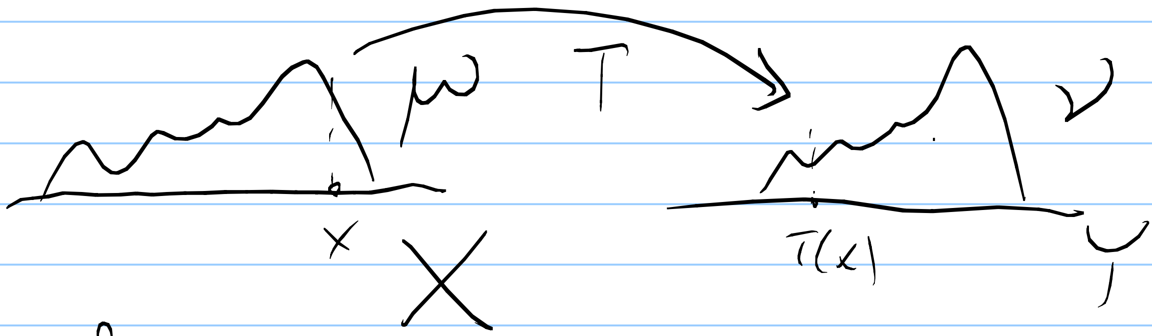
OPTIMAL TRANSPORT



$$\int_X d\mu = \int_Y d\nu = 1$$

$T: X \to Y$ TRANSPORT:

$$T_{\#}\mu = \nu \quad \left(\nu(A) = \mu(T^{-1}(A)) \;\; \forall A \right)$$

$c: X \times Y \to \mathbb{R}$ COST:

$c(x,y) =$ cost from x to y

$$C(T) = \int_X c(x, T(x))\, d\mu(x)$$

<u>Pb</u>: $\inf\limits_{T_{\#}\mu = \nu} C(T)$

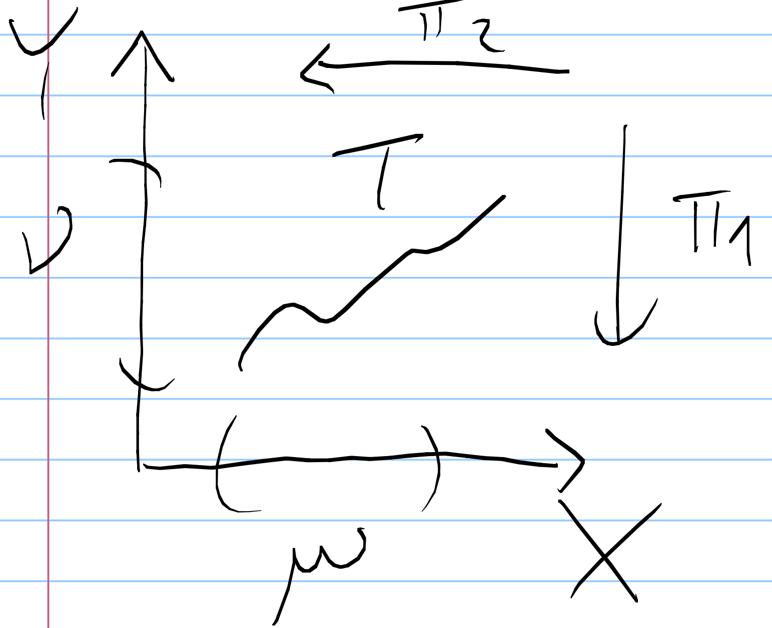
Issues: 1 $\{T : T_\# \mu = \nu\}$

may be $\emptyset$

$(\mu = \delta_x \Rightarrow T_\# \mu = \delta_{T(x)})$

2 The constraint $T_\# \mu = \nu$ is nonlinear

KANTOROVICH 1942



$$\gamma_T := (\mathrm{Id} \times T)_\# \mu$$

$$(\pi_1)_\# \gamma_T = \mu$$

$$(\pi_2)_\# \gamma_T = T_\# \mu = \nu$$

$$\int c(x, T(x))\, d\mu = \int c(x,y)\, d\gamma_T$$

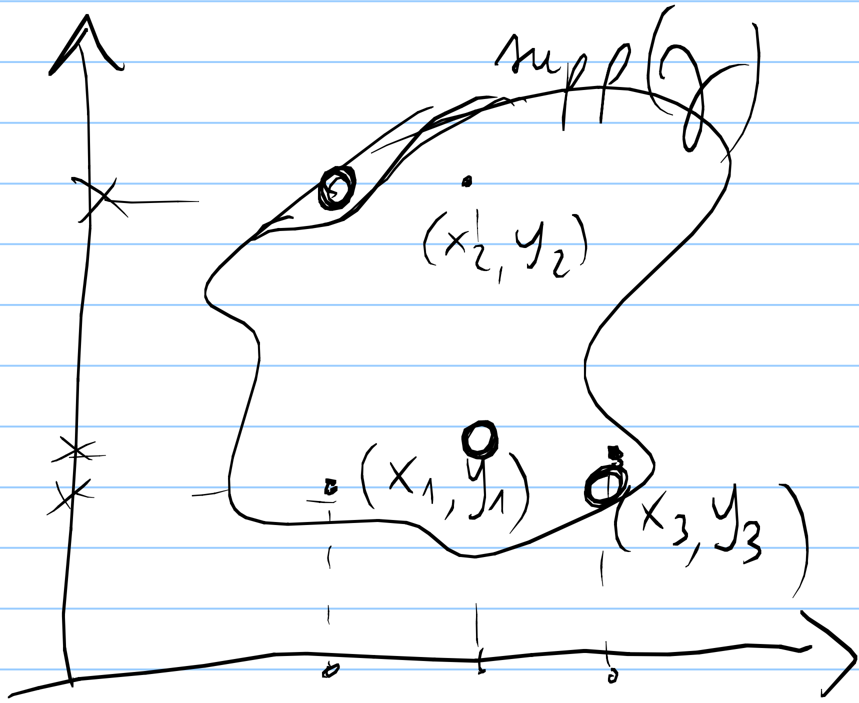
$$\inf_{\substack{(\pi_1)_\# \gamma = \mu \\ (\pi_2)_\# \gamma = \nu}} \int c(x,y)\, d\gamma$$

<u>Rmk</u> $\gamma = \mu \otimes \nu$ is admissible

<u>THM</u>

$c : X \times Y \to \mathbb{R}$, c l.s.c., $c \geq 0 \Rightarrow \exists\, \gamma$ optimal

- Fix γ optimal



<u>DEF</u> $\Gamma \subseteq X \times Y$ is

<u>c-cyclically monotone</u> if

$\forall \{(x_i, y_i)\}_{i=1,\dots,N} \subseteq \Gamma$,

$$\sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_{i+1}, y_i)$$

$x_{N+1} = x_1$

PROP Assume c is cont.

γ opt $\iff$ supp γ c-cycl mon$^{\text{d}}$

Ex $c(x,y) = \frac{|x-y|^2}{2}$, $X = Y = \mathbb{R}^n$

$$\sum_i \frac{|x_i - y_i|^2}{2} \leq \sum_i \frac{|x_{i+1} - y_i|^2}{2}$$

$$\Downarrow$$

$$\sum_i \langle y_i, x_i - x_{i+1} \rangle \geq 0$$

N=2 $\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0$

THM (Rockafellar)

Γ cycl mon (c-cycl mon for $c = \frac{|x-y|^2}{2}$)

$$\iff \exists\, \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$$

s.t. φ convex,

$$\Gamma \subseteq \partial\varphi = \bigcup_x \left(\{x\} \times \partial\varphi(x)\right)$$

$$\left(\partial\varphi(x) := \{p \mid \varphi(y) \geq \varphi(x) + \langle p, y-x \rangle\}\right)$$

DEF

- $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ c-CONVEX if

$$\varphi(x) = \sup_y \{-c(x,y) - \psi(y)\}$$

- The c-SUBDIFF of φ is

$$\partial^c\varphi(x) := \{y \mid \varphi(x) + \tilde{\psi}(y) = c(x,y)\}$$

(RMK Take $\tilde{\psi}(y) = \sup_x \{-c(x,y) - \varphi(x)\}$)

THM

Γ c-cycl monot
$\iff \exists\, \varphi : X \to \mathbb{R} \cup \{+\infty\}$
c-convex, $\Gamma \subseteq \partial^c \varphi$

Pf

$\Rightarrow$) Define

$$\varphi(x) := \sup_N \sup_{\{(x_i,y_i)\}_{i=1}^N \subseteq \Gamma}$$

$$\{ [c(x_1,y_1) - c(x_2,y_1)] + \dots + [c(x_N,y_N) - c(x,y_N)] \}$$

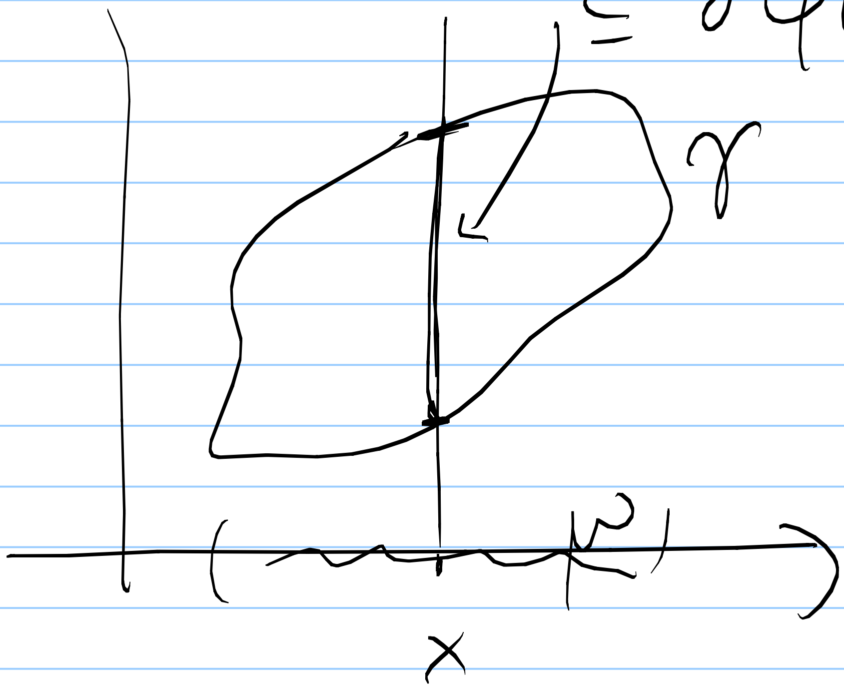
$\square$

Combining all together,

$\mathrm{supp}(\gamma) \subseteq \partial^c \varphi$, φ c-convex

GOAL: show that $\partial^c \varphi(x)$
is a singleton for μ-a.e. x

$\subseteq \partial^c \varphi(x)$



1. $X = Y = \mathbb{R}^n$, $c(x,y) = \frac{|x-y|^2}{2}$

$$\varphi(x) = \sup_y -\frac{|x-y|^2}{2} - \psi(y)$$

$$= -\frac{|x|^2}{2} + \underbrace{\sup_y \left\{ x \cdot y - \frac{|y|^2}{2} - \psi(y) \right\}}_{\text{convex}}$$

$\Downarrow$

φ loc Lip $\Rightarrow \varphi$ diff. a.e.

$\bar{y} \in \partial^c \varphi(\bar{x})$

$\Downarrow$

$$-c(x,\bar{y}) - \psi(\bar{y}) - \varphi(x) \leq 0$$

$=$ if $x = \bar{x}$

If φ is diff at $x = \bar{x}$, we get

$$\nabla\varphi(\bar{x}) = \bar{y} - \bar{x}$$

$$\Downarrow$$

$$\bar{y} = \bar{x} + \nabla\varphi(\bar{x})$$

$$\Downarrow$$

$$\partial^c\varphi(\bar{x}) = \{\bar{x} + \nabla\varphi(\bar{x})\}$$

$$= \left\{ \nabla\left(\frac{|x|^2}{2} + \varphi\right)(\bar{x}) \right\}$$

So, $\partial^c\varphi(x)$ is a singleton for a.e. x

<u>Hyp</u> $\mu \ll dx$

<u>THM</u> (Brenier, 1987)

$X = Y = \mathbb{R}^n$, $c(x,y) = \frac{|x-y|^2}{2}$,

$\mu \ll dx \Rightarrow \exists!$ optimal map T,

$T = \nabla \phi$,

$\phi : \mathbb{R}^n \to \mathbb{R}$ convex

<u>Pf</u>

<u>EXIST</u>

1. Take γ optimal for Kant
2. $\gamma \subseteq \partial^c \varphi$, φ c-convex
3. $\partial^c \varphi(x) = \left\{ \nabla\left(\frac{|x|^2}{2} + \varphi\right)(x) \right\}$ μ-a.e.

(4) So, $\gamma \subseteq \text{graph}(\nabla\phi)$,

$$\phi := \frac{|x|^2}{2} + \varphi \quad \text{convex}$$

So, $\nabla\phi$ is an opt transport.

$$C(\nabla\phi) = \int \frac{|x-y|^2}{2} d\gamma$$

$$= \inf_{\tilde{\gamma}} \int |x-y|^2 d\tilde{\gamma}$$

$$\leq \inf_{T_\# \mu = \nu} C(T)$$

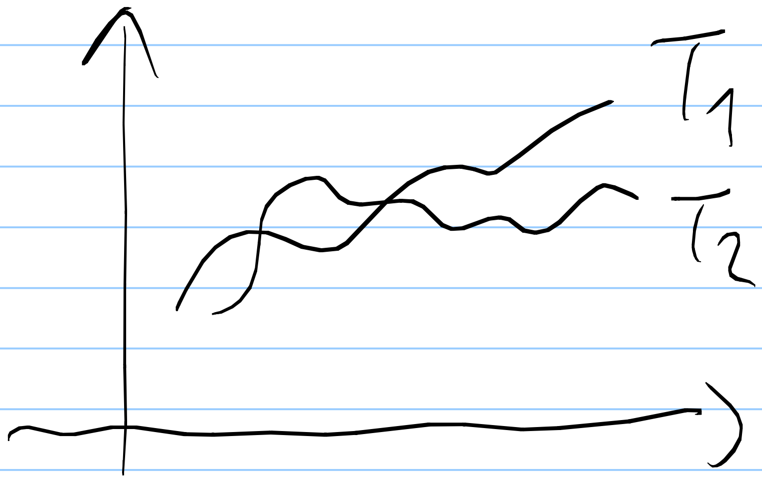
UNIQ If T_1, T_2 are min,

$$\gamma_1 := (Id \times T_1)_\# \mu,$$

$$\gamma_2 := (Id \times T_2)_\# \mu$$

γ_1, γ_2 min for Kant

$\bar{\gamma} := \frac{1}{2}(\gamma_1 + \gamma_2)$ min



$\Downarrow$

$\bar{\gamma} \subseteq \text{graph}(\nabla \bar{\phi})$

$\Downarrow$

$T_1 = T_2$ μ-a.e. $\square$

CHANGE of VARIABLE

$$T_\# \mu = \nu \qquad X = Y = \mathbb{R}^n$$

$$\mu = f\,dx, \quad \nu = g\,dy$$

$$\int \chi(T(x))\, f(x)\, dx = \int \chi(y)\, g(y)\, dy$$

$$= \int \chi(T(x))\, g(T(x))\, |\det \nabla T|\, dx$$

$y = T(x)$ $\qquad \forall \chi$

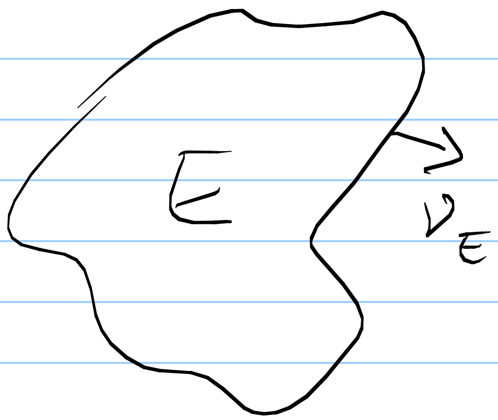
$$|\det \nabla T| = \frac{f(x)}{g(T(x))} \quad \text{a.e.}$$

$$T = \nabla \phi, \quad \phi \text{ convex}$$

$$\det\left(\nabla^2\phi\right) = \frac{f(x)}{g(\nabla\phi(x))} \quad \text{a.e.}$$

MONGE-AMPÈRE EQ.

ISOP. INEQUALITIES



$E \subseteq \mathbb{R}^n$

smooth, bdd

$$P(E) \geq n\,|B_1|^{\frac{1}{n}}\,|E|^{\frac{n-1}{n}}$$

$$= \iff E = x + r B_1 = B_r(x)$$

RMK By scaling, w.l.o.g. $|E| = |B_1|$

STABILITY

DEFICIT: $\delta(E) := P(E) - n|B|^{\frac{1}{n}}|E|^{\frac{n-1}{n}} \geq 0$

Q: $\delta \sim 0 \overset{?}{\Longrightarrow} E \sim \text{ball}$

ASYMM: $A(E) := \inf_x |E \Delta (x + B_1)|$

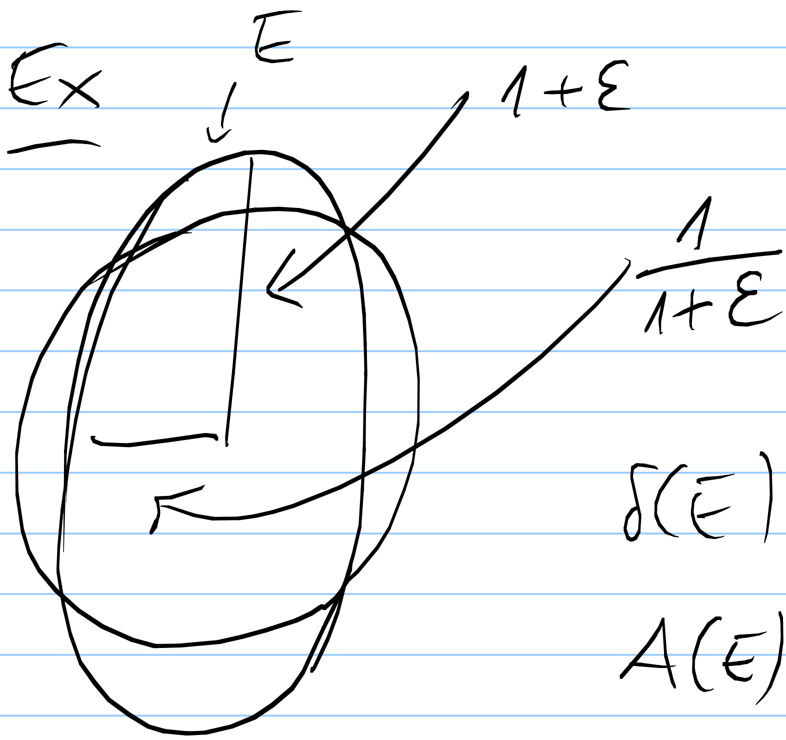
Q: $\delta \sim 0 \overset{?}{\Longrightarrow} A \sim 0$

Yes, in a nonquant form.

Q : Do there exists $C(n)$, $\alpha(n)$ s.t.

$$C(n)\,\delta(E)^{\alpha(n)} \geq A(E)\ ?$$

Ex



$\delta(E) \sim \varepsilon^2$

$A(E) \sim \varepsilon$

$\Downarrow$

$\alpha(n) \leq \frac{1}{2}$.

THM (Hall '81)

$\alpha(n) = \frac{1}{4}$

THM (Fusco - Maggi - Pratelli '08)

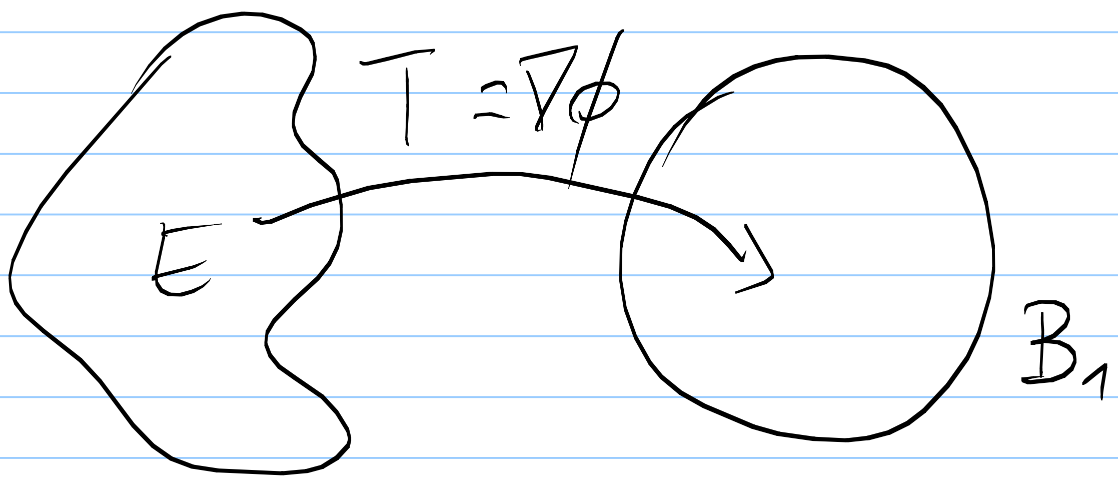
$\alpha(n) = \frac{1}{2}$

THM (F. - Maggi - Pratelli '10)

$\alpha(n) = \frac{1}{2}$, works for general notions of perimeter

Pf of isop ineq (Gromov)

$|E| = |B_1|$



Let $T = \nabla\phi$ be the opt transport sending

$$\mu = \frac{\chi_E \, dx}{|E|} \longrightarrow \nu = \frac{\chi_{B_1} \, dy}{|B_1|}$$

(A) $T(x) \in B_1$ for $x \in E$ $\Rightarrow |T(x)| \leq 1$

(B) $$\frac{\operatorname{div} T}{n} = \frac{\Delta\phi}{n} = \frac{1}{n}\sum_i \lambda_i$$

eigenv

$$\geq \left(\prod_i \lambda_i\right)^{\frac{1}{n}}$$

$\lambda_i \geq 0$

$$= \left(\det \nabla^2\phi\right)^{\frac{1}{n}}$$

$$= \left(\det \nabla T\right)^{\frac{1}{n}}$$

$$\text{C} \quad \det \nabla T = \frac{f}{g \circ T} = \frac{|B_1|}{|E|} = 1$$

$$P(E) = \int_{\partial E} d\sigma$$

$$\overset{A}{\geq} \int_{\partial E} |T(x)| \, d\sigma$$

$$\geq \int_{\partial E} T \cdot \nu_E \, d\sigma$$

$$= \int_E \operatorname{div} T \, dx$$

$$\overset{B}{\geq} \int_E n (\det \nabla T)^{\frac{1}{n}} \, dx$$

$$\overset{C}{=} n|E| = n|B_1|^{\frac{1}{n}} |E|^{\frac{n-1}{n}} \qquad \square$$

EQUALITY CASE

Assume $P(E) = n|B_1|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}$

$$\Downarrow$$

$$\frac{\operatorname{div} T}{n} = \det \nabla T \quad \text{on } E$$

$$\frac{1}{n}\sum_i \lambda_i \qquad \left(\prod_i \lambda_i\right)^{\frac{1}{n}}$$

$$\Downarrow$$

$$\lambda_1(x) = \_\_\_\_ = \lambda_n(x) \quad \text{for } x \in E$$

$$\prod_i \lambda_i(x) = \det \nabla T(x) = 1$$

$$\Downarrow$$

$$\lambda_1(x) = \_\_ = \lambda_n(x) = 1$$

eigen of $D^2\phi$

$$\Downarrow$$

$$DT = D^2\phi(x) = I_n$$

$$\Downarrow$$

$$\left.\begin{array}{l} T(x) = x + x_0 \\ T(E) = B_1 \end{array}\right] \Rightarrow E = B_1 - x_0.$$

QUANTITATIVE ESTIMATE

3 estimates:

1. $$\delta(E) \geq n \int_E \underbrace{\left[\frac{\operatorname{div} T}{n} - (\det \nabla T)^{\frac{1}{n}}\right]}_{\frac{1}{n}\sum_i \lambda_i - \left(\prod_i \lambda_i\right)^{\frac{1}{n}}}$$

$$\prod_i \lambda_i = 1 \qquad \underbrace{\qquad\qquad\qquad}_{\geq c \sum_i (\lambda_i - 1)^2}$$

$$\Downarrow$$

$$\delta(E) \geq c \int_E \underbrace{\sum_i (\lambda_i - 1)^2}_{\geq \|\underbrace{D^2\phi}_{DT} - I\|^2}$$

$$\geq c \int_E \|DT - I\|^2$$

$$\Downarrow \text{ Holder}$$

$$\sqrt{\delta(E)} \geq c \int_E \|DT - I\|$$

2 $\delta(E) \geq \int_{\partial E} (1 - |T(x)|) \, d\sigma$

3 I need a trace ineq:

$$\int_E \|DT - I\| = \int_E \|D(T - x)\|$$

$$\geq c \int_{\partial E} \|T - x\|$$

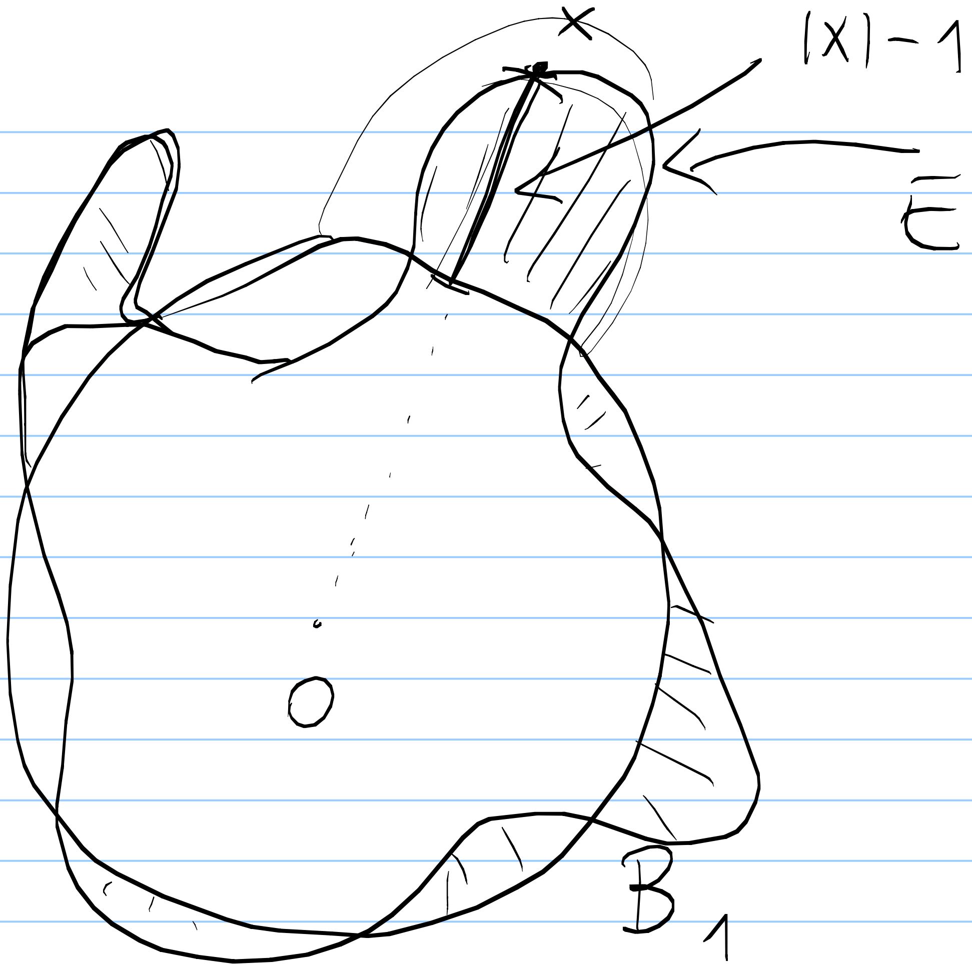
If I have 3,

$$1 + 3 \implies \sqrt{\delta(E)} \geq c \int_{\partial E} \|T - x\|$$

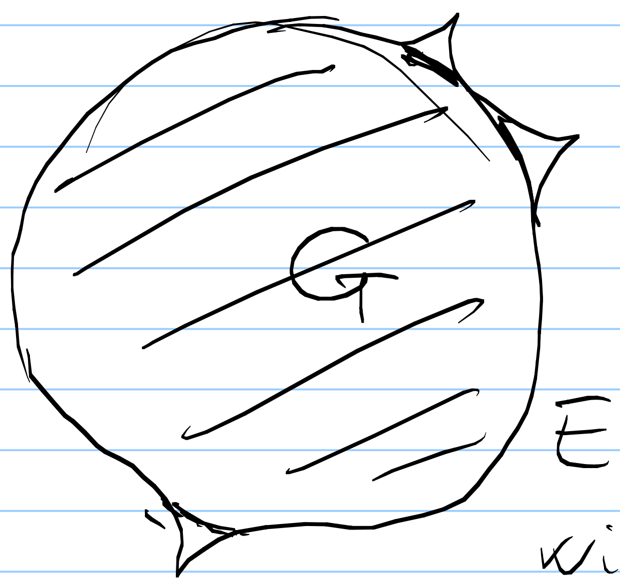
$$2 \implies \delta(E) \geq \int_{\partial E} \big|1 - |T|\big|$$

$$\Downarrow \quad \delta(E) \leq 1$$

$$\sqrt{\delta(E)} \geq c \int_{\partial E} \big||x| - 1\big| \geq c\,|E \Delta B_1|$$



How to handle 3 :



Remove cusps from E to get a set G with a ~~g~~ universal trace ineq

Prove that $\delta(G) \leq C\,\delta(E)$.

Redo the pf with G in place of E.

$$\Downarrow$$

$$|G \Delta B_1| \leq \sqrt{\delta(G)}$$

$$\Downarrow$$

$$|E \Delta B_1| \leq \underbrace{|E \Delta G|}_{\lesssim \sqrt{\delta(E)}} + \underbrace{|G \Delta B_1|}_{\lesssim \sqrt{\delta(G)} \lesssim \sqrt{\delta(E)}}$$

LEMMA

If $\delta(E) \leq \delta(n) \Rightarrow \exists\, G \subseteq E$ s.t.

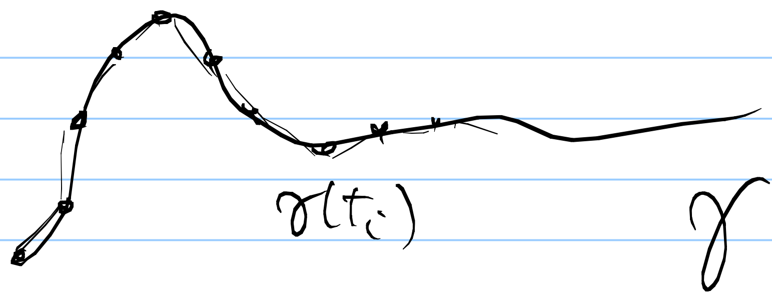
- $\delta(G) \leq C\,\delta(E)$
- $|E \setminus G| \leq C\,\delta(E)$
- G has a univ. trace ineq. $\square$

METRIC and RIEM. GEOM.

- (X, d) complete separable metric space

Given $\gamma : [0,T] \to X$,

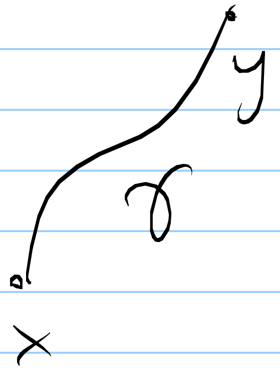
$$L(\gamma) := \sup \left\{ \sum_{i=0}^{N} d(\gamma(t_i), \gamma(t_{i+1})) \;\middle|\; 0 = t_0 \leq t_1 \leq \dots \leq t_{N+1} = 1 \right\}$$



<u>Rmk</u> $L(\gamma)$ is invariant under reparameterization

$\leadsto$ w.l.o.g. $\gamma : [0,1] \to X$

GEODESICS



$\bar{\gamma} : [0,1] \to X$ geod from x to y

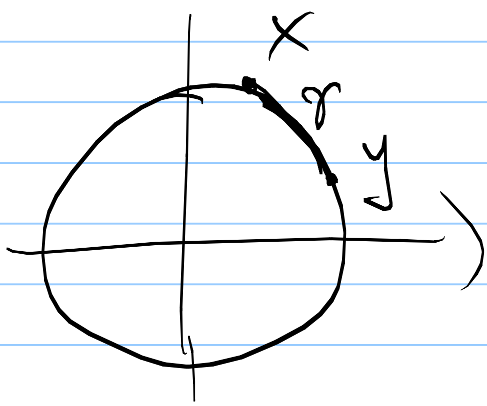
if $L(\bar{\gamma}) \le L(\gamma)$

GEOD. SPACE : (X, d) s.t.

$\forall\, x, y\ \exists\, \gamma$ geod from x to y s.t.

$$L(\gamma) = d(x,y)$$

Ex $X = S^1 \subseteq \mathbb{R}^2$, $d(x,y) = |x-y|$



$$d(x,y) = |x-y|$$

$$L(\gamma) = 2 \arcsin\left(\frac{|x-y|}{2}\right)$$

$$> |x-y|$$

But one can make such a space geod by def

$$\tilde{d}(x,y) := \inf_{\substack{\gamma \text{ from} \\ x \text{ to } y}} L(\gamma)$$

From now on, (X,d) geod space

<u>RMK</u> (X,d), $\gamma : [0,1] \to X$,

$$d(\gamma(s),\gamma(t)) = |t-s|\, d(\gamma(0),\gamma(1)) \quad (*)$$

$$\Downarrow$$

γ is a (constant speed) geod

<u>RMK</u> Any geod can be repar so that $(*)$ holds

RIEMANNIAN GEOM

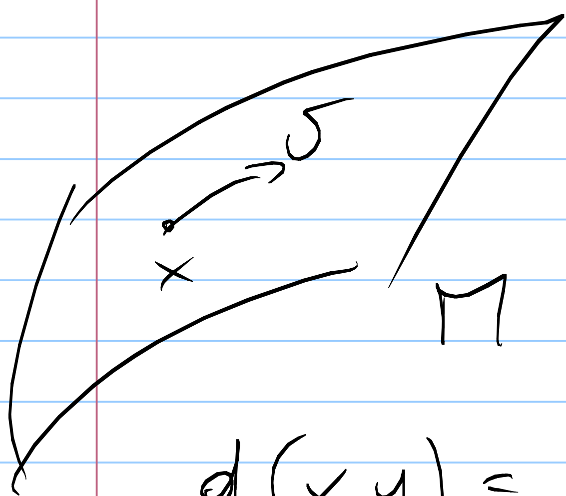
(M, g) is a C^∞ diff manifold

s.t. $\forall x \in M \ \exists g_x : T_xM \times T_xM \to \mathbb{R}$

symmetric pos. def. quadr. form

s.t. $x \mapsto g_x$ is C^∞ in charts

- $v \in T_xM$, $|v|_x := \sqrt{g_x(v, v)}$



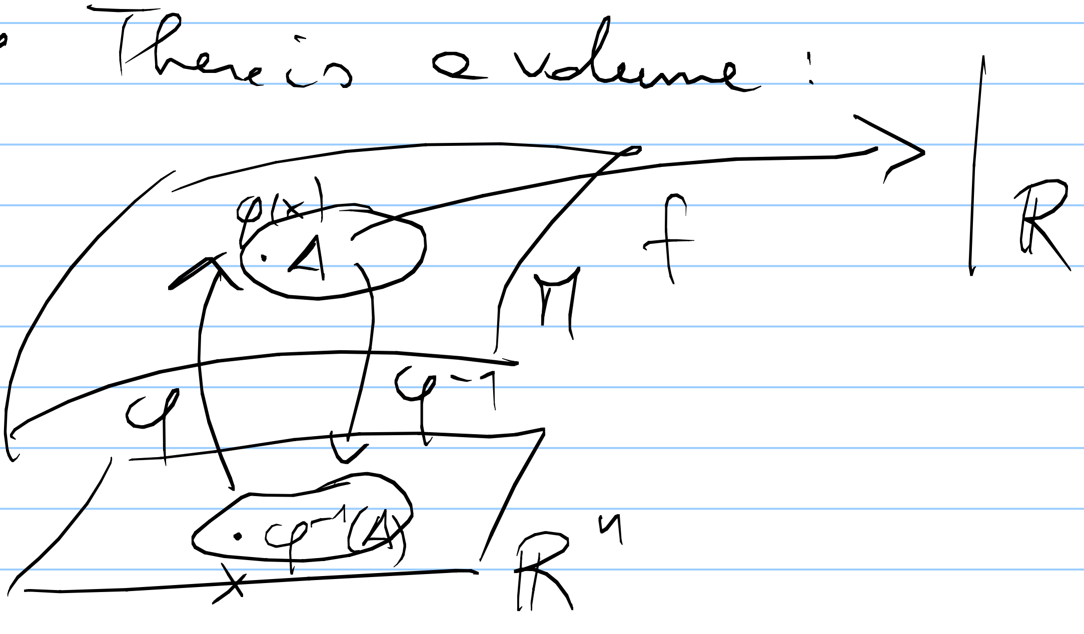
$$d(x,y) = \inf_{\substack{\gamma(0)=x \\ \gamma(1)=y}} \int_0^1 \underbrace{|\dot\gamma(t)|_{\gamma(t)}}_{\sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}} dt$$

Ex

$$d(x,y) = \inf_{\substack{\gamma(0)=x \\ \gamma(1)=y}} \sqrt{\int_0^1 g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))\,dt}$$

Advantage of 2^{nd} def is that minimizers are constant-speed geod.

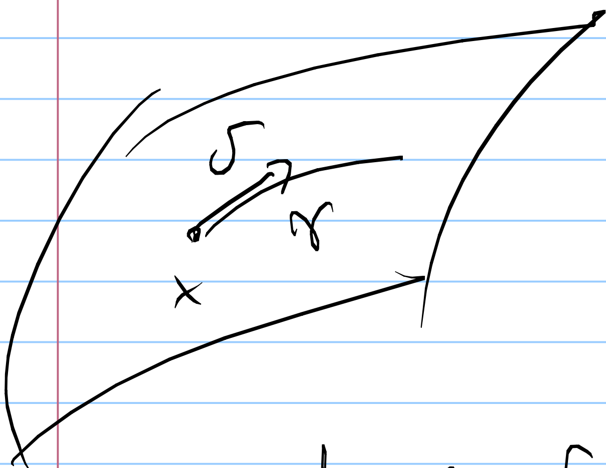
- (M,d) is a ~~geo~~ geod space,
- There is a volume:



$$\int_A f \, d\mathrm{vol} = \int_{\varphi^{-1}(A)} f \circ \varphi \sqrt{\det (g_{ij})} \, dx$$

$$g_{ij}(x) = g_{\varphi(x)}\big(d\varphi(x)[e_i], d\varphi(x)[e_j] \big)$$

DIFF., GRAD, HESSIAN



$$\varphi : M \to \mathbb{R}$$

$$d\varphi : TM \to \mathbb{R}$$

$$d\varphi(x)[\underset{\in T_xM}{v}] = \frac{d}{dt}\Big|_{t=0} \varphi \circ \gamma(t)$$

$$\gamma : [0, \varepsilon] \to M$$

$$\gamma(0) = x, \quad \dot{\gamma}(0) = v$$

$$\nabla\varphi(x) \in T_x M:$$

$$g_x\big(\nabla\varphi(x), v\big) := d\varphi(x)[v]$$

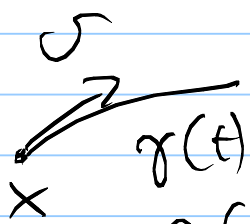
<u>Hessian</u> $\rightsquigarrow$

GEOD satisfy an ODE:

$$\ddot{\gamma}^k + \Gamma^k_{ij}(\gamma)\,\dot{\gamma}^i\dot{\gamma}^j = 0$$

$\Downarrow$

Given $\gamma(0), \dot{\gamma}(0) \rightsquigarrow \exists!$ geod



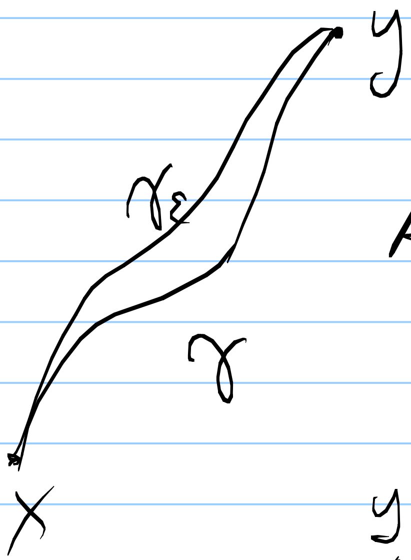
$\gamma(t)$ geod s.t.
$\gamma(0) = x$
$\dot{\gamma}(0) = v$

$$\operatorname{Hess} f(x)[v,v] = \frac{d^2}{dt^2}\Big|_{t=0} \varphi\circ\gamma(t)$$

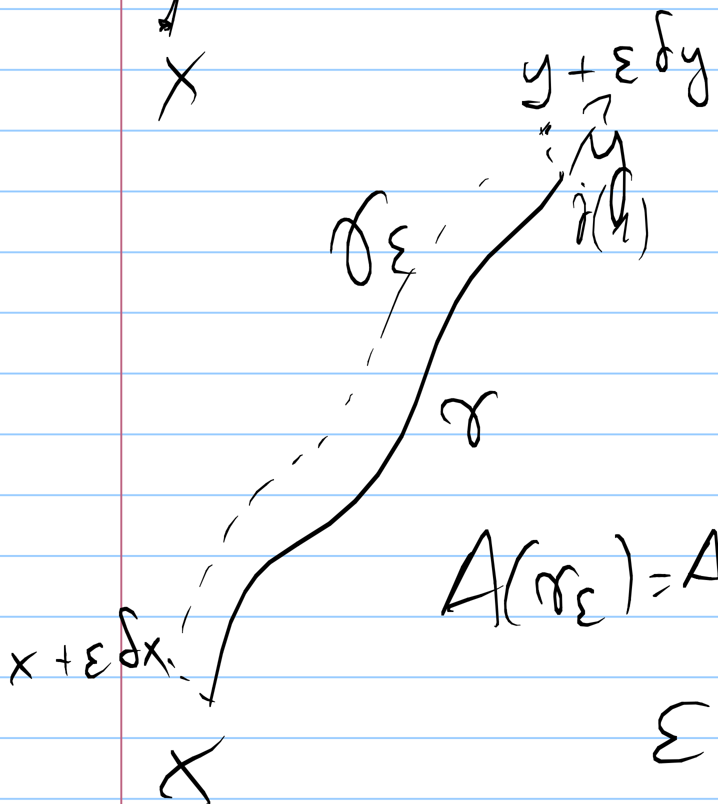
PROPERTIES of DIST FCT

If γ min geod, it minimizes

$$A(\gamma) := \int_0^1 \frac{1}{2}|\dot{\gamma}|^2$$



$$A(\gamma_\varepsilon) = A(\gamma) + O(\varepsilon^2)$$



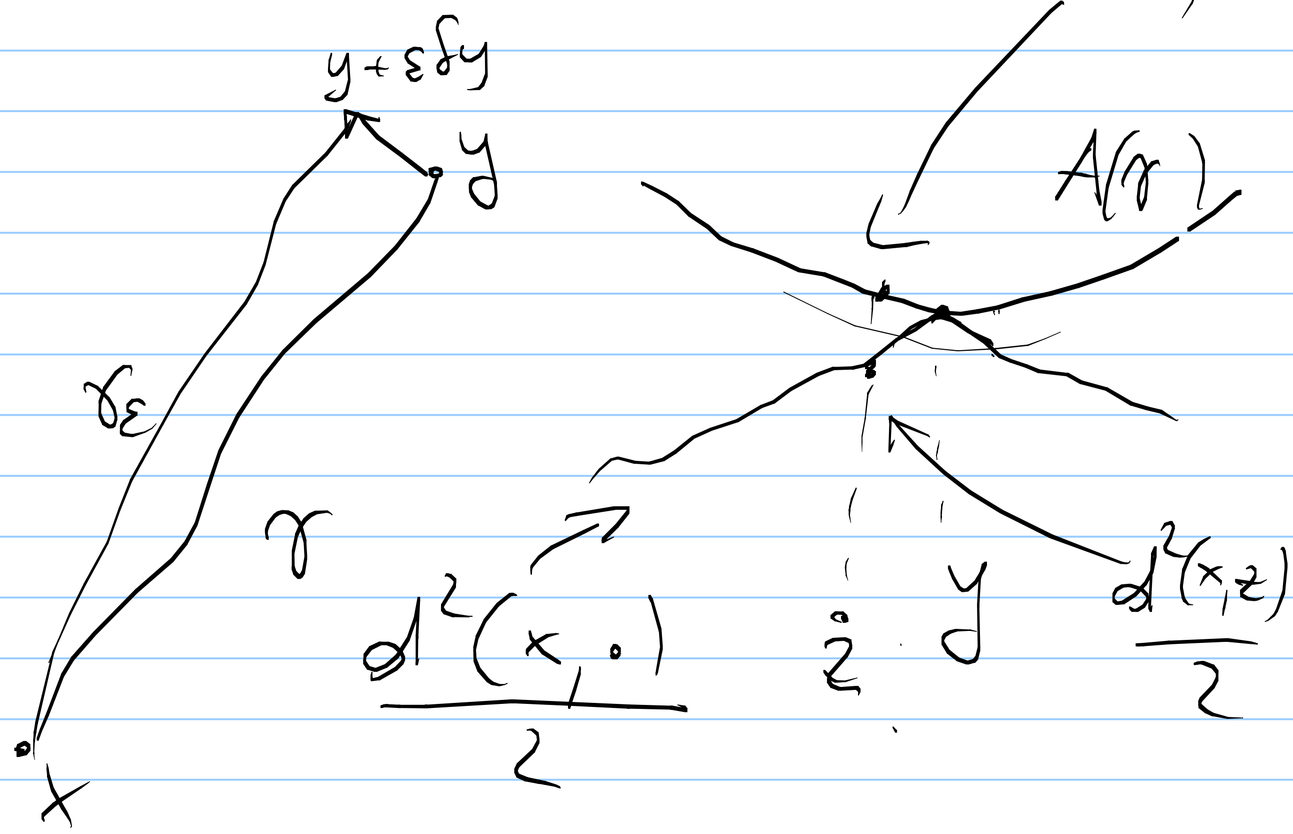
$$A(\gamma_\varepsilon) = A(\gamma) + \varepsilon\left[\langle \delta y, \dot{\gamma}(1)\rangle - \langle \delta x, \dot{\gamma}(0)\rangle\right] + O(\varepsilon^2)$$

Ex Prove it in $\mathbb{R}^n$.

Moreover,

$$A(\gamma_\varepsilon) \geq \frac{d^2(x+\varepsilon\delta x,\, y+\varepsilon\delta y)}{2}$$

Choose $\delta x = 0$

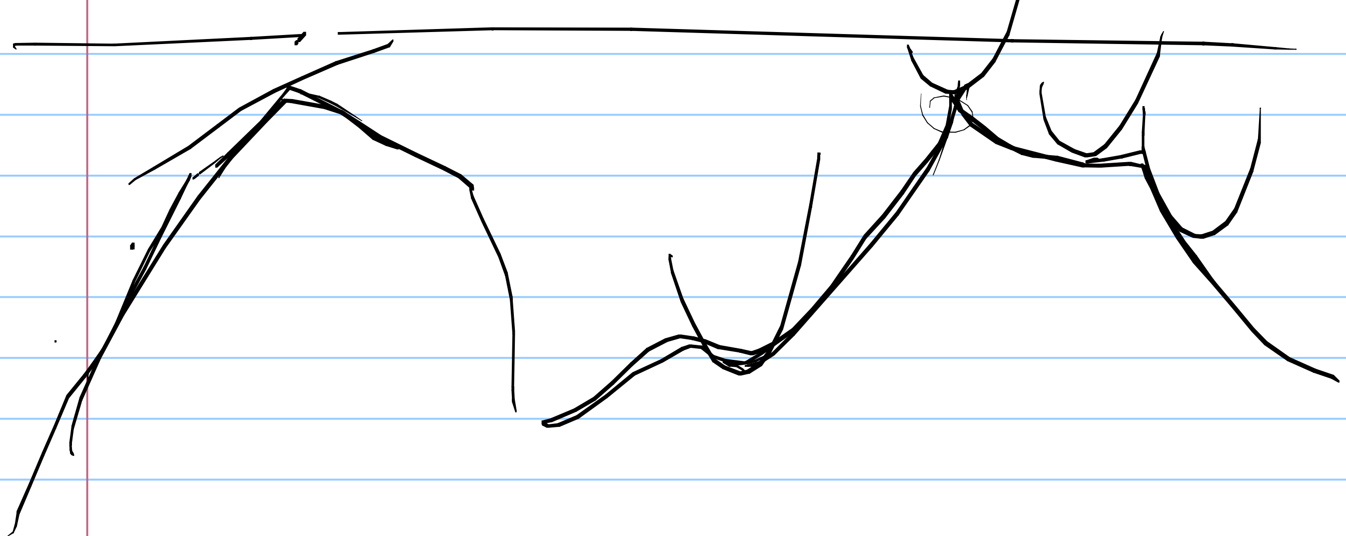


$\forall y\ \exists$ smooth fct which touches $\frac{d^2(x,\cdot)}{2}$ at y from above

$$\Downarrow$$

$\frac{d^2(x,\cdot)}{2}$ is semiconcave,

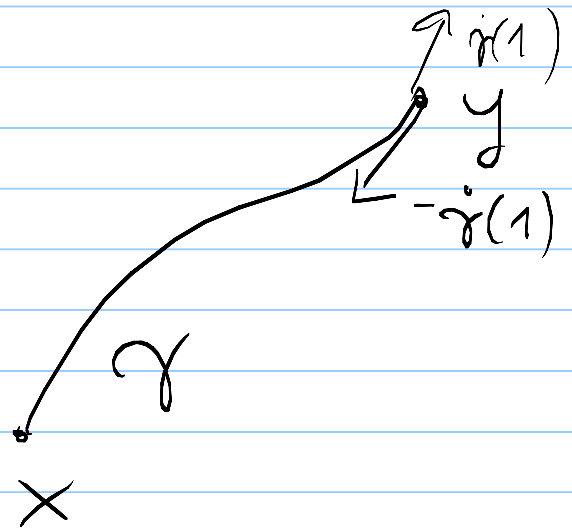
i.e. in charts is equal to concave + C^∞ perturbation



Moreover

$$\frac{d^2(x,\cdot)}{2} \text{ is diff} \iff \exists! \text{ minim geod}$$

$$\text{and } \nabla_y \frac{d^2(x,\cdot)}{2} = -\dot{\gamma}(1)$$



EXISTENCE of OPT. MAPS ON RIEM. MANIFOLDS

$$X = Y = (M, g), \quad c(x,y) = \frac{d^2(x,y)}{2}$$

M cpt

$$\varphi(x) = \underbrace{\sup_y \left\{ \underbrace{-\frac{d^2(x,y)}{2} - \psi(y)}_{\text{Lip in } x,\ \text{unif in } y} \right\}}_{\text{Lip in } x}$$

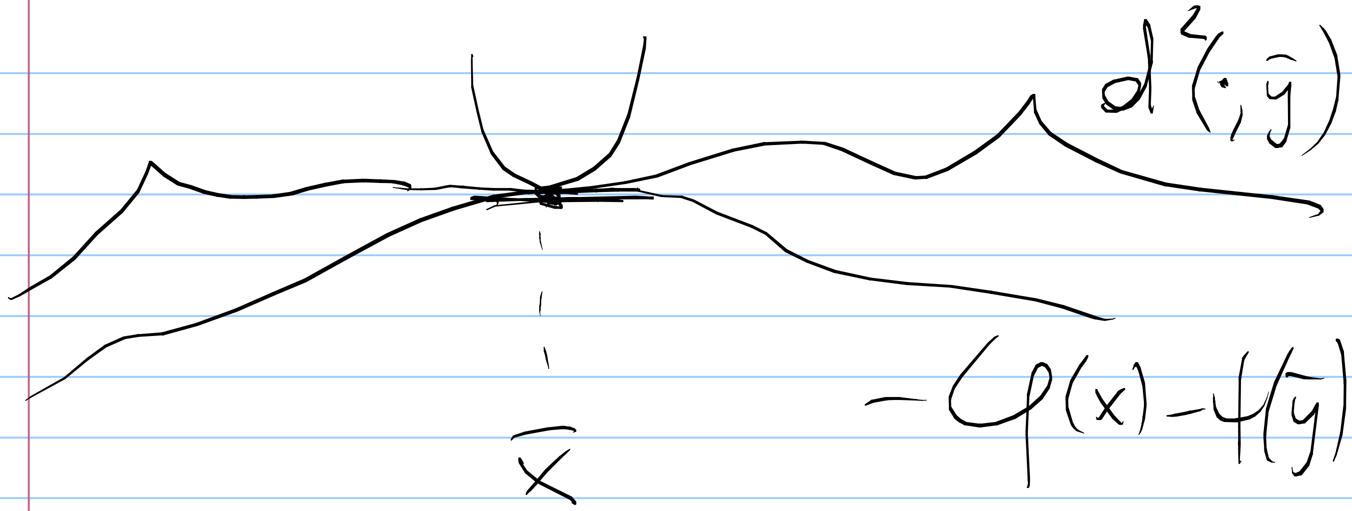
$$\Downarrow$$

φ diff vol-a.e. on M

Fix $\bar{y} \in \partial^c \varphi(\bar{x})$, $\bar{x}$ diff pt for φ

$$\frac{d^2(x,\bar{y})}{2} \geq -\varphi(x) - \psi(\bar{y})$$

$=$ at $x = \bar{x}$



Since $x \mapsto -\varphi(x) - \psi(\bar y)$ diff at $x = \bar x$ and Touches from below the semiconcave fct $x \mapsto \frac{d^2(x,\bar y)}{2}$,

then $x \mapsto \frac{d^2(x,\bar y)}{2}$ is diff at $x = \bar x$.

$$\Downarrow$$

$$\nabla_x \frac{d^2(\bar x,\bar y)}{2} = -\nabla\varphi(\bar x)$$

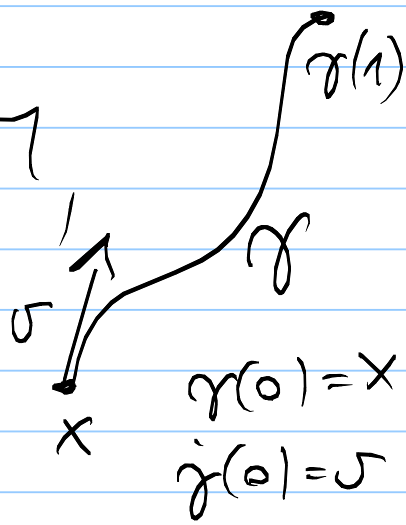
$$\Downarrow$$

$$\bar{y} = \exp_{\bar{x}}\left(\nabla\varphi(\bar{x})\right)$$

<u>RMK</u>

Given $x \in M$, $v \in T_x M$,

$\exp_x(v) = \gamma(1)$



$$\Downarrow$$

$$\partial^c\varphi(\bar{x}) = \left\{ \exp_{\bar{x}}\left(\nabla\varphi(\bar{x})\right) \right\}$$

vol-a.e.

THM (McCann 2001)

$\cancel{X} = Y = (M, g)$, M cpt,

$c(x, y) = \frac{d^2(x,y)}{2}$, $\mu \ll vol$

$\Downarrow$

$\exists!$ opt transp T,

$T(x) = exp_x (\nabla_x \varphi)$,

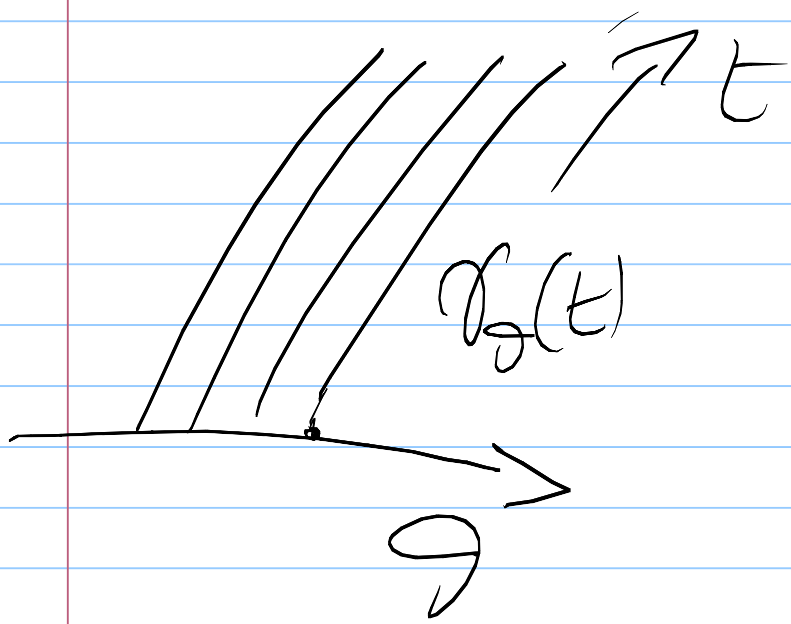
$\varphi : M \to \mathbb{R}$ c-convex.

Moreover, if $\mu = f\, vol$,

$\nu = g\, vol$, then

$$|\det(dT)| = \frac{f(x)\, vol_x}{g(T(x))\, vol_{T(x)}}$$

JACOBI FIELDS

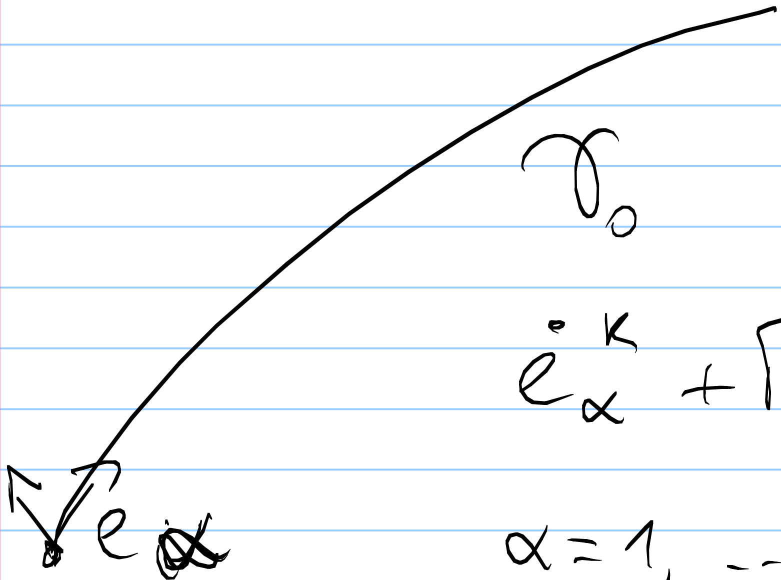


$$\ddot{\gamma}_\theta^k + \Gamma^k_{ij}(\gamma_\theta)\,\dot{\gamma}_\theta^i\,\dot{\gamma}_\theta^j = 0$$

$$\Big\downarrow \; \frac{\partial}{\partial\theta}\Big|_{\theta=0}$$

$$J(t) := \frac{\partial \gamma_\theta}{\partial\theta}\Big|_{\theta=0}(t) \in T_{\gamma(t)}M$$

$$\ddot{J}^k + \frac{\partial\Gamma^k_{ij}}{\partial x^l} J^l \dot{\gamma}^i \dot{\gamma}^j + 2\Gamma^k_{ij}\dot{J}^i\dot{\gamma}^j = 0$$



γ_0

$$\dot{e}^k_\alpha + \Gamma^k_{ij}(\gamma_0)\, e^i_\alpha\, \dot{\gamma}_0^j = 0$$

$$\alpha = 1, \dots n$$

$$e_1 = \dot{\gamma}_0$$

Construct an O.B. along γ_0 by parallel transp

$$\Downarrow$$

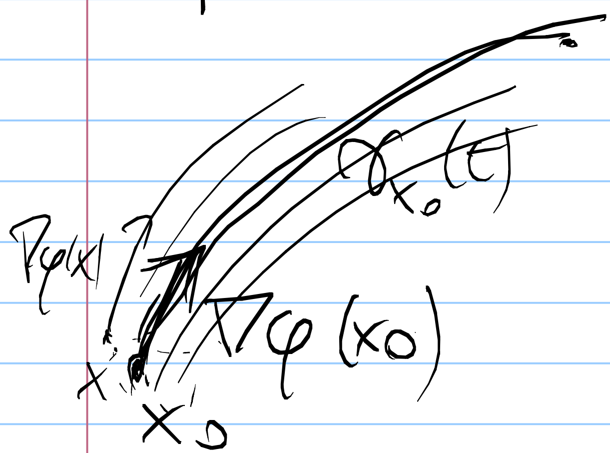
$$J_\alpha(t) := \langle J(t), e_\alpha(t) \rangle$$

$$\ddot{J}_\alpha + R_\alpha^j J_j = 0$$

symmetric matrix

DEF $Ric(\dot\gamma, \dot\gamma) := tr(R^j_\alpha)$

Let us study now the following problem:



φ smooth fct near x_0

$$\underbrace{T_t(x) := \exp_x(t\nabla\varphi(x))}_{\simeq\, x + t\nabla\varphi(x)}$$

$$T_t(x_0) = \gamma_{x_0}(t)$$

Q: What is $\operatorname{Jac}(d_{x_0}T_t)$?

$J(t) := d_{x_0}T_t$ is an array of Jacoby fields:

$$d_{x_0}T_t = \left(\partial_1 T_t(x_0), \ldots, \partial_n T_t(x_0)\right)$$

$$\begin{cases} \ddot{J} + RJ = 0 \\ J(0) = Id \\ \dot{J}(0) = d_{x_0}\left(\frac{d}{dt}\Big|_{t=0} T_t(x)\right) = d_{x_0}(\nabla\varphi) \end{cases}$$

$$= \nabla^2\varphi$$

$$\delta(t) := \det J(t)$$

$\underline{\text{Ex}}$ $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \det(A+\varepsilon B) = \det A \; \mathrm{tr}(BA^{-1})$

$$\frac{d}{dt}\log(\delta(t)) = \frac{\dot{\delta}(t)}{\delta(t)} = \mathrm{tr}(\dot{J}J^{-1})$$

TRUE as long as $\delta(t) > 0$

$$U := \dot J J^{-1}$$

$$\dot U = \underbrace{\ddot J}_{= -R\cancel{J}} \cancel{J^{-1}} - \underbrace{\dot J J^{-1}}_{U}\,\underbrace{\dot J J^{-1}}_{U}$$

$$= -R - U^2$$

$$(\mathrm{tr}\, U)^{\cdot} + \mathrm{tr}(U^2) + \mathrm{Ric}(\dot\gamma,\dot\gamma) = 0$$

CLAIM

U symmetric

Pf U and U^* solve the same ODE (since $R = R^*$) and they satisfy $U(0) = \nabla^2 \varphi = U^*(0)$. □

Hence

$$\operatorname{tr}(U^2) \geq \frac{1}{n}(\operatorname{tr} U)^2$$

PROP

If $T_t(x) = \exp_x(t \nabla_x \varphi)$, then

$s(t) = \det(d_x T_t)$ solves

$$\frac{d^2}{dt^2} \log(s(t)) + \frac{1}{n}\left(\frac{d}{dt} \log s\right)^2 + \operatorname{Ric}(\dot\gamma, \dot\gamma) \leq 0$$

COR

If $\operatorname{Ric} \geq 0 \Rightarrow t \mapsto \log s(t)$ CONCAVE

Actually, more holds

<u>THM</u>

$$\mathrm{Ric} \geq 0 \iff \frac{d^2}{dt^2} \log\left[\det \nabla_x\left(\exp_x(t\nabla\varphi)\right)\right] \leq 0$$

on $[0,1]$

$\forall \varphi : M \to \mathbb{R}$ s.t.

$$\det \nabla_x\left(\exp_x(t\nabla\varphi)\right) > 0$$

on $[0,1]$

<u>W_2 - SPACE</u>

$$(M, g) \rightsquigarrow W_2(\mu, \nu) := \sqrt{\inf_{\substack{(\pi_1)_\# \mu \\ (\pi_2)_\# \nu}} \int d^2(x,y)\, d\gamma}$$

Prob on M

$(P(M), W_2)$ geod space

<u>THM</u>

$\mu_0, \mu_1 \in P(M)$, $\mu_0 \ll vol$

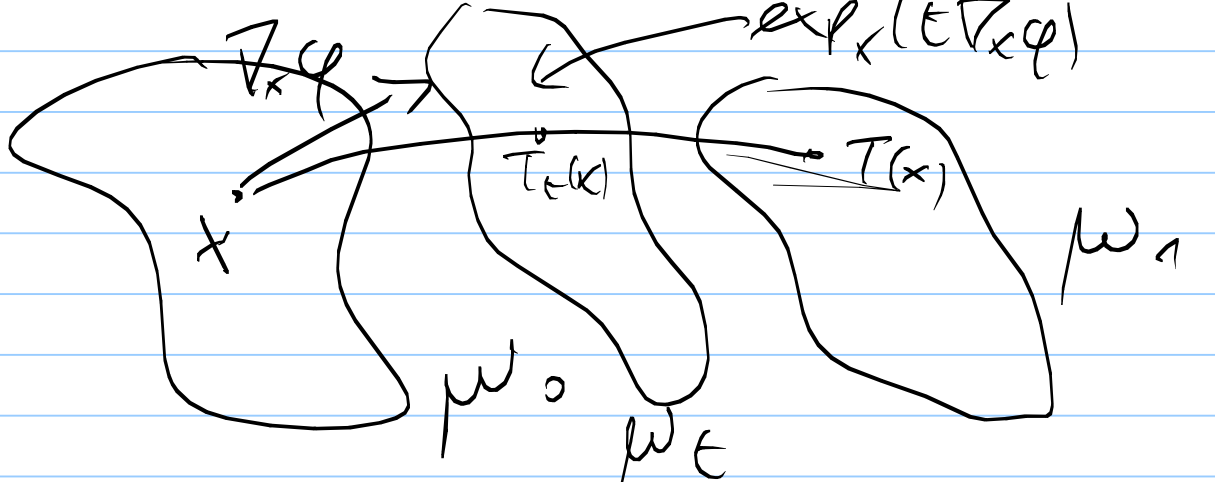
$\Downarrow$

$\exists!$ geod between μ_0 and μ_1,

$$\mu_t := (T_t)_\# \mu_0$$

$$T_t(x) := \exp_x(t \nabla_x \varphi),$$

$T(x) = \exp_x(\nabla_x \varphi)$ is the unique opt. transp from μ_0 to μ_1.



O.T. Reformulation of $\mathrm{Ric} \geq 0$

Let $$H(\mu) := \begin{cases} \int_M \rho \log \rho \, d\mathrm{vol} & \text{if } \mu = \rho \mathrm{vol} \\ +\infty & \text{if } \mu \not\ll \mathrm{vol} \end{cases}$$

THM

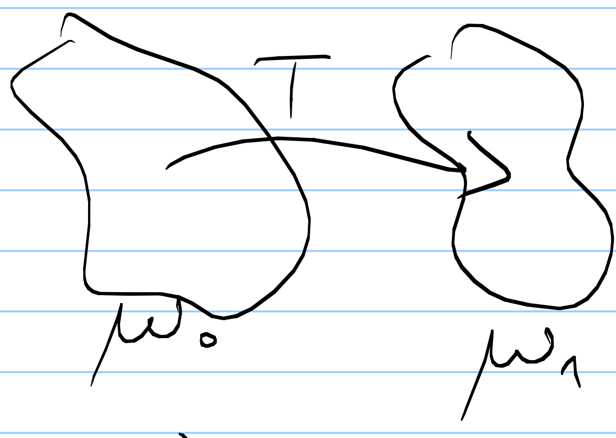
$\mathrm{Ric} \geq 0 \iff \forall (\mu_t)_{0 \leq t \leq 1}$ Wass. geod,

$t \mapsto H(\mu_t)$ convex

Sketch

$\mu_0 = \rho_0 \mathrm{vol}, \quad \mu_1 = \rho_1 \mathrm{vol}$

$T(x) = \exp_x(\nabla_x \varphi)$



$\mu_t := (T_t)_\# \mu_0$

$T_t(x) := \exp_x(t \nabla_x \varphi)$

Let ρ_t be the density of μ_t,

i.e. $\mu_t = \rho_t \, \mathrm{vol}$.

$$(T_t)_\# \mu_0 = \mu_t$$

$$\Downarrow$$

$$|\det d_x T_t| = \frac{\rho_0}{\rho_t \circ T_t}$$

$$H(\mu_t) = \int \log \rho_t \, \underbrace{\rho_t \, d\mathrm{vol}}_{=\mu_t}$$

$$= \int \log \rho_t \, (T_t)_\# (\rho_0 \, d\mathrm{vol})$$

$$= \int \log(\rho_t \circ T_t) \, \rho_0 \, d\mathrm{vol}$$

$$= \int \log\left(\frac{\rho_0}{\det \nabla_x T_t}\right) \rho_0 \, d\mathrm{vol}$$

$$= \int \log \rho_0 \, \rho_0 \, d\mathrm{vol}$$

$$- \int \underbrace{\log(\det \nabla_x T_t)}_{\substack{\text{concave for} \\ \text{a lot of fct } \varphi \\ \Updownarrow \\ Ric \geq 0}} \rho_0 \, d\mathrm{vol}$$

$\square$

STABILITY of RICCI BOUNDS

GH convergence : $(X_k, d_k) \xrightarrow[GH]{cpt} (X, d)$

if $\exists \, f_k : X_k \to X$, $\varepsilon_k \to 0$ s.t.

f_k are ε_k-isometries, i.e.

$$\begin{cases} |d(f_k(x), f_k(y)) - d_k(x,y)| \leq \varepsilon_k & \forall x, y \in X_k \\ \mathrm{dist}_d(f_k(X_k), X) \leq \varepsilon_k \end{cases}$$

ε_k-surjective

<u>IHM</u>

$$(M_k, g_k) \xrightarrow{MGH} (M, g),$$

$$\mathrm{Sect}_{M_k} \geq -K \Rightarrow \mathrm{Sect}_M \geq -K$$

---

measured Gromov-Hausdorff

<u>MGH-conv</u> : $(X_k, d_k, \nu_k) \xrightarrow{MGH} (X, d, \nu)$

if $\exists f_k : X_k \to X$, $\varepsilon_k \to 0$ s.t.

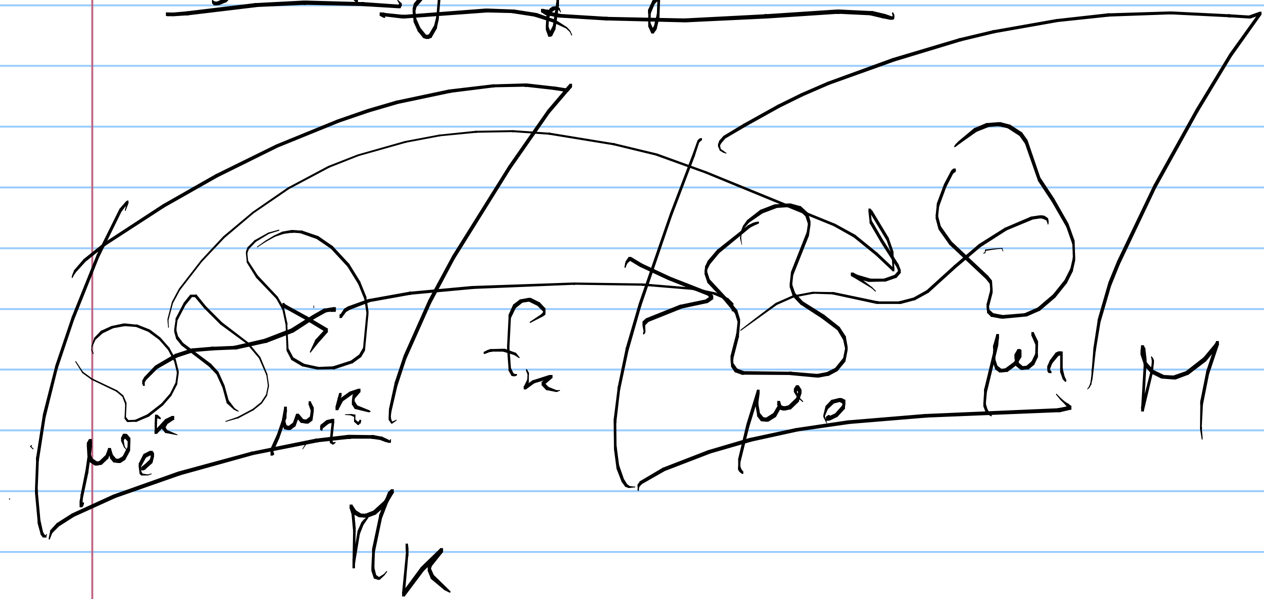
- f_k are ε_k-isom
- $(f_k)_\# \nu_k \longrightarrow \nu$

THM (Cor of Lott–Villani/Sturm)

$$(M_k, g_k, vol_k) \xrightarrow{MGH} (M, g, vol)$$

$$Ric_{M_k} \geq K \Rightarrow Ric_M \geq K.$$

IDEA of Pf for K=0



Given $\mu_0 = \rho_0 \, vol$, $\mu_1 = \rho_1 \, vol$, define

$$\mu_0^k := \frac{\rho_0 \circ f_k \, vol_k}{\int \rho_0 \circ f_k \, vol_k}, \quad \mu_1^k = \frac{\rho_1 \circ f_k \, vol_k}{\int \rho_1 \circ f_k \, vol_k}$$

Take $(\mu_t^k)_{0\le t\le 1}$ Wass geod,

Construct μ_t as lim of

$$(f_k)_\# \mu_t^k \text{ as } k\to\infty.$$

(Actually, μ_t is a geod)

We know

$$(1-t)\,H(\mu_0^k) + t\,H(\mu_1^k) \ge H(\mu_t^k)$$

We want to pass to the limit.

- $$Z_k^0 := \int_{M_k} \rho_0\circ f_k \, d\mathrm{vol}_k$$

$$= \int_M \rho_0 \, d\underbrace{\big((f_k)_\# \mathrm{vol}_k\big)}_{\downarrow\ \mathrm{vol}}$$

$$\downarrow$$

$$1 = \int_M \rho_0 \, d\mathrm{vol} \quad \text{if } \rho_0 \in C^0$$

- $H(\mu\omega_0^k) = \int \frac{\rho_0 \circ f_k}{Z_k} \log\left(\frac{\rho_0 \circ f_k}{Z_k}\right) dvol_k$

$$\simeq \int \rho_0 \circ f_k \log(\rho_0 \circ f_k) \, dvol_k$$

$$= \int \rho_0 \log \rho_0 \, d(\underbrace{(f_k)_\# vol_k}_{\downarrow \, vol})$$

$$\downarrow$$

$$\int \rho_0 \log \rho_0 \, dvol$$

$$\text{if } \rho_0 \in C^0$$

- $H_\nu(\mu) := \int \rho \log \rho \, d\nu \quad \text{if } \mu = \rho\nu$

$\downarrow$

<u>LEMMA</u> $H_{f_\# \nu}(f_\# \mu) \leq H_\nu(\mu)$

$)$

$$\underbrace{H_{(f_k)_\# vol_k}}_{\downarrow\; vol}\Big(\underbrace{(f_k)_\# \mu^k_t}_{\downarrow\;\mu_t}\Big) \le H_{vol_k}(\mu^k_t)$$

LEMMA $(\nu,\mu) \mapsto H_\nu(\mu)$ l.s.c. w.r.t. weak conv

$$\Downarrow$$

$$\liminf_{k\to\infty} H_{(f_k)_\# vol_k}\big((f_k)_\# \mu^k_t\big) \ge H_{vol}(\mu_t)$$

So

$$(1-t)H(\mu^k_0) + t\,H(\mu^k_1) \ge H(\mu^k_t)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \underline{\liminf}$$

$$(1-t)\, H(\mu_0) + t\, H(\mu_1) \geq H(\mu_t)$$

So the ineq holds whenever

$\rho_0, \rho_1 \in C^0$

$\downarrow$

by approx, the convexity ineq holds $\forall \rho_0, \rho_1$. $\square$

COMMENT:

The convexity of $H_\nu(\mu)$ along Wass geod can be used to DEFINE $Ric \geq 0$ ($\geq K$) on metric measure space (X, d, ν) (Lott-Villani Sturm)

Rmk (Petrunin)

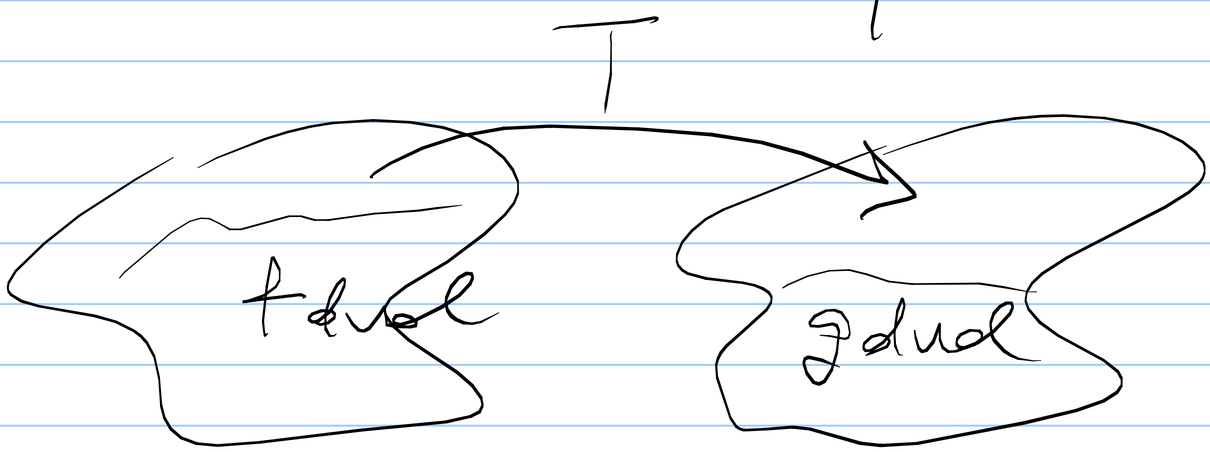
(X,d) Alexandrov space with sect curvature ≥ 0 and finite dim

$$\Downarrow \quad \text{Hausd meas}$$

(X, d, vol) has $Ric \geq 0$

REGULARITY of OPTIMAL TRANSPORT MAPS

Recall the Jacobian eq:



$$\begin{cases} |\det \nabla T| = \dfrac{f(x)}{g \circ T(x)} = h(x, T(x)) \\ T(\operatorname{supp} f) = \operatorname{supp} g \end{cases}$$

$\underline{M = \mathbb{R}^n}$

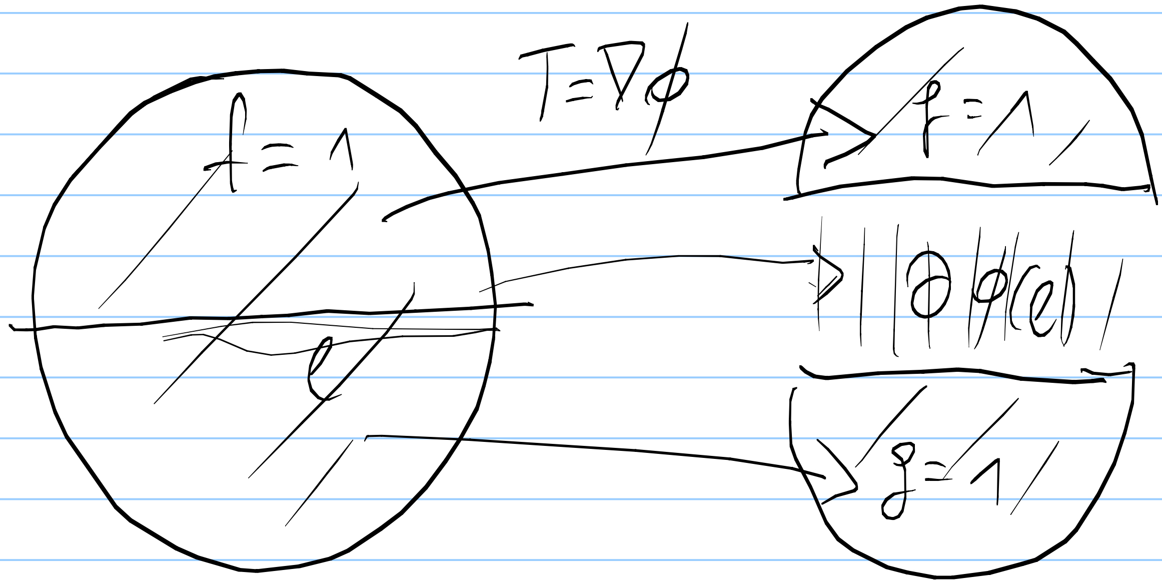
$T = \nabla \phi, \quad \phi$ convex

$$\begin{cases} \det \nabla^2 \phi = \dfrac{f}{g \circ \nabla \phi} = h(x, \nabla \phi) \\ \nabla \phi(\operatorname{supp} f) = \operatorname{supp} g \end{cases}$$

$\underline{Q}$: $f, g \in C^\infty$ on their support

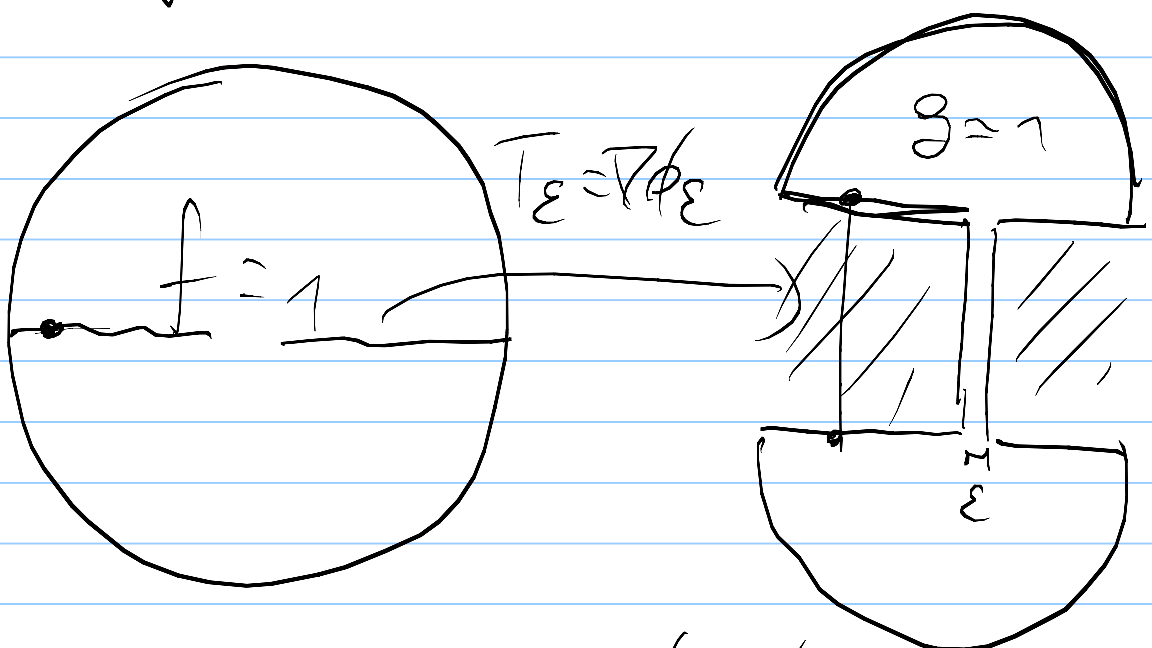
$\Rightarrow T \in C^\infty$

Ex $\mathbb{R}^2$



$T \notin C^0$

Caffarelli '91



If $\varepsilon << 1$, $\phi_\varepsilon \notin C^1$

To be able to "capture" the subdiff, I need to assume that supp g is convex.

<u>THM</u> (Caff, '91) Assume supp g convex

- $\lambda \leq f, g \leq \frac{1}{\lambda}$ on their resp supp

$$\Downarrow$$

$$\phi \in C^{1,\alpha}_{loc} \Longleftrightarrow T \in C^{0,\alpha}_{loc}$$

- If furthermore $f, g \in C^{\infty}$

$$\Downarrow$$

$$\phi \in C^{\infty} \Rightarrow T \in C^{\infty}$$

THM (F., Kim '10)

supp $f = \Omega$, supp $g = \Lambda$ open,

$\lambda \le f \le \frac{1}{\lambda}$ on Ω, $\lambda \le g \le \frac{1}{\lambda}$ on Λ

$\Downarrow$

$\exists\ \Omega' \subseteq \Omega,\ \Lambda' \subseteq \Lambda$ open sets

s.t.

$|\Omega \setminus \Omega'| = |\Lambda \setminus \Lambda'| = 0$

$T : \Omega' \to \Lambda' \in C^{0,\alpha}$

and is homeo.

If further $f, g \in C^\infty$, T smooth.

$$(M, g) \qquad c(x,y) = \frac{d^2(x,y)}{2}$$

$$T(x) = \exp_x(\nabla_x \varphi)$$

$$\Downarrow$$

$$\nabla_x \varphi + \underbrace{\nabla_x c(x, T(x))}_{} = 0$$

$$\nabla_x \left\{ \nabla_x c(x,y) \Big|_{y=T(x)} \right.$$

$$\nabla_{xx} \varphi + \nabla_{xx} c(x, T(x))$$

$$+ \nabla_{xy} c(x, T(x)) \nabla_x T = 0$$

$$\Downarrow$$

$$\det\Big(\underbrace{\nabla_{xx}\varphi + \nabla_{xx} c(x, T(x))}_{\text{pos. def. by c-convexity}} \Big) =$$

$$= |\det \nabla_{xy} c|(x,T(x)) \cdot |\det \nabla_x T|$$

$$= |\det \nabla_{xy} c|(x,T(x)) \, \frac{f(x)}{g(T(x))}$$

$$= h(x,T(x))$$

$$\det\Big(\nabla_{xx}\varphi + \underbrace{\nabla_{xx} c(x,T(x))}_{= A(x,\nabla_x\varphi)}\Big) = \overbrace{h(x,T(x))}^{H(x,\nabla_x\varphi)}$$

$$T(x) = \exp_x(\nabla_x \varphi)$$

$$A(x,p) := \nabla_{xx} c(x, \exp_x(p))$$

$$\begin{cases} \det\left(\nabla_{xx}\varphi + A(x, \nabla_x \varphi)\right) = H(x, \nabla_x \varphi) \\ \text{bdry cond: } T(\operatorname{supp} f) = \operatorname{supp} g \end{cases}$$

RMK

If $M = \mathbb{R}^n$, $A(x,p) = I_n$

$$\nabla_{xx}\varphi + I = \nabla_{xx}\Big(\underbrace{\varphi + \frac{|x|^2}{2}}_{\phi}\Big)$$

Ma – Trudinger – Wang '05:

we want to try to get a-priori estimates on the sol.

$\Big\downarrow$ Fix a direction $\alpha \in S^{n-1}$

Differentiate the eq twice in the direction α:

$$\partial_{\alpha\alpha}\Big(\det\big(\nabla_{xx}\varphi + \underset{A_{ij}(x,\nabla_x\varphi)}{\underset{\shortparallel}{A(x,\nabla_x\varphi)}}\big) = \text{---}\ \text{---}\Big)$$

$$a_{ij}(x)\,\partial_{ij}\underline{(\partial_{\alpha\alpha}\varphi)} + \text{- - -}$$

$$\underbrace{+\,\gamma_{ij}(x)\,\partial_\alpha\Big(D_p A_{ij}(x,\nabla_x\varphi)\nabla_x\partial_\alpha\varphi\Big)}$$

$$\Big\downarrow$$

$$a_{ij}(x)\,D_{pp}A_{ij}(x,\nabla_x\varphi)\left[\nabla_x\partial_\alpha\varphi,\ \nabla_x\partial_\alpha\varphi\right]$$

Fix x_0, α s.t. $\partial_{\alpha\alpha}\varphi(x_0)$ is maximal

$\Downarrow$

try to prove that

$$|\partial_{\alpha\alpha}\varphi(x_0)| \le C$$

<u>COND</u> (MTW):

MTW(0) $\quad D_{p_\eta p_\eta} A_{ij}(x,p)\,\xi^i\xi^j \le 0 \quad \forall \xi \perp \eta$

MTW(K) $\quad D_{p_\eta p_\eta} A_{ij}(x,p)\,\xi^i\xi^j \le -K|\xi|^2|\eta|^2 \quad \forall \xi \perp \eta$

$K > 0$

Thm (Ma-Tr-Wang '05, Tr-W '07)

If $f, g \in C^\infty$, c smooth on $\operatorname{supp} f \times \operatorname{supp} g$, suitable convexity assumptions, MTW(0) holds

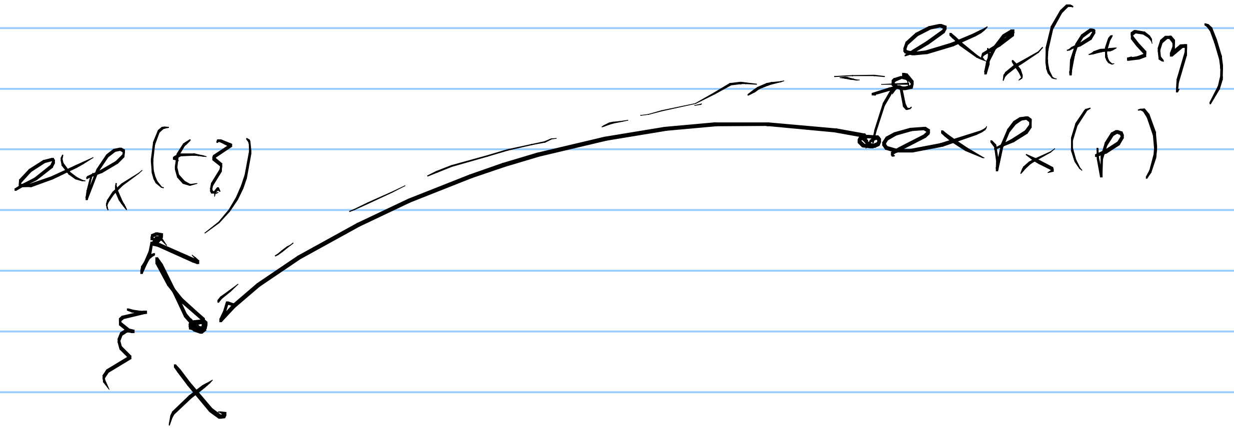
$$\Downarrow$$

$$\varphi \in C^\infty \Rightarrow T \in C^\infty$$

What does MTW mean?

$$A(x,p) = \nabla_{xx} c(x, \exp_x(p))$$

MTW(0)
$$\overbrace{\frac{d^2}{ds^2}\Big|_{s=0} \frac{d^2}{dt^2}\Big|_{t=0} c(\exp_x(t\xi), \exp_x(p+s\eta))}^{\mathfrak{S}_{x,p}(\xi,\eta)} \leq 0$$



RMK (Loeper '07)

If $p=0$, $\quad \sigma_{x,0}(\xi,\eta) = -\frac{3}{2}\,\mathrm{Sect}_x([\xi,\eta])$

THM (Loeper '07)

If $\exists\,(x,p,\xi,\eta)$, $\xi\perp\eta$ s.t.

$$\sigma_{x,p}(\xi,\eta) > 0 \Rightarrow \exists\, f, g \in C^\infty$$

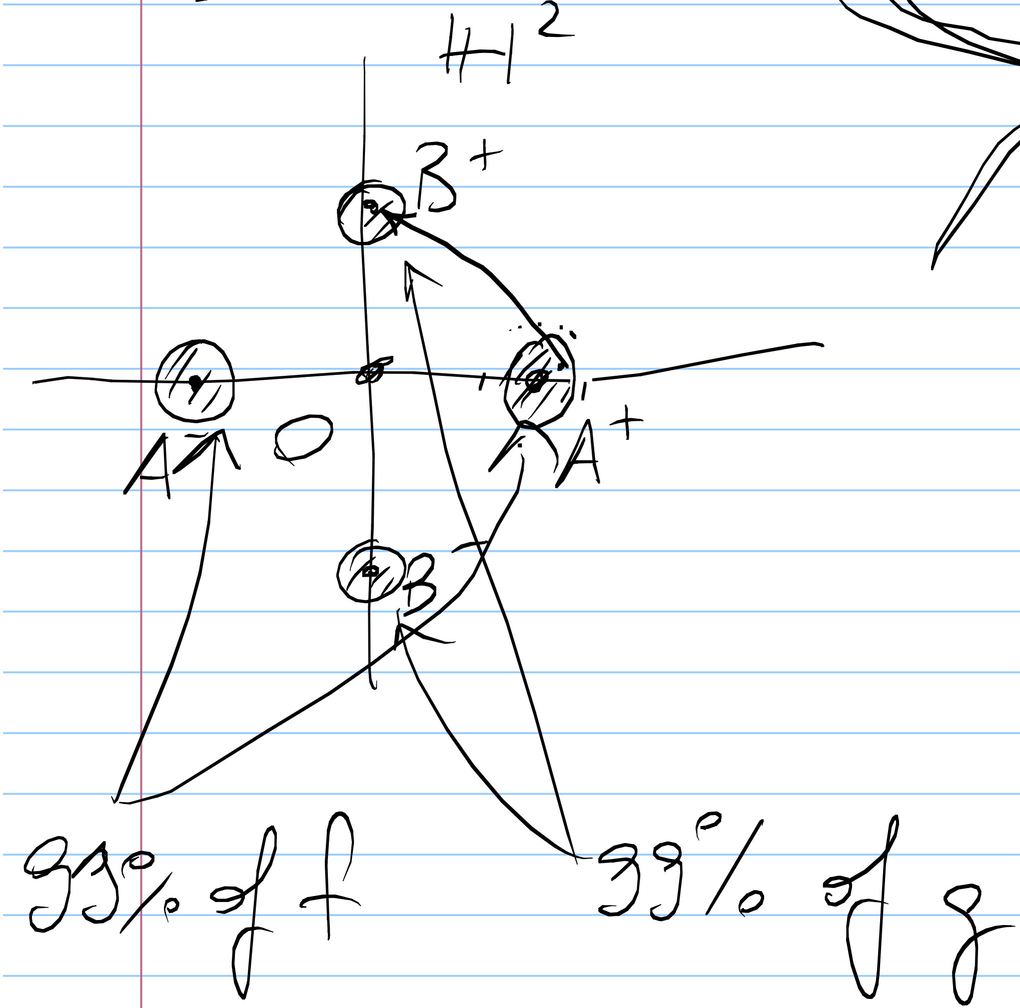
s.t. $T \notin C^0$

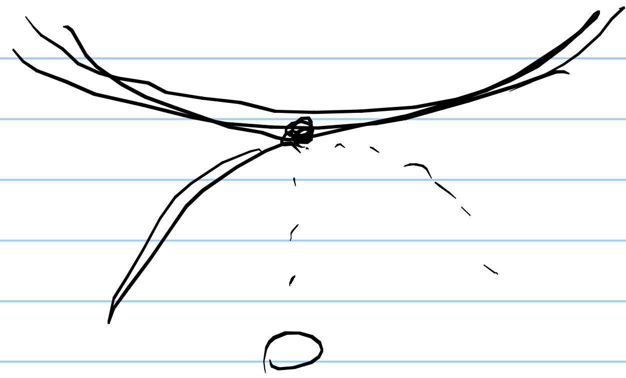
COR

If $\exists\, x\in M$, $\xi,\eta\in T_xM$ s.t.

$\mathrm{Sect}_x([\xi,\eta])<0 \Rightarrow$ NO regr

Example





Assume by contr that $T \in C^0$.

By symm $T(0) = 0$.

Moreover $\exists\, A_\varepsilon^+ \sim A^+$, $B_\varepsilon^+ \sim B^+$ s.t.

$$T(A_\varepsilon^+) = B_\varepsilon^+$$

By negative curv,

$$d^2(A^+,0)+d^2(0,B^+) < d^2(A^+,B^+) + d^2(0,0)$$

$$\Downarrow \quad A_\varepsilon^+ \sim A^+,\ B_\varepsilon^+ \sim B^+$$

$$T(0)=0 \qquad B_\varepsilon^+ = T(A_\varepsilon^+)$$

$$d^2(A_\varepsilon^+, T(0)) + d^2(0, T(A_\varepsilon^+)) < d^2(A_\varepsilon^+, T(A_\varepsilon^+)) + d^2(0, T(0))$$

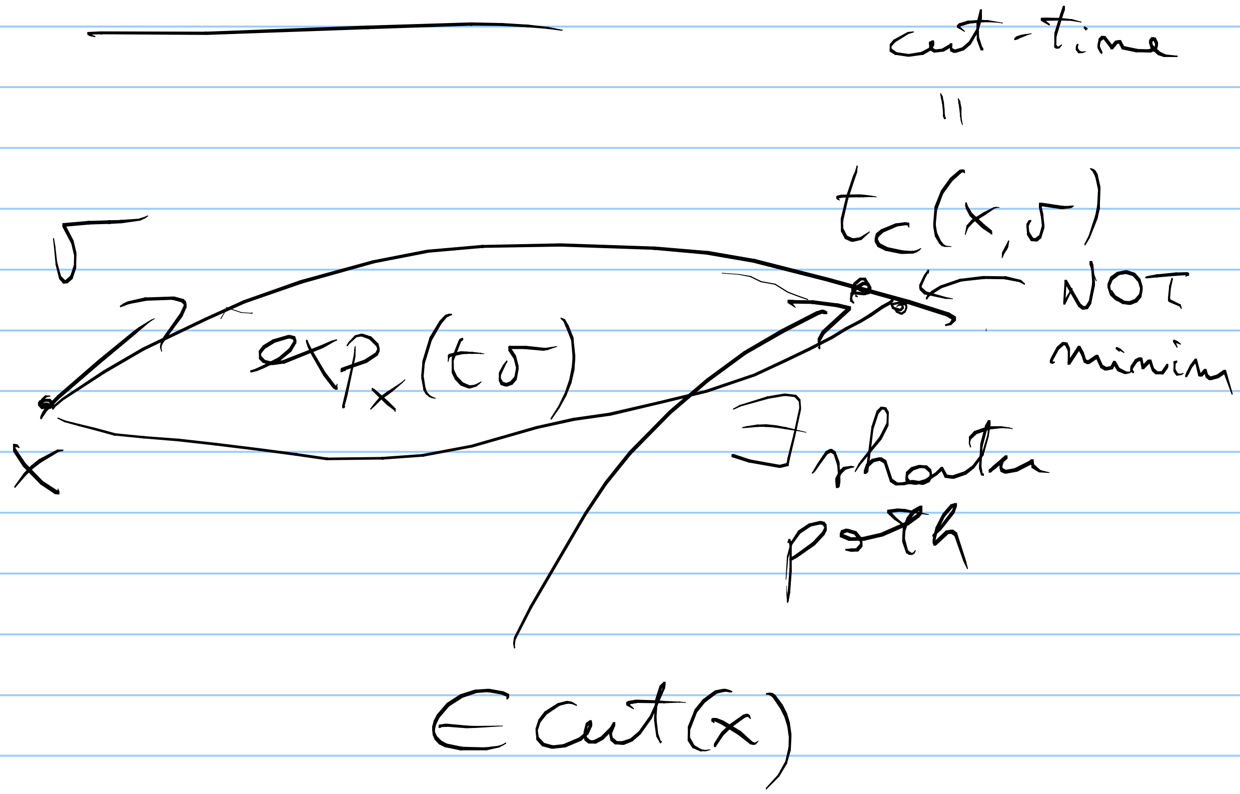
$\Downarrow$

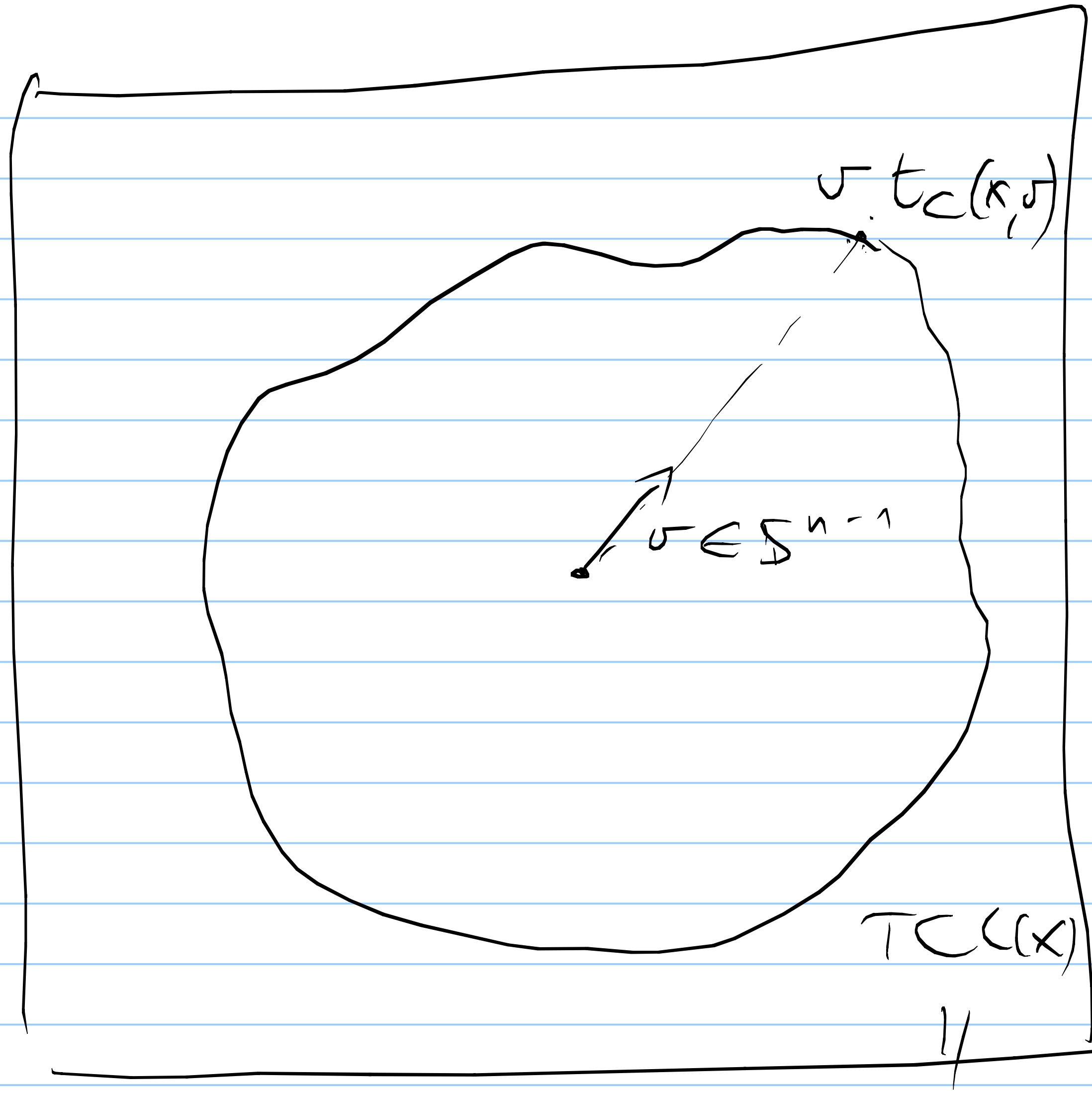
Contradiction to cycl. monot. $\square$

Examples of (M,g) which sat MTW

$\mathbb{R}^n$, $\mathbb{T}^n$, $\mathbb{S}^n$, $\mathbb{S}^{n_1}_{r_1} \times \dots \times \mathbb{S}^{n_k}_{r_k}$, $\mathbb{RP}^n$, $\mathbb{CP}^n$

RELATION of MTW with the CUT-LOCUS

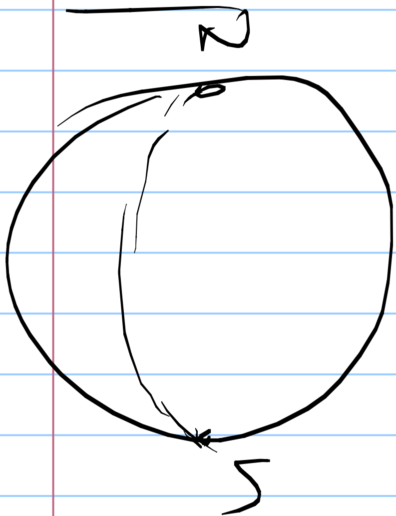




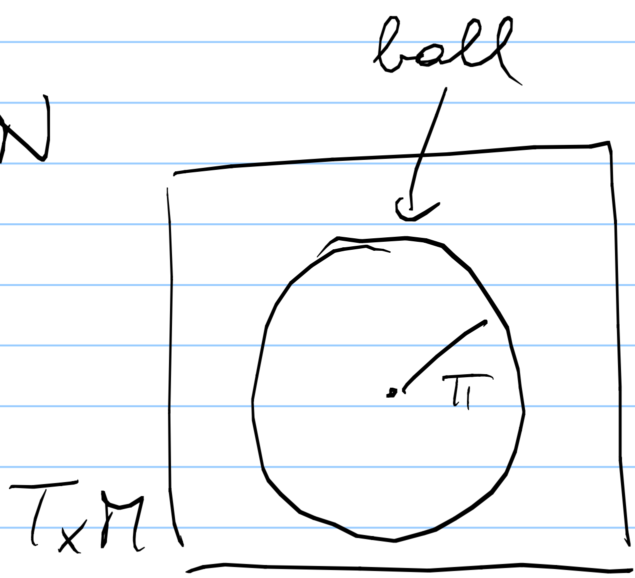
T_xM

$\|$
Tangent cut locus of x

Ex



$\mathrm{cut}(S) = N$



$$x \in \mathrm{cut}(y) \iff \nabla_{x_\xi x_\xi} \frac{d^2(\cdot, y)}{2}\Big|_x = -\infty$$

$$\Downarrow$$

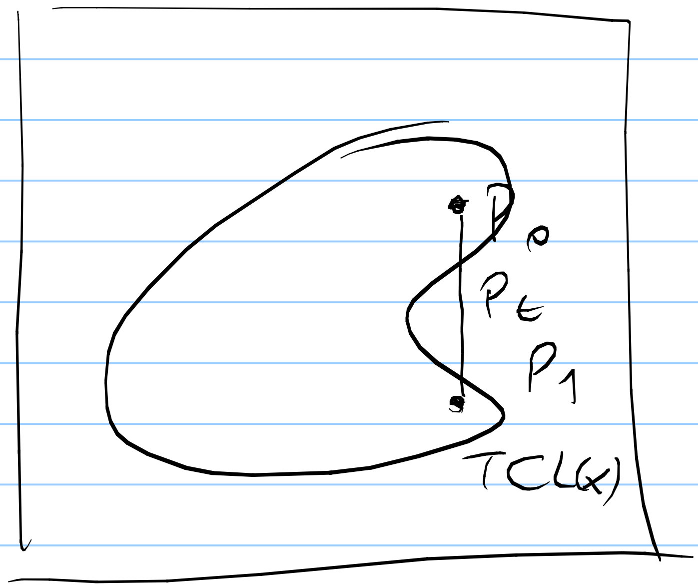
$$y \in \mathrm{cut}(x)$$

MTW(0) $\displaystyle \frac{d^2}{ds^2}\Big|_{s=0} \nabla_{x_\xi x_\xi} c(x, \exp_x(p + s\eta)) \le 0$

$\xi \perp \eta$

Forget $\xi \perp \eta$,

$p \mapsto \nabla_{x_\xi x_\xi} c(x, \exp_x(p))$ concave



$T_x M$

$$\nabla_{x,x} c(x, \exp_x(p_t))$$

$$\geq \underbrace{\min\{ \ldots p_0 , \ldots p_1 \}}_{\geq -C}$$

$$\exp_x(p_t) \notin \mathrm{cut}(x) \iff p_t \notin TCL(x)$$

$$\Downarrow ?$$

TCL(x) convex

STABILITY:

(M, g) satisfies $MTW(K)$
$K>0$

(M, g_ε), $\|g_\varepsilon - g\|_{C^4} << 1$

Q Does (M, g_ε) satisfies $MTW(0)$?

THM ($F.$, Rifford, Villani '10)

(S^n, g_ε), $\|g^{can} - g_\varepsilon\| \leq \varepsilon$, $\varepsilon \leq \varepsilon_0$

$$\Downarrow$$

$TCL(x)$ unif convex $\forall x$