Small transaction costs – Lecture 3s

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Outline

1. Brief Summar
   - Expansion
   - Solution
   - Homethetic case
2. Proof
3. A nearly optimal strategy

Problem

$$v^\epsilon(x,y) := \sup_{(c,k)\in\Theta^\epsilon(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\, U(c_t)dt\right],$$

where c_t is the rate of consumption, $k_t = k^+ - k^-$ is the total amount of transfers, and for any $(X_{0^-}, Y_{0^-}) = (x,y) \in K_\epsilon$, the state equations are,

$$\begin{aligned} dX_t &= \left(-c_t\right)dt + \left(dk_t^+ - (1+\epsilon^3)dk_t^-\right), \\ dY_t &= Y_t\,\frac{dS_t}{S_t} + \left(dk_t^- - (1+\epsilon^3)dk_t^+\right). \end{aligned}$$

Equation

The DPE with our simplifications is,

$$\min\{\ I^\epsilon\ ;\ d_+^\epsilon\ ;\ d_-^\epsilon\ \} = 0, \qquad \text{where}$$

$$\begin{aligned}
I^\epsilon &= \beta v^\epsilon - \mu y v_y^\epsilon - \frac{1}{2}\sigma^2 y^2 v_{yy}^\epsilon - \tilde{U}(v_x^\epsilon), \\
&:= \beta v^\epsilon - \mathcal{L}_y v^\epsilon - \tilde{U}(v_x^\epsilon), \\
d_+^\epsilon &= (1+\epsilon^3) v_x^\epsilon - v_y^\epsilon, \\
d_-^\epsilon &= (1+\epsilon^3) v_y^\epsilon - v_x^\epsilon.
\end{aligned}$$

Form of the Expansion

We postulate the following expansion,

$$v^\epsilon(x,y) = v(z) - \epsilon^2 u(z) - \epsilon^4 w(z,\xi) + \circ(\epsilon^2),$$

where $(z,\xi) =$ is a transformation of $(x,y) \in K_\epsilon$ given by

$$z = x + y, \qquad \xi := \xi_\epsilon(x,y) = \frac{y - \theta(z)}{\epsilon},$$

where $\theta(z)$ is the Merton optimal investment strategy.

And we have substituted this form into the equation to derive equations for the unknowns u and w.

Cell Equation

Set

$$\mathcal{A}u(z) = a(z),$$

and solve for the pair $(w(z,\cdot), a(z))$ solving the following equation with z as a parameter and ξ as the independent variable,

$$\begin{aligned} 0 \quad = \quad & \max\{ \frac{1}{2}\sigma^2\xi^2 v_{zz}(z) - \frac{1}{2}\alpha(z)^2 w_{\xi\xi}(z,\xi) + a(z) \\ & ; \ -v_z(z) + w_\xi \ ; \ -v_z(z) - w_\xi \ \}. \end{aligned}$$

together with $w(z,0) = 0$.

a is exactly the value function of the ergodic problem described earlier and w is its potential function.

Change of variables

$$\bar{w}(z,\rho) := \frac{w(z,\eta(s,z)\rho)}{\eta(z)v_z(z)}, \quad \bar{a}(z) := \frac{a(z)}{\eta(z)v_z(z)}, \quad \bar{\alpha}(z) := \frac{\alpha(z)}{\eta(z)},$$

where $\rho = \xi/\eta(z), \ \eta(z) = -v_z/v_{zz}$. Solve for $(\bar{a}(z), \bar{w}(z,\cdot)) \in \mathbb{R} \times C^2(\mathbb{R})$,

$$\max\Big\{ -\frac{|\sigma\rho|^2}{2} - \frac{1}{2}\bar{\alpha}^2(z)\bar{w}_{\rho\rho}(z,\rho) + \bar{a}(z) \ , \\ -1 + \bar{w}_\rho(z,\rho) \ ; -1 - \bar{w}_\rho(z,\rho)\Big\} \ = \ 0, \quad \forall \ \rho \in \mathbb{R},$$

together with the normalization $\bar{w}(z,0) = 0$. In the power case, the above equation is independent of z. We then use $\bar{a}(z)$ to solve for u,

$$\mathcal{A}u(s,z) = a(s,z) = v_z(s,z)\eta(s,z)\bar{a}(s,z).$$

Solving the Cell Problem

In order to compute the solution explicitely in terms of η, we postulate a solution of the form

$$\bar{w}(\rho) = \begin{cases} k_4\rho^4 + k_2\rho^2; & |\rho| \leq \rho_0, \\ \bar{w}(-\rho_0) - (\rho + \rho_0); & \rho \leq -\rho_0, \\ \bar{w}(\rho_0) + (\rho - \rho_0); & \rho \geq \rho_0. \end{cases}$$

We first determine k_4 and k_2 by imposing that the fourth order polynomial solves the second order equation in $(-\rho_0, \rho_0)$. A direct calculation yields,

$$k_4 = \frac{-\sigma^2}{12\bar{\alpha}^2} \quad \text{and} \quad k_2 = \frac{\bar{a}}{\bar{\alpha}^2}.$$

Cell Problem - cont.

We now impose the smooth pasting condition, namely assume that $\bar{w}$ is C^2 at the points $-\rho_0$ and ρ_0. Then, the continuity of the second derivatives yield,

$$\rho_0^2 = \frac{2\bar{a}}{\sigma^2} \ \text{ implying that } \ \bar{a} \geq 0 \ \text{ and } \ \rho_0 = \left(\frac{2\bar{a}}{\sigma^2}\right)^{1/2}.$$

The continuity of the first derivatives of $\bar{w}$ yield,

$$\begin{aligned} 4k_4(-\rho_0)^3 - 2k_2\rho_0 &= -1, \\ 4k_4(\rho_0)^3 + 2k_2\rho_0 &= 1. \end{aligned}$$

Cell Problem completed.

$$4k_4(\rho_0)^3 + 2k_2\rho_0 = 1,$$

and

$$k_4 = \frac{-\sigma^2}{12\bar{\alpha}^2}, \quad k_2 = \frac{\bar{a}}{\bar{\alpha}^2}, \quad \bar{a} = \frac{\sigma^2}{2}\rho_0^2.$$

Hence,

$$\rho_0 = \left(\frac{3\bar{\alpha}^2}{2\sigma^2}\right)^{1/3}.$$

All coefficients of our candidate are now uniquely determined.

Moreover, we verify that the gradient constraint

$$|\bar{w}_\rho| \leq 1, \qquad \forall\ \rho.$$

Homethetic case

$$U(c) := \frac{c^{1-\gamma}}{1-\gamma}, \qquad c > 0,$$

for some $\gamma > 0$ with $\gamma = 1$ corresponding to the logarithmic utility.

Then,

$$v(z) = \frac{1}{(1-\gamma)} \, \frac{z^{1-\gamma}}{v_M^{\gamma}},$$

with the Merton constant

$$v_M = \frac{\beta - r(1-\gamma)}{\gamma} - \frac{1}{2}\frac{(\mu - r)^2}{\gamma^2\sigma^2}(1-\gamma).$$

Hence the risk tolerance function and the optimal strategies are given by

$$\eta(z) = \frac{z}{\gamma}, \qquad \theta(z) = \frac{\mu - r}{\gamma\sigma^2}\, z := \pi_M z, \qquad \mathbf{c}(z) = v_M z.$$

Homethetic case

Since the diffusion coefficient $\alpha(z) = \sigma\theta(z)(1-\theta_z)$,

$$\bar{\alpha} = \frac{\alpha(z)}{\eta(z)} = \gamma\sigma\pi_M(1-\pi_M).$$

The constants in the solution of the corrector equation are given by,

$$\rho_0 = \left(\frac{3\bar{\alpha}^2}{2\sigma^2}\right)^{1/3},$$

$$a(z) = \eta(z)v'(z)\bar{a} = \frac{\sigma^2(1-\gamma)}{2\gamma}\rho_0^2 \; v(z).$$

Homethetic case

Since

$$\mathcal{A}v(z) = U(\mathbf{c}(z)) = \frac{1}{1-\gamma}(v_M z)^{1-\gamma} = v_M v(z),$$

the unique solution $u(z)$ of the second corrector equation

$$\mathcal{A}u(z) = a(z) = \frac{\sigma^2(1-\gamma)}{2\gamma}\rho_0^2\ v(z)$$

is given by

$$u(z) = \frac{\sigma^2(1-\gamma)}{2\gamma}\rho_0^2 v_M^{-1}\ v(z) = u_0 z^{1-\gamma},$$

where

$$u_0 := (\pi_M(1-\pi_M))^{4/3}\ v_M^{-(1+\gamma)}.$$

Homethetic case

Finally, we summarize the expansion result in the following.

Lemma

For a power utility function U,

$$v^{\epsilon}(x,y) = v(z) - \epsilon^2 u_0 z^{1-\gamma} + O(\epsilon^3).$$

The width of the transaction region for the first correction equation $2\xi_0 = 2\eta(z)\rho_0$ *is given by*

$$\xi_0 = \left(\frac{3}{2\gamma}\right)^{1/3} (\pi_M(1-\pi_M))^{2/3}.$$

The above formulae are exactly as the one computed by Janecek & Shreve.

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Outline of the Proof

Set

$$u^{\epsilon}(x,y) := \frac{v(z) - v^{\epsilon}(x,y)}{\epsilon^2}.$$

Steps of the proof are

- Show that u^{ϵ} is locally uniformly bounded. Since $u^{\epsilon} \geq 0$, we need a uniform upper bound.
- Use the Barles & Perthame methodology to define weak limits $\liminf u^{\epsilon} =: u_*(z) \leq u^*(z) := \limsup u^{\epsilon}$.
- use the Evans technology from homogenization to show that

$$\mathcal{A}u^* \leq a \leq \mathcal{A}u_*.$$

- We then conclude by comparison.

Comments

- The power case is substantially easier.
- Ergodic problem has a solution but may not be as regular as we like. The free boundary in particular.
- Uniform lower bound is the only intersection with the technique of Janecek & Shreve. But in this case, we only need any lower bound of the right order of ϵ (i.e. in ϵ^2). Their approach however, requires the coefficient to be sharp as well, i.e. need a subsolution of the form $v(z) - \epsilon^2 u(z) + \circ(\epsilon^2)$.
- When the corrector is smooth, the lower bound can be obtained via probabilistic techniques.

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Multidimesions

We assume now that

- there are d stocks,
- we can transfer between any any stocks and cash,
- $L_t^{i,j}$ total transfers from i to j, ($i = 0$ is cash,
- power utility,
- proportional cost is $\epsilon^3 \lambda^{i,j}$,
- the normalized cell equation is

$$\max_{0 \le i,j \le d} \max \Big\{ - \frac{|\sigma^{\mathrm{T}} \rho|^2}{2} - \frac{1}{2} \mathrm{Tr} \big[\bar{\alpha} \bar{\alpha}^{\mathrm{T}} D^2 \bar{w}(\rho) \big] + \bar{a} \ , \\ -\lambda^{i,j} + (e_i - e_j) \cdot \hat{D} \bar{w}(\rho) \Big\} \ = \ 0, \quad \forall \ \rho \in \mathbb{R}^d.$$

No Transaction Region

We assume that there is a smooth solution $\bar{w} \in C^2(\mathbb{R}^2)$. Set

$$\mathcal{T} := \left\{ \rho \in \mathbb{R}^d \ : \ -\frac{|\sigma^{\mathrm{T}}\rho|^2}{2} - \frac{1}{2} Tr\left[\bar{\alpha}\bar{\alpha}^{\mathrm{T}} D^2 \bar{w}(\rho]\right) + \bar{a} = 0 \right\}.$$

Formally, we expect that the no-transaction region of the ϵ-problem is asymptotically given by

$$\begin{aligned} \mathcal{C}^\epsilon &:= \left\{(x,y) \in \mathbb{R} \times \mathbb{R}^d \ : \ \left(\frac{y}{z} - \pi_M\right)\gamma \in \epsilon\mathcal{T} \right\} \\ &= \left\{(x,y) \in \mathbb{R} \times \mathbb{R}^d \ : \ \rho \in \mathcal{T} \right\}, \end{aligned}$$

where $\pi_M \in \mathbb{R}^d$ is the Merton proportion.

Strategy

We propose a strategy so that the resulting $(X_t^\epsilon, Y_t^\epsilon)$ are in $\mathcal{C}^\epsilon$. Hence, L acts only at $\partial\mathcal{C}^\epsilon$. In fact, for $i,j = 0,\dots,d$, $L^{i,j}$ acts on the set $(\partial\mathcal{C}^\epsilon)^{i,j}$ which we now define,

$$(\partial\mathcal{T})^{i,j} := \left\{ \rho \in \partial\mathcal{T} \ : \ -\lambda^{i,j} + (e_i - e_j)\cdot \hat{D}\bar{w}(\rho) = 0 \right\}.$$

The boundary $\partial\mathcal{T}$ is covered by $(\partial\mathcal{T})^{i,j}$'s. We now define

$$\begin{aligned}(\partial\mathcal{C}^\epsilon)^{i,j} &:= \left\{(x,y) \in \partial\mathcal{C}^\epsilon \ : \ \left(\frac{y}{z} - \pi_M\right)\gamma \in \epsilon\mathcal{T} \right\} \\ &= \left\{(x,y) \in \mathbb{R}\times\mathbb{R}^d \ : \ \rho \in (\partial\mathcal{T})^{i,j} \right\}.\end{aligned}$$

We once again record the following fact,

$$\partial\mathcal{C}^\epsilon = \cup_{i,j=0}^d (\partial\mathcal{C}^\epsilon)^{i,j}.$$

Strategy

We use the Merton consumption, i.e., $c_t^\epsilon = v_M Z_t^\epsilon$. This proposed investment-consumption strategy yields a portfolio process $(X_t^\epsilon, Y_t^\epsilon) \in \mathcal{C}^\epsilon$ which is a solution of the Skorokhod problem,

$$\begin{cases} dX_t^\epsilon = (rX_t^\epsilon - v_M Z_t^\epsilon)dt + \sum_{j=1}^d \left[dL_t^{0,j} - (1+\lambda^{j,0}\epsilon^3)dL_t^{j,0} \right] \\ dY_t^{i,\epsilon} = Y^{i,\epsilon}\left(\mu^i dt + (\sigma dW_t)^i\right) + \sum_{j=0}^d \left[dL_t^{i,j} - (1+\epsilon^3\lambda^{j,i})dL_t^{j,i} \right] \\ (X_t^\epsilon, Y_t^\epsilon) \in \mathcal{C}^\epsilon, \quad \forall\ t > 0, \\ L_t^{i,j} = \int_0^t \chi_{\{(X_t^\epsilon, Y_t^\epsilon) \in (\partial\mathcal{C}^\epsilon)^{i,j}\}}\ dL_t^{i,j}, \qquad \forall\ t > 0,\ i,j = 0,\dots d, \end{cases}$$

where as usual

$$Z^\epsilon := X^\epsilon + \sum_{i=1}^d Y^{i,\epsilon}.$$

Assumption

Using the Skrokhod equation, we directly calculate that

$$dZ_t^\epsilon = Z_t^\epsilon \left[(r - v_M + \pi_t^\epsilon \cdot (\mu - \vec{r}))\, dt + \pi_t^\epsilon \cdot (\sigma\, dW_t) \right] - \epsilon^3 dL_t^\epsilon,$$

$$\pi_t^\epsilon := \frac{Y_t^\epsilon}{Z_t^\epsilon}, \qquad L_t^\epsilon := \sum_{i,j=0}^{d} L_t^{i,j}.$$

In view of the classical result of Lions & Sznitman, existence of a solution to the above problem requires regularity of the boundary of $\mathcal{C}^\epsilon$. We simply assume that

We assume that there exists a smooth solution $\bar{w} \in C^2(\mathbb{R}^2)$ and a solution $(X_t^\epsilon, Y_t^\epsilon) \in \mathcal{C}^\epsilon \subset \mathbb{R} \times \mathbb{R}^d$ of the Skorokhod problem.

Result

Let

$$J^{\epsilon} := \mathbb{E}\left[\int_0^{\infty} U_{\gamma}(c_t^{\epsilon})dt\right].$$

Theorem

There is a constant $k^* > 0$ so that

$$v(z) - \epsilon^2 u(z) - \epsilon^3 k^* v(z) - \epsilon^4 w(z,\xi) \le J^{\epsilon} \le v^{\epsilon}(z). \qquad (3.1)$$

In view of the upper bound, the investment-consumption $(X_t^{\epsilon}, Y_t^{\epsilon})$ is $\circ(\epsilon^2)$-optimal.

The $\circ(\epsilon^2)$ optimality of the strategy proposed by the cone $\mathcal{C}^{\epsilon}$ is interpreted as an asymptotic shape result.

Three lemmata

Let

$$dZ_t^* = Z_t^* \left[(r - v_M + \pi_M \cdot (\mu - r\mathbf{1}_d))\, dt + \pi_M \cdot (\sigma dW_t) \right]$$

be the Merton optimal wealth. Then, Z^ϵ satisfies

$$\mathbb{E}\left[(Z_t^\epsilon)^{1-\gamma} \right] \le \mathbb{E}\left[(Z_t^*)^{1-\gamma} \right].$$

Lemma 2

$$V^\epsilon(x,y) := v(z) - \epsilon^2 u(z) - \epsilon^4 w(z,\rho) = \left[\frac{U_\gamma(v_M)}{v_M} - \epsilon^2 \frac{\bar a}{\gamma v_M} - \epsilon^4 \frac{\bar w(\rho)}{\gamma}\right] z^{1-\gamma}.$$

Set,

$$I^\epsilon(x,y) := -\beta V^\epsilon(x,y) + \mathcal{L} V^\epsilon(x,y),$$

where $\mathcal{L}$ is the infinitesimal generator of the equations in the region $\mathcal{C}^\epsilon$.

Lemma

There exists a constant $k^* > 0$ such that for all $(x,y) \in \mathcal{C}^\epsilon$,

$$I^\epsilon(x,y) \geq -\left[\epsilon^3 k^* + U_\gamma(v_M)\right] z^{1-\gamma}.$$

Lemma 3

Set

$$\Lambda^{\epsilon}_{i,j} = e_i - e_j + \epsilon^3 \lambda^{i,j} e_i \in \mathbb{R}^{d+1}.$$

Lemma

There is $\epsilon_0 > 0$ so that for all $\epsilon \in (0, \epsilon_0]$,

$$\Lambda^{\epsilon}_{i,j} \cdot DV^{\epsilon}(x,y) \leq 0, \qquad on\ (\partial \mathcal{C}^{\epsilon})^{i,j},$$

for all $i, j = 0, \ldots, d$.

Proof of Theorem

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$$
\begin{aligned}
d\left[e^{-\beta t} V^{\epsilon}(\dots)\right] &= e^{-\beta t}\left[I^{\epsilon}(\dots)dt + d[\text{local martingale}]\right] \\
&\quad - e^{-\beta t}\sum_{i,j=0}^{d}\left[\Lambda^{\epsilon}_{i,j}\cdot DV^{\epsilon}(\dots)\right] dL^{i,j}_t \\
\geq\ & -e^{-\beta t}\left(\left(\epsilon^3 k^* + U_\gamma(v_M)\right)(Z^{\epsilon}_t)^{1-\gamma} + \text{local martingale}\right) \\
& -e^{-\beta t}\sum_{i,j=0}^{d}\left[\Lambda^{\epsilon}_{i,j}\cdot DV^{\epsilon}(\dots)\right]\chi_{\{(X^{\epsilon}_t, Y^{i,\epsilon}_t)\in(\partial\mathcal{C}^{\epsilon})^{i,j}\}}\ dL^{i,j}_t \\
\geq\ & -e^{-\beta t}\left(\epsilon^3 k^*(Z^{\epsilon}_t)^{1-\gamma}\, dt + U_\gamma(c^{\epsilon}_t)dt + d[\text{local martingale}]\right).
\end{aligned}
$$

Proof of Theorem

Localize the local martingale with a stopping time τ. Also we use the fact that

$$z^{1-\gamma} = k^* U(v_M z), \quad \Rightarrow \quad (Z_t^\epsilon)^{1-\gamma} = k^* U_\gamma(c_t^\epsilon),$$

for some constant k^*. The result is,

$$\begin{aligned}
\mathbb{E}\left[e^{-\beta\tau} V^\epsilon\left(X_\tau^\epsilon, Y_\tau^\epsilon\right)\right] &\geq V^\epsilon(x,y) - (1+\epsilon^3 k^*)\mathbb{E}\left[\int_0^\tau e^{-\beta t} U_\gamma(c_t^\epsilon) dt\right] \\
&\geq V^\epsilon(x,y) - (1+\epsilon^3 k^*)\mathbb{E}\left[\int_0^\infty e^{-\beta t} U_\gamma(c_t^\epsilon) dt\right] \\
&= V^\epsilon(x,y) - (1+\epsilon^3 k^*) J^\epsilon.
\end{aligned}$$

Hence

$$V^\epsilon(x,y) \leq (1+\epsilon^3 k^*) J^\epsilon + \mathbb{E}\left[e^{-\beta\tau} V^\epsilon\left(X_\tau^\epsilon, Y_\tau^\epsilon\right)\right].$$

Proof of Theorem

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We let τ to infinity (needs to be done properly),

$$V^{\epsilon}(x,y) \leq (1+\epsilon^3 k^*) J^{\epsilon}.$$

This proves that

$$\begin{aligned} J^{\epsilon} &\geq V^{\epsilon}(x,y) - \epsilon^3 k^* J^{\epsilon} \\ &\geq V^{\epsilon}(x,y) - \epsilon^3 k^* v(z) \\ &= v(z) - \epsilon^2 u(z) - \epsilon^3 k^* v(z) - \epsilon^4 w(z,\rho). \end{aligned}$$

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THANK YOU
FOR YOUR ATTENTION.