SOBOLEV SPACES IN METRIC SPACES

(HAJLASZ, KOSKELA, CHEEGER, HEINONEN, ...)

<u>In $\mathbb{R}^n$</u>

<u>$p=2$</u>

$$W^{1,2}(\mathbb{R}^n) = \left\{ u \cancel{\in L^2}(\mathbb{R}^n) \;\middle|\; \frac{\partial u}{\partial x_i} \in L^2 \quad i=1,\dots,n \right\}$$

$$H^{1,2}(\mathbb{R}^n) = \overline{\{C_c^\infty\}}^{W^{1,2}}$$

" H = W " MEYERS-SERRIN 1860.

B. LEVI 1901 In $\mathbb{R}^2$ (same in $\mathbb{R}^n$)

$u \in N^{1,2}(\mathbb{R}^2)$ iff
a) for a.e. x, $u(x,\cdot) \in AC$
b) for a.e. y, $u(\cdot,y) \in AC$
c) $\iint \left|\frac{\partial u}{\partial y}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2 dx\,dy < \infty$.

THEOREM $N^{1,2} \subset W^{1,2}$ Conversely $\forall u \in W^{1,2}$

$\exists\, \tilde{u}$ in the same Lebesgue class, $\tilde{u} \in N^{1,2}$

$$\tilde{u} = \lim_{\varepsilon \to 0} u * \rho_\varepsilon$$

$$H^{1,2} \longrightarrow \int |\nabla f|_*^2 \, dm$$

$$H^{1,2} \longleftrightarrow \{ f : Ch(f) < \infty \}$$

SHANMUGALINGHAM '00 $\quad \Gamma \subset AC([0,1]; X)$

$$(\mathrm{Mod}_2(\Gamma))^2 = \inf \left\{ \int g^2 \, dm \;\middle|\; \begin{array}{l} g \geq 0 \text{ Borel} \\ \int_\gamma g \geq 1 \;\; \forall \gamma \in \Gamma \end{array} \right\}$$

A property holds for a.e. curve if the set where it fails is Mod_2-negligible.

$$|f(\gamma(0)) - f(\gamma(1))| \leq \int_0^1 |\nabla f|(\gamma(s))\,|\gamma'(s)|\,ds = \int_\gamma |\nabla f|$$

WEAK UPPER GRADIENT of f $\quad G \geq 0 \quad |f(\gamma_0) - f(\gamma_1)| \leq \int_\gamma G$ a.e. γ

$$N^{1,2} = \{ f \mid \exists\, G \text{ w.u.g. of } f \text{ in } L^2 \}$$

CLOSURE OF W.U.G.

$$\begin{cases} f_n \xrightarrow{L^2} f \\ G_n \text{ u.g. of } f_n \\ G_n \rightharpoonup G \text{ in } L^2 \end{cases} \Longrightarrow G \text{ w.u.g. of } f.$$

COROLLARY 1

$\{ G \mid G \text{ w.u.g. of } f \}$ convex closed

$|\nabla f|_S$ = the w.u.g. with minimal L^2 norm.

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THEOREM Any function in $N^{1,2}$ is a.c. along a.e. curve

PROOF $\Gamma = \{ \gamma \mid$ the w.u.g. property fails $\}$ $\underset{\substack{\parallel \\ 0}}{\mathrm{Mod}_2(\Gamma)}$:

$\Gamma_1 = \{ \gamma \mid \exists\ \gamma' \subset \gamma\ ,\ \gamma' \in \Gamma \}$. $\mathrm{Mod}_2(\Gamma_1) = 0$

$\Gamma_2 = \{ \gamma \mid \int_\gamma G = \infty \}$ $\quad g = G/n \quad \mathrm{Mod}_2(\Gamma_2) = 0.$

$$\mathrm{Mod}_2^2\left(\{ \gamma \mid \int_\gamma G \geq n \} \right) \leq \frac{1}{n^2} \int G^2 \longrightarrow 0$$

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$\gamma \in AC \setminus (\Gamma_1 \cup \Gamma_2) \qquad \int_\gamma G < \infty$

$\forall s,t \in [0,1] \qquad |f(\gamma_s) - f(\gamma_t)| \leq \int_{\gamma_{[s,t]}} G$

$\Rightarrow \quad s \to f(\gamma(s)) \in AC.$ $\blacksquare$

Corollary $\quad |\nabla f|_S \leq |\nabla f|_* \quad m\text{-a.e. in } X.$

Proof $\quad |\nabla f|_* = \lim_n |\nabla f_n|, \qquad f_n \in \mathrm{Lip}, \quad f_n \xrightarrow{L^2} f$

$G_n = |\nabla f_n|$

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Closure $\Rightarrow$ $|\nabla f|_*$ is a w.u.g.

Indeed $|\nabla f|_S = |\nabla f|_*$ m-a.e. in X.

Given $f \in N^{1,2}$ we need to find $f_n \in \mathrm{Lip}$, $f_n \to f$ L^2

$$\overline{\lim_n} \int |\nabla f_n|^2 \, dm \leq \int |\nabla f|_S^2$$

CONVENTIONAL STRATEGY

1. $M(x) = \sup_{r>0} \dfrac{\int_{B_r(x)} |\nabla f|_S \, dm}{m(B_r(x))}$

doubling property of the measure

maximal inequalities $\quad M \in L^2$

2. $f\big|_{\{M \le \lambda\}}$ is Lipschitz $\mathrm{lip} \le C\lambda$

Poincaré in.

3 Extend $f\big|_{\{M \leq \lambda\}}$ to a $C\lambda$-Lip. function.

g_λ

$$\lambda^2 m(\{M > \lambda\}) \longrightarrow 0 \qquad \lambda \to \infty.$$

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<u>IDEAS:</u> ENTROPY AND SHARP ENERGY DISSIPATION ESTIMATES

WHY WANT $|\nabla \rho|_S = |\nabla \rho|_*$?

relevant in connection with the L^2 heat flow

relevant in connection with the slope of the entropy.

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$$\mathrm{Ent}_m(f\,m) - \mathrm{Ent}_m(\tilde{f}\,m) \leq \int \left(\log f(x) - \log f(y)\right) d\pi(x,y)$$

π optimal from fm to $\tilde{f}m$.

$$= \int \left(\log f(\gamma_0) - \log f(\gamma_1)\right) d\eta(\gamma)$$

related to $|\nabla \log f|_s$

SUPERPOSITION PRINCIPLE

[AGS], LISINI '07

P. BERNARD "YOUNG MEASURES, SUPERPOSITION AND TRANSPORT"

$e_t(\gamma) = \gamma(t)$

$\{\mu_t\}_{t \in [0,T]} \subset \mathcal{P}(X)$ W_2

$\int_0^T |\dot{\mu}_t|^2 \, dt < \infty$

Then $\exists\, \eta \in \mathcal{P}(C([0,1]; X))$ concentrated on $AC^2([0,1]; X)$ satisfying

(1) $\mu_t = (e_t)_\# \eta \quad \forall t \in [0,T]$

(2) $|\dot{\mu}_t|^2 = \int |\dot{\gamma}_t|^2 \, d\eta(\gamma)$ a.e. t.

$$\int \varphi \, d\mu_t = \int \varphi(\gamma_t) \, d\eta(\gamma) \qquad \forall t, \ \forall \varphi \in C_b$$

<u>**Lemma**</u> $\eta \in \mathcal{P}(C([0,1];X))$ concentrated on AC_2, with bounded marginals (i.e. $(e_t)_\# \eta \leq C m$) and finite 2-action $\int \int_0^1 |\dot\gamma_t|^2 \, dt \, d\eta(\gamma) < \infty$.

Then $\mathrm{Mod}_2(\Gamma) = 0 \implies \eta(\Gamma) = 0$.

<u>Proof</u> Let g be admissible for γ,

$$(\eta(\Gamma))^2 \le \left(\int \int_\gamma \right)^2$$

$$\le \left(\int_0^1 \int g^2(\gamma(s))\, d\eta(\gamma)\, ds \right) \left(\int \right.$$

$$\le \tilde{C} \int_0^1 \int g^2\, dm\, ds$$

$$\le c\tilde{C} \int g^2$$

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$$\left(\eta(\Gamma)\right)^2 \le C\tilde{C}\left(\mathrm{Mod}_2(\Gamma)\right)^2$$

Proof of $|\nabla f|_* \le |\nabla f|_S$.

It is sufficient to show $\exists\, f_n \in D(Ch)$, $f_n \xrightarrow{L^2} f$

$$\overline{\lim_{n\to\infty}}\; Ch(f_n) \le \frac{1}{2}\int |\nabla f|_S^2 \, dm$$

$0 < c \le f \le C < \infty \qquad \int f^2 \, dm = 1$

$h_0 := f^2 \qquad h_t = L^2$-g.f. of Ch starting from h_0

$\mu_t = h_t m \in \mathcal{P}(X)$ SUPERPOSITION PRINCIPLE

$$\Downarrow$$

$$\eta$$

η has bounded marginals 2-action is finite.

f (and also h) is AC along η-a.e. curve.

$$|\nabla h|_S = 2f\,|\nabla f|_S$$

$$|\nabla \log h|_S = 2\,|\nabla f|_S / f.$$

$$\int (h_0 \ln h_0 - h_t \ln h_t)\, dm \leq \int \ln h_0 \,(h_0 - h_t)\, dm$$

$$= \int \left(\ln h_0(\gamma_0) - \ln h_0(\gamma_t)\right) d\eta(\gamma)$$

$$\leq \int \int_0^t |\nabla \log h_0|_S(\gamma_r)\, |\dot{\gamma}_r|\, dr\, d\eta(\gamma)$$

$$\leq \left(\int_0^t \int |\nabla \log h_0|_S^2 (\gamma_r)\, d\eta(\gamma)\, dr \right)^{1/2} \left(\int_0^t \int |\gamma_s'|^2\, d\eta(\gamma)\, ds \right)^{1/2}$$

$$\int_0^t \int |\gamma_s'|^2\, d\eta(\gamma)\, ds = \int_0^t |\dot{\mu}_s|^2\, ds$$

$$\leq$$

$$\int_0^t \int_{\{h_s>0\}} |\nabla h_s|^2 / h_s \, dm \, ds$$

YOUNG $\Rightarrow$

$$\int_0^t \int_{\{h_s>0\}} |\nabla h_s|^2 / h_s \, dm \, ds \le \int_0^t \int_X |\nabla \log h|^2 h_s \, dm \, ds.$$

$$\fint_0^t Ch(\sqrt{h_s}) \, ds \le \fint_0^t \int |\nabla f|_S^2 / f^2 \, h_s \, dm \, ds.$$

$\exists\, s_i \downarrow 0$ s.t.

$$\overline{\lim_{i\to\infty}} \; Ch(\sqrt{h_{s_i}}) \le \int |\nabla f|_S^2$$

LOWER BOUNDS ON RICCI CURVATURE IN METRIC MEASURE SPACES

LOTT - VILLANI STURM (DESCRIPTION OF GROMOV-HAUS. LIMITS OF MANIFOLDS WITH RICCI ≥ 0)

Ricci ≥ 0 $\iff$ Ent_m is convex in $\mathcal{P}(M)$

1) CONSISTENCY

2) STABILITY

$\Longrightarrow$ BISHOP-GROMOV POINCARÉ

Ent_m is convex in all $\cancel{(\mathbb{R}^n, \|\ \|)}$

IS THERE A MORE RESTRICTIVE CONDITION?

- STRONG CONVEXITY OF ENT_m + LINEARITY OF THE HEAT FLOW

$$\Updownarrow$$

(EVI) property of gradient flow of Ent_m

$x'(t) \in -\partial F(x(t))$ $\quad$ F convex lsc

H

$$\Updownarrow$$

$\forall z \in D(F)$ $\quad$ $\frac{d}{dt} \frac{|x(t) - z|^2}{2} + F(x(t)) \leq F(z)$ a.e. t