Título da nota 14-06-2011

· W_2 - GRADIENT FLOW OF Ent_m

ABSOLUTELY CONTINUOUS CURVE AND METRIC DERIVATIVE

$$f : [0,1] \longrightarrow \mathbb{R} \qquad f(x) = f(y) + \int_x^y f'(r)\,dr$$

$$f' \in L^1(0,1)$$

$$|f(x) - f(y)| \leq \int_x^y |f'|\,dr \qquad \forall\ [x,y] \subset [0,1].$$

Def $f : [0,1] \to X$ is AC if $\exists\, g \in L^1(0,1)$ s.t.

$$d(f(x), f(y)) \leq \int_x^y g\,dr \qquad \forall\ [x,y] \subset [0,1].$$

Theorem If $f \in AC([0,1], X)$, then the limit

$$\lim_{h \to 0} \frac{d(f(t+h), f(t))}{|h|} =: |f'|(t) \quad \text{METRIC DERIVATIVE.}$$

exists a.e. in $(0,1)$, and provides the least possible g.

SKETCH $f([0,1]) =: K \subset X$ cpt. $\{z_n\} \subset K$ dense.

$$\varphi_n(t) = d(f(t), z_n) \qquad |\varphi_n(s) - \varphi_n(t)| \leq \int_s^t g$$

Título da nota 14-06-2011

$\exists\ \varphi_n'(t)$ a.e. $\qquad m(t) := \sup_n |\varphi_n'(t)|$

1. $m \in L^1(0,1)$ $\quad$ (indeed, $|\varphi_n'| \le g$ a.e.)

2. $\liminf_{|h|\to 0} \dfrac{d(f(t+h), f(t))}{|h|} \geq m(t)$ a.e.

$$d(f(t+h), f(t)) \geq |\varphi_n(t+h) - \varphi_n(t)|$$

$$\Rightarrow \underline{\lim} \frac{d(f(t+h), f(t))}{|h|} \geq |\varphi_n'(t)| \quad \text{a.e.}$$

3 limsup (EXERCISE) $\blacksquare$

GRADIENT FLOW IN A METRIC SPACE (Y, d_Y)

DE GIORGI '83 LOOK AT THE RATE OF ENERGY DISSIPATION

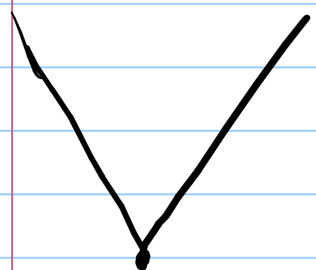
$$x' = -\nabla F(x) \qquad x(t) \in \mathbb{R}^m, \qquad t > 0.$$

Consider <u>any</u> curve $y(t)$

$$\frac{d}{dt} F(y(t)) = \nabla F(y(t)) \cdot y'(t) \qquad \Big(= \text{iff } \underset{-\nabla F}{\overset{\|}{y'}}\,\Big)$$

$$\geq -|\nabla F|(y(t))\,|y'(t)|$$

$$\geq -\frac{1}{2}\underbrace{|\nabla F|^2}_{|\nabla^- F|}(y(t)) - \frac{1}{2}\underbrace{|y'(t)|^2}_{\text{metric derivative.}} \qquad \Big(= \text{iff } |\,| = |\,|\Big)$$



$$|\nabla^- F|(x) = \limsup_{y \to x} \frac{(F(x)-F(y))^+}{d(x,y)}$$

$(|\nabla^+ \varphi|$ for K.p. $\varphi)$

(EDI) $$F(x(t)) + \int_0^t \frac{1}{2}|\nabla^- F|^2(x(r)) + \frac{1}{2}|x'(r)|^2 \, dr \le F(x(0))$$

(EDE) $$=$$

Título da nota 14-06-2011

<u>REMARK</u> IF F is sufficiently regular

$$F(x(0)) \leq F(x(t)) + \int_0^t |x'(r)|\,|\bar{\nabla} F(x(r))|\,dr \qquad (*)$$

$\Rightarrow$ (EDE) out of (EDI), $|x'| = |\bar{\nabla} F|(x)$,

$t \to F(x(t))$ is AC, $\frac{d}{dt} F(x(t)) = -|\bar{\nabla} F|^2(x(t))$ a.e. t.

F convex $\Rightarrow$ $(*)$

Título da nota 14-06-2011

W_2 GF of Ent_m

$$\mathrm{Ent}(\mu_t) + \int_0^t \frac{1}{2}|\dot{\mu}_r|^2 + \frac{1}{2}|\nabla^- \mathrm{Ent}_m|^2(\mu_r)\,dr \le \mathrm{Ent}_m(\mu_0)$$

Lema (KUWADA) $f_0 \in L^2(X,m)$, $f_0 \geq 0$, $\int f_0\,dm = 1$

$f_t = H_t(f_0)$ $\quad$ ($H_t = L^2$-g.f. of Ch)

$\mu_t = f_t m \in \mathcal{P}(X)$

Then $$|\dot{\mu}_t|^2 \leq \int_{\{f_t>0\}} |\nabla f_t|_*^2 / f_t \, dm \quad \text{a.e. } t.$$

$$= 4 \int |\nabla \sqrt{f_t}|_*^2 \, dm.$$

<u>Proof</u>

- duality formula for W_2^2
- $\frac{d^+}{dt} Q_t \varphi + \frac{1}{2} |\nabla Q_t \varphi|^2 \leq 0$

Integral formulation $\quad 0 < t < s \qquad \Delta = (s-t)$

Título da nota 14-06-2011

$$W_2^2(\mu_t,\mu_s) \le \Delta \int_t^s \left(\int_{\{f_r>0\}} |\nabla f_r|_*^2 / f_r \, dm \right) dr.$$

$$\frac{1}{2} W_2^2(\mu_t,\mu_s) = \int \psi \, d\mu_t + \int \psi^c \, d\mu_s$$

$$\psi \in \mathrm{Lip}(X)$$

$\varphi = -\psi$

$$= \int -\varphi \, d\mu_t + \int Q_1 \varphi \, d\mu_s.$$

$$r \longrightarrow \int Q_{r/\Delta} \varphi \, d\mu_{t+r} \qquad r \in (0,\Delta)$$

$$\overset{\shortparallel}{f_{t+r}}\, m$$

Título da nota 14-06-2011

$$\frac{d}{dr} \int Q_{r/\Delta}\varphi \, f_{t+r}\, dm = \frac{1}{\Delta} \int \xi_{r/\Delta} f_{t+r} + \int Q_{r/\Delta}\varphi\, \Delta f_{t+r}$$

$$\xi_t = \frac{d^+}{dt} Q_t \varphi$$

$$\leq -\frac{1}{2\Delta} \int |\nabla Q_{r/\Delta}\varphi|_*^2 f_{t+r} + \int |\nabla Q_{r/\Delta}\varphi|_* |\nabla f_{t+r}|_*$$

$$\leq -\frac{1}{2\Delta} \int |\nabla Q_{r/\Delta}\varphi|_*^2 f_{t+r} + \frac{1}{2\Delta} \int |\nabla Q_{r/\Delta}\varphi|_*^2 f_{t+r}$$

Título da nota 14-06-2011

$$+ \frac{\Delta}{2} \int |\nabla \rho_{t+r}|^2 / \rho_{t+r}$$

IDENTIFICATION OF GRADIENT FLOWS

TRADITIONAL $\quad \{ \text{g. f. of } \mathrm{Ent}_m \} \subset \{ L^2\text{-g. f. of } \mathrm{Dir} \}$

$$=$$

NEW VIEWPOINT $\quad \{ L^2 \text{ g. f. of } \mathrm{Ch} \} \subset \{ W_2\text{-g. f. of } \mathrm{Ent}_m \}$

GIGLI: UNIQUENESS OF W_2-g. f.

Título da nota 14-06-2011

$\mu_t = f_t m \in \mathcal{P}(X)$ $\quad$ f_t solves L^2 heat flow

- $\frac{d}{dt} \int f_t \ln f_t \, dt = - \int_{\{f_t > 0\}} |\nabla f_t|_*^2 / f_t \, dm$ a.e.t

- (KUWADA) $\quad |\dot{\mu}_t|^2 \leq \int_{\{f_t > 0\}} |\nabla f_t|_*^2 / f_t$ a.e.t.

- (VILLANI) $\quad |\bar{\nabla} \mathrm{Ent}_m (fm)|^2 \leq \int_{\{f > 0\}} |\nabla f|_*^2 / f \, dm$

Proof of the inclusion (EDI) (for convex entropy)

WANT: $\mathrm{Ent}(f_t m) + \int_0^t \frac{1}{2}|\dot{\mu}_r|^2 + \frac{1}{2}|\bar{\nabla}\,\mathrm{Ent}_m(\mu_r)|^2 \, dr \leq \mathrm{Ent}(f_0 m)$

$$\leq \frac{1}{2} \int_{\{f_r>0\}} |\nabla f_r|_*^2 / f_r$$

$$\leq \frac{1}{2} \int_{\{f_r>0\}} |\nabla f_r|_*^2 / f_r \, dm$$

<u>REMARK</u> A closer analysis of this proof allows to reverse the inequality between slope of Ent_m and $|\nabla\sqrt{f_t}|_*^2$.

$$|\nabla^- \mathrm{Ent}_m|^2(fm) \leq \int_{\{f>0\}} |\nabla f|_*^2/f \, dm.$$

CONVEXITY IMPLIES A NICE FORMULA FOR THE SLOPE.

$$|\nabla^- F|(x) = \sup_{y \neq x} \frac{(F(x) - F(y))^+}{d(x,y)}$$

$x \longrightarrow (F(x) - F(y))^+ / d(x,y)$ are lsc.

Hence $|\nabla^- F|$ is lsc.

$$|\nabla^- \mathrm{Ent}(\underbrace{f m}_{=\mu})|^2 \leq \int \frac{|\nabla f|^2}{f}$$

<u>$f \in \mathrm{Lip}$</u>

$\inf f > 0$

$\sup f < \infty$

$\tilde{\mu} = \tilde{f} m$ $\qquad\qquad$ $\tilde{\pi}$ optimal from μ to $\tilde{\mu}$

$$\mathrm{Ent}_m(f m) - \mathrm{Ent}_m(\tilde{f} m) = \int (f \log f - \tilde{f} \log \tilde{f}) \, dm$$

$$\leq \int \log f \, (f - \tilde{f}) \, dm$$

$$= \int (\log f(x) - \log f(y)) \, d\tilde{\pi}(x,y)$$

$$\leq \int \left(|\nabla^- \log f|(x) + \omega_x(d(x,y))\right) d(x,y) \, d\tilde{\pi}(x,y)$$

$$\leq W_2(fm, \tilde{f}m) \left(\int \left(|\nabla^- \log f|(x) + \omega_x(d)\right)^2 d\tilde{\pi} \right)^{1/2}$$

$$\overline{\lim_{\tilde{\mu} \to \mu}} \int \left(|\nabla^- \log f| + \cancel{\omega_x(d)}\right)^2 d\tilde{\pi} \leq ?$$

$$\log f \ \text{lip} \Rightarrow \omega_x \leq C$$

$$\int |\bar{\nabla} \log f|^2 \, d\mu = \int \frac{|\bar{\nabla} f|^2}{f^2} \, d\mu$$

$$\leq \int \frac{|\nabla f|^2}{f} \, dm$$