

## SOBOLEV SPACES IN METRIC SPACES

( HASLHASZ, KOSKELA, CHEEGER, HEINONEN, ... )

$$\underbrace{W^{1,2}(\mathbb{R}^m)}_{p=2} = \left\{ u \in \cancel{L^2(\mathbb{R}^m)} \mid \frac{\partial u}{\partial x_i} \in L^2, i=1, \dots, m \right\}$$

$$H^{1,2}(\mathbb{R}^m) = \underbrace{\{C_c^\infty\}}_{W^{1,2}}$$

" H = W " METENS - SERIN 1860.

B. LEVI 1901  $\Gamma_m \mathbb{R}^2$  (nome em  $\mathbb{R}^n$ )

$u \in N^{1,2}(\mathbb{R}^2)$  v.p.

a) p.a.a.  $x$ ,  $u(x, \cdot) \in AC$

b) p.a.a.  $y$ ,  $u(\cdot, y) \in AC$

c)  $\iint \left| \frac{\partial u}{\partial y} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 \, dx dy < \infty$ .

THEOREM  $N^{1,2} \subset W^{1,2}$  Conversely  $\forall u \in W^{1,2}$

$\exists \tilde{u}$  in the sense Lebesgue sense,  $\tilde{u} \in N^{1,2}$

$$\tilde{u} = \lim_{\varepsilon \rightarrow 0} u * \rho_\varepsilon$$

$$H^{1,2} \rightarrow \int_{\Omega} |\nabla \varphi|^2 \, dx$$

$$H^{1,2} \leftrightarrow \{ \varphi : \mathcal{D}_h(\varphi) < \infty \}$$

SHANMUGALINGHAM '00  $\Gamma \in AC(\bar{\Omega}, \mathbb{R}; \chi)$

$$(\text{Mod}_2(\Gamma))^2 = \inf \left\{ \int_{\Omega} g^2 \, dx \mid \begin{array}{l} g \geq 0 \text{ Bred} \\ \int_{\Omega} g \geq 1 \forall \gamma \in \Gamma \end{array} \right\}$$

A property holds for a.s. curve of the net where  
it fails in Montz - negligible.

$$|f(x_0) - f(x_1)| \leq \int_0^1 |\nabla f(x(s))| |x'(s)| ds = \int_0^1 |\nabla f|$$

WEAK UPPER GRADIENT  $G \geq 0$   $|f(x_0) - f(x_1)| \leq \int_0^1 G$   
of  $f$  a.s.  $x$

$$N^{1,2} = \{f \mid \exists G \text{ w.u.g. of } f \text{ in } L^2\}$$

CLOSURE OF W.U.G.

$$\left\{ \begin{array}{l} P_m \xrightarrow{L^2} P \\ G_m \text{ u.g. of } P_m \\ G_m \rightarrow G \text{ in } L^2 \end{array} \right. \Rightarrow G \text{ w.u.g. of } P.$$

COROLLARY 1  $\mathcal{B} \{ G \mid G \text{ w.u.g. of } P \}$  convex closed

$|\nabla P|_S =$  the w.u.g. with minimal  $L^2$  norm.

THEOREM Any function in  $N^{1,2}$  is a.e. along a.e. curve

PROOF  $\Gamma = \{ \gamma \mid \text{the w.u.g. property fails} \}$   $\text{Mod}_2(\Gamma) = 0$

$$\Gamma_1 = \{ \gamma \mid \exists \gamma' \subset \gamma, \gamma' \in \Gamma \}. \quad \text{Mod}_2(\Gamma_1) = 0$$

$$\Gamma_2 = \{ \gamma \mid \sum_{\gamma} G = \infty \} \quad g = G/m \quad \text{Mod}_2(\Gamma_2) = 0.$$

$$\text{Mod}_2^2(\{ \gamma \mid \sum_{\gamma} G \geq m \}) \leq \frac{1}{m^2} \sum G^2 \rightarrow 0$$

$$\gamma \in AC \setminus (I_1 \cup I_2) \quad \int_{\gamma} G < \infty$$

$$\forall \gamma, t \in [0, 1] \quad |P(\gamma_t) - P(\gamma_{t+1})| \leq \int_{\gamma} G$$

$$\Rightarrow \gamma \rightarrow P(\gamma_t) \in AC.$$

□

Corollary  $| \nabla P |_S \leq | \nabla P |_*$  m.-a.e. in  $X$ .

Proof  $| \nabla P |_* = \lim_n | \nabla P_n |$ ,  $f_n \in \text{lip}$ ,  $f_n \xrightarrow{L^2} P$   
 $G_n = | \nabla P_n |$

Classe  $\Rightarrow \|\nabla p\|_*$  is a w.u.g.

Indeed  $\|\nabla p\|_S = \|\nabla p\|_*$  m.o.e. in  $X$ .

Given  $p \in \mathcal{N}^{1,2}$  we need to find  $p_m \in \text{lip}$   
 $p_m \rightarrow p$  in  $\mathcal{L}^2$

$$\overline{\lim}_m \int \|\nabla p_m\|_{dm}^2 \leq \int \|\nabla p\|_S^2$$



CONVENTIONAL STRATEGY

$$\textcircled{1} \quad M(x) = \sup_{r > 0} \underbrace{\int_{B_r(x)} |\nabla \rho|_g \, dv}_{m(B_r(x))}$$

checking  
property  
of the  
measure

maximal inequalities

$$M \in L^2$$

$\textcircled{2}$

$f$

is Lipschitz  $\text{lip} \leq C\lambda$

$\{M \leq \lambda\}$

Poincaré in.

③ Extend  $f$  to a  $C^1$ -lip. function.

$g_\lambda$

$$\lambda^2 m(\{M > \lambda\}) \rightarrow 0 \quad \lambda \rightarrow \infty.$$

IDEAS: ENTROPY AND SHARP ENERGY  
DISSIPATION  
ESTIMATES

WHY WANT  $|\nabla p|_s = |\nabla p|_*$  ?

↑  
relevant in connection with the  
L<sup>2</sup> heat flow

↓  
relevant in connection with the  
slope of the entropy.

$$E_{\pi} \text{Ent}_m(f_m) - \text{Ent}_m(\tilde{f}_m) \leq \int (\log f(x) - \log f(y)) d\pi(x, y)$$

$\pi$  optimal from  $f_m$  to  $\tilde{f}_m$ .

$$= \int (\log f(x_0) - \log f(x_1)) d\eta(x)$$



related to  $|\nabla \log f|_s$

# SUPERPOSITION PRINCIPLE [AGS], LISINI '07

P. BERNARD "YOUNG MEASURES,  
SUPERPOSITION AND  
TRANSPORT"

$$\rho_t(x) = \rho(x)$$

$$\left\{ \mu_t \right\}_{t \in [0, T]} \subset \mathcal{P}(x) \quad W_2$$

$$\sum_0^T |\dot{\mu}_t|^2 < \infty$$

Then  $\exists \eta \in \mathcal{D}([0, T]; x)$   
concentrated on  $AC^2([0, T]; x)$   
satisfying

$$(1) \quad \mu_t = (\rho_t) \# \eta \quad \forall t \in [0, T]$$

$$(2) \quad |\dot{\mu}_t|^2 = \int |\dot{\sigma}_t|^2 d\eta(x) \quad \text{a.e. } t.$$

$$\int \varphi d\mu_\varepsilon = \int \varphi(x_\varepsilon) d\eta(x) \quad \forall t, \forall \varphi \in C_0$$

Lemma  $\eta \in \mathcal{P}(C([0, T]; \mathbb{R}^d))$  concentrated on  $AC_2$ ,

with bounded marginals (i.e.  $(\mathcal{L}_t)_\# \eta \leq C m$ )  
and finite 2-action  $\int_0^1 \int \|\dot{x}_\varepsilon\|_t^2 dt d\eta(x) < \infty$ .

$$\text{Then} \quad \text{Mod}_2(\Gamma) = 0 \Rightarrow \eta(\Gamma) = 0.$$

Proof Let  $g$  be admissible for  $\gamma$ ,

$$\begin{aligned} (\eta^a(\Gamma))^2 &\leq \left( \int \int_{\gamma} g \right)^2 \\ &\leq \left( \int_0^1 \int_{-\infty}^{\infty} g^2(\gamma(s)) d\eta(\gamma) ds \right) \left( \int_0^1 \int_{-\infty}^{\infty} g^2(\gamma(s)) d\eta(\gamma) ds \right) \\ &\leq C^2 \int_0^1 \int_{-\infty}^{\infty} g^2(\gamma(s)) d\eta(\gamma) ds \\ &\leq C^2 \int_0^1 \int_{-\infty}^{\infty} g^2(\gamma(s)) d\eta(\gamma) ds \end{aligned}$$

$$(y(\Gamma))^2 \cong C\tilde{C}(\text{Mod}_2(\Gamma))^2$$



Proof of  $\|\nabla \rho\|_* \leq \|\nabla \rho\|_S$ .

It is sufficient to show  $\exists \rho_m \in \Delta(\partial \rho)$ ,  $\rho_m \xrightarrow{L^2} \rho$

$$\overline{\lim}_{m \rightarrow \infty} \partial \rho(\rho_m) \leq \frac{1}{2} \int \|\nabla \rho\|_S^2 \, \text{dvol}$$

$$0 < c \leq \rho \leq C < \infty \quad \int \rho^2 \, \text{dvol} = 1$$

$$\rho_0 := \rho^2 \quad \rho_m = L^2\text{-g.f. of } \partial \rho \text{ starting from } \rho_0$$

$\mu_E = h_E m \in \mathcal{D}(x)$  SUPERPOSITION PRINCIPLE

⇓

$\eta$

$\eta$  has bounded marginals 2-action is finite.

$f$  (and also  $h$ ) is AC along  $\eta$ -a.s. curve.

$$|\nabla h|_S = 2P |\nabla f|_S$$

$$|\nabla \log h|_S = 2 |\nabla f|_S / f.$$

$$\int (h_0 \mu_{h_0} - h_T \mu_{h_T}) \, d\mu \leq \int \mu_{h_0} (h_0 - h_T) \, d\mu$$

$$= \int (\mu_{h_0}(x_0) - \mu_{h_0}(x_T)) \, d\eta(x)$$

$$\leq \int_0^T \int \log h_0 |x'(x)| \, dx \, d\eta(x)$$

$$\leq \left( \int_0^T \int \log h_0 |x'(x)|^2 \, dx \right)^{1/2} \left( \int_0^T \int |x'|^2 \, d\eta(x) \right)^{1/2}$$

$$\leq \int_0^T |\dot{\mu}_n|^2 \, dx$$

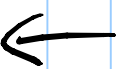
$$\int_0^t \int_{\{R_s > 0\}} |V_{R_s}|^2 / R_s \, d\mu \, ds$$

YOUNG  $\Rightarrow$

$$\int_0^t \int_{\{R_s > 0\}} |V_{R_s}|^2 / R_s \, d\mu \, ds \leq \int_0^t \int_{\{R_s > 0\}} |V_{R_s}|^2 R_s \, d\mu \, ds.$$

$$\int_0^t \mathcal{Q}_n(\sqrt{R_s}) \, ds \leq \int_0^t \int_{\{R_s > 0\}} |V_{R_s}|^2 / R_s \, d\mu \, ds.$$

$\exists$  n.i.d.o n.t.



$$\lim_{n \rightarrow \infty} \mathcal{Q}_n(\sqrt{R_{s_i}}) \leq \int |V_{R_s}|^2$$

# LOWER BOUNDS ON RICCI CURVATURE

IN METRIC MEASURE SPACES

LOTT - VILLANI STURM (DESCRIPTION OF GRONOV-HAUS. UNITS OF MANIFOLDS WITH RICCI  $\geq 0$ )

Ricci  $\geq 0 \iff \text{Emb}_m$  is convex in  $\mathcal{P}(M)$

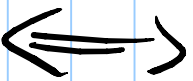
1) CONSISTENCY

GRONOV - GRONOV  
 $\implies$  POINCARÉ 2) STABILITY

$E_{\text{ent}_m}$  is convex in  $\mathcal{R}(\mathbb{R}^n, \|\cdot\|)$

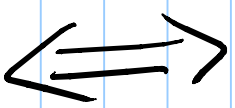
IS THERE A MORE RESTRICTIVE CONDITION?

- STRONG CONVEXITY OF  $E_{\text{NT}_m}$  + LINEARITY OF THE HEAT FLOW



(EVI) property of gradient flow of  $E_{\text{ent}_m}$

$x'(t) \in -\partial F(x(t))$        $F$  convex lsc



$H$

$\forall z \in D(F) \quad \frac{d}{dt} \frac{|x(t) - z|^2}{2} + F(x(t)) \leq F(z) \quad \text{e.a.t.}$