_i, v_i) + (P + \Delta)(x_{i+1} - x_i) + \overline{H}(P + \Delta).$$

By subtraction we obtain

$$\begin{aligned} O(1) &\leq \sum_{i=0}^{n-1} \Delta(x_{i+1} - x_i) + \overline{H}(P + \Delta) - \overline{H}(P) = \\ &= \Delta(x_n - x_0) + n\Delta D_P \overline{H}(P) + o(n|\Delta|). \end{aligned}$$

Thus

$$\Delta \frac{x_n - x_0 + D_P \overline{H}(P)n}{n} = O\left(\frac{1}{n}\right) + o(|\Delta|)$$

thus sending $n \rightarrow \infty$ and $\Delta \rightarrow 0$ we obtain the result. ■

7. Convergence in the continuous limit. In this section we show that these discretizations are approximations to the continuous Mather problem. The Euler method for an ODE

$$\dot{x} = v(t)$$

yields the discrete dynamics

$$x_{n+1} = x_n + hv_n.$$

To study the limit $h \rightarrow 0$, it is convenient to consider a rescaled problem in which the constraint (1.2) is replaced by

$$\frac{1}{h} \int_{\Omega} w(x + vh) - w(x) d\mu^h = 0.$$

As $h \rightarrow 0$ this constraint converges formally to

$$\int_{\Omega} v D_x w(x) d\mu^0 = 0.$$

The dual problem is then

$$\inf_{\psi} \sup_{(x,v) \in \Omega} \frac{\psi(x + hv) - \psi(x)}{h} - L(x, v),$$

which formally tends, as $h \rightarrow 0$, to

$$\inf_{\psi} \sup_{(x,v) \in \Omega} v D_x \psi - L(x, v).$$

The corresponding discrete Hamilton-Jacobi is then

$$u(x) = \inf_v [u(x + hv) + hL(x, v) + \bar{H}h] \quad (7.18)$$

There are several questions that we would like to address

1. $\bar{H}^h \rightarrow \bar{H}^0$
2. $u^h \rightarrow u$, in which u is a viscosity solution of

$$H(D_x u, x) = \bar{H}^0.$$

3. $\mu^h \rightharpoonup \mu^0$

The first point, the convergence of \bar{H}^h can be addressed in two parts - it is clear that the sequence \bar{H}^h is uniformly bounded in h , therefore through some subsequence it converges to a limit. To see that this number is indeed \bar{H}^0 we need to address the second point.

The main problem in dealing with these convergence issues is that the proof of proposition 5.1 for the semiconcavity constant for a solution of (7.18) is $O(\frac{1}{h})$, therefore we need a slightly improved proof:

Proposition 7.1. *Let u be a solution of (7.18). Then u is semiconcave, that is*

$$u(x + y) - 2u(x) + u(x - y) \leq C|y|^2,$$

in which the constant is independent of h .

PROOF. To simplify the proof, suppose $h = 1/n$ for some integer $n > 0$. Let

$$y_j = \frac{j}{n}y \quad 1 \leq j \leq n.$$

Then

$$u(x) = u(w) + \frac{1}{n} \sum_{j=1}^n (L(x_j, v_j) + \bar{H})$$

and

$$u(x \pm y) \leq u(w) + \frac{1}{n} \sum_{j=1}^n (L(x_j \pm y_j, v_j \mp y) + \bar{H}),$$

Therefore

$$\begin{aligned} u(x + y) - 2u(x) + u(x - y) &\leq \\ &\leq \frac{1}{n} \sum_{j=1}^n [L(x_j + y_j, v_j - y) - 2L(x_j, v_j) + L(x_j - y_j, v_j + y)] \leq \\ &\leq C|y|^2, \end{aligned}$$

in which the constant is independent of n . ■

As a consequence of the previous proposition the optimal controls are bounded since

$$D_x u(x + v) = D_v L(x, v)$$

yields uniformly bounded v if u is uniformly Lipschitz.

Proposition 7.2. *Through some subsequence, u^h converges to a function u^0 . This function is a viscosity solution of*

$$H(D_x u^0, x) = \bar{H}.$$

PROOF. Suppose ϕ is a C^1 function and $u - \phi$ has a strict local minimum at a point x . Then for h sufficiently small $u^h - \phi$ has a strict local minimum at a point $x_h \rightarrow x$, as $h \rightarrow 0$. Then

$$\begin{aligned} -\bar{H}^h &= \inf_v \frac{u^h(x_h + hv) - u^h(x)}{h} + L(x, v) \geq \\ &\geq \inf_v \frac{\phi^h(x_h + hv) - \phi^h(x)}{h} + L(x, v). \end{aligned}$$

Taking the limit $h \rightarrow 0$

$$-\bar{H} \geq \inf_v v \cdot D_x \phi(x) + L(x, v) = -H(D_x \phi, x),$$

that is,

$$H(D_x \phi, x) \geq \bar{H}.$$

In the case that $u - \phi$ has a strict local maximum is similar. ■

Proposition 7.3. *Suppose $\mu^h \rightarrow \mu^0$. Then*

$$\int v D_x \phi(x) d\mu_0 = 0,$$

for all $\phi(x) \in C^1(\mathbb{T}^n)$.

PROOF. It suffices to prove the claim for a dense class. For instance, if $\phi \in C^2(\mathbb{T}^n)$ we have

$$\frac{\phi(x + hv) - \phi(x)}{h} \rightarrow v D_x \phi(x),$$

uniformly, since the support of μ^h is uniformly bounded in h . Therefore

$$0 = \lim_{h \rightarrow 0} \int \frac{\phi(x + hv) - \phi(x)}{h} d\mu^h = \int v D_x \phi(x) d\mu^0.$$

Since $\bar{H}^h \rightarrow \bar{H}^0$, and ■

$$\int L d\mu^h \rightarrow \int L d\mu^0 = -\bar{H}^0,$$

we conclude:

Theorem 7.4. *μ^0 is a Mather measure, that is, it minimizes*

$$\int L d\mu,$$

among all positive probability measures that satisfy

$$\int v D_x \phi(x) d\mu = 0,$$

for all $\phi(x)$.

If the projection of the measure μ has a smooth density $\theta(x)$, and the viscosity solution u is smooth then the constraint in the last theorem reads

$$\nabla \cdot (\theta D_x u) = 0,$$

as the Mather measure is supported on the graph $(x, v) = (x, D_x u(x))$.

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