Homogenization and asymptotics for small transaction costs

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Goal

Let $\epsilon > 0$ be a small parameter related to transaction costs and let $v^\epsilon(s,x,y)$ be the maximum utility starting with initial bond position x and stock position y and the stock price s. Then, it is clear that

$$\lim_{\epsilon \downarrow 0} v^\epsilon(s,x,y) = v_{merton}(s,z), \qquad z = x + y,$$

where v_{merton} is the value of the classical Merton problem. To understand this convergence we look for an expansion of the form

$$v^\epsilon(s,x,y) = v_{merton}(s,z) - \epsilon^2 u(s,z) + \circ(\epsilon^2).$$

Shreve & Janecek (one stock or future) and Bichuch & Shreve (two futures) analyzed the power utility case.

Outline

1. Overview
   - Context
   - Results
   - Problem
   - Properties
   - Merton
2. Derivation
   - Corrector
   - Power Case
3. Proof

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Brief History

- Magill & Constantinides (1976), Constantinides (1986) initiates the problem ;
- Taksar, Klass & Asaf (1988) rigorous study of the ergodic problem within no consumption ;
- Dumas & Luciano (1991) further studies the ergodic problem ;
- Davis & Norman (1990) puts into the modern framework ;
- Shreve & Soner (1994) introduces viscosity theory ;
- Cvitanic & Karatzas (1996) duality and martingale approach ;
- Oksendal & Sulem (2002) problem with fixed costs ;

Previous results on asymptotics

- Constantinides (1986) numerically considers the problem ;
- Shreve (1994) is the first rigorous result ;
- Janecek & Shreve (2004, 2010) one dimensional problem ;
- Bichuch & Shreve (preprint) first two dimensional result ;
- Gerhold, Muhle-Karbe, Schachermayer (2011) and also with Guassoni (2011) consider the simpler Dumas-Luciano problem but give more detailed explicit results.
- Several formal results, Whaley & Willmont (1997) , Atkinson & Mokkhavesa (2004),..., Goodman & Ostrov(2010).

Why ?

No explicit result available and the small parameter makes the numerical approach difficult and/or unstable. The asymptotic approach provides us with

- Explicit formulae in one dimensions ;
- Substantial simplification for instance in the fixed costs ;
- Usable portfolio rebalancing rules : Suppose one solves a maximization problem with no transaction costs and would like to understand the first order effect of transaction costs both on the maximum value and also on the trading strategy. However, since the zero-cost problem could be quite general, a robust approach is desirable.

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Rebalancing

Suppose an optimization problem dictates that we want θ fraction of our wealth in the stock(s). In a short time stock moves randomly and bond account moves deterministically. Hence, the fraction moves randomly and we need to rebalance. This is not desirable because

- not realistically feasible ;
- transaction costs would be very high ;

Then, one would want to create a band around *theta* and wait until we reach the boundary of this band and then rebalance. The question is then, to determine this band.

Our Approach

The main observation is to use the techniques from

- formal matched asymptotics ;
- homogenization and correctors ;
- perturbed test function technique of Evans together with the weak limits of Barles & Perthame.

Although there are similarities with the theory of homogenization,

- homogenization has is a given fast variable, here there is none ;
- here we do an inner expansion as it is done in fast reaction – slow diffusion problems like Ginzburg-Landau or super-conductivity.

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Expansion

Main Theorem. *As ϵ tends to zero,*

$$u^{\epsilon}(x,y) := \frac{v_{merton}(x+y) - v^{\epsilon}(x,y)}{\epsilon^2} \to u(s,z) := \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} a(\hat{Z}_t^z) dt\right],$$

locally uniformly, where a will be defined in the next slide and $\{\hat{Z}_t^{s,z}, t \geq 0\}$ is the optimal wealth process . Hence,

$$v^{\epsilon}(x,y) = v_{merton}(z) - \epsilon^2 u^{\epsilon}(x,y) = v(z) - \epsilon^2 u(z) + \circ(\epsilon^2).$$

Notation

We assume that $v := v_{merton}$ is smooth and let

$$\eta(z) := -\frac{v_z(z)}{v_{zz}(z)}$$

be the corresponding risk tolerance. The solution of the Merton problem also provides us an optimal feedback portfolio strategy $\theta(z)$ and an optimal feedback consumption function $\mathbf{c}(z)$.

An Ergodic Problem

$$a(z) := \eta(z) v_z(z) \bar{a}(z),$$

and for every fixed $z > 0$,

$$\bar{a}(z) := \inf_M J(z, M),$$

$$J(z, M) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T \frac{|\sigma(s)\xi_t|^2}{2} + \|M\|_T\right],$$

where M is a bounded variation process with variation $\|M\|$, and the controlled process ξ satisfies (with a Brownian motion B),

$$d\xi_t = [\theta(z)(1 - \theta_z(z))] dB_t + dM_t.$$

This is an ergodic *monotone follower problem*.

Asymptotic Transaction Region

Let

$$\mathcal{I}(z) := \text{no transaction region of the ergodic problem.}$$

Then, we expect and can prove under some additional assumptions that the region

$$\mathcal{C}^{\epsilon} := \{(x, y) \mid y - \theta(z) \in \epsilon\, \mathcal{I}(z)\},$$

provides an $\circ(\epsilon^2)$ optimal investment strategy.

Comment on the general case

In the multi-dimensional case with non-constant coefficients, same result holds but the functions

$$v(s,z), \quad u(s,z), \quad a(s,z), \quad \theta(s,z), \quad \mathbf{c}(s,z)$$

all depend on the initial stock value s as well.
Moreover, in the ergodic problem like z, s also appears but as a parameter. Therefore, the expansion has the form,

$$v^{\epsilon}(s,x,y) = v(s,z) - \epsilon^2 u^{\epsilon}(s,x,y) = v(s,z) - \epsilon^2 u(s,z) + \circ(\epsilon^2).$$

Simplify

For this talk, I will take all coefficients to be constant and assume that there is only one stock.

Also I take the proportional costs to be same in both directions.

Market

We assume a financial market with constant interest rate $r = 0$ and one stock with geometric Brownian motion dynamics,

$$dS_t = S_t \left[\mu dt + \sigma dW_t \right],$$

where W is a Brownian motion and $\mu > r$, $\sigma > 0$ are the coefficients of instantaneous mean return and volatility.

Transaction costs

Suppose we decide to transfer $\ell > 0$ dollars from stock to cash or the other way around. Then, this incurs a transaction cost of

$$\epsilon^3 \ell,$$

where $\epsilon > 0$ is a small parameter.

We let k_t to be the total dollar transfers between the cash and stock positions. Due to the transaction costs, k needs to be a function of bounded variation. Hence,

$$k_t = k_t^+ - k_t^-,$$

where $k^{\pm}$ are non decreasing with

$$k_{0-} = 0.$$

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Wealth Equations

As it is standard, let c_t, be the rate of consumption, and let $k_t = k^+ - k^-$ be the total amount of transfers and k^+ from cash to stock and k^- in the other direction.

For any initial position $(X_{0-}, Y_{0-}) = (x, y) \in K_\epsilon$, the portfolio position of the investor are given by the following state equation,

$$\begin{aligned} dX_t &= (-c_t)dt + \left(dk_t^+ - (1+\epsilon^3)dk_t^-\right), \\ dY_t &= Y_t \frac{dS_t}{S_t} + \left(dk_t^- - (1+\epsilon^3)dk_t^+\right). \end{aligned}$$

Total wealth

Set $Z_t := X_t + Y_t$. Since

$$
\begin{aligned}
dX_t &= (rX_t - c_t)dt + \left(dk_t^+ - (1+\epsilon^3)k_t^-\right), \\
dY_t &= Y_t \frac{dS_t}{S_t} + \left(dk_t^- - (1+\epsilon^3)dk_t^+\right).
\end{aligned}
$$

we derive that

$$
dZ_t = Z_t \pi_t (\mu dt + \sigma dB_t) - c_t - \epsilon^3 [dk_t^+ + dk_t^-].
$$

Transaction costs

We define the solvency region K_ϵ is defined as the set of all portfolio positions which can be transferred into portfolio positions with nonnegative entries through an appropriate portfolio rebalancing.

$(x, y) \in K_\epsilon$ if and only if there are $k^\pm \geq 0$ so that

$$x + k^+ - (1+\epsilon^3)k^- \geq 0, \qquad y + k^- - (1+\epsilon^3)k^+ \geq 0,$$

Then,

$$K_\epsilon = \{(x,y) \in \mathbb{R}^2 \;:\; x + (1+\epsilon^3)y \geq 0, \;\; y + (1+\epsilon^3)x \geq 0\}.$$

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Admissible strategies

The above solution depends on the initial condition (x, y) and on the control $\nu = (c, k)$. Let $(X, Y)^{\nu,x,y}$ be the solution of the above equation. Then, a consumption-investment strategy ν is said to be admissible for the initial position (x, y), and denoted by $\Theta^{\epsilon}(x, y)$ if

$$(X, Y)_t^{\nu,x,y} \in K_{\epsilon}, \qquad \forall\ t \geq 0, \quad \mathbb{P} - \text{a.s.}$$

Utility and the value

$$v^{\epsilon}(x,y) := \sup_{(c,L)\in\Theta^{\epsilon}(x,y)} \mathbb{E}\left[\int_0^{\infty} e^{-\beta t}\, U(c_t)dt\right],$$

where $U : (0,\infty) \mapsto \mathbb{R}$ is a utility function that is C^2, increasing, strictly concave, and its convex conjugate is given by,

$$\tilde{U}(\tilde{c}) := \sup_{c>0} \left\{ U(c) - c\tilde{c} \right\}, \qquad \tilde{c} \in \mathbb{R}.$$

Then $\tilde{U}$ is a C^2 convex function. It is well known that the value function is a viscosity solution of the corresponding dynamic programming equation.

Concavity

The above is an optimal control problem with following properties

- linear dynamics ;
- concave reward ;
- maximization.

Then, the resulting value function is a concave function of the initial condition.

The proof is straightforward : Let $\nu_i \in \Theta^\epsilon(x_i, y_i)$. Set ν and (x, y) be the mid points. Then, easy to check $\nu \in \Theta^\epsilon(x, y)$. Moreover,

$$2U(c_t) \geq U((c_1)_t) + U((c_2)_t).$$

Concavity - consequences

Any concave function is Lipschitz in its domain of definition, i.e, on K_ϵ. Some work proves that it is continuous all the way up to the boundary.

Subdifferentials. For a concave function ϕ

$$\partial\phi(x) := \{p \ : \ \phi(x) + p \cdot (y - x) \geq \phi(y), \ \forall \ y\}.$$

It is as good as differentiable!

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Gradient Constraint

Consider an initial data

$$X_{0-} = x, \;\; Y_{0-} = y.$$

We can immediately move our position to

$$X_0 = x + m, \;\; Y_0 = y - (1 + \epsilon^3)m,$$

for any $m \geq 0$ and not to large. Hence,

$$v^\epsilon(x, y) \geq v^\epsilon(x + m, y - (1 + \epsilon^3)m).$$

Differentiate formally to conclude

$$v_x^\epsilon(x, y) - (1 + \epsilon^3)v_y^\epsilon(x, y) \leq 0.$$

This is formal and be made rigorous with sub differentials.

Gradient Constraint - rigorous

We know

$$v^{\epsilon}(x,y) \geq v^{\epsilon}(x+m, y-(1+\epsilon^3)m).$$

Change (x,y) to $(x-m, y+(1+\epsilon^3))$, hence

$$\begin{aligned} v^{\epsilon}(x-m, y+(1+\epsilon^3)) & \\ &\geq v^{\epsilon}((x-m)+m, (y+(1+\epsilon^3))-(1+\epsilon^3)m) \\ &= v^{\epsilon}(x,y). \end{aligned}$$

Gradient Constraint - rigorous, cont.

Let $(p_x, p_y) \in \partial v^\epsilon(x, y)$. Then, for any h,

$$v^\epsilon(x, y) + (p_x, p_y) \cdot (-1, 1 + \epsilon^3)h \geq v^\epsilon(x - h, y + (1 + \epsilon^3)h).$$

Choose $h = m$ and combine the two inequalities :

$$\begin{aligned} v^\epsilon(x, y) + (p_x, p_y) \cdot (-1, 1 + \epsilon^3)m &\geq v^\epsilon(x - m, y + (1 + \epsilon^3)m) \\ &\geq v^\epsilon(x, y), \quad \forall\ m \geq 0. \end{aligned}$$

Hence, for all

$$(p_x, p_y) \cdot (-1, 1 + \epsilon^3) \geq 0, \qquad \forall\ (p_x, p_y) \in \partial v^\epsilon(x, y).$$

Other Gradient Constraint

We move money from can to bond. the result is,

$$(p_x, p_y) \cdot (1+\epsilon^3, -1) \geq 0, \qquad \forall\ (p_x, p_y) \in \partial v^\epsilon(x,y).$$

When differentiable,

$$v_y^\epsilon(x,y) - (1+\epsilon^3) v_x^\epsilon(x,y) \leq 0.$$

Scaling

For any positive constant $\lambda > 0$, the linearity of the state equations imply that

$$\nu \in \Theta^\epsilon(x,y) \quad \Leftrightarrow \quad \lambda\nu \in \Theta^\epsilon(\lambda(x,y)).$$

If

$$U(c) = c^{1-\gamma}/(1-\gamma),$$

then,

$$v^\epsilon(\lambda(x,y)) = \lambda^{1-\gamma} v^\epsilon(x,y).$$

Cones

$$K_\epsilon = \{(x, y) \in \mathbb{R}^2 \ : \ y/z \in (-1/\epsilon^3, 1 + 1/\epsilon^3) \ \}, \qquad z = x + y.$$

Lemma

There are

$$-1/\epsilon^3 \leq a_\epsilon < b_\epsilon \leq 1 + 1/\epsilon^3,$$

so that for any $p \in \partial v^\epsilon(x, y)$,

$$(p_x, p_y) \cdot (1 + \epsilon^3, -1) = 0, \qquad \forall \ y/z \in (-1/\epsilon^3, a_\epsilon),$$

$$(p_y, p_x) \cdot (1 + \epsilon^3, -1) = 0, \qquad \forall \ y/z \in (b_\epsilon, 1 + 1/\epsilon^3).$$

Equation

The DPE with our simplifications is,

$$\min\{\ I^\epsilon\ ;\ d_+^\epsilon\ ;\ d_-^\epsilon\ \} = 0, \qquad \text{where}$$

$$\begin{aligned} I^\epsilon &= \beta v^\epsilon - \mu y v_y^\epsilon - \frac{1}{2}\sigma^2 y^2 v_{yy}^\epsilon - \tilde{U}(v_x^\epsilon), \\ &:= \beta v^\epsilon - \mathcal{L}_y v^\epsilon - \tilde{U}(v_x^\epsilon), \\ d_+^\epsilon &= (1+\epsilon^3) v_x^\epsilon - v_y^\epsilon, \\ d_-^\epsilon &= (1+\epsilon^3) v_y^\epsilon - v_x^\epsilon. \end{aligned}$$

Viscosity Property

Suppose $v^\epsilon < \infty$, then it is the unique continuous viscosity solution of

$$\min\{\ I^\epsilon\ ;\ d_+^\epsilon\ ;\ d_-^\epsilon\ \} = 0.$$

The proof is now standard.

When is the value function finite ? Easy way is

$$v^\epsilon(x,y) < v(x+y),$$

and we know exactly when the Merton value is finite. But more detailed analysis is carried out by Choi, Sirbu & Zitkovic recently. They gave the complete characterization of the finiteness of the value function.

Optimal strategy

The structure is,

No transact and consume $= \{l^\epsilon = 0\}$.

Buy stock $= \{d_+^\epsilon = 0\}$,

Buy bond $= \{d_-^\epsilon = 0\}$.

Optimal strategy

The structure is,

No transact and consume $= \{l^\epsilon = 0\} = \{y/z \in (a_\epsilon, b_\epsilon)\}$.

Buy stock $= \{d_+^\epsilon = 0\} = \{y/z \in (-1/\epsilon^3, a_\epsilon)\}$,

Buy bond $= \{d_-^\epsilon = 0\} = \{y/z \in (b_\epsilon, 1 + 1/\epsilon^3)\}$.

The optimal strategy is at time zero to move into the no transaction region and then stay there by local time at the boundaries. Consumption is given as a feedback strategy.

$\epsilon = 0$

Then, the value function is a function of z alone, $v(z)$, and the DPE is

$$\beta v(z) - \sup_{\pi \in \mathbb{R}} \{ \mathcal{L}_{\pi z} v(z) \} - \tilde{U}(v_z(z)) = 0,$$

where or recall that

$$\mathcal{L}_y \phi(z) = \mu y \phi_z(z) + \frac{1}{2}\sigma^2 y^2 \phi_{zz}.$$

$$\tilde{U}(p) = \sup_{c>0}\{U(c) - cp\} = \frac{\gamma}{1-\gamma} p^{(\gamma-1)/\gamma}.$$

$\epsilon = 0$, cont.

Then,

$$\begin{aligned}
\beta v(z) - \mathcal{L}_y v(z) - \tilde{U}(v_z(z)) \\
&= \beta v(z) - \sup_{\pi \in \mathbb{R}} \{ \mathcal{L}_{\pi z} v(z) \} - \tilde{U}(v_z(z)) \\
&\quad + \sup_{\pi \in \mathbb{R}} \{ \mathcal{L}_{\pi z} v(z) \} - \mathcal{L}_y v(z) \\
&= \frac{1}{2}(-v_{zz}(z)) \, (y - \theta(z))^2
\end{aligned}$$

where $\theta(z) = \pi^*(z)z$ is the maximizer. In the power case,

$$\pi^* = \mu / \gamma \sigma^2 .$$

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3 Proof

Form of the Expansion

We postulate the following expansion,

$$v^\epsilon(x,y) = v(z) - \epsilon^2 u(z) - \epsilon^4 w(z,\xi) + \circ(\epsilon^2),$$

where $(z,\xi) =$ is a transformation of $(x,y) \in K_\epsilon$ given by

$$z = x + y, \qquad \xi := \xi_\epsilon(x,y) = \frac{y - \theta(z)}{\epsilon},$$

where $\theta(z)$ is the Merton optimal investment strategy. In our postulate, we have also introduced two functions

$$u : \mathbb{R}_+ \mapsto \mathbb{R}, \text{ and } w : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}.$$

The main goal is to derive equations for these two functions.

Why the corrector w ?

$$v^{\epsilon}(x,y) = v(z) - \epsilon^2 u(z) - \epsilon^4 w(z,\xi) + \circ(\epsilon^2),$$

Notice that the above expansion is assumed to hold up to ϵ^2, i.e. the $\circ(\epsilon^2)$ term. Therefore, the reason for having an higher term like $\epsilon^4 w(z,\xi)$ explicitly in the expansion may not be clear.

However, this term contains the fast variable ξ and its second derivative is of order ϵ^2. And the introduction of w in the expansion is crucial ; without the formal calculations do not match.

This is also the case in the pioneering work of Lions, Papanicolaou and Varadhan in the theory of homogenization.

Equation

We now substitute the above postulated form into the dynamic programming equation (DPE) and derive the equations for u and w. We assume that $\xi = (y - \theta(z))/\epsilon$ is order one.

The DPE with our simplifications is,

$$
\begin{aligned}
0 &= \min\{ \, I^\epsilon \, ; \, d_+^\epsilon \, ; \, d_-^\epsilon \, \}, \\
I^\epsilon &= \beta v^\epsilon - \mu y v_y^\epsilon - \frac{1}{2}\sigma^2 y^2 v_{yy}^\epsilon - \tilde{U}(v_x^\epsilon), \\
&:= \beta v^\epsilon - \mathcal{L}_y v^\epsilon - \tilde{U}(v_x^\epsilon), \\
d_+^\epsilon &= (1+\epsilon^3) v_x^\epsilon - v_y^\epsilon, \\
d_-^\epsilon &= (1+\epsilon^3) v_y^\epsilon - v_x^\epsilon.
\end{aligned}
$$

Calculation – d's

Recall that $\xi = (y - \theta(z))/\epsilon$ and

$$v^\epsilon(x,y) = v(z) - \epsilon^2 u(z) - \epsilon^4 w(z,\xi), \qquad z = x + y.$$

Hence, we directly calculate that

$$(w_x(z,\xi), w_y(z,\xi)) = (w_z, w_z) + \frac{1}{\epsilon}(-\theta_z, (1-\theta_z))w_\xi,$$

$$d_+^\epsilon = (1+\epsilon^3)v_x^\epsilon - v_y^\epsilon = \epsilon^3 \left[v_z(z) + w_\xi\right] + 0(\epsilon^5).$$

d's continued

Similarly

$$d_-^\epsilon = (1+\epsilon^3)v_y^\epsilon - v_x^\epsilon = \epsilon^3 \left[v_z(z) - w_\xi\right] + 0(\epsilon^5).$$

Hence, the DPE can be approximately rewritten as

$$\begin{aligned} 0 &= \min\{\ I^\epsilon\ ;\ d_+^\epsilon\ ;\ d_-^\epsilon\ \} \\ &\approx \min\{\ I^\epsilon\ ;\ v_z(z) + w_\xi\ ;\ v_z(z) - w_\xi\ \}. \end{aligned}$$

Calculation – I

We use the Merton equation satisfied by v :

$$\begin{aligned} I^\epsilon &:= \beta v^\epsilon - \mathcal{L}_y v^\epsilon - \tilde{U}(v_x^\epsilon) \\ &= \beta v - \mathcal{L}_y v - \tilde{U}(v_z) \\ &\quad + \left(\tilde{U}(v_z) - \tilde{U}\left(v_z - \epsilon^2 u_z + O(\epsilon^3)\right)\right) - \epsilon^2 \left(\beta u - \mathcal{L}_{\theta(z)} u\right) \\ &\quad + \frac{\epsilon^4}{2}[\sigma^2 y^2 w_{yy}] + O(\epsilon^3), \\ &= \epsilon^2 \left[-\frac{1}{2}\sigma^2 \xi^2 v_{zz}(z) + \frac{1}{2}\alpha(z)^2 w_{\xi\xi}(z,\xi) - \mathcal{A}u(z) \right] + O(\epsilon^3), \end{aligned}$$

where

/ Calculation continued

$$\alpha(z) = \theta(z)(1 - \theta_z(z)),$$

and $\mathcal{A}$ is the infinitesimal generator of the Merton optimal wealth process, i.e.,

$$\mathcal{A}u(z) = [\theta(z)\mu - \mathbf{c}(z)]\, u_z(z) + \frac{1}{2}\sigma^2\theta(z)^2 u_{zz}(z).$$

We used the fact that

$$\tilde{U}'(v_z(z)) = \mathbf{c}(z).$$

Corrector Equations

We calculated that

$$
\begin{aligned}
0 &= -\min\{\ I^\epsilon\ ;\ d_+^\epsilon\ ;\ d_-^\epsilon\ \} \\
&\approx -\min\{\ I^\epsilon\ ;\ v_z(z) + w_\xi\ ;\ v_z(z) - w_\xi\ \} \\
&\approx \max\{\ \frac{1}{2}\sigma^2\xi^2 v_{zz}(z) - \frac{1}{2}\alpha(z)^2 w_{\xi\xi}(z,\xi) + \mathcal{A}u(z) \\
&\qquad ;\ -v_z(z) + w_\xi\ ;\ -v_z(z) - w_\xi\ \}.
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Note that the above is an equation for w alone. Hence in the above equation z is just a parameter. We solve the above equation with boundary condition $w(z,0) = 0$.

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Solving the Corrector

Set

$$\mathcal{A}u(z) = a(z),$$

and solve for the pair $(w(z,\cdot), a(z))$ solving the following equation with z as a parameter and ξ as the independent variable,

$$0 = \max\{ \frac{1}{2}\sigma^2\xi^2 v_{zz}(z) - \frac{1}{2}\alpha(z)^2 w_{\xi\xi}(z,\xi) + a(z) \; ; \; -v_z(z) + w_\xi \; ; \; -v_z(z) - w_\xi \}.$$

together with $w(z,0) = 0$.

a is exactly the value function of the ergodic problem described earlier and w is its potential function.

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Solving the Corrector - continued

The solution is obtained in two steps;

1. Fix z (and s if the problem depends on it). Solve for the pair $(w(z,\cdot), a(z))$. In fact, we rescale the equation consistent with the power utility case and solve that equation. This form is reported earlier.
2. Once we have $a(z)$, we then solve the equation

$$\mathcal{A}u(z) = a(z).$$

Since $\mathcal{A}$ is related to the infinitesimal generator of the optimal wealth process, the solution is simply an expectation.

Solution of the Corrector

- in one dimension corrector equation is solved by using smooth fit. The solution is an explicit function of the Merton value function and its derivatives.
- In the case a power utility, the equation has the same homothety and is solved explicitly in one space dimension.
- In higher dimensions we only know its existence and compute it numerically.

Comments

- In addition to the first correction u, the corrector equation also provides the asymptotic shape of the no-transaction region.
- But the function w is not then used.
- In the theory of homogenization, a similar corrector equation is always needed. This equation is always related to an ergodic control problem

Explicit Solutions

Explicit solutions exist only in power case in one dimension or independent Brownian motions.

We could also derive explicit expansions in the case of fixed transaction costs as well.

Homothety of a, u and w

$U(c) = c^{1-\gamma}/(1-\gamma)$ for some $\gamma > 0$ with constant coefficients.

$$\begin{aligned} a(z) &= \text{constant } z^{1-\gamma}, \\ u(z) &= \text{constant } z^{1-\gamma}, \\ w(z,\xi) &= z^{1-\gamma}\ W\left(\frac{(y/z) - \pi_{merton}}{\epsilon}\right), \\ \mathcal{C}(z) &= z\,\mathcal{C}(1). \end{aligned}$$

In one dimensions one can compute all the constants explicitly. And they agree with the formulae computed in Shreve & Janecek.

Outline

1 Overview
- Context
- Results
- Problem
- Properties
- Merton

2 Derivation
- Corrector
- Power Case

3 Proof

Outline

Set

$$u^\epsilon(x,y) := \frac{v(z) - v^\epsilon(x,y)}{\epsilon^2}.$$

Steps of the proof are

- Show that u^ϵ is locally uniformly bounded. Since $u^\epsilon \geq 0$, we need a uniform upper bound.
- Use the Barles & Perthame methodology to define weak limits $\liminf u^\epsilon =: u_*(z) \leq u^*(z) := \limsup u^\epsilon$.
- use the Evans technology from homogenization to show that

$$\mathcal{A}u^* \leq a \leq \mathcal{A}u_*.$$

- We then conclude by comparison.

Comments

- The power case is substantially easier.
- Ergodic problem has a solution but may not be as regular as we like. The free boundary in particular.
- Uniform lower bound is the only intersection with the technique of Shreve & Janecek. But in this case, we only need any lower bound of the right order of ϵ (i.e. in ϵ^2). Their approach however, requires the coefficient to be sharp as well, i.e. need a subsolution of the form $v(z) - \epsilon^2 u(z) + \circ(\epsilon^2)$.

Homogenization and asymptotics for small transaction costs,
H. Mete Soner, Nizar Touzi
arXiv :1202.6131.

Large liquidity expansion of super-hedging costs
Dylan Possamai, H. Mete Soner, Nizar Touzi. (2011)
Aysmptotic Analysis, forthcoming.

THANK YOU
FOR YOUR ATTENTION.