Modifications and possible uses of Perron's method in stochastic control and finance

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Abstract/objective

The goal is to present, with examples, a particular point of view on dynamic programming in stochastic control and finance. Starting from the best scenario when the Hamilton-Jacobi-Bellman (HJB) equation has a smooth solution, it leads to a modification of Perron's method that can substitute for such verification, in case smoothness is not available.

What is the modification of Perron's method?

New look at an old (set of) problem(s).

Disclaimer:

- not trying to "reinvent the wheel" but provide a different view (and a new tool)

Questions:

- why a new look?
- how/the tool we propose

Summary

- Brief overview of Dynamic Programming (DP) and the possible uses and modifications of Perron's method (preview of the following parts).
- Examples of stochastic control problems (in finance) where the HJB has a smooth solution: [4], [5]. Special emphasis on verification (since the closed form solution is well known).
- The Dynamic Programming Principle (DPP) and viscosity solutions. Examples of problems where the viscosity solution is actually smooth: [7], [6].
- Perron's method: a possible bypass of the (DPP) to obtain a smooth solution (if such solution exists) Example: [3].
- Stochastic Perron's method in linear and non-linear problems: [2], [1].

Overview and Preview

Stochastic Control Problems

State equation

$$\begin{cases} dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t \\ X_s = x. \end{cases}$$

$X \in \mathbb{R}^n, W \in \mathbb{R}^d$

Cost functional $J(s, x, \alpha) = \mathbb{E}[\int_s^T R(t, X_t, \alpha_t)dt + g(X_T)]$

Value function

$$v(s, x) = \sup_{\alpha} J(s, x, \alpha).$$

Comments: all formal, no filtration, admissibility, etc. Also, we have in mind other classes of control problems as well.

(My understanding of) Continuous-time DP and HJB's

Two possible approaches

1. analytic (direct/constructive)
2. probabilistic (study the properties of the value function)

The Analytic approach

1. write down the DPE/HJB

$$\begin{cases} u_t + \sup_\alpha \left\{ L_t^\alpha u + R(t,x,\alpha) \right\} = 0 \\ u(T,x) = g(x) \end{cases}$$

2. solve it i.e.
   - prove existence of a smooth solution u
   - (if lucky) find a closed form solution u
3. go over verification arguments
   - proving existence of a solution to the closed-loop SDE
   - use Itô's lemma and uniform integrability, to conclude $u = v$ and the solution of the closed-loop eq. is optimal

Analytic approach cont'd

Conclusions: the existence of a smooth solution of the HJB (with some properties) implies

1. $u = v$ (uniqueness of the smooth solution)
2. (DPP)

$$v(s,x) = \sup_{\alpha} \mathbb{E}[\int_s^{\tau} R(t, X_t, \alpha_t)dt + v(\tau, X_\tau)]$$

3. $\alpha(t,x) = \arg\max$ is the optimal feedback

Complete description: Fleming and Rishel

smooth sol of (DPE) $\rightarrow$ (DPP)+value fct is the unique sol

Probabilistic/Viscosity Approach

1. prove the (DPP)
2. show that (DPP) $\longrightarrow$ v is a viscosity solution
3. IF viscosity comparison holds, then v is the unique viscosity solution

$$\boxed{\text{(DPP)+visc. comparison} \rightarrow v \text{ is the unique visc sol(DPE)}}$$

Meta-Theorem If the value function is the unique viscosity solution, then finite difference schemes approximate the value function and the optimal feedback control (approximate backward induction works).

Comments on probabilistic approach

1. quite hard (actually very hard compared to deterministic case)
   1.1 by approx with discrete-time or smooth problems (Krylov)
   1.2 work directly on the value function (El Karoui, Borkar, Hausmann, Bouchard-Touzi for a weak version)
2. non-trivial, but easier than 1: Fleming-Soner, Bouchard-Touzi
3. has to be proved separately (analytically) anyway

Probabilistic/Viscosity Approach pushed further

Sometimes we are lucky:

- using the specific structure of the HJB can prove that a viscosity solution of the DPE is actually smooth!
- if that works we can just come back to the Analytic approach and go over step 3, i.e. we can perform verification using the smooth solution v (the value function) to obtain
  1. the (DPP)
  2. Optimal feedback control $\alpha(t, x)$

(DPP)$\rightarrow$ v is visc. sol $\rightarrow$ v is smooth sol $\rightarrow$ (DPP) +opt. controls

Examples: Shreve and Soner, Pham (Part 3)

Viscosity solution is smooth, cont'd

- the first step is hardest to prove
- the program seems circular

Question: can we just avoid the first step, proving the (DPP)?
Answer: yes, we can use (Ishii's version of) Perron's method to construct (quite easily) a viscosity solution.

Lucky case, revisited

Perron $\rightarrow$ visc. sol $\rightarrow$ smooth sol $\rightarrow$ unique+(DPP) +opt. controls

Example: Janeček, S.

Comments:

- old news for PDE
- the new approach is analytic/direct

Perron's method

General Statement: sup over sub-solutions and inf over super-solutions are solutions.

$$v^- = \sup_{w \in \mathscr{U}^-} w, v^+ = \inf_{w \in \mathscr{U}^+} w \textit{ are solutions}$$

Ishii's version of Perron (1984): sup over viscosity sub-solutions and inf over viscosity super-solutions are viscosity solutions.

$$v^- = \sup_{w \in \mathscr{U}^{-,visc}} w, v^+ = \inf_{w \in \mathscr{U}^{+,visc}} w \textit{ are viscosity solutions}$$

Question: why not inf/sup over classical super/sub-solutions?
Answer: Because one cannot prove (in general/directly) the result is a viscosity solution. The classical solutions are not enough (the set of classical solutions is not stable under max or min).

Objective

Provide a method/tool to replace the first two steps in the program

Perron $\rightarrow$ visc. sol $\rightarrow$ smooth sol $\rightarrow$ unique+(DPP) +opt. controls

in case one cannot prove viscosity solutions are smooth ("the unlucky case")

New method/tool $\rightarrow$ **construct** a visc. sol $u \rightarrow u = v$ +(DPP)

Why not try a version of Perron's method?

Perron's method, recall

(Ishii's version) Provides viscosity solutions of the HJB

$$v^- = \sup_{w \in \mathscr{U}^{-,visc}} w, v^+ = \inf_{w \in \mathscr{U}^{+,visc}} w$$

Problem:

- w does NOT compare to the value function v UNLESS one proves v is a viscosity solutions already AND the viscosity comparison
- if we ask w to be classical semi-solutions, we cannot prove that the inf/sup are viscosity solutions

Main Idea

Perform Perron's Method over a class of semi-solutions which are

- weak enough to conclude (in general/directly) that v^-, v^+ are viscosity solutions
- strong enough to compare with the value function **without studying the properties of the value function**

We know that

classical sol $\rightarrow$ (DPP) $\rightarrow$ viscosity sol

Actually, we have

classical semi-sol $\rightarrow$ half-(DPP) $\rightarrow$ viscosity semi-sol

Translation
"half (DPP)= stochastic semi-solution"
Main property: stochastic sub and super-solutions DO compare with the value function v!

Stochastic Perron Method, quick summary

General Statement:

- supremum over stochastic sub-solutions is a viscosity (super)-solution

$$v_* = \sup_{w \in \mathscr{U}^{-,stoch}} w \leq v$$

- infimum over stochastic super-solutions is a viscosity (sub)-solution

$$v^* = \inf_{w \in \mathscr{U}^{+,stoch}} w \geq v$$

Conclusion:

$$v_* \leq v \leq v^*$$

IF we have a viscosity comparison result, then v is the unique viscosity solution!

(SP)+visc comp $\rightarrow$ (DPP)+ v is the unique visc sol of (DPE)

Some comments

- the Stochastic Perron Method plus viscosity comparison substitute for (large part of) verification (in the analytic approach)
- this method represents a "probabilistic version of the analytic approach"
- loosely speaking, stochastic sub and super-solutions amount to sub and super-martingales
- stochastic sub and super-solution have to be carefully defined (depending on the control problem) as to obtain viscosity solutions as sup/inf (and to retain the comparison build in)

An example of how to use the analytic approach: the Merton problem

The Merton Problem

- money market paying constant interest $r = 0$
- stock

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

Investment strategies θ:

$$dX_t = \theta_t \frac{dS_t}{S_t}, \quad X_0 = x > 0$$

Admissible strategies:

$$X > 0$$

Can think in terms of proportions:

$$\theta \leftrightarrow \pi = \theta / S.$$

The optimization problem

Fix $X_0 = x$, find the optimal θ or π in

$$\max_{\theta} \mathbb{E}[U(X_T)]$$

where

$$U(x) = \frac{x^{1-p}}{1-p}, \quad p > 0$$

The HJB

Look for $u(t,x)$ solution of

$$u_t + \sup_{\pi}\{\pi\mu x u_x + \frac{1}{2}\pi^2\sigma^2 x^2 u_{xx}\} = 0$$

$$u(x,T) = U(x)$$

Formally, the argmax

$$\hat{\pi}(x) = -\frac{\mu x u_x}{\sigma^2 x^2 u_{xx}}$$

is optimal feedback.

Closed form solution for power utility

Expect $u(t,x) = a(t)\frac{x^{1-p}}{1-p}$, so we get

$$a'(t) + Ca(t) = 0, \quad a(T) = 1$$

for a constant C.
We therefore have the smooth solution of the HJB

$$u(t,x) = e^{C(T-t)} \times \frac{x^{1-p}}{1-p}$$

and

$$\hat{\pi} = \frac{\mu}{p\sigma^2}.$$

The problem is completely solved, no?

No, we still need to go over verification

Recall, verification means

- proving existence of a solution to the closed-loop SDE
- use Itô's lemma and uniform integrability, to conclude $u = v$ (v is the value function) and the solution of the closed-loop eq. is optimal

The closed loop SDE

The "candidate" optimal proportion $\hat{\pi}$ is a constant, and the SDE is

$$\begin{cases} \frac{dX_t}{X_t} = \hat{\pi}(\mu dt + \sigma dW_t) \\ X_s = x. \end{cases}$$

This is actually a very particular case, when even the closed loop eq has a closed form solution: the Geometric Brownian Motion (with initial time s, if we really start at s)

Verification, cont'd

Since $u(t,x)$ satisfies the HJB, and $\hat{\pi}$ is the argmax, we have

- $u(t, X_t)$ is a local super-martingale
- $u(t, \hat{X}_t)$ is a local martingale

(between times s and T)
Full verification means showing the super/martingale property in between times s (initial time) and T.

- $u(s,x) \geq \mathbb{E}[u(T, X_T)]$ for each π
- $u(s,x) = \mathbb{E}[u(T, \hat{X}_T)]$

Last part of verification

For this very particular example, $u(t, \hat{X}_t)$ is actually a geometric Brownian Motion (should double check!), so it is a true martingale. In general, we need bounds on u to get

$$u(s, x) = (\leq)\mathbb{E}[u(T, \hat{X}_T)]$$

Last part of verification, cont'd

Left to prove

$$u(s,x) \geq \mathbb{E}[u(T,X_T)] \ (\forall) \ \pi \leftrightarrow \theta$$

Case 1: $0 < p < 1$, we have a positive local martingale. Done!

Case 2: $p > 1$, don't have a bound below. Small trick: add ε.

What do we gain? $X + \varepsilon$ is an admissible wealth process (with the same θ but different π) starting at $x + \varepsilon$. Now, we have $u(t, X + \varepsilon)$ is a local mart bounded below, so it is a supermartingale.

$$u(s, x+\varepsilon) \geq \mathbb{E}[U(T, X_T + \varepsilon)] \geq \mathbb{E}[U(T, X_T)].$$

Let $\varepsilon \searrow 0$, to obtain

$$u(s,x) \geq \mathbb{E}[U(T, X_T)].$$

Conclusion of verification

We get

$$u(s,x) = v(s,x) := \sup_{\theta} \mathbb{E}[U(X_T^{s,x,\theta})],$$

and $\hat{\pi}$ is optimal.

Actually, we can go over identical verification arguments up to time τ. We have

$$u(s,x) = \sup_{\theta} \mathbb{E}[u(\tau, X_\tau^{s,x,\theta})].$$

This is the DPP (since we just proved above that $u = v$).

An example of constructing a smooth viscosity solution using Perron's method:

Optimal investment with high-watermark performance fee

(joint work with Karel Janeček)

Objective

- build and analyze a model of optimal investment and consumption where the investment opportunity is represented by a hedge-fund using the "two-and-twenty rule"
- analyze the impact of the high-watermark fee on the investor

Previous work on hedge-funds and high-watermarks

All existing work analyzes the impact/incentive of the high-watermark fees on fund managers

- extensive finance literature
  - Goetzmann, Ingersoll and Ross, Journal of Finance 2003
  - Panagea and Westerfield, Journal of Finance 2009
  - Agarwal, Daniel and Naik Journal of Finance, forthcoming
  - Aragon and Qian, preprint 2007
- recently studied in mathematical finance
  - Guasoni and Obloj, preprint 2009

A model of profits from dynamically investing in a hedge-fund

- the investor chooses to hold θ_t in the fund at time t
- the value of the fund F_t is given exogenously
- denote by P_t the accumulated profit/losses up to time t

Evolution of the profit

- without high-watermark fee

$$dP_t = \theta_t \frac{dF_t}{F_t}, \quad P_0 = 0$$

- with high-watermark proportional fee $\lambda > 0$

$$\begin{cases} dP_t = \theta_t \frac{dF_t}{F_t} - \lambda dP_t^*, \quad P_0 = 0 \\ P_t^* = \max_{0 \le s \le t} P_s \end{cases}$$

High-watermark of the investor

$$P_t^* = \max_{0 \le s \le t} P_s.$$

Observation: can be also interpreted as taxes on gains, paid right when gains are realized (pointed out by Paolo Guasoni)

Path-wise solutions

(same as Guasoni and Obloj)
Denote by I_t the paper profits from investing in the fund

$$I_t = \int_0^t \theta_u \frac{dF_u}{F_u}$$

Then

$$P_t = I_t - \frac{\lambda}{\lambda + 1} \max_{0 \le s \le t} I_s$$

The high-watermark of the investor is

$$P_t^* = \frac{1}{\lambda + 1} \max_{0 \le s \le t} I_s$$

Observations:

- the fee λ can exceed 100% and the investor can still make a profit
- the high-watemark is measured before the fee is paid

Connection to the Skorohod map (Part of work in progress with Gerard Brunick)

Denote by $Y = P^* - P$ the distance from paying fees. Then Y satisfies the equation:

$$\begin{cases} dY_t = -\theta_t \frac{dF_t}{F_t} + (1+\lambda) dP^*_t \\ Y_0 = 0, \end{cases}$$

where $Y \geq 0$ and

$$\int_0^t \mathbb{I}_{\{Y_s \neq 0\}} dP^*_s = 0, \quad (\forall)\ t \geq 0.$$

Skorohod map

$$I_{\cdot} = \int_0^{\cdot} \theta_u \frac{dF_u}{F_u} \to (Y, P^*) \approx (P, P^*).$$

Remark: Y will be chosen as state in more general models.

The model of investment and consumption

An investor with initial capital $x > 0$ chooses to

- have θ_t in the fund at time t
- consume at a rate γ_t
- finance from borrowing/investing in the money market at zero rate

Denote by $C_t = \int_0^t \gamma_s ds$ the accumulated consumption. Since the money market pays zero interest, then

$$X_t = x + P_t - C_t \leftrightarrow P_t = (X_t + C_t) - x$$

Therefore, the fees (high-watermark) is computed tracking the wealth and accumulated consumption

$$P_t^* = \max_{0 \le s \le t} \left\{ X_s + \int_0^s \gamma_u du \right\} - x$$

Can think that the investor leaves all her wealth (including the money market) with the investor manager.

Evolution equation for the wealth

The evolution of the wealth is

$$\begin{cases} dX_t = \theta_t \frac{dF_t}{F_t} - \gamma_t dt - \lambda dP_t^*, \quad X_0 = x \\ P_t^* = \max_{0 \le s \le t} \left\{ X_s + \int_0^s \gamma_u du \right\} - x \end{cases}$$

- consumption is a part of the running-max, as opposed to the literature on draw-dawn constraints
  - Grossman and Zhou
  - Cvitanic and Karatzas
  - Elie and Touzi
  - Roche
- we still have a similar path-wise representation for the wealth in terms of the "paper profit" I_t and the accumulated consumption

Optimal investment and consumption

Admissible strategies

$$\mathscr{A}(x) = \{(\theta, \gamma) : X > 0\}.$$

Can represent investment and consumption strategies in terms of proportions

$$c = \gamma/X, \quad \pi = \theta.$$

Obervation:

- no closed form path-wise solutions for X in terms of (π, c) (unless $c = 0$)

Optimal investment and consumption:cont'd

Maximize discounted utility from consumption on infinite horizon

$$\mathscr{A}(x) \ni (\theta, \gamma) \to \mathrm{argmax}\, \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(\gamma_t) dt\right].$$

Where $U : (0, \infty) \to \mathbb{R}$ is the CRRA utility

$$U(\gamma) = \frac{\gamma^{1-p}}{1-p}, \quad p > 0.$$

Finally, choose a geometric Brownian-Motion model for the fund share price

$$\frac{dF_t}{F_t} = \alpha dt + \sigma dW_t.$$

Dynamic programming: state processes

Fees are paid when $P = P^*$. This can be translated as $X + C = (X + C)^*$ or as

$$X = (X + C)^* - C.$$

Denote by

$$N \triangleq (X + C)^* - C.$$

The (state) process (X, N) is a two-dimensional controlled diffusion $0 < X \leq N$ with reflection on $\{X = N\}$.
The evolution of the state (X, N) is given by

$$\begin{cases} dX_t = \big(\theta_t \alpha - \gamma_t\big) dt + \theta_t \sigma dW_t - \lambda dP_t^*, \; X_0 = x \\ dN_t = -\gamma_t dt + dP_t^*, \quad N_0 = x. \end{cases}$$

Recall we have path-wise solutions in terms of (θ, γ).

Dynamic Programming: Objective

- we are interested to solve the problem using dynamic programing. We are only interested in the initial condition (x, n) for $x = n$ but we actually solve the problem for all $0 < x \leq n$. This amounts to setting an initial high-watemark of the investor which is larger than the initial wealth.
- expect to find the two-dimensional value function $v(x, n)$ as a solution of the HJB, and find the (feed-back) optimal controls.

Dynamic programming equation

Use Itô and write formally the HJB

$$\sup_{\gamma \geq 0, \theta} \left\{ -\beta v + U(\gamma) + (\alpha\theta - \gamma) v_x + \frac{1}{2}\sigma^2\theta^2 v_{xx} - \gamma v_n \right\} = 0$$

for $0 < x < n$ and the boundary condition

$$-\lambda v_x(x, x) + v_n(x, x) = 0.$$

(Formal) optimal controls

$$\hat{\theta}(x, n) = -\frac{\alpha}{\sigma^2} \frac{v_x(x, n)}{v_{xx}(x, n)}$$

$$\hat{\gamma}(x, n) = I(v_x(x, n) + v_n(x, n))$$

HJB cont'd

Denote by $\tilde{U}(y) = \frac{p}{1-p} y^{\frac{p-1}{p}}, \quad y > 0$ the dual function of the utility. The HJB becomes

$$-\beta v + \tilde{U}(v_x + v_n) - \frac{1}{2}\frac{\alpha^2}{\sigma^2}\frac{v_x^2}{v_{xx}} = 0, \quad 0 < x < n$$

plus the boundary condition

$$-\lambda v_x(x,x) + v_n(x,x) = 0.$$

Observation:

- if there were no v_n term in the HJB, we could solve it closed-form as in Roche or Elie-Touzi using the (dual) change of variable $y = v_x(x, n)$
- no closed-from solutions in our case (even for power utility)

Reduction to one-dimension

Since we are using power utility

$$U(x) = \frac{x^{1-p}}{1-p}, \quad p > 0$$

we can reduce to one-dimension

$$v(x, n) = x^{1-p} v(1, \frac{n}{x})$$

and

$$v(x, n) = n^{1-p} v(\frac{x}{n}, 1)$$

- first is nicer economically (since for $\lambda = 0$ we get a constant function $v(1, \frac{n}{x})$)
- the second gives a nicer ODE (works very well if there is a closed form solution, see Roche)

There is no closed form solution, so we can choose either one-dimensional reduction.

Reduction to one-dimension cont'd

We decide to denote $z = \frac{n}{x} \geq 1$ and

$$v(x,n) = x^{1-p} u(z).$$

Use

$$\begin{aligned} v_n(x,n) &= u'(z) \cdot x^{-p}, \\ v_x(x,n) &= \Big((1-p)u(z) - zu'(z)\Big) \cdot x^{-p}, \\ v_{xx}(x,n) &= \Big(-p(1-p)u(z) + 2pzu'(z) + z^2 u''(z)\Big) \cdot x^{-1-p}, \end{aligned}$$

to get the reduced HJB

$$-\beta u + \tilde{U}\big((1-p)u - (z-1)u'\big) - \frac{1}{2}\frac{\alpha^2}{\sigma^2}\frac{\big((1-p)u - zu'\big)^2}{-p(1-p)u + 2pzu' + z^2 u''} = 0$$

for $z > 1$ with boundary condition

$$-\lambda(1-p)u(1) + (1+\lambda)u'(1) = 0$$

(Formal) optimal proportions

$$\hat{\pi}(z) = \frac{\alpha}{p\sigma^2} \cdot \frac{(1-p)u - zu'}{(1-p)u - 2zu' - \frac{1}{p}z^2 u''},$$

$$\hat{c}(z) = \frac{(v_x + v_n)^{-\frac{1}{p}}}{x} = \left((1-p)u - (z-1)u'\right)^{-\frac{1}{p}}$$

Optimal amounts (controls)

$$\hat{\theta}(x,n) = x\hat{\pi}(z), \quad \hat{\gamma}(x,n) = x\hat{c}(z)$$

Objective: solve the HJB analytically and then do verification

Solution of the HJB for $\lambda = 0$

This is the classical Merton problem. The optimal investment proportion is given by

$$\pi_0 \triangleq \frac{\alpha}{p\sigma^2},$$

while the value function equals

$$v_0(x, n) = \frac{1}{1-p}\, c_0^{-p} x^{1-p}, \quad 0 < x \leq n,$$

where

$$c_0 \triangleq \frac{\beta}{p} - \frac{1}{2}\,\frac{1-p}{p^2} \cdot \frac{\alpha^2}{\sigma^2}$$

is the optimal consumption proportion. It follows that the one-dimensional value function is constant

$$u_0(z) = \frac{1}{1-p}\, c_0^{-p}, \qquad z \geq 1.$$

Solution of the HJB for $\lambda > 0$

If $\lambda > 0$ we expect that (additional boundary condition)

$$\lim_{z \to \infty} u(z) = u_0.$$

(For very large high-watermark, the investor gets almost the Merton expected utility)

Existence of a smooth solution

Theorem 1 The HJB has a smooth solution.

Idea of solving the HJB:

- find a viscosity solution using an adaptation of Perron's method. Consider infimum of concave supersolutions that satisfy the boundary condition. Obtain as a result a concave viscosity solution. The subsolution part is more delicate. Have to treat carefully the boundary condition.

Proof of existence: cont'd

- show that the viscosity solution is C^2 (actually more). Concavity, together with the subsolution property implies C^1 (no kinks). Go back into the ODE and formally rewrite it as

  $$u'' = f(z, u(z), u'(z)) \triangleq g(z).$$

  Compare locally the viscosity solution u with the classical solution of a similar equation

  $$w'' = g(z)$$

  with the same boundary conditions, whenever u, u' are such that g is continuous. The difficulty is to show that u, u' always satisfy this requirement.

Avoid defining the value function and proving the Dynamic Programming Principle.

Verification, Part I

Theorem 2 The closed loop equation

$$\begin{cases} dX_t = \hat{\theta}(X_t, N_t)\frac{dF_t}{F_t} - \hat{\gamma}(X_t, N_t)dt - \lambda(dN_t + \gamma_t dt), \quad X_0 = x \\ N_t = \max_{0\leq s\leq t}\left\{X_s + \int_0^s \hat{\gamma}(X_u, N_u)du\right\} - \int_0^t \hat{\gamma}(X_u, N_u)du \end{cases}$$

has a unique strong solution $0 < \hat{X} \leq \hat{N}$.

Ideea of proof:

- use the path-wise representation

$$(Y, L) \rightarrow (\hat{\theta}(Y, L), \hat{\gamma}(Y, L)) \rightarrow (X, N)$$

  together with the Itô-Picard theory to obtain a unique global solution $X \leq N$.
- use the fact that the optimal proportion $\hat{\pi}$ and $\hat{c}$ are bounded to compare $\hat{X}$ to an exponential martingale and conclude

$$\hat{X} > 0$$

Verification, Part II

Theorem 3 The controls $\hat{\theta}(\hat{X}_t, \hat{N}_t)$ and $\hat{\gamma}(\hat{X}_t, \hat{N}_t)$ are optimal.

Idea of proof:

- use Itô together with the HJB to conclude that

$$e^{-\beta t} V(X_t, N_t) + \int_0^t e^{-\beta s} U(\gamma_s) ds, \quad 0 \leq t < \infty,$$

  is a local supermartingale in general and a local martingale for the candidate optimal controls (the obvious part)
- uniform integrability. Has to be done separately for $p < 1$ and $p > 1$ (the harder part, requires again the use of $\hat{\pi}$ and $\hat{c}$ bounded, and comparison to an exponential martingale).

The impact of fees

Everything else being equal, the fees have the effect of

- reducing rate of return
- reducing initial wealth

Certainty equivalent return

We consider two investors having the same initial wealth, risk-aversion, who invest in two funds with the same volatility

- one invests in a fund with return α, and pays fees $\lambda > 0$. The initial high-watermark is $n = xz \geq x$
- the other invests in a fund with return $\tilde{\alpha}$ but pays no fees

Equate the expected utilities:

$$u_0\big(\tilde{\alpha}(z), \cdot\big) = u_\lambda(\alpha, z).$$

Can be solved as

$$\tilde{\alpha}^2(z) = 2\sigma^2 \frac{p^2}{1-p} \left(\frac{\beta}{p} - \big((1-p)u_\lambda(z)\big)^{-\frac{1}{p}} \right), \quad z \geq 1.$$

The relative size of the certainty equivalent excess return is therefore

$$\frac{\tilde{\alpha}(z)}{\alpha} = \frac{\sqrt{2}\sigma p}{\alpha} \left(\frac{\frac{\beta}{p} - \big((1-p)u_\lambda(z)\big)^{-\frac{1}{p}}}{1-p} \right)^{\frac{1}{2}}, \quad z \geq 1.$$

Certainty equivalent initial wealth

We consider two investors having the same risk-aversion, who invest in the same fund

- one has initial wealth x, initial high-watermark $n = xz \geq x$ and pays fees $\lambda > 0$
- the other has initial wealth $\tilde{x}$ but pays no fees

Equate the expected utilities:

$$\tilde{x}(z)^{1-p} u_0(\cdot) = v_0(\tilde{x}(z), \cdot) = v_\lambda(x, n) = x^{1-p} u_\lambda(z)$$

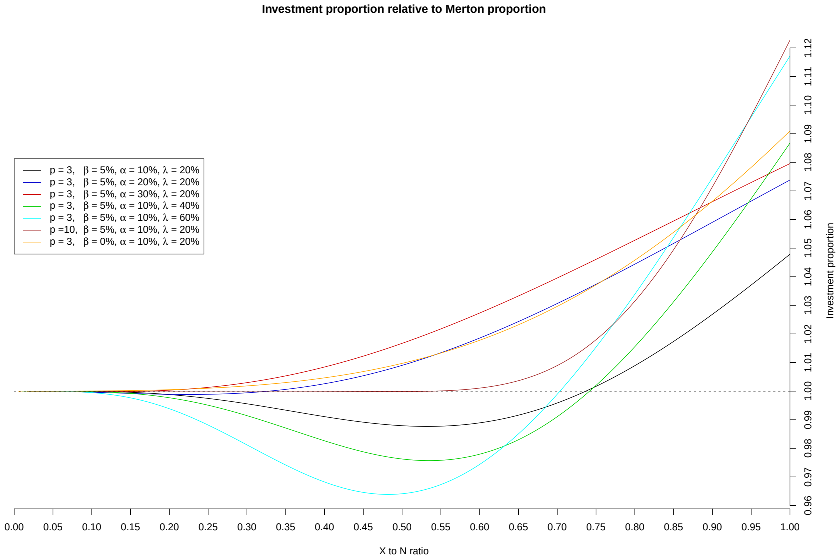
all other parameters being equal. Can be solved as

$$\tilde{x}(z) = x \cdot \left(\frac{u_\lambda(z)}{u_0} \right)^{\frac{1}{1-p}} = x \cdot \left((1-p) c_0^p \, u_\lambda(z) \right)^{\frac{1}{1-p}}, \quad z \geq 1.$$

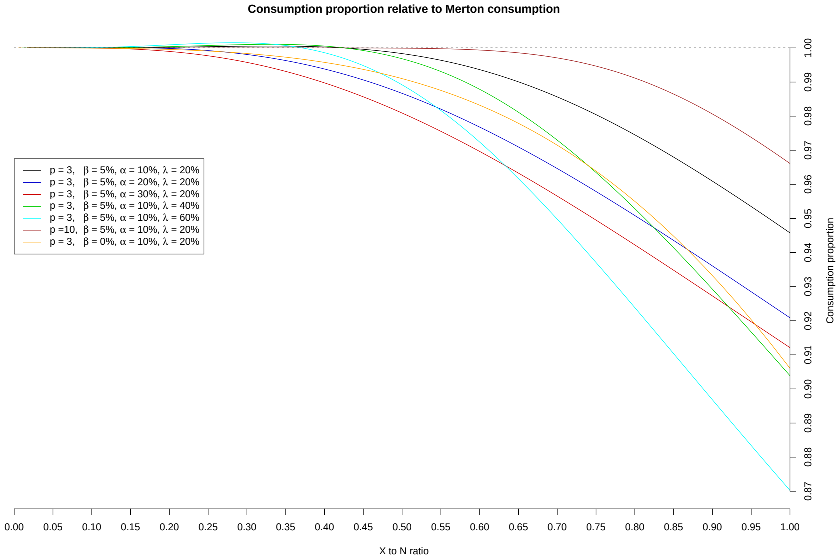
The quantity

$$\frac{\tilde{x}(z)}{x} = \left(\frac{u_\lambda(z)}{u_0} \right)^{\frac{1}{1-p}} = \left((1-p) c_0^p \, u_\lambda(z) \right)^{\frac{1}{1-p}}, \quad z \geq 1,$$

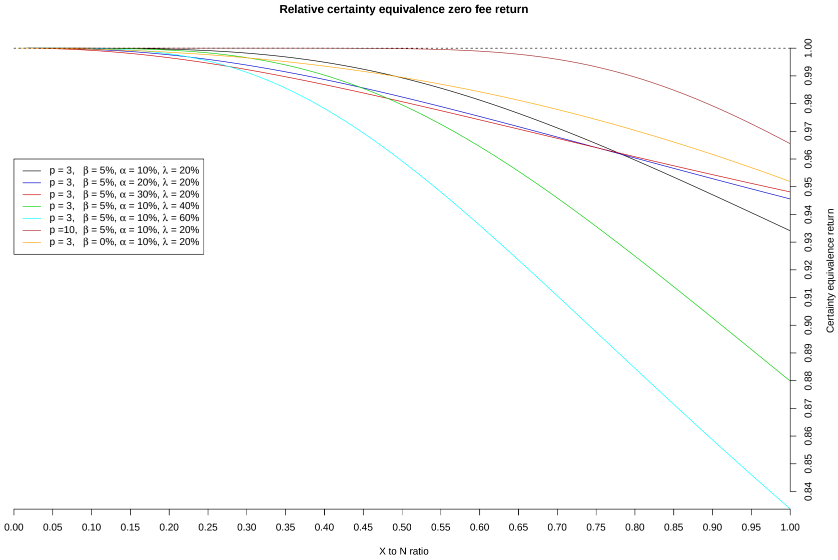
is the relative certainty equivalent wealth.



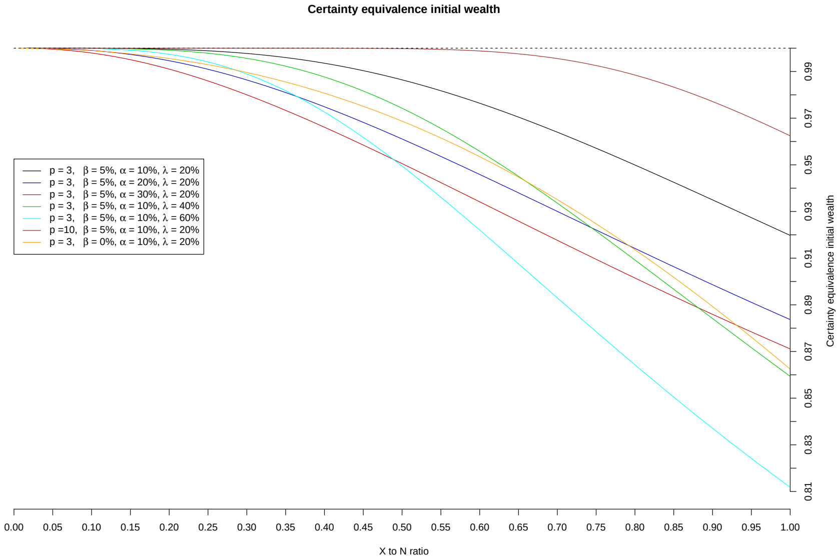
Investment proportion relative to Merton proportion
p = 3, β = 5%, α = 10%, λ = 20%
p = 3, β = 5%, α = 20%, λ = 20%
p = 3, β = 5%, α = 30%, λ = 20%
p = 3, β = 5%, α = 10%, λ = 40%
p = 3, β = 5%, α = 10%, λ = 60%
p =10, β = 5%, α = 10%, λ = 20%
p = 3, β = 0%, α = 10%, λ = 20%
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95 1.00
X to N ratio
0.96 0.97 0.98 0.99 1.00 1.01 1.02 1.03 1.04 1.05 1.06 1.07 1.08 1.09 1.10 1.11 1.12
Investment proportion



Consumption proportion relative to Merton consumption
p = 3, β = 5%, α = 10%, λ = 20%
p = 3, β = 5%, α = 20%, λ = 20%
p = 3, β = 5%, α = 30%, λ = 20%
p = 3, β = 5%, α = 10%, λ = 40%
p = 3, β = 5%, α = 10%, λ = 60%
p =10, β = 5%, α = 10%, λ = 20%
p = 3, β = 0%, α = 10%, λ = 20%
Consumption proportion
X to N ratio
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95 1.00
0.87 0.88 0.89 0.90 0.91 0.92 0.93 0.94 0.95 0.96 0.97 0.98 0.99 1.00



Relative certainty equivalence zero fee return
p = 3, β = 5%, α = 10%, λ = 20%
p = 3, β = 5%, α = 20%, λ = 20%
p = 3, β = 5%, α = 30%, λ = 20%
p = 3, β = 5%, α = 10%, λ = 40%
p = 3, β = 5%, α = 10%, λ = 60%
p =10, β = 5%, α = 10%, λ = 20%
p = 3, β = 0%, α = 10%, λ = 20%
0.84 0.85 0.86 0.87 0.88 0.89 0.90 0.91 0.92 0.93 0.94 0.95 0.96 0.97 0.98 0.99 1.00
Certainty equivalence return
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95 1.00
X to N ratio


Certainty equivalence initial wealth
p = 3, β = 5%, α = 10%, λ = 20%
p = 3, β = 5%, α = 20%, λ = 20%
p = 3, β = 5%, α = 30%, λ = 20%
p = 3, β = 5%, α = 10%, λ = 40%
p = 3, β = 5%, α = 10%, λ = 60%
p =10, β = 5%, α = 10%, λ = 20%
p = 3, β = 0%, α = 10%, λ = 20%
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95 1.00
X to N ratio
0.81 0.83 0.85 0.87 0.89 0.91 0.93 0.95 0.97 0.99
Certainty equivalence initial wealth

Conclusions

Point of view of Finance:

- model optimal investment with high-watermark fees from the point of view of the investor
- analyze the impact of the fees

Point of Mathematics:

- an example of controlling a two-dimensional reflected diffusion
- solve the problem using direct dynamic programming: first find a smooth solution of the HJB and then do verification

"Meta Conclusion":

- whenever one can prove enough regularity for the viscosity solution to do verification, the viscosity solution can/should be constructed analytically, using Perron's method, and avoiding DPP altogether

Work in progress and future work

with Gerard Brunick and Karel Janeček

- presence of (multiple and correlated) traded stocks, interest rates and hurdles: can still be modeled as a two-dimensional diffusion problem using X and $Y = P - P^*$ as state processes (reduced to one-dimension by scaling)
- analytic approximations when λ is small
- more than one fund: genuinely multi-dimensional problem with reflection
- stochastic volatility, jumps, etc

Where does it all go?

Investor

- can either invest in a number of assets $(S_1, \ldots, S_n)$ with transaction costs
- invest in the hedge-fund F paying profit fees.

The hedge-fund

- can invest in the assets with lower (even zero for mathematical reasons) transaction costs, and produce the fund process F.

For certain choices of F (time-dependent combinations of the stocks and money market), one can compare the utility of the investor in the two situations: this should the existence of hedge-funds (from the point of view of the investor).

Actually, the whole situation should be modeled as a game between the investor and the hedge fund.

Stochastic Perron's method
(joint work with Erhan Bayraktar)

Linear case

Want to compute $v(s,x) = \mathbb{E}[g(X_T^{s,x})]$, for

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_s = x. \end{cases}$$

Assumption: continuous coefficients with linear growth
There exist (possibly non-unique) weak solutions of the SDE.

$$\Big((X_t^{s,x})_{s\le t\le T}, (W_t^{s,x})_{s\le t\le T}, \Omega^{s,x}, \mathscr{F}^{s,x}, \mathbb{P}^{s,x}, (\mathscr{F}_t^{s,x})_{s\le t\le T}\Big),$$

where the $W^{s,x}$ is a d-dimensional Brownian motion on the stochastic basis

$$(\Omega^{s,x}, \mathscr{F}^{s,x}, \mathbb{P}^{s,x}, (\mathscr{F}_t^{s,x})_{s\le t\le T})$$

and the filtration $(\mathscr{F}_t^{s,x})_{s\le t\le T}$ satisfies the usual conditions. We denote by $\mathscr{X}^{s,x}$ the non-empty set of such weak solutions.

Which selection of weak solutions to consider?

Just take sup/inf over all solutions.

$$v_*(s,x) := \inf_{X^{s,x} \in \mathscr{X}^{s,x}} \mathbb{E}^{s,x}[g(X_T^{s,x})]$$

and

$$v^*(s,x) := \sup_{X^{s,x} \in \mathscr{X}^{s,x}} \mathbb{E}^{s,x}[g(X_T^{s,x})].$$

The (linear) PDE associated

$$\begin{cases} -v_t - L_t v = 0 \\ v(T,x) = g(x), \end{cases} \tag{1}$$

Assumption: g is bounded (and measurable).

Stochastic sub and super-solutions

Definition

A stochastic sub-solution of (1) $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. lower semicontinuous (LSC) and bounded on $[0, T] \times \mathbb{R}^d$. In addition $u(T, x) \leq g(x)$ for all $x \in \mathbb{R}^d$.
2. for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and each weak solution $X^{s,x} \in \mathscr{X}^{s,x}$, the process $(u(t, X_t^{s,x}))_{s \leq t \leq T}$ is a submartingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$.

Denote by $\mathscr{U}^-$ the set of all stochastic sub-solutions.

Semi-solutions cont'd

Symmetric definition for stochastic super-solutions $\mathscr{U}^+$.

Definition

A stochastic super-solution $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. upper semicontinuous (USC) and bounded on $[0, T] \times \mathbb{R}^d$. In addition $u(T, x) \geq g(x)$ for all $x \in \mathbb{R}^d$.
2. for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and each weak solution $X^{s,x} \in \mathscr{X}^{s,x}$, the process $(u(t, X_t^{s,x}))_{s \leq t \leq T}$ is a supermartingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$.

About the semi-solutions

- if one choses a Markov selection of weak solutions of the SDE (and the canonical filtration), super an sub solutions are the time-space super/sub-harmonic functions with respect to the Markov process X
- we use the name associated to Stroock–Varadhan. In Markov framework, sub+ super-solution is a stochastic solution in the definition of Stroock-Varadhan.

The definition of semi-solutions are strong enough to provide comparison to the expectation(s).
For each $u \in \mathscr{U}^-$ and each $w \in \mathscr{U}^+$ we have

$$u \leq v_* \leq v^* \leq w.$$

Define

$$v^- := \sup_{u \in \mathscr{U}^-} u \leq v_* \leq v^* \leq v^+ := \inf_{w \in \mathscr{U}^+} w.$$

We have (need to be careful about point-wise inf)

$$v^- \in \mathscr{U}^-, \quad v^+ \in \mathscr{U}^+.$$

Linear Stochastic Perron

Theorem

(Stochastic Perron's Method) *If g is bounded and LSC then v^- is a bounded and LSC viscosity supersolution of*

$$\begin{cases} -v_t - L_t v \geq 0, \\ v(T, x) \geq g(x). \end{cases} \tag{2}$$

If g is bounded and USC then v^+ is a bounded and USC viscosity subsolution of

$$\begin{cases} -v_t - L_t v \leq 0, \\ v(T, x) \leq g(x). \end{cases} \tag{3}$$

Comment: new method to construct viscosity solutions (recall v^- and v^+ are anyway stochastic sub and super-solutions).

Verification by viscosity comparison

Definition

Condition $CP(T, g)$ is satisfied if, whenever we have a bounded (USC) viscosity sub-solution u and a bounded LSC viscosity super-solution w we have $u \leq w$.

Theorem

Let g be bounded and continous. Assume $CP(T, g)$. Then there exists a unique bounded and continuous viscosity solution v to (1), and

$$v_* = v = v^*.$$

In addition, for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and each weak solution $X^{s,x} \in \mathscr{X}^{s,x}$, the process $(v(t, X^{s,x}))_{s \leq t \leq T}$ is a martingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$.

Comments:

- v is a stochastic solution (in the Markov case)
- if comparison holds for all T and g, then the diffusion is actually Markov (but we never use that explicitly)

Idea of proof

Similar to Ishii.
To show that v^- is a super-solution

- touch v^- from below with a smooth test function φ
- if the viscosity super-solution property is violated, then φ is locally a smooth sub-solution
- push it to $\varphi_\varepsilon = \varphi + \varepsilon$ slightly above, to still keep it still a smooth sub-solution (locally)
- Itô implies that φ_ε is also (locally wrt stopping times) a submartingale along X
- take $\max\{v^-, \varphi_\varepsilon\}$, still a stochastic-subsolution (need to "patch" sub-martingales along a sequence of stopping times)

Comments: why don't we need Markov property? Because we only use Itô, which does not require the diffusion to be Markov.

Obstacle problems and Dynkin games

First example of non-linear problem.

Same diffusion framework as for the linear case. Choose a selection of weak solutions $X^{s,x}$ to save on notation.

$g : \mathbb{R}^d \to \mathbb{R}$, $l, u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ *bounded* and measurable, $l \leq u$, $l(T, \cdot) \leq g \leq u(T, \cdot)$.

Denote by $\mathscr{T}^{s,x}$ the set of stopping times τ (with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$) which satisfy $s \leq \tau \leq T$.

The first player (ρ) *pays* to the second player (τ) the amount

$$J(s, x, \tau, \rho) :=$$

$$= \mathbb{E}^{s,x} \left[\mathbb{I}_{\{\tau < \rho\}} l(\tau, X_\tau^{s,x}) + \mathbb{I}_{\{\rho \leq \tau, \rho < T\}}) u(\rho, X_\rho^{s,x}) + \mathbb{I}_{\{\tau = \rho = T\}} g(X_T^{s,x}) \right].$$

Dynkin games, cont'd

Lower value of the Dynkin game

$$v_*(s,x) := \sup_{\tau \in \mathscr{T}^{s,x}} \inf_{\rho \in \mathscr{T}^{s,x}} J(s,x,\tau,\rho)$$

and the *upper value of the game*

$$v^*(s,x) := \inf_{\rho \in \mathscr{T}^{s,x}} \sup_{\tau \in \mathscr{T}^{s,x}} J(s,x,\tau,\rho).$$

$$v_* \leq v^*$$

Remark: we could appeal directly to what is known about Dynkin games to conclude $v_* \leq v^*$, but this is exactly what we wish to avoid.

DPE equation for Dynkin games

$$\begin{cases} F(t, x, v, v_t, v_x, v_{xx}) = 0, & \text{on } [0, T) \times \mathbb{R}^d, \\ u(T, \cdot) = g, \end{cases} \tag{4}$$

where

$$\begin{aligned} F(t, x, v, v_t, v_x, v_{xx}) &:= \\ \max\{v - u, \min\{-v_t - L_t v, v - l\}\} & \\ = \min\{v - l, \max\{-v_t - L_t v, v - u\}\}. & \end{aligned} \tag{5}$$

Super and Subsolutions

Definition

$\mathscr{U}^+$, is the set of functions $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. are continuous (C) and bounded on $[0, T] \times \mathbb{R}^d$. $w \geq l$ and $w(T, \cdot) \geq g$.
2. for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and any stopping time $\tau_1 \in \mathscr{T}^{s,x}$, the function w along the solution of the SDE is a super-martingale in between τ_1 and the first (after τ_1) hitting time of the upper stopping region $\mathscr{S}^+(w) := \{w \geq u\}$. More precisely, for any $\tau_1 \leq \tau_2 \in \mathscr{T}^{s,x}$, we have

$$w(\tau_1, X^{s,x}_{\tau_1}) \geq \mathbb{E}^{s,x}\left[w(\tau_2 \wedge \rho^+, X^{s,x}_{\tau_2 \wedge \rho^+})|\mathscr{F}^{s,x}_{\tau_1}\right] \; - \mathbb{P}^{s,x} \; a.s.$$

   where the stopping time ρ^+ is defined as

$$\rho^+(v, s, x, \tau_1) = \inf\{t \in [\tau_1, T] : X^{s,x}_t \in \mathscr{S}^+(w)\}.$$

Question: why the starting stopping time? No Markov property.

Stochastic Perron for obstacle problems

Define symmetrically sub-solutions $\mathscr{U}^-$. Now define, again

$$v^- := \sup_{w \in \mathscr{U}^-} w \leq v_* \leq v^* \leq v^+ := \inf_{w \in \mathscr{U}^+} w.$$

Cannot show $v^- \in \mathscr{U}^-$ or $v^+ \in \mathscr{U}^+$, but it is not really needed. All is needed is stability with respect to max/min, not sup/inf (and this is the reason why we can assume continuity).

Theorem

- *v^- is viscosity super-solution of the (DPE)*
- *v^+ is viscosity sub-solution of the (DPE)*

Verification by comparison for obstacle problems

Theorem

- *if comparison holds, then there exists a unique and continuous viscosity solution v, equal to $v^- = v_* = v^* = v^+$*
- *the first hitting times are optimal for both players*

In the Markov case, Peskir showed (with different definitions for sub, super-solutions, which actually involve the value function) that

$$v^- = v^+$$

by showing that $v^- =$ "*value function*" $= v^+$. Peskir generalizes the characterization of value function in optimal stopping problems.

What about optimal stopping $u = \infty$?

Classic work of El Karoui, Shiryaev: in the Markov case, the value function is the least excessive function. In our notation

$$v^+ := \inf_{w \in \mathscr{U}^+} w = v.$$

Comment: the proof requires to actually show that $v \in \mathscr{U}^+$. We avoid that, showing that

$$v^- \leq v \leq v^+,$$

and then using comparison.
We provide a short cut to conclude the value function is the continuous viscosity solution of the free-boundary problem (study of continuity in Bassan and Ceci)

Back to the original control problem

work in progress (blackboard details)

- can define the classes of stochastic super and sub-solutions such that
- the Stochastic Perron's method (existence part) works well (at least away from T)

Left to do:

- study the possible boundary layer at T
- go over verification by comparison (easy once the first step is done)

Conclusions

- new method to construct viscosity solutions as sup/inf of stochastic sub/super-solutions
- compare directly with the value function
- if we have viscosity comparison, then the value fct is the unique continuous solution of the (DPE) and the (DPP) holds

Conjecture

Any PDE that is associated to a stochastic optimization problem can be approached by Stochastic Perron's Method.

Even (zero sum) games should work, with Isaasc condition (basically any problem with a single value function)

The general approach

- can choose stochastic semi-solutions continuous (learned from the Dynkin games)
- choose definition of semi-solutions which is
  - weak enough to have stability (for $v \wedge w$ and $v \vee w$)
  - strong enough to be able to follow the proof of Ishii pasting martingales

E. Bayraktar and M. Sîrbu, *Stochastic perron's method and verification without smoothness using viscosity comparison: obstacle problems and dynkin games*, Proceedings of the AMS (2012), accepted, available at http://arxiv.org/abs/1112.4904.

———, *Stochastic perron's method and verification without smoothness using viscosity comparison: the linear case*, Proceedings of the AMS **140** (2012), 3645–3554.

K. Janeček and M. Sîrbu, *Optimal investment with high-watermark performance fee*, SIAM Journal on Control and Optimization **50** (2012), no. 2, 790–819.

R. C. Merton, *Lifetime portfolio selection under uncertainty: the continuous-time case*, Rev. Econom. Statist. (1969), 247–257.

———, *Optimum consumption and portfolio rules in a continuous-time model*, J. Economic Theory (1971), 373–413.

H. Pham, *Continuous-time stochastic control and optimization with financial applications*, Springer New York, 2009.