Trading costs in discrete time – Lecture 4

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Outline

1. Model
   - Problem
   - Previous Results
   - Main results
2. Proofs
   - Duality
   - Convergence
3. Approximation
   - Simple Example

Overview

These results are joint Yan Dolinksy (2011 preprint) and with Selim Gökay (Math Fin., 2011), and with Erdinç Akyildirim (preprint).

These results are related to continuous time limits of discrete models with friction.

The main question is to understand the effect of discretization and to derive the dual in a direct manner.

Outline

Processes

We use the following notation for $n = 0, \ldots, N$,

- $S_n \in \mathbb{R}^+$ is the spot price at time n in a Binomial model.
- $Z_n \in \mathbb{R}$ is the number of shares. This is the predictable control process.
- $Y_n \in \mathbb{R}$ is the "wealth" like process. It can be the mark-to-market value, or after-liquidation value or possibly another choice. It is the main state process.
- $\mathcal{F}_n$ is the filtration generated by the stock price.

All these process depend on the time step, $1/N$. And we will be interested in letting $N \to \infty$.

Wealth dynamics

We postulate that Y follows

$$Y_{n+1} = Y_n + Z_{n+1}\left[S_{n+1} - S_n\right] - g\left(\frac{n}{N}, S_n, Z_{n+1} - Z_n\right),$$

where g is an adapted and

$$g : \Omega \times [0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+,$$

is the cost-of-trading function. We assume that it is adapted, convex with $g(x) \geq g(0) = 0$. Examples are

- Proportional transaction costs : $g(s,x) = \lambda s|x|$.
- Illiquidity (or price impact) : $g(t,s,x) = \lambda(t,s)x^2$.

Super-hedging

For a given adapted portfolio process Z and an initial wealth y, let $Y^{y,Z}$ be the corresponding wealth process, i.e., it solves

$$Y_{n+1} = Y_n + Z_{n+1}\left[S_{n+1} - S_n\right] - g\left(\frac{n}{N}, S_n, Z_{n+1} - Z_n\right), \; n = 0, \ldots, N-1,$$

with initial data $Y_0 = y$. For an $\mathcal{F}_N$-measurable given liability ξ, the super-replication cost is given by

$$V^N(\xi) := \inf\left\{ y \;:\; \exists\, Z \text{ such that } Y_N^{y,Z} \geq \xi \;a.s.\right\}.$$

Questions

1. **Duality.** The dual of the problem for fixed N. One expects to compute this dual directly as the problem is in fact a standard convex program.

2. **Convergence.** We take

$$S_{n+1} = S_n \left[1 + \frac{\sigma_0}{\sqrt{N}} \varepsilon_{n+1} \right],$$

where $\varepsilon \in \{\pm 1\}$ and $\sigma_0 > 0$ is the given volatility.

We also assume that (in fact without loss of generality)

$$\text{“}\xi = \varphi(\{S_n\}_{n=0,\dots,N})\text{”}.$$

Then we want to compute the limit as N tends to infinity.

Previous results : Kusuoka

In 1995 Kusuoka studied the proportional transaction costs. It is well known that, for fix cost the limit would be trivial. So he takes

$$g(s,x) = \frac{c\ s}{\sqrt{N}}\ |x| \quad \text{and proved that}$$

$$V^{\infty}(\xi) = \sup_{\sigma^2 \in I(c)} \mathbb{E}\left[\varphi(\{S_t^{\sigma}\}_{t\in[0,1]})\right] = \sup \mathbb{E}^{\mathbb{P}^{\sigma}}\left[\varphi(\{S_t\}_{t\in[0,1]})\right],$$

where $I(c) = [\sigma_0(\sigma_0 - 2c), \sigma_0(\sigma_0 + 2c)]$, $dS_t^{\sigma} = S_t^{\sigma}\sigma_t dW_t$.

This is related to 2BSDE's (Cheredito, S. Touzi, Victoir, general 2BSDE's S. Touzi, Zhang, capacity Denis, Martini, G-expectation Peng, random G-expectation Nutz and uniform Doob-Meyer decomposition of Nutz, Soner.

This connection was further developed by Dolinsky, Nutz, S.

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Previous results : Illiquidity

The case $g(x) = \lambda x^2$ with $\xi = \Psi(S_N)$ (i.e. the Markov case) was studied by Gökay & Soner by PDE methods. In the case when Ψ is convex, the limiting function $V(t,s)$ is the solution of

$$-V_t - \frac{\sigma_0^2 s^2}{2} V_{ss}\left[1 + 2\lambda V_{ss}\right] = 0,$$

with boundary condition $V(1,s) = \Psi(s)$.

With $\lambda = 0$, this is exactly the classical Black & Scholes equation.

And for $\lambda > 0$, there is a premium over the classical price.

In fact, we have $Z_n \approx V_s(n/N, S_n)$, $S_{n+1} = S_n(1 \pm \sigma_0/\sqrt{N})$ and

$$g\left(Z_{n+1} - Z_n\right) \approx \lambda \left(\frac{1}{N} V_{st} \pm \frac{\sigma_0 S_n}{\sqrt{N}} V_{ss}\right)^2 \approx \lambda \left(V_{ss}\right)^2 \frac{\sigma_0^2 S_n^2}{N}.$$

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Main results : Dual

This is proved in Dolinsky & Soner (2011).

Let $\mathcal{Q}^N$ be the set of all probability measures on the discrete probability space. Then

$$V^N(\xi) = \sup_{\mathbb{P}\in\mathcal{Q}^N} \mathbb{E}^{\mathbb{P}} \left[\xi - \sum_{k=0}^{n-1} G\left(\mathbb{E}^{\mathbb{P}}\left[S_N \mid \mathcal{F}_k\right] - S_k \right) \right],$$

where $\mathbb{E}^{\mathbb{P}}$ denotes the expectation with respect to the probability measure $\mathbb{P}$ and G is the convex conjugate of g, i.e.,

$$G(y) = \sup_{x\in\mathbb{R}} \left(xy - g(x)\right).$$

Truncation

Motivated by the Kusuoka's result, we truncate the cost function by

$$G^c(t,s,y) := \begin{cases} G(t,s,y) & \text{if } \ |y| \le cs/\sqrt{N}, \\ +\infty & \text{else.} \end{cases} ,$$

where $c > 0$ is arbitrary and s is the stock price.
Let $V^{N,c}$ be the minimal super-replication cost with a trading cost function g^c which is the convex dual of G^c.

Convergence result stated

$$\lim_{N\to\infty} V^{N,c} = \sup_{\sigma\in I(c)} J(S^\sigma),$$

$$J(S^\sigma) := \mathbb{E}\left[\varphi(S^\sigma) - \int_0^1 \widehat{G}\left(\frac{\sigma_t^2-\sigma_0^2}{2\sigma_0}\, S_t^\sigma\right) dt\right],$$

where as in Kusuoka's case

$$dS_t^\sigma = \sigma_t S_t^\sigma dW_t,$$

W is a standard Brownian motion and

$$I(c) = [\sigma_0(\sigma_0 - 2c), \sigma_0(\sigma_0 + 2c)]$$

I will define $V^{N,c}$, $\widehat{G}$ in the next slide.

$\hat{G}$

For $\gamma \geq 1$, let

$$g_\gamma(\nu) = \frac{1}{\gamma}\ |\nu|^\gamma.$$

Then, for $\gamma > 1$

$$G_\gamma(y) = \frac{1}{\gamma^*}\ |y|^{\gamma^*}, \quad \gamma^* = \frac{\gamma}{\gamma - 1}.$$

For $\gamma = 1$, $G_1(y) = 0$ for $|y| \leq 1$ and is equal to infinity otherwise.
We always $\widehat{G}_\gamma(0) = 0$ and for $y \neq 0$,

$$\widehat{G}_\gamma(y) := \lim_{n \to \infty}\ nG_\gamma\left(\frac{y}{\sqrt{n}}\right) = \begin{cases} G_2(t, y), & \text{if } \gamma = 2, \\ 0, & \text{if } \gamma \in [1, 2), \\ +\infty, & \text{if } \gamma > 2. \end{cases}$$

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Duality theorem, restated

Recall that $V^N(\xi)$ is the minimal super-replication cost of the liability ξ. Then

$$\begin{aligned} V^N(\xi) &= \inf \left\{ y \ : \ \exists \ Z \text{ such that } Y_N^{y,Z} \geq \xi \ a.s. \right\} \\ &= \sup_{\mathbb{P} \in \mathcal{Q}^N} \mathbb{E}^{\mathbb{P}} \left[\xi - \sum_{k=0}^{n-1} G\left(\mathbb{E}^{\mathbb{P}} \left[S_N \mid \mathcal{F}_k \right] - S_k \right) \right], \end{aligned}$$

where $\mathcal{Q}^N$ is the set of all probability measures on the discrete probability space, $\mathbb{E}^{\mathbb{P}}$ denotes the expectation with respect to the probability measure $\mathbb{P}$ and G is the convex conjugate of g.

One inequality

We sum the wealth equation over k to arrive at

$$\begin{aligned} Y_N &= y + \sum_{k=0}^{N-1} \left(Z_{k+1}[S_{k+1} - S_k] - g\left(Z_{k+1} - Z_k\right)\right) \\ &= y + \sum_{k=0}^{N-1} \left(\left[Z_{k+1} - Z_k\right] \left[S_N - S_k\right] - g\left(Z_{k+1} - Z_k\right)\right). \end{aligned}$$

Let $\mathbb{P}$ be a probability measure in $\mathcal{Q}_n$. We take the conditional expectations and use the definition of the dual function G to obtain,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[Y_N] &= y + \mathbb{E}^{\mathbb{P}} \left(\sum_{k=0}^{N-1} \left[Z_{k+1} - Z_k\right] \left[S_N - S_k\right] - g\left(Z_{k+1} - Z_k\right) \right) \\ &\leq y + \mathbb{E}^{\mathbb{P}} \left(\sum_{k=0}^{N-1} G\left(\mathbb{E}^{\mathbb{P}}(S_N | \mathcal{F}_k) - S_k \right) \right). \end{aligned}$$

One inequality, contd

If Z is a super-replicating strategy with initial wealth y, then $Y_N^{y,Z} \geq \xi$ and

$$y \quad \geq \quad \mathbb{E}^{\mathbb{P}}\left(\xi - \sum_{k=0}^{N-1} G\left(\mathbb{E}^{\mathbb{P}}(S_N|\mathcal{F}_k) - S_k\right)\right).$$

Since $\mathbb{P} \in \mathcal{Q}_N$ is arbitrary, the above calculation proves that

$$V^N \geq \sup_{\mathbb{P}\in\mathcal{Q}^N} \mathbb{E}^{\mathbb{P}}\left(\xi - \sum_{k=0}^{N-1} G\left(\mathbb{E}^{\mathbb{P}}(S_N|\mathcal{F}_k) - S_k\right)\right).$$

The opposite inequality is proved using the "standard duality".

Outline of the argument

By definition, $G^{N,c}(y)$ is infinity for $|y|$ greater than a constant times $1/\sqrt{N}$. This allows us to prove tightness of any sequence of probability measure with a finite dual value. Then using the dual representation we show that

$$\limsup_{N\to\infty} V^{N,c} \leq V^c.$$

For the opposite inequality, we use the construction of Kusuoka to obtain a sequence of discrete martingales approximating S^σ for an arbitrary $\sigma \in I(c)$. But, we not only need to approximate the process but also its quadratic variation.

lim sup part

By the dual representation, there is a sequence $\mathbb{P}^N \in \mathcal{Q}^N$ such that

$$\limsup_{N\to\infty} V^{N,c} = \limsup_{N\to\infty} \mathbb{E}^{\mathbb{P}^N}\left[\xi - \sum_{k=0}^{N-1} G^c\left(\frac{\xi_k}{\sqrt{N}}\,\alpha_k^N\, S_k\right)\right],$$

where $\xi_k \in \{-1,+1\}$'s are the i.i.d. sequence as before and

$$\alpha_k^N := \sqrt{N}\xi_k\left(\frac{M_k^N - S_k}{S_k}\right) \qquad M_k^N := \mathbb{E}^{\mathbb{P}^N}(S_N|\mathcal{F}_k).$$

Notice that the above definition is equivalent to

$$M_k^N = \mathbb{E}^{\mathbb{P}^N}(S_N|\mathcal{F}_k) = S_k\left[1 + \frac{\alpha_k^N}{\sqrt{N}}\xi_k\right].$$

Also by the truncation of G to G^c,

$$\left|\alpha_k^N\right| \le c, \quad \forall\ k, a.s.$$

Weak convergence

The uniform boundedness of α^N's allow us to construct a subsequence, a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ and a martingale M so that

$$\left(\frac{1}{S}, M^N\right) \to \left(\frac{1}{M}, M\right), \quad \tilde{\mathbb{P}} - a.s.$$

Then, we define

$$Y_n^N := \sum_{k=1}^{n} \frac{M_k^N - M_{k-1}^N}{S_{k-1}},$$

so that it also converges to $\int dM/M$. Hence $dM = MdY$.

We show that α^N converges to

$$\frac{(\langle Y \rangle)'(t) - \sigma_0^2}{2\sigma_0}.$$

lim sup completed

By the choice of $\mathbb{P}^N$ and the scaling assumption,

$$\limsup_{N\to\infty} V^{N,c} = \limsup_{N\to\infty} \mathbb{E}^{\mathbb{P}^N}\left[\xi - \sum_{k=0}^{N-1} G^c\left(\frac{\xi_k}{\sqrt{N}}\ \alpha_k^N\ S_k\right)\right],$$

$$= \limsup_{N\to\infty} \mathbb{E}^{\mathbb{P}^N}\left[\xi - \int_0^1 \widehat{G}(t, S_t, \alpha_t^N S_t)dt\right].$$

We also have

$$S \to M, a.s. \quad \alpha^N \to \frac{(\langle Y\rangle)'(t) - \sigma_0^2}{2\sigma_0} a.s. \quad \text{and} \quad dM = MdY.$$

Hence (with $\xi = \varphi(S.)$)

$$\limsup_{N\to\infty} V^{N,c} \leq J(M) = \mathbb{E}\left[\varphi(M) - \int_0^1 \widehat{G}\left(t, M_t, \frac{(\langle \ln M\rangle)' - \sigma_0^2}{2\sigma_0} M_t\right) dt\right]$$

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Geometric Binomial Tree

Consider a sequence of processes $\{S^{(N)}\}_{N=1,2,\ldots}$,

$$S_{n+1}^{(N)} = S_n^{(N)} \left[1 + \sigma_* \sqrt{h}\, \xi_{n+1}\right], \qquad n = 0, 1, \ldots,$$

$$h := 1/N,$$

where ξ_n takes values in $\{\pm 1\}$ and $\sigma_* > 0$ is the given volatility. We call this process as the "standard" geometrical Binomial process.

Let $\mathcal{Q}^{(N)}$ be the set of all probability measures on this discrete probability space.

Approximation question

For a given "nice" process σ_t, let S^σ be the continuous process,

$$dS_t^\sigma = \sigma_t S_t^\sigma dW_t,$$

where W is a standard Brownian motion. Then, the general question is

QUESTION : *Can you find a sequence of probability measures $\mathbb{P}^{(N)} \in \mathcal{Q}^{(N)}$ such that the standard geometric Binomial process under these measures weakly converges to the above process ?*

Kusuoka's Result

Kusuoka asked this question in a paper related to the formal calculations of Leland for option pricing with transaction costs.

For a general process σ that is path-dependent but smooth, he constructed a sequence of probability measures $\mathbb{P}^{(N)}$ so that under these measures $S^{(N)}$ converges weakly to S^{σ}.

This construction is, however, not algorithmic. Next we want to take a structured volatility and construct the approximation.

One dimensional diffusion

Consider the simple constant coefficient case

$$dX_t = \sigma dW_t,$$

where $\sigma > 0$ is constant and W is a standard Brownian motion. We want to approximate this by the random walk

$$X_n^{(N)} := x_0 + \sigma_* \sqrt{h} \sum_{k=1}^{n} \xi_k, \qquad h := 1/N.$$

where ξ_k's take values in $\{-1, +1\}$.

The point here is that the volatility of the tree is σ_* and it is different than the volatility σ of the continuous process.

Approach

We allow ourselves to choose any probability measure on the tree and want to get weak convergence of $X^{(N)}$ under this measure to the continuous process X.

We do this by choosing the conditional probabilities

$$\mathbb{P}^{(N)}\left(X_{k+1}^{(N)} = X_k^{(N)} + \sigma_* \sqrt{h}\, \xi_{k+1} \;\middle|\; \mathcal{F}_k \right)$$

as a function of $X_k^{(N)}$ and ξ_k. In other words, we construct the pair $(X^{(N)}, \xi)$ as a Markov chain.

First Try

To get weak convergence the standard approach is to match conditional moments.

However, here the only free parameter we have is the conditional up probability. And this allows us to match only one moment. However, for weak convergence we have to match the first two moments.

Hence we need another free parameter and we need to modify something !

Modification of $X^{(N)}$

For a constant α, to be chosen, we set

$$\hat{X}_k^{(N)} := X_k^{(N)} + \alpha\sqrt{h}\,\xi_k .$$

We will construct probabilities so that $\hat{X}^{(N)}$ weakly converges to X. This would also imply the convergence of $X^{(N)}$ to X as well.

Let $\mathbb{P} = \mathbb{P}^{(N)}$ be the measure we want to construct and let $\mathbb{E}[\,\cdot\,] := \mathbb{E}^{\mathbb{P}}[\,\cdot\,]$. And let $\mathbb{E}_k$ and $\mathbb{P}_k$ be conditional expectation and probability.

How to choose α ?

We match the first two conditional moments of $\hat{X}^{(N)}$ and X :

$$\mathbb{E}_k\left[\hat{X}^{(N)}_{k+1} - \hat{X}^{(N)}_k\right] = 0, \quad \mathbb{E}_k\left[\left(\hat{X}^{(N)}_{k+1} - \hat{X}^{(N)}_k\right)^2\right] = \sigma^2 h.$$

Since

$$\frac{1}{\sqrt{h}}\left(\hat{X}^{(N)}_{k+1} - \hat{X}^{(N)}_k\right) = (\sigma_* + \alpha)\xi_{k+1} - \alpha\xi_k,$$

the above equations are equivalent to

$$(\sigma_* + \alpha)\mathbb{E}_k[\xi_{k+1}] = \alpha\xi_k,$$

$$\sigma^2 = (\sigma_* + \alpha)^2(\xi_{k+1})^2 - (\sigma_* + \alpha)\mathbb{E}_k[\xi_{k+1}]\,\alpha\xi_k + \alpha^2(\xi_k)^2.$$

Hence,

$$\alpha = \frac{\sigma^2 - {\sigma_*}^2}{2}.$$

Transition probability

Set

$$p_k := \mathbb{P}_k(\xi_{k+1} = 1).$$

Then,

$$\mathbb{E}_k[\xi_{k+1}] = 2p_k - 1 = \frac{\alpha}{1+\alpha}\,\xi_k = \frac{\sigma^2 - \sigma_*^2}{\sigma^2 + \sigma_*^2}\,\xi_k.$$

Hence,

$$p_k = \frac{1}{2} + \frac{\sigma^2 - \sigma_*^2}{2(\sigma^2 + \sigma_*^2)}\,\xi_k.$$

The volatility is increased by momentum, i.e., if $\xi_k = 1$ then the probability ξ_{k+1} is higher then $1/2$. Opposite is needed to decrease the volatility.

Summary of results

- We considered the super-hedging cost of a discrete financial model. This models lends itself to analysis more easily. In particular, the duality is proved directly.
- In the Markov case, one can use dynamic programming to compute. This is done by Gökay & Soner.
- This discretization (and therefore the computational method) has the advantage of a true financial market.
- Dual representation and Kusuoka's powerful approximation technique can be used to compute the continuous time limit.

Concluding

Liquidity in a Binomial market,

Selim Gökay, H. Mete Soner , Mathematical Finance (2012)

Duality and Convergence for Binomial Markets with Friction,

Yan Dolinsky, H. Mete Soner,

math archive, arXiv :1106.2095

Approximating stochastic volatility by recombinant trees,

Erdinç Akyildirim, Yan Dolinsky, H. Mete Soner,

math archive, arXiv :1205.3555

THANK YOU FOR YOUR ATTENTION.