OVERVIEW OF STOCHASTIC PORTFOLIO THEORY

IOANNIS KARATZAS

INTECH Investment Technologies LLC, Princeton
and
Department of Mathematics, Columbia University

(Builds on work of E.R. Fernholz, as well as A. Banner, C. Kardaras, S. Pal, V. Papathanakos, T. Ichiba, D. Fernholz, J. Ruf, some of it joint.)

Talk at Summer School in Lisabon, Portugal July 2012

SYNOPSIS

The purpose of these lectures is to offer an overview of **Stochastic Portfolio Theory**, a rich and flexible framework introduced by E.R. Fernholz (2002) for analyzing portfolio behavior and equity market structure.

This theory is descriptive as opposed to normative, is consistent with observable characteristics of actual markets and portfolios, and provides a theoretical tool which is useful for practical applications.

As a theoretical tool, this framework provides fresh insights into questions of market structure and arbitrage, and can be used to construct portfolios with controlled behavior.

As a practical tool, Stochastic Portfolio Theory has been applied to the analysis and optimization of portfolio performance and has been the basis of successful investment strategies for close to 20 years.

More importantly, SPT explains under what conditions it becomes possible to **outperform a cap-weighted benchmark index** – and then, exactly how to do this by means of *simple* investment rules.

These typically take the form of adjusting the capitalization weights of an index portfolio to more efficient combinations.

They do it by exploiting the natural *volatilities* of stock prices, and need no forecasts of mean rates of return.

SOME REFERENCES

- Fernholz, E.R. (2002). *Stochastic Portfolio Theory.* Springer-Verlag, New York.
- Fernholz, E.R., Karatzas, I. & Kardaras, C. (2005). Diversity and arbitrage in equity markets. *Finance & Stochastics* **9**, 1-27.
- Fernholz, E.R. & Karatzas, I. (2005). Relative arbitrage in volatility-stabilized markets. *Annals of Finance* **1**, 149-177.
- Karatzas, I. & Kardaras, C. (2007). The numéraire portfolio and arbitrage in semimartingale markets. *Finance & Stochastics* **11**, 447-493.
- Fernholz, E.R. & Karatzas, I. (2009) Stochastic Portfolio Theory: An Overview. *Handbook of Numerical Analysis, volume "Mathematical Modeling and Numerical Methods in Finance" (A. Bensoussan, ed.)* 89-168.

1. THE FRAMEWORK

Equity market framework (Bachelier, Samuelson...) of the form

$$\mathrm{d}B(t) = B(t)r(t)\,\mathrm{d}t, \qquad B(0) = 1, \tag{1}$$

$$\mathrm{d}X_i(t) = X_i(t)\left(\beta_i(t)\,\mathrm{d}t + \sum_{\nu=1}^{N} \sigma_{i\nu}(t)\,\mathrm{d}W_\nu(t)\right), \qquad i = 1, \ldots, n.$$

Money-market $B(\cdot)$ and n stocks with strictly positive capitalizations $X_1(\cdot), \cdots, X_n(\cdot)$.

Driven by the Brownian motion $W(\cdot) = (W_1(\cdot), \cdots, W_N(\cdot))'$ with $N \geq n$. Probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

All processes are assumed to be progressively measurable with respect to a filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ which represents the "flow of information" in the market.

Not much needs to be assumed at this point about it... .

We shall take $r(\cdot) \equiv 0$ until further notice: investing in the money-market will amount to hoarding, whereas borrowing from the money-market will incur no interest.

Arithmetic mean rates of return $\beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_n(\cdot))'$ and *Variance rates* $(\alpha_{ii}(\cdot))_{1 \le i \le n}$ satisfy for every $T \in (0, \infty)$ the integrability condition

$$\sum_{i=1}^{n} \int_0^T \big(|\beta_i(t)| + \alpha_{ii}(t) \big) \, \mathrm{d}t < \infty \,, \qquad \text{a.s.}$$

Here $\sigma(\cdot) = (\sigma_{ij}(\cdot))_{1 \le i \le n, \, 1 \le j \le N}$ is the $(n \times N)-$matrix of local volatility rates, and $\alpha(\cdot) = \sigma(\cdot)\sigma'(\cdot)$ is the $(n \times n)-$matrix of *Covariance rates*

$$\alpha_{ij}(t) := \sum_{\nu=1}^{N} \sigma_{i\nu}(t)\sigma_{j\nu}(t) = \frac{1}{X_i(t)X_j(t)} \cdot \frac{\mathrm{d}}{\mathrm{d}\,t} \langle X_i, X_j \rangle(t) \,.$$

2. STRATEGIES and PORTFOLIOS

A **small investor** decides, at each time t and for every $1 \leq i \leq n$, which proportion $\pi_i(t)$ of his current wealth $V(t)$ to invest in the i^{th} stock.

We require that each $\pi_i(t)$ be $\mathcal{F}(t)-$measurable. The proportion $1 - \sum_{i=1}^{n} \pi_i(t)$ gets invested in the money market.

The wealth $V(\cdot) \equiv V^{v,\pi}(\cdot)$ corresponding to an initial capital $v \in (0, \infty)$ and an investment strategy $\pi(\cdot) = \big(\pi_1(\cdot), \cdots, \pi_n(\cdot)\big)'$ satisfies $V(0) = v$ and the linear MARKOWITZ equation

$$\frac{\mathrm{d}\, V(t)}{V(t)} = \sum_{i=1}^{n} \pi_i(t) \, \frac{\mathrm{d}X_i(t)}{X_i(t)} + \left(1 - \sum_{i=1}^{n} \pi_i(t)\right) \frac{\mathrm{d}B(t)}{B(t)}$$

$$= \pi'(t) \big[\, \beta(t)\mathrm{d}t + \sigma(t) \, \mathrm{d}W(t) \, \big] \, .$$

Equivalently,

$$\frac{\mathrm{d}\, V(t)}{V(t)} = \beta^{\pi}(t)\mathrm{d}t + \sum_{\nu=1}^{N} \sigma_{\nu}^{\pi}(t)\, \mathrm{d}W_{\nu}(t)$$

where

$$\beta^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t)\beta_i(t)\,, \qquad \sigma_{\nu}^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t)\sigma_{i\nu}(t)\,,$$

are, respectively, the portfolio's arithmetic rate-of-return, and the portfolio's volatilities.

- We shall call *investment strategy* an $\mathbb{F}-$progressively measurable process $\pi : [0,\infty) \times \Omega \to \mathbb{R}^n$ which satisfies for each $T \in (0,\infty)$ the integrability condition

$$\int_0^T \Big(\big|\beta^{\pi}(t)\big| + \big(\sigma^{\pi}(t)\big)'\sigma^{\pi}(t) \Big)\, \mathrm{d}t < \infty\,, \quad \text{a.s.}$$

The collection of all investment strategies will be denoted by $\boldsymbol{\Pi}$.
The wealth process corresponding to an investment strategy $\pi(\cdot) \in \boldsymbol{\Pi}$ and an initial capital $v > 0$ is strictly positive:

$$V^{v,\pi}(t) \,=\, v\,\exp\left\{\int_0^t \Big(\beta^\pi(s) - \frac{1}{2}\,(\sigma^\pi(s))'\sigma^\pi(s)\Big)\mathrm{d}s + \int_0^t (\sigma^\pi(s))'\mathrm{d}W(s)\right\} > 0\,.$$

- An investment strategy $\pi(\cdot) \in \boldsymbol{\Pi}$ with

$$\sum_{i=1}^n \pi_i(t) \,=\, 1\,, \qquad \forall\;\; 0 \le t < \infty$$

almost surely, will be called *portfolio*.

A portfolio never invests in the money market, and never borrows from it. We shall say that a portfolio is *bounded*, if there exists a real constant $\mathcal{K} > 0$ such that $\| \pi(t, \omega) \| \leq \mathcal{K}$ holds for all $(t, \omega) \in [0, \infty) \times \Omega$.

The collection of all portfolios will be denoted by by $\mathfrak{P}$.

We shall call $\pi(\cdot)$ a *long-only portfolio*, if it satisfies almost surely

$$\pi_1(t) \geq 0, \cdots, \pi_n(t) \geq 0, \qquad \forall \ 0 \leq t < \infty.$$

Every long-only portfolio is also bounded.

3. THE MARKET PORTFOLIO

Consider now the *market portfolio* $\mu(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_n(\cdot)\big)'$ given by

$$\mu_i(t) := \frac{X_i(t)}{X(t)}\,, \qquad i = 1, \ldots, n\,,$$

where

$$X(t) := X_1(t) + \ldots + X_n(t)\,.$$

This invests in all stocks in proportion to their relative capitalization weights.

Such an investment amounts to "owning the entire market": the wealth process becomes

$$V^{v,\mu}(\cdot) = vX(\cdot)/X(0)\,.$$

4. RELATIVE ARBITRAGE

Given a real number $T > 0$ and any two investment strategies $\pi(\cdot) \in \mathbf{\Pi}$ and $\rho(\cdot) \in \mathbf{\Pi}$, we shall say that $\pi(\cdot)$ is an *arbitrage relative to* $\rho(\cdot)$ over $[0, T]$, if we have

$$\mathbb{P}\left(V^{1,\pi}(T) \geq V^{1,\rho}(T)\right) = 1 \quad \text{and} \quad \mathbb{P}\left(V^{1,\pi}(T) > V^{1,\rho}(T)\right) > 0\,.$$

- *Strong* relative arbitrage: $\mathbb{P}\left(V^{1,\pi}(T) > V^{1,\rho}(T)\right) = 1\,.$

- No arbitrage is possible relative to a strategy $\varrho^*(\cdot) \in \mathbf{\Pi}$ that has the so-called *"numéraire property"*:

$$V^{1,\pi}(\cdot)/V^{1,\varrho^*}(\cdot) \quad \text{is a supermartingale, for every } \pi(\cdot) \in \mathbf{\Pi}\,.$$

4.a: Market Price of Risk (Optional)

Suppose there exists a *market price of risk* (or "relative risk") $\vartheta : [0, \infty) \times \Omega \to \mathbb{R}^N$, an $\mathbb{F}-$progressively measurable process that satisfies for each $T \in (0, \infty)$ the requirements

$$\sigma(t)\vartheta(t) = \beta(t)\,, \quad \forall \;\; 0 \le t \le T \qquad \text{and} \qquad \int_0^T \|\vartheta(t)\|^2 \,\mathrm{d}t < \infty\,.$$

- Whenever it exists, this $\vartheta(\cdot)$ allows us to introduce a corresponding "delfator" $Z^{\vartheta}(\cdot) \equiv Z(\cdot)$. This is an exponential local martingale and supermartingale

$$Z(t) := \exp\left\{ -\int_0^t \vartheta'(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t \|\vartheta(s)\|^2\,\mathrm{d}s \right\}\,, \;\; 0 \le t < \infty\,.$$

A martingale, if and only if $\mathbb{E}(Z(T)) = 1\,, \;\forall\;\; T \in (0, \infty)$.

The existence of this deflator proscribes *immediate* (or *egregious*, or *scalable*) arbitrage opportunities, a.k.a. UP's BR (Unbounded Profits with Bounded Risk).

• For the purposes of these talks, as we shall see, it is important to allow $Z(\cdot)$ to be a *strict local martingale,* that is, not to exclude the possibility $\mathbb{E}(Z(T)) < 1$ for some (possibly all...) horizons $T \in (0, \infty)$. This means, we still keep the door open to the existence of arbitrage opportunities that cannot be scaled, that is to say, to some (small) profits with bounded risk.

• Recall that the wealth process corresponding to an investment strategy $\pi(\cdot) \in \mathbf{\Pi}$ and an initial capital $v > 0$ is strictly positive:

$$V^{v,\pi}(t) = v \exp\left\{ \int_0^t \Big(\beta^\pi(s) - \frac{1}{2} \, (\sigma^\pi(s))' \sigma^\pi(s)\Big) \mathrm{d}s + \int_0^t (\sigma^\pi(s))' \mathrm{d}W(s) \right\} > 0 \, .$$

Whereas, in the presence of a market of risk process $\vartheta(\cdot)$ we have in addition

$$\frac{\mathrm{d}\, V^{v,\pi}(t)}{V^{v,\pi}(t)} = \pi'(t)\sigma(t)\big[\,\mathrm{d}W(t) + \vartheta(t)\mathrm{d}t\,\big]\,.$$

Let us pair this with the equation

$$\mathrm{d}Z(t) = -Z(t)\,\vartheta'(t)\mathrm{d}W(t)$$

for the deflator $Z(\cdot)$

$$Z(\cdot) = \exp\left\{-\int_0^{\cdot} \vartheta'(t)\,\mathrm{d}W(t) - \frac{1}{2}\int_0^{\cdot} \|\vartheta(t)\|^2\,\mathrm{d}t\right\}$$

we introduced a couple of slides ago. We see that the "deflated wealth process" $Z(\cdot)V^{v,\pi}(\cdot)$ is also a positive local martingale and a supermartingale for every $\pi(\cdot) \in \mathbf{\Pi}$, $v > 0$, namely

$$Z(t)V^{v,\pi}(t) = v + \int_0^t Z(s)V^{v,\pi}(s)\,\big(\sigma'(s)\pi(s) - \vartheta(s)\big)'\,\mathrm{d}W(s)\,.$$

- Suppose that the covariance process $\alpha(\cdot)$ satisfies, for some real number $L > 0$, the a.s. boundedness condition

$$\xi'\alpha(t)\xi = \xi'\sigma(t)\sigma'(t)\xi \leq L\|\xi\|^2, \quad \forall \ t \in [0,\infty), \ \ \xi \in \mathbb{R}^n . \qquad (2)$$

If $\pi(\cdot)$ is arbitrage relative to $\rho(\cdot)$ and both are **bounded portfolios**, then $Z(\cdot)$ and $Z(\cdot)V^{v,\rho}(\cdot)$ are *strict local martingales:*

$$\mathbb{E}\big[Z(T)\big] < 1\,, \qquad \mathbb{E}\,[\,Z(T)V^{v,\rho}(T)\,] \,<\, v\,.$$

  - In particular, if there exists a bounded portfolio $\pi(\cdot)$ which is arbitrage relative to $\mu(\cdot)$, we have

$$\mathbb{E}\big[Z(T)\big] < 1\,, \quad \mathbb{E}\,[\,Z(T)X(T)\,] \,<\, X(0)\,, \quad \mathbb{E}\,[\,Z(T)X_i(T)\,] \,<\, X_i(0)\,.$$

Relative arbitrage becomes a "machine" for generating strict local martingales.

5. REMARKS

- Suppose there exists a real constant $h > 0$ for which we have

$$\boxed{\sum_{i=1}^{n} \mu_i(t)\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{i=1}^{n} \mu_i(t)\alpha_{ij}(t)\mu_j(t) \ \geq \ h \ ,} \quad \forall \ 0 \leq t < \infty \, . \tag{3}$$

Under this condition, we shall see that, for a sufficiently large real constant $c > 0$, the long-only *modified entropic portfolio*

$$\mathfrak{E}_i^{(c)}(t) \ = \ \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\sum_{j=1}^{n} \mu_j(t)\big(c - \log \mu_j(t)\big)} \ , \qquad i = 1, \cdots, n \tag{4}$$

is a strong arbitrage relative to the market portfolio $\mu(\cdot)$ over any time-horizon $[0, T]$ with $T > (2 \log n)/h$.

- It is still an open question, whether such relative arbitrage can be constructed over *arbitrary* time-horizons under (3).

• Another condition guaranteeing the existence of arbitrage relative to the market is, as we shall see, that there exist a real constant $h > 0$ with

$$\boxed{\big(\mu_1(t)\cdots\mu_n(t)\big)^{1/n}\left[\sum_{i=1}^{n}\alpha_{ii}(t)-\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(t)\right]\;\geq\; h\,,}\quad \forall\;\, t\geq 0\,. \tag{5}$$

Then with $\mathbf{M}(t) := (\mu_1(t)\cdots\mu_n(t))^{1/n}$ and for $c > 0$ large enough, the long-only *modified equally-weighted portfolio*

$$\varphi_i^{(c)}(t) \;=\; \frac{c}{c+\mathbf{M}(t)}\cdot\frac{1}{n} \;+\; \frac{\mathbf{M}(t)}{c+\mathbf{M}(t)}\cdot\mu_i(t)\,, \qquad i=1,\cdots,n\,, \tag{6}$$

(a convex combination of equal-weighting and the market), is strong arbitrage relative to $\mu(\cdot)$ over any time horizon $[0,T]$ with $T > (2\,n^{1-(1/n)})/h$.

- Consider now the a.s. strong non-degeneracy condition

$$\xi' \alpha(t) \xi = \xi' \sigma(t) \sigma'(t) \xi \geq \varepsilon \|\xi\|^2, \quad \forall \;\; t \in [0, \infty)\,, \quad \xi \in \mathbb{R}^n \tag{7}$$

for some real number $\varepsilon > 0$, on the covariance process $\alpha(\cdot)$.

Suppose the conditions (2), (3), namely

$$\xi' \alpha(t) \xi = \xi' \sigma(t) \sigma'(t) \xi \leq L \|\xi\|^2, \quad \forall \;\; t \in [0, \infty)\,, \quad \xi \in \mathbb{R}^n\,,$$

$$\sum_{i=1}^{n} \mu_i(t) \alpha_{ii}(t) - \sum_{i=1}^{n} \sum_{i=1}^{n} \mu_i(t) \alpha_{ij}(t) \mu_j(t) \;\geq\; h\,, \quad \forall \;\; 0 \leq t < \infty$$

and (7) all hold.

Then, as we shall see, for any given constant $p \in (0,1)$, the long-only *diversity-weighted portfolio*

$$\boxed{\mathfrak{D}_i^{(p)}(t) = \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}}, \qquad i = 1, \cdots, n \qquad (8)$$

is again a strong arbitrage relative to the market portfolio over sufficiently long time-horizons.

- **Appropriate modifications of this diversity-weighted portfolio** do **yield such arbitrage over** *any* **time-horizon** $[0, T]$. This takes some work to prove. And the shorter the time-horizon, the bigger the amount of initial capital that is required to achieve the extra basis point's worth of outperformance:

$$v \geq v(T) \equiv \frac{q(T)}{(\mu_1(0))^{q(T)}} - 1, \quad q(T) := 1 + \big(2/\varepsilon\,\delta\,T\big) \log\big(1/\mu_1(0)\big).$$

• *Please note that these portfolios (entropic, equally-weighted, modified equally-weighted, diversity-weighted) are determined entirely from the market weights* $\mu_1(t), \cdots, \mu_n(t)$ *.*

These market weights are perfectly easy to observe and to measure.

• Construction of these portfolios does not assume *any* knowledge about the exact structure of market parameters, such as the mean rates of return $\beta_i(\cdot)$'s, or the local covariance rates $\alpha_{ij}(\cdot)$'s.

To put it a bit more colloquially: *does not require us to take these particular features of the model "too seriously".* Only as a general "framework"... .

To put it in the parlance of finance practice: *these portfolios are completely "passive"* (their construction requires neither estimation nor optimization).

6. GROWTH RATES

- Equivalent way of representing the positive Itô process $X(\cdot)$ of (1), namely

$$\mathrm{d}X_i(t) \;=\; X_i(t)\left(\beta_i(t)\,\mathrm{d}t + \sum_{\nu=1}^{N}\sigma_{i\nu}(t)\,\mathrm{d}W_\nu(t)\right), \quad i = 1, \ldots, n,$$

for the stock-price

$$\underbrace{\mathrm{d}\big(\log X_i(t)\big) \;=\; \gamma_i(t)\,\mathrm{d}t + \sum_{\nu=1}^{N}\sigma_{i\nu}(t)\,\mathrm{d}W_\nu(t)\,,}$$

with the *logarithmic mean rate of return* for the i^{th} stock

$$\boxed{\gamma_i(t) \;:=\; \beta_i(t) - \frac{1}{2}\,\alpha_{ii}(t)}\;.$$

EXAMPLE

Stock XYZ doubles in good years (+100%) and halves in bad years (-50%). Years good and bad alternate independently and equally likely (to wit, with probability 0.50), thus

$$\beta = \frac{1}{2}(+100\%) + \frac{1}{2}(-50\%) = \frac{1}{2} - \frac{1}{4} = 0.25\,,$$

$$\gamma = \frac{1}{2}(\log 2) + \frac{1}{2}\left(\log \frac{1}{2}\right) = 0\,.$$

On the other hand, $\log 2 \simeq 0.7$, so the variance is

$$\alpha = \sigma^2 = \frac{1}{2}(0.7)^2 + \frac{1}{2}(-0.7)^2 \simeq 0.50\,,$$

and indeed

$$(0.25) = 0 + (1/2)(0.50) \qquad \text{or} \qquad \beta = \gamma + (1/2)\,\alpha\,.$$

- This logarithmic rate of return can be interpreted also as a *growth-rate*, in the sense that

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(t) - \int_0^T \gamma_i(t)\mathrm{d}t\right) = 0 \quad \text{a.s.}$$

holds, under the assumption $\alpha_{ii}(\cdot) \le L < \infty$ on the variance of the stock; recall

$$\gamma_i(t) := \beta_i(t) - \frac{1}{2}\alpha_{ii}(t).$$

More generally, under the condition

$$\lim_{T\to\infty}\left(\frac{\log\log T}{T^2}\int_0^T \alpha_{ii}(t)\mathrm{d}t\right) = 0, \qquad \text{a.s.}$$

- Similarly, the solution of the linear equation

$$\frac{\mathrm{d}\, V(t)}{V(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{\mathrm{d}X_i(t)}{X_i(t)} + \left(1 - \sum_{i=1}^{n} \pi_i(t)\right) \frac{\mathrm{d}B(t)}{B(t)}$$

$$= \pi'(t) \big[\beta(t)\mathrm{d}t + \sigma(t)\, \mathrm{d}W(t) \big]$$

for the wealth $V(\cdot) \equiv V^{v,\pi}(\cdot)$ corresponding to an initial capital $v \in (0, \infty)$ and portfolio $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot))'$, is given as

$$\boxed{\mathrm{d}\left(\log V^{v,\pi}(t)\right) = \gamma^{\pi}(t)\, \mathrm{d}t + \sum_{\nu=1}^{N} \sigma^{\pi}_{\nu}(t)\, \mathrm{d}W_{\nu}(t)} \qquad (9)$$

Portfolio growth-rate and volatilities

$$\gamma^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_*^{\pi}(t)\,, \qquad \sigma_\nu^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t)\sigma_{i\nu}(t).$$

Excess growth-rate

$$\underbrace{\gamma_*^{\pi}(t) := \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\,\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\,\alpha_{ij}(t)\,\pi_j(t)\right)}.$$

Portfolio variance

$$\mathrm{a}^{\pi\pi}(t) := \sum_{\nu=1}^{N} (\sigma_\nu^{\pi}(t))^2 \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\alpha_{ij}(t)\pi_j(t)\,.$$

7. RELATIVE COVARIANCE STRUCTURE

• *Variance/Covariance Process*, not in absolute terms, but relative to the portfolio $\pi(\cdot)$:

$$\mathfrak{A}^{\pi}_{ij}(t) := \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \sigma^{\pi}_{\nu}(t)\big)\big(\sigma_{j\nu}(t) - \sigma^{\pi}_{\nu}(t)\big) \,, \quad 1 \le i,j \le n \,.$$

If the covariance matrix $\alpha(t)$ is positive-definite, than the relative covariance matrix

$$\mathfrak{A}^{\pi}(t) = \{\mathfrak{A}^{\pi}_{ij}(t)\}_{1 \le i,j \le n}$$

has rank $n-1$ and its null space is spanned by the vector $\pi(t)$.

- The excess growth-rate

$$\gamma_*^\pi(t) := \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\,\alpha_{ii}(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\,\alpha_{ij}(t)\pi_j(t)\right)$$

has, for *any* two portfolios $\pi(\cdot) \in \mathfrak{P}$, $\rho(\cdot) \in \mathfrak{P}$, the **invariance property**

$$\gamma_*^\pi(t) = \tfrac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\,\mathfrak{A}_{ii}^\rho(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\,\mathfrak{A}_{ij}^\rho(t)\,\pi_j(t)\right),$$

and consequently (read the above with $\rho(\cdot) \equiv \pi(\cdot)$, and recall that the null space of the relative covariance matrix $\{\mathfrak{A}_{ij}^\pi(t)\}_{1\le i,j\le n}$ is spanned by $\pi(t)$):

$$\boxed{\gamma_*^\pi(t) = \frac{1}{2}\sum_{i=1}^n \pi_i(t)\,\mathfrak{A}_{ii}^\pi(t)\,.}$$

In particular, we have $\gamma_*^\pi(\cdot) \ge 0$ for a long-only portfolio.

- Now let us consider the market portfolio $\pi \equiv \mu$. The excess growth rate of the market portfolio

$$\gamma_*^\mu(t) = \tfrac{1}{2}\Big(\sum_{i=1}^n \mu_i(t)\alpha_{ii}(t) - \sum_{i=1}^n \sum_{i=1}^n \mu_i(t)\alpha_{ij}(t)\mu_j(t) \Big)$$

can then be interpreted as a measure of *intrinsic volatility* available in the market:

$$\boxed{\gamma_*^\mu(t) = \tfrac{1}{2} \sum_{i=1}^n \mu_i(t)\, \mathfrak{A}_{ii}^\mu(t)} \,,$$

where

$$\mu_i(t) := \frac{X_i(t)}{X(t)}\,, \qquad \sigma_\nu^\mu(t) := \sum_{i=1}^n \mu_i(t)\sigma_{i\nu}(t)\,,$$

$$\mathfrak{A}_{ij}^\mu(t) := \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \sigma_\nu^\mu(t)\big)\big(\sigma_{j\nu}(t) - \sigma_\nu^\mu(t)\big) = \frac{\mathrm{d}\langle \mu_i, \mu_j\rangle(t)}{\mu_i(t)\mu_j(t)\mathrm{d}t}\,.$$

Thus the excess growth rate of the market portfolio

$$\gamma_*^{\mu}(t) \;=\; \tfrac{1}{2} \sum_{i=1}^{n} \mu_i(t)\, \mathfrak{A}_{ii}^{\mu}(t)$$

is also a weighted average, according to capitalization, of the variances of individual stocks – not in absolute terms, but *relative to the market*. This quantity will be very important in what follows.

- Related to the dynamics of the log-market-weights

$$\mathrm{d}(\log \mu_i(t)) \;=\; \big(\gamma_i(t) - \gamma^{\mu}(t)\big)\, \mathrm{d}t + \sum_{\nu=1}^{N} \big(\sigma_{i\nu}(t) - \sigma_{\nu}^{\mu}(t)\big)\, \mathrm{d}W_{\nu}(t)$$

for all stocks $i = 1, \ldots, n$.

Equivalently, in arithmetic terms

$$\frac{\mathrm{d}\mu_i(t)}{\mu_i(t)} = \Big(\gamma_i(t) - \gamma^{\mu}(t) + \frac{1}{2}\,\mathfrak{A}^{\mu}_{ij}(t)\Big)\,\mathrm{d}t + \sum_{\nu=1}^{N} \big(\sigma_{i\nu}(t) - \sigma^{\mu}_{\nu}(t)\big)\mathrm{d}W_{\nu}(t)\,. \tag{10}$$

It is now clear from this, that

$$\frac{\mathrm{d}\langle \mu_i, \mu_j\rangle(t)}{\mu_i(t)\mu_j(t)\mathrm{d}t} = \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \sigma^{\mu}_{\nu}(t)\big)\big(\sigma_{j\nu}(t) - \sigma^{\mu}_{\nu}(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\,\langle \log \mu_i, \log \mu_j\rangle(t) = \mathfrak{A}^{\mu}_{ij}(t)\,.$$

TWO-STOCK EXAMPLE (or "PARABLE")

Suppose there are only two, perfectly negatively correlated, stocks A and B. We toss a fair coin, independently from day to day; when the toss comes up heads, stock A doubles and stock B halves in price (and vice-versa, if the toss comes up tails).

Clearly, each stock has a growth rate of zero: holding any one of them produces nothing in the long term.

• What happens if we hold both stocks? Suppose we invest \$100 in each; one of them will rise to \$200 and the other fall to \$50, for a guaranteed total of \$250, representing a net gain of 25%; the winner has gained more than the loser has lost.

If we **rebalance** to \$125 in each stock (so as to maintain the equal proportions we started with), and keep doing this day after day, we lock in a long-term growth rate of 25%.

Indeed, taking $n = 2$ and

$$\gamma_1 = \gamma_2 = 0\,, \qquad \alpha_{11} = \alpha_{22} = -\alpha_{12} = -\alpha_{21} = 0.50$$

and

$$\pi_1 \;=\; \pi_2 \;=\; 0.50$$

in

$$\gamma^\pi \;=\; \sum_{i=1}^{n} \pi_i\,\gamma_i + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i\,\alpha_{ii} - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i\,\alpha_{ij}\,\pi_j\right)$$

$$= \;\frac{1}{2}\Big(\pi_1\big(1-\pi_1\big)\alpha_{11} + \pi_2\big(1-\pi_2\big)\alpha_{22}\Big) - \pi_1\pi_2\alpha_{12}$$

we get the same growth rate that we computed a moment ago:

$$\gamma^\pi \;=\; \gamma^\pi_* \;=\; 0.25\,.$$

MORAL OF THIS TALE

- In the presence of "sufficient intrinsic volatility", setting target weights and rebalancing to them can **capture this volatility** and **turn it into growth**.

(And this can occur even without precise estimates of model parameters, and without refined optimization.)

We have encountered several variations on this parable already, and will encounter a few more below. In particular, we shall quantify what "sufficient intrinsic volatility" means.

8. PORTFOLIO DIVERSIFICATION AND MARKET VOLATILITY AS DRIVERS OF GROWTH

Now let us suppose that, for some real number $\varepsilon > 0$, condition (7) holds:

$$\xi' \alpha(t) \xi = \xi' \sigma(t) \sigma'(t) \xi \geq \varepsilon \|\xi\|^2, \quad \forall \;\; t \in [0, \infty), \quad \xi \in \mathbb{R}^n .$$

That is, we have a strictly nondegenerate covariance structure. Then an elementary computation shows

$$\gamma^\pi(t) - \sum_{i=1}^{n} \pi_i(t) \gamma_i(t) = \gamma_*^\pi(t) \geq (\varepsilon/2) \Big(1 - \max_{1 \leq i \leq n} \pi_i(t) \Big) \geq (\varepsilon/2) \eta > 0 \, ,$$

as long as for some $\eta \in (0,1)$ we have

$$\max_{1 \leq i \leq n} \pi_i(t) \leq 1 - \eta \, .$$

To wit, such a portfolio's growth rate $\gamma^{\pi}(t)$ will dominate, and strictly, the average growth rate of the constituent assets

$$\sum_{i=1}^{n} \pi_i(t)\, \gamma_i(t)$$

(Fernholz & Shay, *Annals of Finance* (1982)):

$$\gamma^{\pi}(t) \geq \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + (\varepsilon/2)\eta \, .$$

In words: Under the above condition of "sufficient volatility", even *the slightest bit of portfolio diversification can not only decrease the portfolio's variance, as is well known, but also enhance its growth.*

We shall see below additional incarnations of this principle.

¶ To see just how significant such an enhancement can be, consider any fixed-proportion, long-only portfolio $\pi(\cdot) \equiv \mathfrak{p}$, for some vector $\mathfrak{p} \in \Delta^n$ with $1 - \max_{1 \le i \le n} \mathfrak{p}_i =: \eta > 0$, and with

$$\Delta^n := \left\{ (p_1, \cdots, p_n) : p_1 \ge 0, \cdots, p_n \ge 0, \quad p_1 + \cdots + p_n = 1 \right\}.$$

For any portfolio $\pi(\cdot)$ and $T \in (0, \infty)$, we have the identity

$$\boxed{\log \left(\frac{V^{1,\pi}(t)}{V^{1,\mu}(t)} \right) = \int_0^T \gamma_*^\pi(t)\, \mathrm{d}t + \sum_{i=1}^n \int_0^T \pi_i(t)\, \mathrm{d} \log \mu_i(t) \,.} \tag{11}$$

(This is a rather simple consequence of

$$\mathrm{d} \left(\log V^{\nu,\pi}(t) \right) = \gamma^\pi(t)\, \mathrm{d}t + \sum_{\nu=1}^N \sigma_\nu^\pi(t)\, \mathrm{d}W_\nu(t)$$

in (9), slide 25.)

From

$$\boxed{\log\left(\frac{V^{1,\pi}(t)}{V^{1,\mu}(t)}\right) = \int_0^T \gamma_*^{\pi}(t)\,\mathrm{d}t + \sum_{i=1}^{n}\int_0^T \pi_i(t)\,\mathrm{d}\log\mu_i(t)\,,}$$

of the previous slide, we get the a.s. comparisons

$$\frac{1}{T}\log\left(\frac{V^{1,\mathfrak{p}}(T)}{V^{1,\mu}(T)}\right) - \sum_{i=1}^{n}\frac{\mathfrak{p}_i}{T}\log\left(\frac{\mu_i(T)}{\mu_i(0)}\right) =$$

$$= \frac{1}{T}\int_0^T \gamma_*^{\mathfrak{p}}(t)\,\mathrm{d}t \geq \frac{\varepsilon\eta}{2} > 0\,.$$

Suppose now that the market is *coherent*, meaning that no individual stock crashes relative to the rest of the market:

$$\lim_{T\to\infty} \frac{1}{T} \log \mu_i(T) = 0 \,, \quad \forall \;\; i = 1, \cdots , n \,.$$

Then passing to the limit as $T \to \infty$ in

$$\frac{1}{T} \log \left(\frac{V^{1,\mathfrak{p}}(T)}{V^{1,\mu}(T)} \right) - \sum_{i=1}^{n} \frac{\mathfrak{p}_i}{T} \log \left(\frac{\mu_i(T)}{\mu_i(0)} \right) \geq \frac{\varepsilon\eta}{2} > 0$$

we see that the wealth corresponding to any such fixed-proportion, long-only portfolio, grows exponentially at a rate **strictly higher** than that of the overall market:

$$\liminf_{T\to\infty} \frac{1}{T} \log \left(\frac{V^{1,\mathfrak{p}}(T)}{V^{1,\mu}(T)} \right) \geq \frac{\varepsilon\eta}{2} > 0 \,, \qquad \text{a.s.}$$

REMARK:

Tom Cover's (1991) "universal portfolio"

$$\Pi_i(t) := \frac{\int_{\Delta^n} \mathfrak{p}_i \, V^{1,\mathfrak{p}}(t) \, \mathrm{d}\mathfrak{p}}{\int_{\Delta^n} V^{1,\mathfrak{p}}(t) \, \mathrm{d}\mathfrak{p}}, \quad i = 1, \cdots, n$$

has value

$$V^{1,\Pi}(t) = \frac{\int_{\Delta^n} V^{1,\mathfrak{p}}(t) \, \mathrm{d}\mathfrak{p}}{\int_{\Delta^n} \mathrm{d}\mathfrak{p}} \sim \max_{\mathfrak{p} \in \Delta^n} V^{1,\mathfrak{p}}(t) \,.$$

Note the "total agnosticism" of this portfolio regarding the details of the particular, underlying model.

✠ Up to now we have not even tried to select portfolios in an "optimal" fashion. Here then are some Portfolio Optimization problems; some of them are classical, while for others very little is known.

9. PORTFOLIO OPTIMIZATION

Problem #1: Quadratic criterion, linear constraint (Markowitz, 1952). Minimize the portfolio variance

$$\mathrm{a}^{\pi\pi}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) \alpha_{ij}(t) \pi_j(t) \tag{12}$$

among all portfolios $\pi(\cdot) \in \mathfrak{P}$ that keep the rate-of-return at least equal to a given constant: $\beta^{\pi}(t) = \sum_{i=1}^{n} \pi_i(t) \beta_i(t) \geq \beta_0$.

Problem #2: Quadratic criterion, quadratic constraint. Minimize the portfolio variance $\mathrm{a}^{\pi\pi}(t)$ of (12) among all portfolios $\pi(\cdot) \in \mathfrak{P}$ with growth-rate at least equal to a given constant γ_0 :

$$\sum_{i=1}^{n} \pi_i(t) \beta_i(t) \geq \gamma_0 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) \alpha_{ij}(t) \pi_j(t) .$$

Problem #3: *Maximize the probability of reaching a given "ceiling" $\mathfrak{c}$ before reaching a given "floor" $\mathfrak{f}$, with $0 < \mathfrak{f} < 1 < \mathfrak{c} < \infty$.*

More specifically, maximize over $\pi(\cdot) \in \mathfrak{P}$ *the probability*

$$\mathbb{P}\,[\,\mathfrak{T}^\pi_\mathfrak{c} < \mathfrak{T}^\pi_\mathfrak{f}\,]\,, \quad \text{with} \quad \mathfrak{T}^\pi_c := \inf\{\, t \ge 0 \,:\, X^\pi(t) = c\,\}\,.$$

In the case of constant coëfficients γ_i and α_{ij}, and with

Γ_n the collection of vectors $p \in \mathbb{R}^n$ with $p_1 + \cdots + p_n = 1$,

the solution to this problem is given by the vector $\pi \in \Gamma^n$ that maximizes the mean-variance, or *signal-to-noise*, ratio (Pestien & Sudderth, *Mathematics of Operations Research* 1985):

$$\frac{\gamma^\pi}{\mathrm{a}^{\pi\pi}} = \frac{\sum_{i=1}^n \pi_i(\gamma_i + \frac{1}{2}\alpha_{ii})}{\sum_{i=1}^n \sum_{j=1}^n \pi_i \alpha_{ij} \pi_j} - \frac{1}{2}\,.$$

Open Question: *How about (more) general coëfficients?*

Problem #4: *Maximize the probability* $\mathbb{P}\,[\,\mathfrak{T}^{\pi}_{\mathfrak{c}} < T \wedge \mathfrak{T}^{\pi}_{\mathfrak{f}}\,]$ *of reaching a given "ceiling"* $\mathfrak{c}$ *before reaching a given "floor"* $\mathfrak{f}$ *with* $0 < \mathfrak{f} < 1 < \mathfrak{c} < \infty$ *, by a given "deadline"* $T \in (0, \infty)$.

Always with constant coëfficients, suppose there is a vector $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_n)'$ that maximizes **both** the signal-to-noise ratio **and** the variance,

$$\frac{\gamma^{\pi}}{\mathrm{a}^{\pi\pi}} = \frac{\sum_{i=1}^{n} \pi_i (\gamma_i + \frac{1}{2}\alpha_{ii})}{\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \, \alpha_{ij} \, \pi_j} - \frac{1}{2} \qquad \text{and} \qquad \mathrm{a}^{\pi\pi} = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \, \alpha_{ij} \, \pi_j \, ,$$

over all $\pi_1 \geq 0, \ldots, \pi_n \geq 0$ with $\sum_{i=1}^{n} \pi_i = 1$.

Then the portfolio $\hat{\pi}$ is optimal for the above criterion (Sudderth & Weerasinghe, *Mathematics of Operations Research*, 1989).

This is a huge assumption; it is satisfied, for instance, under the (very stringent) condition that, for some $\beta \le 0$, we have

$$\beta_i \;=\; \gamma_i + \tfrac{1}{2}\,\alpha_{ii} \;=\; \beta\,, \quad \text{for all } \; i = 1, \ldots, n\,.$$

Open Question: *As far as I can tell, nobody seems to know the solution to this problem, if such "simultaneous maximization" is not possible.*

Problem #5: *Minimize over portfolios* $\pi(\cdot)$ *the expected time* $\mathbb{E}\,[\,\mathfrak{T}^{\pi}_{\mathfrak{c}}\,]$ *until a given "ceiling"* $\mathfrak{c} \in (1, \infty)$ *is reached.*

Again with constant coëfficients, it turns out that it is enough to maximize, over vectors $\pi \in \mathbb{R}^n$ with $\sum_{i=1}^{n} \pi_i = 1$, the drift in the equation for $\log X^{\pi}(\cdot)$, namely the portfolio growth-rate

$$\gamma^{\pi} = \sum_{i=1}^{n} \pi_i \left(\gamma_i + \tfrac{1}{2}\alpha_{ii}\right) - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \alpha_{ij} \pi_j \,.$$

(See Heath, Orey, Pestien & Sudderth, *SIAM Journal on Control & Optimization*, 1987.)

Again, how about (more) general coëfficients?

Problem #6: Suppose we can find a portfolio $\varrho^*(\cdot) \in \mathfrak{P}$ that satisfies

$$\big(p - \varrho^*(t)\big)'\big(\beta(t) - \alpha(t)\,\varrho^*(t)\big) \le 0\,, \qquad \forall \;\; p \in \Gamma_n\,, \;\; 0 \le t < \infty$$

Then for every $\pi(\cdot) \in \mathfrak{P}$ we have the numéraire-like property

$$V^{1,\pi}(\cdot)\,/\,V^{1,\varrho^*}(\cdot) \quad \text{is a supermartingale,}$$

as well as

$$\limsup_{T\to\infty} \frac{1}{T}\,\log\left(\frac{V^{1,\pi}(T)}{V^{1,\varrho^*}(T)}\right) \le 0\,, \quad \text{a.s.,}$$

$$\mathbb{E}\big[\,\log V^{1,\pi}(T)\,\big] \le \mathbb{E}\big[\,\log V^{1,\varrho^*}(T)\,\big]\,, \qquad \forall \;\; T \in (0,\infty)\,.$$

Unlike Cover's universal (and *"agnostic"*) portfolio, this one needs considerable $\gamma\nu\widetilde{\omega}\sigma\iota\varsigma$ (knowledge) of both the covariance structure of the market and its growth rates.

Discussion:

In none of these problems did we need to assume the existence of an equivalent martingale measure – or even of a deflator $Z(\cdot)$. (As Constantinos Kardaras showed in his dissertation, the solvability of quite general utility maximization problems only needs the existence of a numéraire portfolio $\varrho^*(\cdot)$.)

In most of them, we needed to "take our model quite seriously", to the extent that the solution assumed knowledge ($\gamma\nu\widetilde{\omega}\sigma\iota\varsigma$) of both the covariance structure of the market and of the assets' growth rates.

And in some (rather special) such problems, the solution only needs estimates of the covariance structure of the market – not a trivial task, but much easier than estimating growth rates of individual assets.

FUNCTIONALLY GENERATED PORTFOLIOS

Let us recall the expression

$$\log\left(\frac{V^{1,\pi}(t)}{V^{1,\mu}(t)}\right) = \int_0^T \gamma_*^{\pi}(t)\,\mathrm{d}t + \sum_{i=1}^{n}\int_0^T \pi_i(t)\,\mathrm{d}\log\mu_i(t)$$

of (11) for the relative performance of an arbitrary portfolio $\pi(\cdot)$ with respect to the market.

There is a class of very special portfolios $\pi(\cdot)$, described solely in terms of the market weights $\mu_1(\cdot),\ldots,\mu_1(\cdot)$ *and nothing else,* for which the stochastic integrals disappear completely from the right-hand side and the remaining (Lebesgue) also depend solely on market weights and have a particular sign. This allows for pathwise comparisons of relative performance; or, to put it a bit differently, for the construction of arbitrage relative to the market, under appropriate conditions.

We start with a concave, smooth function $\mathbf{S} : \Delta_+^n \to \mathbb{R}_+$, and consider *the portfolio* $\pi^{\mathbf{S}}(\cdot)$ *generated by it:*

$$\pi_i^{\mathbf{S}}(t) \;:=\; \mu_i(t) \left(D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) \cdot D_j \log \mathbf{S}(\mu(t)) \right) . \tag{13}$$

Then an application of Itô's rule gives the "master equation"

$$\log \left(\frac{V^{\pi^{\mathbf{S}}}(T)}{V^{\mu}(T)} \right) = \log \left(\frac{\mathbf{S}(\mu(T))}{\mathbf{S}(\mu(0))} \right) + \int_0^T \mathfrak{g}(t)\, dt \, . \tag{14}$$

Here, thanks to our assumptions, the quantity $\mathfrak{g}(\cdot)$ is nonnegative:

$$\mathfrak{g}(t) \;:=\; \frac{-1}{\mathbf{S}(\mu(t))} \sum_i \sum_j D_{ij}^2 \mathbf{S}(\mu(t)) \cdot \mu_i(t) \mu_j(t)\, \mathfrak{A}_{ij}^{\mu}(t) \, . \tag{15}$$

$$\pi_i^{\mathbf{S}}(t) := \mu_i(t) \left(D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) \cdot D_j \log \mathbf{S}(\mu(t)) \right)$$

$$\mathfrak{g}(t) := \frac{-1}{\mathbf{S}(\mu(t))} \sum_i \sum_j D^2_{ij} \mathbf{S}(\mu(t)) \cdot \mu_i(t) \mu_j(t)\, \mathfrak{A}^{\mu}_{ij}(t)$$

Recall the entries of the relative covariance matrix:

$$\mathfrak{A}^{\mu}_{ij}(t) = \sum_{\nu=1}^{d} \left(\sigma_{i\nu}(t) - \sigma^{\mu}_{\nu}(t)\right)\left(\sigma_{j\nu}(t) - \sigma^{\mu}_{\nu}(t)\right) = \frac{\mathrm{d}\langle \mu_i, \mu_j \rangle(t)}{\mu_i(t)\mu_j(t)\mathrm{d}t}.$$

Significance: *Stochastic integrals have been excised in (14)* , i.e.,

$$\log \left(\frac{V^{\pi^{\mathbf{S}}}(T)}{V^{\mu}(T)} \right) = \log \left(\frac{\mathbf{S}(\mu(T))}{\mathbf{S}(\mu(0))} \right) + \int_0^T \mathfrak{g}(t)\, dt\,,$$

and we can begin to make comparisons that are valid with probability one (a.s.)...

Proof of the "Master Equation" (14): To ease notation we set

$$h_i(t) := D_i \log \mathbf{S}(\mu(t)) \qquad \text{and} \qquad N(t) := \sum_{j=1}^{n} \mu_j(t)\, h_j(t)\,,$$

so (13), that is

$$\pi_i(t) \,=\, \mu_i(t) \left(D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) \cdot D_j \log \mathbf{S}(\mu(t)) \right) \,,$$

reads:

$$\pi_i(t) = \big(h_i(t) + N(t)\big)\mu_i(t)\,, \qquad i = 1, \cdots n\,.$$

Then the terms on the right-hand side of

$$\mathrm{d}\log\left(\frac{V^{\pi}(t)}{V^{\mu}(t)}\right) = \sum_{i=1}^{n}\frac{\pi_i(t)}{\mu_i(t)}\,\mathrm{d}\mu_i(t) - \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i(t)\pi_j(t)\mathfrak{A}_{ij}^{\mu}(t)\right)\mathrm{d}t\,,$$

an equivalent version of

$$\log\left(\frac{V^{1,\pi}(t)}{V^{1,\mu}(t)}\right) = \int_0^T \gamma_*^{\pi}(t)\,\mathrm{d}t + \sum_{i=1}^{n}\int_0^T \pi_i(t)\,\mathrm{d}\log\mu_i(t)$$

in (11), become

$$\sum_{i=1}^{n}\frac{\pi_i(t)}{\mu_i(t)}\,\mathrm{d}\mu_i(t) = \sum_{i=1}^{n} h_i(t)\,\mathrm{d}\mu_i(t) + N(t)\cdot\mathrm{d}\Big(\sum_{i=1}^{n}\mu_i(t)\Big)$$

$$= \sum_{i=1}^{n} h_i(t)\,\mathrm{d}\mu_i(t)\,,$$

whereas $\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i(t)\pi_j(t)\mathfrak{A}_{ij}^{\mu}(t)$ becomes

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\big(h_i(t)+N(t)\big)\big(h_j(t)+N(t)\big)\mu_i(t)\mu_j(t)\mathfrak{A}_{ij}^{\mu}(t)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} h_i(t)h_j(t)\mu_i(t)\mu_j(t)\mathfrak{A}_{ij}^{\mu}(t)\,.$$

(Again, because $\mu(t)$ spans the null subspace of $\{\mathfrak{A}_{ij}^{\mu}(t)\}_{1\le i,j\le n}$.) Thus, using the dynamics of market weights in (10), the above equation gives

$$\mathrm{d}\log\left(\frac{V^{\pi}(t)}{V^{\mu}(t)}\right) = \sum_{i=1}^{n} h_i(t)\,\mathrm{d}\mu_i(t) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} h_i(t)h_j(t)\mu_i(t)\mu_j(t)\mathfrak{A}_{ij}^{\mu}(t)\,\mathrm{d}t\,. \qquad (16)$$

On the other hand, we have

$$D^2_{ij} \log \mathbf{S}(x) \;=\; \big(D^2_{ij}\mathbf{S}(x)/\mathbf{S}(x)\big) - D_i \log \mathbf{S}(x) \cdot D_j \log \mathbf{S}(x)\,,$$

so we get

$$\mathrm{d} \log \mathbf{S}(\mu(t)) = \sum_{i=1}^{n} h_i(t)\,\mathrm{d}\mu_i(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} D^2_{ij} \log \mathbf{S}(\mu(t))\,\mathrm{d}\langle \mu_i, \mu_j\rangle(t)$$

$$= \sum_{i=1}^{n} h_i(t)\mathrm{d}\mu_i(t)$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{D^2_{ij}\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(t))} - h_i(t)h_j(t)\Big)\mu_i(t)\mu_j(t)\mathfrak{A}^{\mu}_{ij}(t)\mathrm{d}t$$

by Itô's rule. Comparing this last expression with (16) and recalling the notation of (15), we deduce (14), namely:

$$\mathrm{d} \log \mathbf{S}(\mu(t) = \mathrm{d} \log \big(V^{\pi}(t)/V^{\mu}(t)\big) - \mathfrak{g}(t)\mathrm{d}t\,.$$

For instance:

- $\mathbf{S}(\cdot) \equiv w$, a positive constant, generates the *market* portfolio.
- $\mathbf{S}(x) = w_1 x_1 + \cdots + w_n x_n$ generates the *passive* portfolio that buys at time $t = 0$, and holds up until time $t = T$, a fixed number of shares w_i in each asset $i = 1, \cdots, n$.

  (The market portfolio corresponds to the special case

  $$w_1 = \cdots = w_n = w$$

  of equal numbers of shares across assets.)

- The geometric mean

$$\mathbf{S}(x) \equiv \mathbf{G}(x) \ := \ (x_1 \cdots x_n)^{1/n}$$

generates the *equal weighted* portfolio

$$\varphi_i(\cdot) \equiv 1/n\,, \qquad i = 1, \cdots, n\,,$$

with drift equal to the excess growth rate:

$$\mathfrak{g}^{\varphi}(\cdot) \equiv \gamma^*_{\varphi}(\cdot) \ = \ \frac{1}{2n} \left(\sum_{i=1}^{n} \alpha_{ii}(\cdot) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}(\cdot) \right) .$$

Discussion on Equal Weighting:

The equal weighted portfolio $\varphi(\cdot)$ maintains the same weights in all stocks at all times; it accomplishes this by selling those stocks whose price rises relative to the rest, and by buying stocks whose price falls relative to the others. Because of this built-in aspect of *"buying-low-and-selling-high"*, the equally weighted portfolio can be used as a simple prototype for studying systematically the performance of *statistical arbitrage* strategies in equity markets; see Fernholz & Maguire (2006) for details.

It has been observed empirically, that such a portfolio can outperform the market (we shall see a rigorous result along these lines in a short while). Of course, implementing such a strategy necessitates very frequent trading and can incur substantial transaction costs for an investor who is not a broker/dealer.

It can also involve considerable risk: whereas the second term on the right-hand side of

$$\log V^{1,\varphi}(T) = \frac{1}{n} \log \left(\frac{X_1(T) \cdots X_n(T)}{X_1(0) \cdots X_n(0)} \right) + \int_0^T \gamma_\varphi^*(t)\, dt \,,$$

or of

$$\log \left(\frac{V^{1,\varphi}(T)}{V^{1,\mu}(T)} \right) = \frac{1}{n} \log \left(\frac{\mu_1(T) \cdots \mu_n(T)}{\mu_1(0) \cdots \mu_n(0)} \right) + \int_0^T \gamma_\varphi^*(t)\, dt \,,$$

is increasing it T, the first terms on the right-hand sides of these expressions can fluctuate quite a bit.

- The diversity-weighted portfolio $\mathfrak{D}^{(p)}(\cdot)$ of

$$\mathfrak{D}_i^{(p)}(t) = \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}\,, \qquad i = 1, \cdots, n$$

with $0 < p < 1$, stands between these two extremes, of capitalization weighting (as in S&P 500) and of equal weighting (as in the Value-Line Index). It is generated by the concave function

$$\mathbf{S}^{(p)}(x) := \left(x_1^p + \cdots + x_n^p\right)^{1/p},$$

and has drift proportional to the excess growth rate:

$$\mathfrak{g}(\cdot) \equiv (1-p)\,\gamma_*^{\mathfrak{D}^{(p)}}(\cdot)\,.$$

$$\mathfrak{D}_i^{(p)}(t) = \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}, \qquad i = 1, \cdots, n$$

With $p = 0$ this becomes equal weighting $\varphi_i(\cdot) \equiv 1/n$, $1 \le i \le n$.
With $p = 1$ we get the market portfolio $\mu(\cdot)$.

This portfolio *over-weighs the small-cap stocks and under-weighs the large-cap stocks,* relative to the market weights (*the valleys are exalted,* at least somewhat, *and the mountains and hills made low*(er).). It tries to capture some of the "buy-low/sell-high" characteristics of equal weighting, but without deviating too much from market capitalizations and without incurring a lot of trading costs or excessive risk.

It can be viewed as an "enhanced market portfolio" or "enhanced capitalization index", in this sense.

- Another way to "bridge" the extremes of equal-weighting and capitalization-weighting goes as follows. Consider the *geometric mean*

$$\mathbf{G}(x) := (x_1 \cdots x_n)^{1/n}$$

and, for any given $c \in (0, \infty)$, its modification

$\mathbf{G}_c(x) := c + \mathbf{G}(x)$, which satisfies: $c < \mathbf{G}_c(x) \leq c + (1/n)$.

This modified geometric mean function generates the *modified equally-weighted portfolio*

$$\varphi_i^{(c)}(t) = \frac{c}{c + \mathbf{G}(\mu(t))} \cdot \frac{1}{n} + \frac{\mathbf{G}(\mu(t))}{c + \mathbf{G}(\mu(t))} \cdot \mu_i(t),$$

for $i = 1, \cdots, n$ that we saw already in (6). These weights are convex combination of the equal-weighting and market portfolios; and we have

$$\mathfrak{g}^{\varphi^{(c)}}(t) = \frac{\mathbf{G}(\mu(t))}{c + \mathbf{G}(\mu(t))} \gamma_\varphi^*(t).$$

- In a similar spirit, consider the *entropy function*

$$\mathbf{H}(x) := -\sum_{i=1}^{n} x_i \log x_i \,, \quad x \in \Delta_+^n \,.$$

This entropy function generates the *entropic* portfolio $\mathfrak{E}(\cdot)$, with weights

$$\mathfrak{E}_i(t) = \frac{-\mu_i(t)\,\log \mu_i(t)}{\mathbf{H}(\mu(t))}\,, \qquad i = 1, \cdots, n$$

and drift-process

$$\mathfrak{g}^{\mathfrak{E}}(t) = \frac{\gamma_\mu^*(t)}{\mathbf{H}(\mu(t))}\,.$$

- Now take again the *entropy function*

$$\mathbf{H}(x) := -\sum_{i=1}^{n} x_i \log x_i \,, \quad x \in \Delta_+^n$$

and, for any given $c \in (0, \infty)$, look at its modification

$$\mathbf{S}_c(x) := c + \mathbf{H}(x), \quad \text{which satisfies:} \quad c < \mathbf{S}_c(x) \le c + \log n \,.$$

This modified entropy function generates the *modified entropic* portfolio $\mathfrak{E}^{(c)}(\cdot)$ of (4), with weights

$$\mathfrak{E}_i^{(c)}(t) = \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{c + \mathbf{H}(\mu(t))} \,, \qquad i = 1, \cdots, n$$

and drift-process given by

$$\mathfrak{g}^{\mathfrak{E}^{(c)}}(t) = \frac{\gamma_\mu^*(t)}{c + \mathbf{H}(\mu(t))} \,.$$

11. SUFFICIENT INTRINSIC VOLATILITY LEADS TO ARBITRAGE RELATIVE TO THE MARKET

Broadly accepted practitioner wisdom upholds that *sufficient volatility creates growth opportunities in a financial market.*

We have already encountered an instance of this phenomenon in section 8: we saw there that, in the presence of a strong non-degeneracy condition on the market's covariance structure, "reasonably diversified" long-only portfolios with constant weights represent superior long-term growth opportunities relative to the overall market.

We shall examine in Proposition 1 below another instance of this phenomenon.

More precisely, we shall try again to put the above intuition on a precise quantitative basis, by identifying now the *excess growth rate of the market portfolio* – which also measures the market's *intrinsic volatility* – as a *driver of growth*; to wit, as a quantity whose "availability" or "sufficiency" (boundedness away from zero) can lead to opportunities for strong arbitrage and for superior long-term growth, relative to the market.

Proposition 1: *Assume that over* $[0, T]$ *there is "sufficient intrinsic volatility" (excess growth):*

$$\int_0^T \gamma_*^{\mu}(t)dt \ge hT\,, \quad \text{or} \quad \gamma_*^{\mu}(t) \ge h\,, \quad 0 \le t \le T$$

holds a.s., for some constant $h > 0$*. Take*

$$T > T_* := \frac{\mathbf{H}(\mu(0))}{h}\,, \qquad \text{and} \qquad \mathbf{H}(x) := -\sum_{i=1}^{n} x_i \log x_i$$

the entropy function. Then the modified entropic portfolio (from a couple of slides ago)

$$\mathfrak{E}_i^{(c)}(t) := \frac{\mu_i(t)\,(c - \log \mu_i(t))}{\sum_{j=1}^{n} \mu_j(t)\,(c - \log \mu_j(t))}\,, \quad i = 1, \cdots, n$$

is generated by the function

$$\mathbf{S}_c(x) := c + \mathbf{H}(x)$$

on Δ_+^n*; and for* $c > 0$ *sufficiently large, it represents a strong arbitrage relative to the market.*

- *Sketch of Argument for Proposition 1:* Note that the function $\mathbf{S}_c(\cdot) := c + \mathbf{H}(\cdot)$ is bounded both from above and below:

$$0 < c < \mathbf{S}_c(x) \leq c + \log n \,, \qquad x \in \Delta^n_+ \,.$$

The master equation now shows that

$$\log\left(\frac{V^{1,\mathfrak{E}^{(c)}}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{c + \mathbf{H}(\mu(T))}{c + \mathbf{H}(\mu(0))}\right) + \int_0^T \mathfrak{g}^{\mathfrak{E}^{(c)}}(t)\,\mathrm{d}t$$

is strictly positive, provided

$$T > \frac{1}{h}\big(c + \log n\big) \log\left(1 + \frac{\log n}{c}\right) \longrightarrow \frac{\log n}{h}$$

as $c \to \infty$.

This is because the first term on the right-hand side of

$$\log\left(\frac{V^{1,\mathfrak{E}(c)}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{c+\mathbf{H}(\mu(T))}{c+\mathbf{H}(\mu(0))}\right) + \int_0^T \mathfrak{g}^{\mathfrak{E}(c)}(t)\,\mathrm{d}t$$

dominates

$$-\log\left(\frac{c+\log n}{c}\right)$$

and, under the conditions of the proposition, the second term

$$\int_0^T \mathfrak{g}^{\mathfrak{E}(c)}(t)\,\mathrm{d}t = \cdots = \int_0^T \frac{\gamma_*^{\mu}(\cdot)}{c+\mathbf{H}(\cdot)}\,\mathrm{d}t \geq \int_0^T \frac{\gamma_*^{\mu}(\cdot)}{c+\log n}\,\mathrm{d}t$$

dominates $hT\,/\,(c+\log n)$.

To put it a bit differently: *if you have a constant wind on your back, sooner all later you'll overtake any obstacle* – e.g., the constant $\log\big((c+\log n)/c\big)$.

This leads to strong arbitrage relative to the market, for sufficiently large $T > 0$; indeed to

$$\mathbb{P}\Big(V^{1,\mathfrak{E}^{(c)}}(T) > V^{1,\mu}(T) \Big) = 1 \,.$$

(Intuition, as before: you can generate such relative arbitrage if there is "enough intrinsic volatility" in the market... .)

Major Open Question: Is such relative arbitrage possible over arbitrary time-horizons, under the conditions of Proposition 1 ?

We shall discuss below two special cases, where the answer to this question is known – and is affirmative.

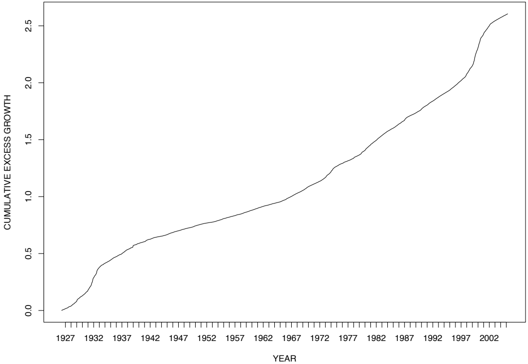

Figure 1: Cumulative Excess Growth $\int_0^{\cdot} \gamma_*^{\mu}(t)\,\mathrm{d}t$ for the U.S. Stock Market during the period 1926-1999.

The previous figure plots the cumulative excess growth $\int_0^{\cdot} \gamma_\mu^*(t)\, \mathrm{d}t$ for the U.S. equities market over most of the twentieth century. Note the conspicuous bumps in the curve, first in the Great Depression period in the early 1930s, then again in the "irrational exuberance" period at the end of the century. The data used for this chart come from the monthly stock database of the Center for Research in Securities Prices (CRSP) at the University of Chicago.

The market we construct consists of the stocks traded on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX) and the NASDAQ Stock Market, after the removal of all REITs, all closed-end funds, and those ADRs not included in the S&P 500 Index. Until 1962, the CRSP data included only NYSE stocks. The AMEX stocks were included after July 1962, and the NASDAQ stocks were included at the beginning of 1973. The number of stocks in this market varies from a few hundred in 1927 to about 7500 in 2005.

Proposition 2:

Introduce the "modified intrinsic volatility"

$$\zeta_*(t) := \big(\mu_1(t)\cdots\mu_n(t)\big)^{1/n}\left[\sum_{i=1}^n \alpha_{ii}(t) - \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \alpha_{ij}(t)\right]$$

and assume that over the given horizon $[0,T]$ *we have a.s.:*

$$\int_0^T \zeta_*(t)\mathrm{d}t \geq h\,T\,, \quad \text{or} \quad \boxed{\zeta_*(t) \geq h\,, \quad 0 \leq t \leq T}$$

for some constant $h > 0$. *Then, with* $\mathbf{Q}(t) := \big(\mu_1(t)\cdots\mu_n(t)\big)^{1/n}$ *and for sufficiently large* $c > 0$, *the modified equally-weighted portfolio of (6)*

$$\varphi_i^{(c)}(t) = \frac{c}{c+\mathbf{Q}(t)}\cdot\frac{1}{n} + \frac{\mathbf{Q}(t)}{c+\mathbf{Q}(t)}\cdot\mu_i(t)\,, \qquad i = 1,\cdots,n\,,$$

is an arbitrage relative to the market over $[0,T]$, *provided* $T > (2n^{1-(1/n)})/h$.

The proof is rather similar to that of Proposition 1. The modified equally-weighted portfolio is generated by the function $c+\mathbf{Q}(\cdot)$, and we use the "master formula" as before.

12. NOTIONS OF MARKET DIVERSITY

Major Open Question: Is such relative arbitrage possible over arbitrary time-horizons, under the conditions of Proposition 1 ?

$$\int_0^T \gamma_*^\mu(t)\,\mathrm{d}t \geq hT\,, \quad \text{or} \quad \gamma_*^\mu(t) \geq h\,, \;\; 0 \leq t \leq T\,.$$

Partial Answer #1: YES, if the variance/covariance matrix $\alpha(\cdot) = \sigma(\cdot)\sigma'(\cdot)$ has all its eigenvalues bounded away from zero and infinity: to wit, if we have (a.s.)

$$\kappa ||\,\xi||^2 \;\leq\; \xi'\alpha(t)\xi \;\leq\; \mathcal{K}||\,\xi||^2\,, \qquad \forall\; t \geq 0\,, \;\; \xi \in \mathbb{R}^d \tag{17}$$

for suitable constants $0 < \kappa < \mathcal{K} < \infty$.

In this case one can show (Bob Fernholz, Kostas Kardaras)

$$\boxed{\frac{\kappa}{2}\left(1-\pi_{(1)}(t)\right) \le \gamma_*^{\pi}(t) \le 2\mathcal{K}\left(1-\pi_{(1)}(t)\right)} \tag{18}$$

for the maximal weight of any long-only portfolio $\pi(\cdot)$, namely

$$\pi_{(1)}(t) := \max_{1\le i\le n} \pi_i(t)\,.$$

Thus, under the structural assumption

$$\kappa||\,\xi||^2 \le \xi'\alpha(t)\xi \le \mathcal{K}||\,\xi||^2\,, \qquad \forall\; t\ge 0\,,\quad \xi\in\mathbb{R}^d$$

of (17), the "sufficient intrinsic volatility" (a.s.) condition of Proposition 1, namely

$$\int_0^T \gamma_*^\mu(t)dt \geq hT\,, \qquad \text{or} \qquad \gamma_*^\mu(t) \geq h\,, \;\; 0 \leq t \leq T\,,$$

is equivalent to the (a.s.) requirement of **Market Diversity**

$$\int_0^T \mu_{(1)}(t)dt \leq (1-\delta)T\,, \qquad \text{or} \qquad \boxed{\max_{0\leq t\leq T} \mu_{(1)}(t) \leq 1-\delta}$$

for some $\delta \in (0,1)$ (weak diversity and strong diversity, respectively).

(The maximal relative capitalization never gets above a certain percentage. In the S&P 500 universe, no company has ever been more than 15% of the total market capitalization; in the last 40 years, this has been more like 6%.)

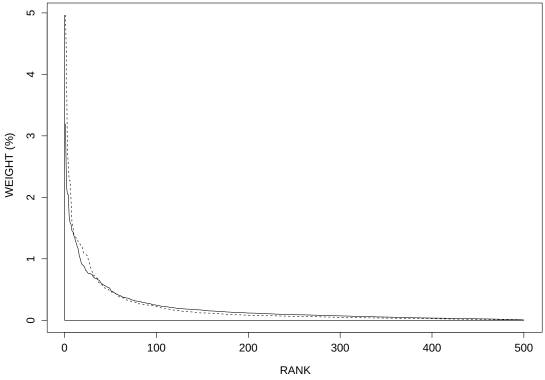

Figure 2: Capital Distribution for the S&P 500 Index. December 30, 1997 (solid line), and December 29, 1999 (broken line).

Proposition 3:

Suppose (weak) diversity prevails, and the lowest eigenvalue of the covariance matrix is bounded away from zero. For fixed $p \in (0,1)$ *, consider the simple "diversity-weighted" portfolio*

$$\mathfrak{D}_i^{(p)}(t) \equiv \mathfrak{D}_i(t) := \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}, \quad \forall \;\; i = 1, \ldots, n,$$

generated by the concave function

$$\mathbf{S}^{(p)}(x) \equiv \mathbf{S}(x) = (x_1^p + \cdots + x_n^p)^{1/p}.$$

Then this portfolio leads to arbitrage relative to the market, over sufficiently long time horizons.

With $p = 0$ this becomes equal weighting $\varphi_i(\cdot) \equiv 1/n, \; 1 \leq i \leq n$.
With $p = 1$ we get the market portfolio $\mu(\cdot)$.

(Recall in this vein the modified equal-weighted portfolio of (6), which "interpolates" between equal-weighting and cap-weighting in a rather different manner.)

With respect to the market portfolio, this "diversity-weighted" portfolio

$$\mathfrak{D}_i^{(p)}(t) \equiv \mathfrak{D}_i(t) := \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}\,, \qquad \forall \;\; i = 1, \ldots, n\,,$$

de-emphasizes the "upper (big cap) end" of the market, and over-emphasizes the "lower (small cap) end" – but observes all relative rankings. It does all this in a completely passive way, without estimating or optimizing anything.

. Appropriate modifications of this rule generate such arbitrage over *arbitrary* time-horizons; for detais, see FKK (2005).

For extensive discussion of the actual performance of this "diversity-weighted portfolio" as well as of the "pure entropic portfolio" (with $c = 0$) we saw before, see Fernholz (2002).

Proof of Proposition 3: For this *"diversity-weighted"* portfolio $\mathfrak{D}^{(p)}(\cdot)$ we have from the "master equation" (14) the formula

$$\boxed{\log\left(\frac{V^{1,\mathfrak{D}^{(p)}}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{\mathbf{S}^{(p)}(\mu(T))}{\mathbf{S}^{(p)}(\mu(0))}\right) + (1-p)\int_0^T \gamma_*^{\mathfrak{D}^{(p)}}(t)\,dt\ .}$$

- First term on RHS tends to be mean-reverting, and is certainly bounded:

$$1 = \sum_{j=1}^{n} x_j \leq \sum_{j=1}^{n} (x_j)^p = \Big(\mathbf{S}^{(p)}(x)\Big)^p \leq n^{1-p}\,.$$

Measure of Diversity: minimum occurs when one company is the entire market, maximum when all companies have equal relative weights.

• We remarked already, that the biggest weight of $\mathfrak{D}^{(p)}(\cdot)$ does not exceed the largest market weight:

$$\mathfrak{D}^{(p)}_{(1)}(t) := \max_{1 \le i \le n} \mathfrak{D}^{(p)}_i(t) = \frac{\big(\mu_{(1)}(t)\big)^p}{\sum_{k=1}^n \big(\mu_{(k)}(t)\big)^p} \le \mu_{(1)}(t)\,.$$

By weak diversity over $[0, T]$, there is a number $\delta \in (0,1)$ for which

$$\int_0^T \big(1 - \mu_{(1)}(t)\big)\,\mathrm{d}t > \delta\, T$$

holds.

Thus, from the strict non-degeneracy of the covariance matrix we have a.s.

$$\boxed{\frac{\kappa}{2}\left(1-\pi_{(1)}(t)\right) \le \gamma_*^{\pi}(t)}$$

as in (18), and thus:

$$\frac{2}{\kappa}\int_0^T \gamma_*^{\mathfrak{D}^{(p)}}(t)\,\mathrm{d}t \ge \int_0^T \left(1-\mathfrak{D}_{(1)}^{(p)}(t)\right)\mathrm{d}t \ge \int_0^T \left(1-\mu_{(1)}(t)\right)\mathrm{d}t > \delta T.$$

- From these two observations we get

$$\log\left(\frac{V^{1,\mathfrak{D}^{(p)}}(T)}{V^{1,\mu}(T)}\right) > (1-p)\left[\frac{\kappa T}{2}\cdot\delta - \frac{1}{p}\cdot\log n\right],$$

so for a time-horizon

$$T > T_* := (2\log n)/(p\kappa\delta)$$

sufficiently large, the RHS is strictly positive. □

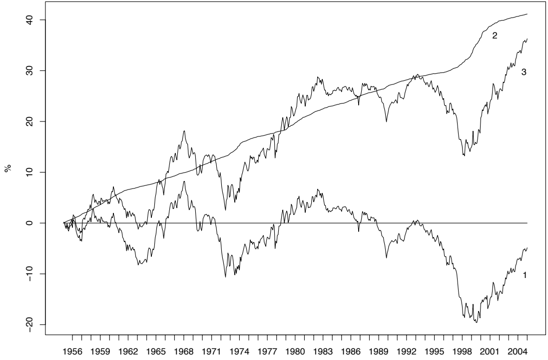

Figure 3: Simulation of a diversity-weighted portfolio, 1956–2005.
(1: generating function; 2: drift process; 3: relative return.)

$$\log\left(\frac{V^{1,\mathfrak{D}^{(p)}}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{\mathbf{S}^{(p)}(\mu(T))}{\mathbf{S}^{(p)}(\mu(0))}\right) + (1-p)\int_0^T \gamma_*^{\mathfrak{D}^{(p)}}(t)\,dt \ .$$

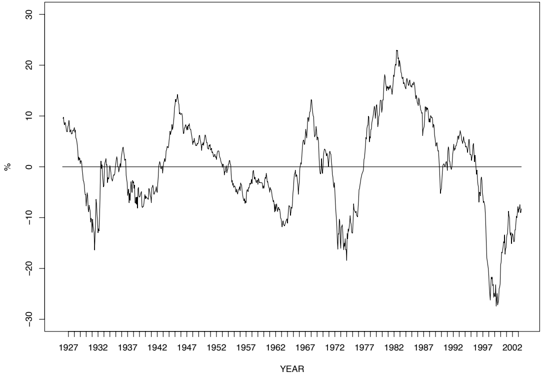

Figure 4: Cumulative Change in Market Diversity, 1927-2004.

- **Remark:** *Consider a market that satisfies the strong non-degeneracy condition as in (7), for some* $\kappa \in (0, \infty)$ *:*

$$\xi' \alpha(t) \xi = \xi' \sigma(t) \sigma'(t) \xi \geq \kappa \|\xi\|^2, \quad \forall \;\; t \in [0, \infty)\,, \quad \xi \in \mathbb{R}^n\,.$$

If all its stocks $i = 1, \ldots, n$ *have the same growth-rate* $\gamma_i(\cdot) \equiv \gamma(\cdot)$*, then*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \gamma_*^\mu(t)\, \mathrm{d}t \; = \; 0, \quad \text{a.s.}$$

In particular, such a market *cannot be diverse on long time horizons*: once in a while a single stock dominates such a market, then recedes; sooner or later another stock takes its place as absolutely dominant leader; and so on.

The same can be seen to be true for a market that satisfies the above strong nondegeneracy condition as in (7) and its assets have *constant,* though not necessarily equal, growth rates.

- Here is a quick argument: from $\gamma_i(\cdot) \equiv \gamma(\cdot)$ and $X(\cdot) = X_1(\cdot) + \cdots X_n(\cdot)$ we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X(T) - \int_0^T \gamma^\mu(t)\mathrm{d}t\right) = 0\,,$$

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma(t)\mathrm{d}t\right) = 0$$

for all $1 \le i \le n$. But then

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_{(1)}(T) - \int_0^T \gamma(t)\mathrm{d}t\right) = 0\,, \quad \text{holds a.s.}$$

for the biggest stock $X_{(1)}(\cdot) := \max_{1\le i\le n} X_i(\cdot)$, and we note $X_{(1)}(\cdot) \le X(\cdot) \le n\, X_{(1)}(\cdot)$.

Therefore, from $X_{(1)}(\cdot) \leq X(\cdot) \leq n X_{(1)}(\cdot)$ we deduce

$$\lim_{T\to\infty} \frac{1}{T} \Big(\log X(T) - \log X_{(1)}(T) \Big) = 0 , \quad \text{thus}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma^\mu(t) - \gamma(t)\big)\, \mathrm{d}t \;=\; 0 .$$

But

$$\gamma^\mu(t) = \sum_{i=1}^n \mu_i(t)\gamma(t) + \gamma_*^\mu(t) = \gamma(t) + \gamma_*^\mu(t) ,$$

because all growth rates are equal. □

13. STABILIZATION BY VOLATILITY

Major Open Question: Is such relative arbitrage possible over arbitrary time-horizons, under the conditions of Proposition 1 ?

$$\int_0^T \gamma_*^\mu(t)dt \geq hT\,, \quad \text{or} \quad \gamma_*^\mu(t) \geq h\,, \;\; 0 \leq t \leq T\,.$$

Partial Answer #2: YES, for the (non-diverse) so-called *VOLATILITY-STABILIZED* model that we broach now.

Consider the abstract market model

$$\boxed{\mathrm{d}\Big(\log X_i(t)\Big) = \frac{\alpha\,\mathrm{d}t}{2\,\mu_i(t)} + \frac{1}{\sqrt{\mu_i(t)}}\,\mathrm{d}W_i(t)}$$

for $i = 1, \cdots, n$ with $d = n \geq 2$ and $\alpha \geq 0$.

In other words, we assign the largest volatilities and the largest log-drifts to the smallest stocks.

This model amounts to solving in the positive orthant of $\mathbb{R}^n$ the system of degenerate stochastic differential equations, for $i = 1, \cdots, n$:

$$\mathrm{d}X_i(t) = \frac{1+\alpha}{2}\Big(X_1(t) + \cdots + X_n(t)\Big)\,\mathrm{d}t + \sqrt{X_i(t)\Big(X_1(t) + \cdots + X_n(t)\Big)} \cdot \mathrm{d}W_i(t)\,.$$

General theory: Bass & Perkins (TAMS 2002). Shows this system has a weak solution, unique in distribution, so the model is well-posed.

Better still: It is possible to describe this solution fairly explicitly, in terms of Bessel processes.

- Since we have

$$\alpha_{ij}(t) \,=\, \frac{\delta_{ij}}{\mu_i(t)}$$

in this model, an elementary computation gives the quantities

$$\gamma_*^{\mu}(t) \,=\, \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\big(1-\mu_i(t)\big)\,\alpha_{ij}(t) \,=\, \frac{n-1}{2} \,=:\, h > 0\,,$$

$$\mathrm{a}^{\mu\mu}(\cdot) \,\equiv\, 1$$

for the market portfolio $\mu(\cdot)$, and

$$\gamma^{\mu}(\cdot) \,\equiv\, \frac{(1+\alpha)n-1}{2} \,=:\, \gamma > 0\,.$$

Despite the erratic, widely fluctuating behavior of individual stocks, the overall market performance is remarkably stable. In particular, the total market capitalization is

$$X(t) = X_1(t) + \ldots + X_n(t) = x \cdot e^{\gamma t + B(t)},$$

for the scalar Brownian motion

$$B(t) := \sum_{\nu=1}^{n} \int_0^t \sqrt{\mu_\nu(s)}\, \mathrm{d}W_\nu(s), \qquad 0 \le t < \infty.$$

• We call this phenomenon **stabilization by volatility**: big volatility swings for the smallest stocks, and smaller volatility swings for the largest stocks, end up stabilizing the overall market by producing constant, positive overall growth and variance.

(Note $\kappa = 1$ but $\mathcal{K} = \infty$, so

$$\kappa ||\, \xi ||^2 \le \xi' \alpha(t) \xi \le \mathcal{K} ||\, \xi ||^2, \quad \forall\ t \ge 0,\ \xi \in \mathbb{R}^d$$

in (17), slide 73, fails.)

• We saw that the condition $\gamma_*^\mu(\cdot) \geq h > 0$ of Proposition 1 is satisfied here, with $h = (n-1)/2$. Thus the model admits arbitrage relative to the market, at least on time-horizons $[0, T]$ with

$$T > T_*, \qquad \text{where} \qquad T_* := \frac{2\,\mathbf{H}(\mu(0))}{n-1} < \frac{2 \log n}{n-1}.$$

The upper estimate $(2 \log n)/(n-1)$ is a rather small number if $n = 5000$ as in Wilshire 5000.

• This makes plausible the earlier claim, proved by A. Banner and D. Fernholz (*Annals of Finance* 2008) not just for the volatility-stabililized model but for quite general growth rates in

$$\mathrm{d}\Big(\log X_i(t)\Big) = \gamma_i(t)\,\mathrm{d}t + \frac{1}{\sqrt{\mu_i(t)}}\,\mathrm{d}W_i(t), \quad i = 1, \cdots, n$$

that such arbitrage is now possible on *any* time-horizon.

- On the other hand, the condition

$$\zeta_*(\cdot) \geq h > 0$$

of Proposition 2 (slide 66) is also satisfied here, with $h = n - 1$. This follows from the geometric mean / harmonic mean inequality

$$\zeta_*(t) \,=\, \big(\mu_1(t)\cdots\mu_n(t)\big)^{1/n} \left[\sum_{i=1}^n \alpha_{ii}(t) - \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \alpha_{ij}(t) \right]$$

$$= \big(\mu_1(t)\cdots\mu_n(t)\big)^{1/n} \cdot \sum_{i=1}^n \left(1 - \frac{1}{n}\right) \alpha_{ii}(t)$$

$$\geq \frac{n}{\frac{1}{\mu_1(t)} + \cdots + \frac{1}{\mu_n(t)}} \cdot \frac{n-1}{n} \sum_{i=1}^n \frac{1}{\mu_i(t)} = n - 1\,.$$

- *What is the long-term-growth behavior of an* **individual** *stock?*

A little bit of Stochastic Analysis provides the Representations

$$X_i(t) \,=\, \Big(\mathfrak{R}_i(\Lambda(t)) \Big)^2 , \qquad 0 \le t < \infty \,, \quad i = 1, \cdots , n$$

and

$$X(t) \,=\, X_1(t) + \cdots + X_n(t) = x \, e^{\,\gamma t + B(t)} \,=\, \Big(\mathfrak{R}(\Lambda(t)) \Big)^2 .$$

Here

$$4 \, \Lambda(t) \,:=\, \int_0^t X(s) \, ds \,=\, x \int_0^t e^{\,\gamma s + B(s)} \, \mathrm{d}s \,,$$

whereas $\mathfrak{R}_1(\cdot), \cdots , \mathfrak{R}_n(\cdot)$ are *independent* Bessel processes in dimension $m := 2(1+\alpha)$, and

$$\mathfrak{R}(u) \,:=\, \sqrt{ \big(\mathfrak{R}_1(u) \big)^2 + \, \cdots \, + \big(\mathfrak{R}_n(u) \big)^2 } \,.$$

That is, with $\widehat{W}_1(\cdot), \cdots, \widehat{W}_n(\cdot)$ independent scalar Brownian motions, we have

$$\mathrm{d}\mathfrak{R}_i(u) = \frac{m-1}{2\,\mathfrak{R}_i(u)}\,\mathrm{d}u + \mathrm{d}\widehat{W}_i(u)\,, \quad i = 1, \cdots, n\,.$$

Finally,

$$\mathfrak{R}(u) := \sqrt{\big(\mathfrak{R}_1(u)\big)^2 + \cdots + \big(\mathfrak{R}_n(u)\big)^2}$$

is Bessel process in dimension mn.

We are led to the *skew representation* (Irina Goia, Soumik Pal)

$$\mathfrak{R}_i^2(u) \;=\; \mathfrak{R}^2(u) \cdot \mu_i \left(4 \int_0^u \frac{\mathrm{d}v}{\mathfrak{R}^2(v)} \right) , \quad 0 \le u < \infty \,.$$

Here the vector $\mu(\cdot) = \big(\mu_1(\cdot), \ldots, \mu_n(\cdot)\big)$ of market-weights

$$\mu_i(\cdot) \;=\; \big(\mathfrak{R}_i^2 / \mathfrak{R}^2\big)\big(\Lambda(\cdot)\big)$$

is *independent* of the Bessel process $\mathfrak{R}(\cdot)$, thus also of the change-of-clock $\Lambda(\cdot)$ which is defined in terms of this Bessel process $\mathfrak{R}(\cdot)$ via the integral equation

$$4\,\Lambda(\cdot) \;=\; \int_0^{\cdot} \mathfrak{R}^2\left(\Lambda(t)\right) \mathrm{d}t \,, \quad \text{equivalently} \quad \Lambda^{-1}(\cdot) \;=\; 4 \int_0^{\cdot} \frac{\mathrm{d}v}{\mathfrak{R}^2(v)} \,.$$

This vector $\mu(\cdot) = (\mu_i(\cdot))_{i=1}^n$ of market-weights is a so-called vector *Jacobi process* with values in Δ_+^n and the dynamics

$$\mathrm{d}\mu_i(t) = (1+\alpha)\big(1 - n\mu_i(t)\big)\mathrm{d}t \,+\, \big(1 - \mu_i(t)\big)\sqrt{\mu_i(t)}\,\mathrm{d}\beta_i(t) \\ -\mu_i(t)\sum_{j \neq i}\sqrt{\mu_j(t)}\,\mathrm{d}\beta_j(t)\,,$$

for $i = 1, \cdots, n$.

Here, $\beta_1(\cdot), \cdots, \beta_n(\cdot)$ are independent, standard Brownian motions.

In particular, the processes $\mu_1(\cdot), \cdots, \mu_n(\cdot)$ have local variances $\mu_i(t)(1 - \mu_i(t))$ and covariances $-\mu_i(t)\,\mu_j(t)$.

This structure suggests that the invariant measure for the Δ_+^n −valued diffusion $\mu(\cdot) = (\mu_i(\cdot))_{i=1}^n$ of market weights, is the distribution of the vector

$$\left(\frac{Q_1}{Q_1 + \cdots + Q_n} \, , \, \cdots \, , \frac{Q_n}{Q_1 + \cdots + Q_n} \right) ,$$

where $Q_1, \cdots , Q_n$ are independent random variables with common distribution

$$\frac{2^{-(1+\alpha)}}{\Gamma(1+\alpha)} \; q^{\alpha} \, e^{-q/2} \, dq \, , \qquad 0 < q < \infty \, ,$$

(chi-square with "$2(1+\alpha)$-degrees-of-freedom").

- From these representations, one obtains the (a.s.) *long-term growth rates of the entire market and of the largest stock*

$$\boxed{\lim_{T\to\infty} \frac{1}{T} \log X(T) \,=\, \lim_{T\to\infty} \frac{1}{T} \log \left(\max_{1\le i\le n} X_i(T) \right) \,=\, \gamma} \,;$$

the *long-term growth rates for individual stocks*

$$\lim_{T\to\infty} \frac{1}{T} \log X_i(T) \,=\, \gamma\,, \quad i = 1, \cdots, n \tag{19}$$

for $\alpha > 0$.

The long-term volatilities

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{dt}{\mu_i(t)} = \frac{2\gamma}{\alpha} = n + \frac{n-1}{\alpha}$$

(for $\alpha > 0$, using the Birkhoff ergodic theorem); that this model is not diverse; and much more...

NOTE: When $\alpha = 0$, the equation (19) holds only *in probability;* the (a.s.) limit-superior is γ, whereas the (a.s.) limit-inferior is $-\infty$. (Spitzer's 0-1 law for planar Brownian motion).

Crashes.... Failure of diversity... .

13.a: Some Concluding Remarks

We have exhibited simple conditions, such as "sufficient level of intrinsic volatility" and "diversity", which lead to arbitrages relative to the market.

These conditions are **descriptive** as opposed to normative, and can be tested from the predictable characteristics of the model posited for the market. In contrast, familiar assumptions, such as the existence of an equivalent martingale measure (EMM), are **normative** in nature, and *cannot* be decided on the basis of predictable characteristics in the model; see example in [Karatzas & Kardaras] (2007).

14. RANK, LEAKAGE, AND THE SIZE EFFECT

An important generalization of the ideas and methods in this section concerns generating functions that record market weights not according to their name (or index) i, but according to their rank. To present this generalization, let us start by recalling the order statistics notation

$$\max_{1 \le i \le n} \pi_i(t) =: \pi_{(1)}(t) \ge \pi_{(2)}(t) \ge \ldots \ge \pi_{(n)}(t) := \min_{1 \le i \le n} \pi_i(t)$$

and consider for each $0 \le t < \infty$ the random permutation $\big(\mathfrak{p}_t(1), \cdots, \mathfrak{p}_t(n)\big)$ of $(1, \cdots, n)$ with

$$\mu_{\mathfrak{p}_t(k)}(t) = \mu_{(k)}(t), \qquad \text{and} \;\; \mathfrak{p}_t(k) < \mathfrak{p}_t(k+1) \;\; \text{if} \;\; \mu_{(k)}(t) = \mu_{(k+1)}(t)$$

for $k = 1, \ldots, n$. In words: $\mathfrak{p}_t(k)$ is the name (index) of the stock that occupies the k^{th} rank in terms of relative capitalization at time t; ties are resolved by resorting to the lowest index.

Using Itô's rule for convex functions of semimartingales one can obtain the following analogue of

$$\frac{\mathrm{d}\mu_i(t)}{\mu_i(t)} = \Big(\gamma_i(t) - \gamma^\mu(t) + \frac{1}{2}\,\mathfrak{A}^\mu_{ij}(t)\Big)\,\mathrm{d}t + \sum_{\nu=1}^{N} \big(\sigma_{i\nu}(t) - \sigma^\mu_\nu(t)\big)\mathrm{d}W_\nu(t)$$

in (10) for the dynamics of the *ranked market-weights*

$$\begin{aligned}\frac{\mathrm{d}\mu_{(k)}(t)}{\mu_{(k)}(t)} &= \Big(\gamma_{\mathfrak{p}_t(k)}(t) - \gamma^\mu(t) + \frac{1}{2}\,\mathfrak{A}^\mu_{(kk)}(t)\Big)\,\mathrm{d}t \\ &\quad + \frac{1}{2}\Big(\mathrm{d}\mathfrak{L}^{k,k+1}(t) - \mathrm{d}\mathfrak{L}^{k-1,k}(t)\Big) \qquad (20) \\ &\quad + \sum_{\nu=1}^{N} \big(\sigma_{\mathfrak{p}_t(k)\nu}(t) - \sigma^\mu_\nu(t)\big)\,\mathrm{d}W_\nu(t)\end{aligned}$$

for each $k = 1, \ldots, n-1$.

Here the quantity $\mathfrak{L}^{k,k+1}(t) \equiv \Lambda^{\Xi_k}(t)$ is the semimartingale local time at the origin, accumulated by the nonnegative process

$$\Xi_k(t) := \log \mu_{(k)}(t) - \log \mu_{(k+1)}(t)\,, \qquad 0 \le t < \infty \qquad (21)$$

up to the calendar time t; it measures the cumulative effect of the changes, or *turnover,* that have occurred during the time-interval $[0, t]$ between ranks k, $k+1$. We are also setting

$$\mathfrak{L}^{0,1}(\cdot) \equiv 0\,, \qquad \mathfrak{L}^{m,m+1}(\cdot) \equiv 0$$

and

$$\mathfrak{A}^{\mu}_{(k\ell)}(\cdot) := \mathfrak{A}^{\mu}_{\mathfrak{p}_t(k)\mathfrak{p}_t(\ell)}(\cdot)\,.$$

A derivation of this result, under appropriate conditions that we do not broach here, can be found on pp. 76-79 of Fernholz (2002); see also Banner & Ghomrasni (2007) for generalizations.

With this setup, we have then the following generalization of the "master equation" : consider a function $\mathbf{G} : \Delta_+^n \to (0, \infty)$ exactly as assumed there, written in the form

$$\mathbf{G}(x_1, \cdots, x_n) = \mathfrak{G}\big(x_{(1)}, \cdots, x_{(n)}\big), \qquad \forall \quad x \in U$$

for some $\mathfrak{G} \in \mathcal{C}^2(U)$ and U an open neighborhood of Δ_+^n . The effective domain of definition of this function is $U \cap \mathbb{W}^n$, where

$$\mathbb{W}^n := \big\{ (x_{(1)}, \cdots, x_{(n)}) \in \mathbb{R}^n \, : \, x_{(1)} \geq \cdots \geq x_{(n)} \big\}$$

is the Weyl chamber of descending order statistics in $\mathbb{R}^n$.

Then with the shorthand $x_{(\cdot)} := (x_{(1)}, \cdots, x_{(n)})'$ and

$$\mu_{(\cdot)}(t) := \big(\mu_{(1)}(t), \cdots, \mu_{(n)}(t)\big)', \qquad \mathfrak{A}^{\mu}_{(k\ell)}(t) := \mathfrak{A}^{\mu}_{\mathfrak{p}_t(k)\mathfrak{p}_t(\ell)}(t),$$

let us consider the rank-based portfolio $\widetilde{\varpi}(\cdot)$ defined for $1 \le k \le n$, by analogy with (13), as

$$\frac{\widetilde{\varpi}_{\mathfrak{p}_t(k)}(t)}{\mu_{(k)}(t)} = D_k \log \mathfrak{G}\big(\mu_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t) D_\ell \log \mathfrak{G}\big(\mu_{(\cdot)}(t)\big). \tag{22}$$

It can be can shown that the performance of this portfolio, relative to the market, is given for any $0 \le T < \infty$ and by analogy with (14), as

$$\boxed{\log\left(\frac{V^{\widetilde{\varpi}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathfrak{G}\big(\mu_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\mu_{(\cdot)}(0)\big)}\right) + \Gamma(T)\,,} \tag{23}$$

where we have set

$$\Gamma(T) := -\int_0^T \sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{D^2_{k\ell}\,\mathfrak{G}\big(\mu_{(\cdot)}(t)\big)}{2\mathfrak{G}\big(\mu_{(\cdot)}(t)\big)}\,\mu_{(k)}(t)\mu_{(\ell)}(t)\,\mathfrak{A}^{\mu}_{(k\ell)}(t)\,\mathrm{d}t \tag{24}$$
$$+\frac{1}{2}\sum_{k=1}^{n-1}\big(\widetilde{\varpi}_{\mathfrak{p}_t(k+1)}(t) - \widetilde{\varpi}_{\mathfrak{p}_t(k)}(t)\big)\,\mathrm{d}\mathfrak{L}^{k,k+1}(t)\,.$$

We say that $\widetilde{\varpi}(\cdot)$ is the *rank-based portfolio generated by the function* $\mathfrak{G}(\cdot)$. (Functional generation by rank, as opposed to by name.) For details see Fernholz (2002), pp. 79-83.

- For instance,

$$\mathfrak{G}(x_{(\cdot)}) = x_{(1)}$$

generates in (22) the portfolio

$$\widetilde{\varpi}_{\mathfrak{p}_t(k)}(t) = \delta_{1k}\,, \qquad k = 1, \cdots, n\,, \quad 0 \le t < \infty$$

that invests only in the largest stock at all times.
The relative performance

$$\log\left(\frac{V^{1,\widetilde{\varpi}}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{\mu_{(1)}(T)}{\mu_{(1)}(0)}\right) - \frac{1}{2}\,\mathfrak{L}^{1,2}(T)\,, \quad 0 \le T < \infty$$

of this portfolio will suffer (leakage) in the long run, if there are many changes in leadership: in order for the biggest stock to do well relative to the market, it must crush all competition!

14.a: The Size Effect

This is the tendency of small stocks to have higher long-term returns relative to their larger brethren. The formula of (23) offers a simple, structural explanation of this observed phenomenon, thus:

- Fix an integer $m \in \{2, \cdots, n-1\}$ and consider the functions

$$\mathcal{G}_L(x) = x_{(1)} + \cdots + x_{(m)} \qquad \text{and} \qquad \mathcal{G}_S(x) = x_{(m+1)} + \cdots + x_{(n)}\,.$$

These generate, respectively, a *large-stock portfolio*

$$\zeta_{\mathfrak{p}_t(k)}(t) = \frac{\mu_{(k)}(t)}{\mathcal{G}_L(\mu(t))}\,,\;\; 1 \le k \le m \quad \text{and} \quad \zeta_{\mathfrak{p}_t(k)}(t) = 0,\;\; m{+}1 \le k \le n$$

and a *small-stock portfolio*

$$\eta_{\mathfrak{p}_t(k)}(t) = \frac{\mu_{(k)}(t)}{\mathcal{G}_S(\mu(t))}\,,\;\; m{+}1 \le k \le n \quad \text{and} \quad \eta_{\mathfrak{p}_t(k)}(t) = 0,\;\; 1 \le k \le m.$$

According to (23) and (22), the performances of these portfolios, relative to the market, are given respectively by

$$\log\left(\frac{V^{1,\zeta}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{\mathcal{G}_L(\mu(T))}{\mathcal{G}_L(\mu(0))}\right) - \frac{1}{2}\int_0^T \zeta_{(m)}(t)\,\mathrm{d}\mathfrak{L}^{m,m+1}(t)\,,$$

$$\log\left(\frac{V^{1,\eta}(T)}{V^{1,\mu}(T)}\right) = \log\left(\frac{\mathcal{G}_S(\mu(T))}{\mathcal{G}_S(\mu(0))}\right) + \frac{1}{2}\int_0^T \eta_{(m)}(t)\,\mathrm{d}\mathfrak{L}^{m,m+1}(t)\,.$$

Therefore,

$$\log\left(\frac{V^{1,\eta}(T)}{V^{1,\zeta}(T)}\right) = \log\left(\frac{\mathcal{G}_S(\mu(T))\,\mathcal{G}_L(\mu(0))}{\mathcal{G}_L(\mu(T))\,\mathcal{G}_S(\mu(0))}\right) + \frac{1}{2}\int_0^T \left(\zeta_{(m)}(t) + \eta_{(m)}(t)\right)\mathrm{d}\mathfrak{L}^{m,m+1}(t)\,. \tag{25}$$

If there is "stability" in the market, in the sense that the ratio of the relative capitalization of small to large stocks does not change much over time, then the first term on the right-hand side of (25) does not change much, whereas the second term keeps increasing and accounts for the better relative performance of the small stocks.

Note that this argument does not need to invoke any assumption about the putative greater riskiness of the smaller stocks at all.

Note also that the expression of (25) can be used to estimate the local times $\mathfrak{L}^{m,m+1}(\cdot)$.

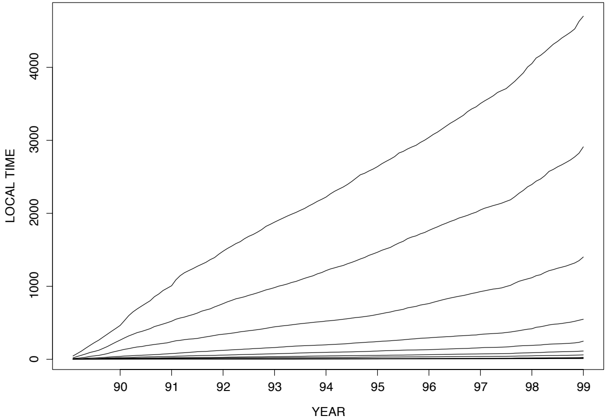

Figure 5: The estimated local time processes $\mathfrak{L}^{m,m+1}(\cdot)$ for $m = 10, 20,$ 40, ... , 5120.