LECTURE II

Yesterday: $\displaystyle Q_t f(x) = \inf_y \left\{ \frac{1}{2t} d^2(x,y) + f(y) \right\}$ HOPF-LAX

$$\frac{d^+}{dt} Q_t f + \frac{1}{2} |\nabla Q_t|^2 \leq 0 \quad \text{in } (0,\infty) \times X$$

$$\begin{cases} \varphi^c(y) = \inf_x \frac{1}{2} d^2(x,y) - \varphi(x) \\ \psi^c(x) = \inf_y \frac{1}{2} d^2(x,y) - \psi(y) \end{cases}$$

$(\varphi^c)^c = \varphi$

φ c-concave

$\leq$ trivial

$$\begin{cases} \varphi + \varphi^c = c & \gamma\text{-a.e.} \\ \varphi \text{ } c\text{-concave} & \end{cases}$$

KANTOROVICH POTENTIAL

$$\varphi = (\varphi^c)^c = Q_1(-\varphi^c) \implies |\nabla^+ \varphi|(x) \leq d(x,y) \quad \gamma\text{-a.e.}$$

In particular $\displaystyle\int_X |\nabla^+ \varphi|^2 \, d\mu \leq W_2^2(\mu, \nu)$

More connections with OTT

Geodesics in $\mathcal{P}_2(X)$

$(\mathcal{P}(X), W_2)$ is a metric space, geodesic of X is geodesic.

(Y, d_Y) $\quad\quad \gamma : [0,1] \to Y$

$$d_Y(\gamma(s), \gamma(t)) = |t-s| \, d_Y(\gamma(0), \gamma(1)) \quad \forall s,t.$$

$Geo(Y)$

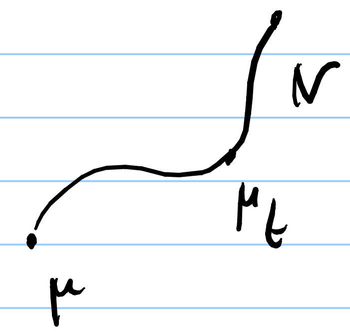
$Geo(\mathcal{P}(X))$??

SIMPLEST CASE: $X = \mathbb{R}^n$ $\mu \ll \mathcal{L}^n$

$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^n)$

$\exists !\, T$ optimal $T_\# \mu = \nu$

$T_t = (1-t) I + t T$ $t \in [0,1]$



$\mu_t := T_{t\#} \mu$ the unique geodesic from μ to ν.

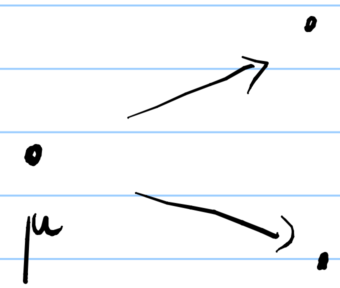
$\mu = \delta_x$ $\nu = \delta_y$

$\tilde{\mu}_t = (1-t)\mu + t\nu$

$\mu_t = \delta_{(1-t)x + ty}$

<u>Exercise</u> $\tilde{\mu}_t$ has infinite length! (w.r.t. W_2)

$\mu \not\ll \mathcal{L}^n$



transport plans.

(X, d) Given x, y there might be <u>many</u> geodesics from x to y.

$\gamma \in \mathcal{P}(X \times X)$ transport plan.

$\eta \in \mathcal{P}(\mathrm{Geo}(X))$ geodesic transport plan

Given $\eta \longrightarrow$ $\gamma = (e_0, e_1)_{\#}\eta \in \mathcal{P}(X \times X)$

$e_t : \mathrm{Geo}(X) \longrightarrow X$ $e_t(\gamma) = \gamma(t)$ ev. map at time t.

<u>Theorem</u> If (X, d) is geodesic, then $\mathcal{P}(X)$ is geodesic.

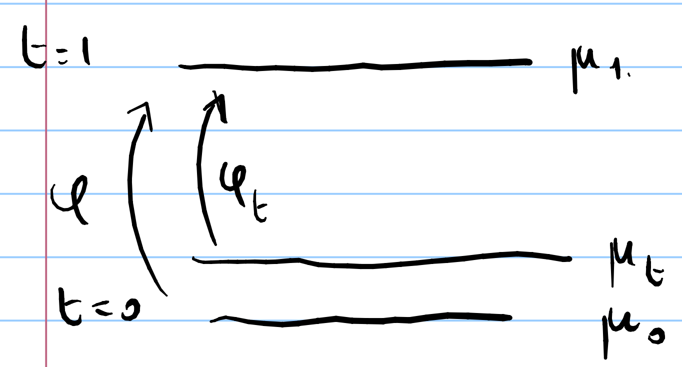
Moreover, geodesics in $\mathcal{P}(X)$ can be built as

$$\mu_t := (e_t)_{\#}\eta \qquad \eta \text{ optimal geodesic plan.}$$

Behaviour of K potential along geodesisc.

(BERNARD-BUFFONI, JEMS, '07)

(BENAMOU-BRENIER, NUM. MAT. '00)



<u>Theorem</u> If we define $\varphi_t = Q_t(-\varphi^c)$ then φ_t is a K-potential relative to <u>scaled</u> cost $c_t = c/t$, from μ_{1-t} to μ_1.

(SKETCH OF PROOF) $\quad \varphi_t + \varphi^c \leq c_t \quad$ obvious

η optimal from μ_0 to μ_1

$$\varphi(\gamma_0) + \varphi^c(\gamma_1) = c(\gamma_0, \gamma_1) \qquad \eta\text{-a.e. } \gamma$$

$$\Downarrow$$

$$\varphi_t(\gamma_t) + \varphi^c(\gamma_1) = c_t(\gamma_t, \gamma_1) \qquad \eta\text{-a.e. } \gamma$$

$\blacksquare$

Cheeger energy and relaxed gradients

(X, d, m)

$$\mathrm{Ch}(f) = \inf\left\{ \liminf_{n\to\infty} \frac{1}{2}\int_X |\nabla f_n|^2 \, dm \;\middle|\; \begin{array}{l} f_n \in \mathrm{Lip}(X) \\ f_n \to f \quad L^2(X,m) \end{array} \right\}$$

$$: L^2(X,m) \to [0,+\infty]$$

- Ch L^2-l.s.c. , convex.

RELAXED GRADIENT

G is a r.g. if $\quad G \geq w\text{-}L^2 \text{ limit } |\nabla f_n|$
for some sequence $f_n \to f$ in L^2.

$RG(f)$ convex set, is also a closed set.

G is a r.g. iff $\iff$ $G \geq \text{strong-}L^2 \text{ limit of } |\nabla f_n|$
$f_n \to f \quad L^2$.

Minimal Relaxed gradient $\quad |\nabla f|_* =$ the element with least L^2 norm in $RG(f)$

__Theorem__ $f \in L^2$ $\mathrm{Ch}(f) < \infty$. Then

(1) $\mathrm{Ch}(f) = \int_X |\nabla f|_*^2 \, dm$

(2) $G_1, G_2 \in RG(f) \Rightarrow G_1 \wedge G_2 \in RG(f)$

(3) $|\nabla f|_* \leq G$ m-a.e. $G \in RG(f)$

(4) $g = f$ a.e. B Borel $\Rightarrow |\nabla f|_* = |\nabla g|_*$ a.e. on B

(5) $|\nabla \phi(f)|_* \leq |\phi|'(f) \, |\nabla f|_*$ $\phi \in C^1$ on the image of f

$=$ if $\phi' \geq 0$

Título da nota 13-06-2011

<u>Proof of 2</u> $\quad \chi_{X \setminus B} G_1 + \chi_B G_2 \in RG(f) \quad \forall B$ Borel.

<u>B closed</u> $\quad \rho(x) = \mathrm{dist}(x, B) \qquad \chi_r = \min\{1, \rho/r\}$

$$\chi_r \sim \chi_B \quad \text{as } r \to 0.$$

$$|\nabla f_{m,1}| \xrightarrow[L^2]{} \tilde{G}_1 \le G_1 \quad , \quad |\nabla f_{m,2}| \xrightarrow[L^2]{} \tilde{G}_2 \le G_2$$

$$|\nabla(\chi_r f_{m,1} + (1-\chi_r) f_{m,2})| \le \chi_r |\nabla f_{m,1}| + (1-\chi_r)|\nabla f_{m,2}| +$$

$$+ \, \mathrm{lip}(\chi_n) \cancel{|\varrho_{m,1} - \varrho_{m,2}|}$$

$$\chi_n G_1 + (1-\chi_n) G_2 \in RG(\varrho) \qquad n \to \infty.$$

3

$$|\nabla\varrho|_* \in RG(\varrho) \qquad m\big(\underbrace{G < |\nabla\varrho|_*}_{=B}\big) > 0.$$

$$G \in RG(\varrho)$$

$$\tilde{G} := \min\{G, |\nabla\varrho|_*\} \in RG(\varrho) \qquad \|\tilde{G}\|_{L^2} < \| |\nabla\varrho|_* \|_{L^2}$$

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L^2-HEAT FLOW AND LAPLACIAN IN (X, d, m)

GRADIENT FLOWS FOR CONVEX LSC FUNCTIONALS IN
Ch

HILBERT SPACES

L^2

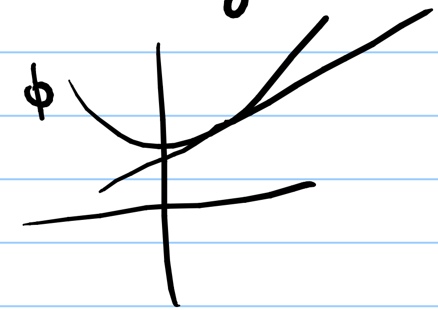
(BREZIS '73)

H $\qquad \phi : H \to \mathbb{R} \cup \{+\infty\}$ convex lsc.

$$D(\phi) = \{x \in H \mid \phi(x) < \infty\} \qquad D(\phi) \neq \emptyset$$

GRADIENT FLOW: A locally a.c. $x : (0,\infty) \to H$ satisfying $x'(t) \in -\partial\phi(x(t))$ a.e. t.

$$\partial\phi(x) = \{ p \in H \mid \phi(y) \geq \phi(x) + \langle p, y-x \rangle \ \forall y \}.$$



$x(t)$ starts from $\bar{x} \in H$ &

$$\lim_{t \to 0} x(t) = \bar{x}$$

__Theorem__ (Existence and uniqueness) $\forall \bar{x} \in \overline{D(\phi)}$ $\exists!$ g.f. $x(t)$ starting from $\bar{x}$. Moreover the induced $S_t : [0,\infty) \times \overline{D(\phi)} \to \overline{D(\phi)}$ is a contraction, i.e.

$$|S_t \bar{x} - S_t \bar{y}| \leq |\bar{x} - \bar{y}| \qquad \forall t \geq 0.$$

(Regularization effects) $S_t \bar{x} \in D(\phi)$, more precisely

$$\phi(S_t \bar{x}) \leq \inf_y \left\{ \phi(y) + \frac{1}{2t} d^2(y, \bar{x}) \right\}$$

$$x'_+(t) = -\nabla\phi(x(t)) \qquad \forall t>0$$

$\nabla\phi$ the element with smallest H norm in $\partial^-\phi$

$$t \to |\nabla\phi|(x(t)) \searrow \quad \text{in } (0,\infty)$$

$$\frac{d}{dt}\phi(x(t)) = -|\nabla\phi|^2(x(t)) = -|x'(t)|^2 \quad \text{a.e. } t.$$

Our framework: $H = L^2(X, m)$

$\Phi = \mathrm{Ch}$ $\qquad \overline{\mathrm{Ch}}^{L^2} = L^2.$

L^2 - heat flow. $\qquad H_t(f)$

<u>Laplacian</u> $\qquad f$ s.t. $\partial^- \mathrm{Ch}(f) \neq \emptyset$

$-\Delta f =$ the element with smallest L^2 norm in $\partial^- \mathrm{Ch}(f)$.

$$\frac{d}{dt} H_t(f) = \Delta H_t(f)$$

Remarks about Δ

depends on d

in $\mathbb{R}^n$

1) $\Delta = \mathrm{div}\,(\nabla f)$

depends on m

$$\int (\mathrm{div}\, F)\, g \, dx = \int - F \cdot \nabla g \, dx$$

2) Δ <u>not</u> a linear quota in general.

$X = \mathbb{R}^n$, $m = \mathcal{L}^n$, $\| \ \|$ (for instance the ℓ_1)

$|\nabla f|_* = \|\nabla f\|^*$ (ℓ_∞ norm)

$Ch(f) = \frac{1}{2} \int_X (\|\nabla f\|^*)^2 \, dx$ <u>not</u> a quadratic form.

"WEAK" INTEGRATIONS BY PARTS.

$$- \int g \, \Delta f \, dm \leq \int |\nabla g|_* |\nabla f|_* \, dm.$$

$$- \int \phi(f) \Delta f \, dm = \int |\nabla f|_*^2 \, \phi'(f) \, dm$$

$$-\Delta f \in \partial \mathrm{Ch}(f)$$

$$\mathrm{Ch}(f+\varepsilon g) - \mathrm{Ch}(f) \geq \int_X \varepsilon g(-\Delta f)\, dm \qquad \varepsilon > 0$$

$$\mathrm{Ch}(f+\varepsilon g) \leq \frac{1}{2}\int (|\nabla f|_* + \varepsilon |\nabla g|_*)^2\, dm$$

$$= \mathrm{Ch}(f) + \varepsilon \int |\nabla f|_* |\nabla g|_*\, dm + o(\varepsilon)$$

PROPERTIES OF THE L^2-HEAT FLOW

(1) (COMPARISON PRINCIPLE) $\quad f \le g \implies H_t(f) \le H_t(g)$

(2) (CONTRACTIVITY IN ALL L^p SPACES) $\quad \| H_t(f) - H_t(g) \|_p \le \| f - g \|_{L^p}$

$1 \le p \le \infty \qquad f, g \in L^p$

(3) $e : J \to \mathbb{R} \cup \{+\infty\}$ convex, $C^{1,1}$, $J \supset$ image of f_0.

$$\int_X e(f_t)\,dm = \int_X e(f_0)\,dm - \int_0^t \int_X e''(f_s) |\nabla f_s|_*^2 \,dm\,ds$$

(4) (MASS PRESERVATION)

$$\int \rho_t \, dm = \int \rho_0 \, dm \quad \forall t \geq 0.$$

<u>Remark</u> An important case is $e(z) = z \ln z$

$$\frac{d}{dt} \int \rho_t \ln \rho_t \, dm = -\int_{\{\rho_t > 0\}} \frac{|\nabla \rho_t|_*^2}{\rho_t} \, dm$$