Título da nota 13-06-2011

f lsc bounded

$$Q_t f(x) = \inf_y \left\{ \frac{1}{2t} d^2(x,y) + f(y) \right\}$$

<u>Th</u> 1) $Q_t f \uparrow f$ as $t \downarrow 0$

2) $Q_t(Q_s f) \geq Q_{t+s} f$

3) $\frac{d}{dt} Q_t f + \frac{1}{2} |\nabla Q_t|^2 \leq 0$

with countably many exceptions

(everywhere if $\frac{d^+}{dt} Q_t f$)

4) $Q_t f \big|_{(\varepsilon,\infty) \times X}$ lip $\forall \varepsilon > 0$

If (X,d) is geodesic, then $=0$

in 3) and $Q_{t+s} = Q_t(Q_s)$

Proof $\quad Q_t(Q_s f)(x)$

$$\inf_y \left(\inf_z f(z) + \frac{1}{2s} d^2(z,y) \right) + \frac{1}{2t} d^2(y,x)$$

$$\inf_z f(z) + \inf_y \frac{1}{2s} d^2(z,y) + \frac{1}{2t} d^2(y,x)$$

$$\geq \inf_z f(z) + \frac{1}{2(s+t)} d^2(z,x)$$

Proof of 3

$$\Delta^+(x,t) = \max\{\, d(x,y) \mid y \text{ minimizer} \}$$

$$\Delta^-(x,t) = \min\{\, d(x,y) \mid y \text{ minimizer} \}$$

$\Delta^- \leq \Delta^+$ $\quad$ Δ^+ is usc

Δ^- is lsc

$\Delta^\pm(x,\cdot)$ are $\nearrow$

$\Delta^+(x,t) \leq \Delta^-(x,s) \qquad t<s$

$\Rightarrow \Delta^+(x,\cdot) = \Delta^-(x,\cdot)$ out of a countable set.

$$\frac{d^\pm}{dt} Q_t(x) = -\frac{1}{2t^2}\left(\Delta^\pm(x,t)\right)^2$$

Proof for the right derivative.

$s > t$

$$Q_s f(x) - Q_t f(x)$$

$$\le \cancel{f(x_t)} + \frac{1}{2s} d^2(x_t, x) - \cancel{f(x_t)} - \frac{1}{2t} d^2(x, x_t)$$

$$= \frac{1}{2} \underbrace{d^2(x, x_t)}_{(D^+(x,t))^2} \left(\frac{1}{s} - \frac{1}{t} \right)$$

$$\ge \frac{(D^-(x,s))^2}{2} \left(\frac{1}{s} - \frac{1}{t} \right)$$

$D^-(x,s) \downarrow D^+(x,t)$ as $s \downarrow t$

To conclude 3), we need

$$|\nabla Q_t f|^2(x) \le \left(D^+(x,t)\right)^2/t^2$$

$$\|$$

$$\overline{\lim_{y \to x}} \frac{|Q_t f(y) - Q_t f(x)|^2}{d^2(y,x)}$$

$$Q_t f(x) - Q_t(y) \le \cancel{d(x,y)\left(\frac{D^-(y,t)}{t}\right)}$$

$$d(x,y)\left(\frac{D^-(y,t)}{t} + \frac{d(x,y)}{2t}\right)$$

Divide both sides by $d(x,y)$ and use

$$\overline{\lim_{y \to x}} \, D^{-}(y,t) \leq \overline{\lim_{y \to x}} \, D^{+}(y,t)$$

$$\leq D^{+}(x,t)$$

<u>Remark</u>

$$|\nabla^{+} f|(x) = \overline{\lim_{y \to x}} \frac{\left(f(y) - f(x)\right)^{+}}{d(x,y)}$$

$$|\nabla^{-} f|(x) = \overline{\lim_{y \to x}} \frac{\left(f(x) - f(y)\right)^{+}}{d(x,y)}$$

$$|\nabla^- f| = |\nabla^+(-f)|$$

$$|\nabla f| = \max\{|\nabla^+ f|, |\nabla^- f|\}.$$

If we keep y fixed and let $x \to y$ we get

$$|\nabla^+ Q_t f|(y) \leq D^-(y,t)/t.$$

CONNECTION WITH OTT

$c =$ cost function : $X \times Y \to [0, \infty)$

$\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is <u>c-concave</u>

if $\varphi = \psi^c$ for some $\psi : Y \to \mathbb{R} \cup \{-\infty\}$.

<u>c transform ψ^c</u> $\quad \psi : Y \to \mathbb{R} \cup \{-\infty\}$

$$\psi^c(x) = \inf_y \; c(x,y) - \psi(y)$$

$$c = \frac{1}{2} d^2 \Rightarrow \psi^c = Q_1(-\psi)$$

KANTOROVICH POTENTIAL

$\mu, \nu \in \mathcal{P}(X)$ $\qquad c = \frac{1}{2} d^2$

Then $\exists\ \varphi : X \to \mathbb{R}$ c-concave (lipschitz) satisfying

$$\varphi + \varphi^c = c \qquad \gamma\text{-a.e. in } X \times X$$

for any optimal plan $\gamma \in \mathcal{P}(X \times X)$.

$\forall f \quad f + f^c \leq c$ by def.

OPTIMAL PLAN

$\gamma \in \mathcal{P}(X \times X)$ having μ and ν as marginals $\quad \Gamma(\mu, \nu)$

$$\gamma(A \times X) = \mu(A) \quad \forall A \in \mathcal{B}(X)$$
$$\gamma(X \times B) = \nu(B) \quad \forall B \in \mathcal{B}(X)$$

$$(\pi^1)_\# \gamma = \mu \qquad (\pi^2)_\# \gamma = \nu$$

$$\pi^1(x, y) = x, \qquad \pi^2(x, y) = y$$

$$\min \left\{ \int d^2(x,y)\, d\gamma \;\middle|\; \gamma \in \Gamma(\mu,\nu) \right\}$$

$$\| \\ W_2^2(\mu,\nu)$$

Theorem φ K potential.

Then $\left(\int |\nabla^+ \varphi|^2 d\mu \leq W_2^2(\mu,\nu) \right)$

$$|\nabla^+ \varphi|(x) \leq d(x,y) \quad \gamma\text{-a.e.}$$

for any optimal γ.

Proof $\varphi = (\varphi^c)^c = Q_1(-\varphi^c)$

$$|\nabla^+ \varphi|(x) \leq \bar{D}(x,1)$$

On the other hand $\left(c = \frac{1}{2} d^2\right)$

$$\varphi + \varphi^c = c \quad \gamma\text{-a.e.}$$

$$\Rightarrow \quad \Delta(x,1) \leq d(x,y) \quad \gamma\text{-a.e.}$$

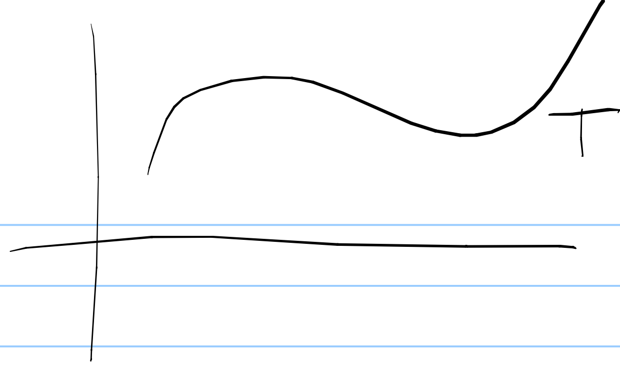
$\blacksquare$

BRENIER THEOREM $X = Y = \mathbb{R}^n$

$$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^n), \quad \mu \ll \mathcal{L}^n$$

Under this assumption, any optimal plan is induced by a map T $\quad y = T(x)$

$$\gamma\text{-a.e.} \qquad y = T(x)$$



$$T(x) = x - \nabla\varphi(x) \qquad \mu\text{-a.e.}$$

$$|\nabla\varphi|^2 = |x - T(x)|^2 \qquad \mu\text{-a.e.}$$

METRIC BRENIER THEOREM

$$\varphi \qquad |\nabla^+\varphi|(x) = d(x,y) \quad \gamma\text{-a.e.}$$

<u>EXAMPLE</u> $X = [0,1]$ $\quad d = |x-y|$

$$\mu = \delta_0 \qquad \nu = \mathcal{L}^1|_{[0,1]}$$

$$\gamma = \delta_0 \times \nu = \mu \times \nu$$

$$\varphi(x) = \frac{x^2}{2} - x \qquad |\nabla^+\varphi|(0) = 0$$

$$|\nabla^-\varphi|(0) = \underline{1}$$

$$W_2^2(\mu,\nu) = \int_0^1 s^2\,ds = \frac{1}{3}$$

$$\int_X |\nabla^+\varphi|^2\,d\mu < W_2^2(\mu,\nu)$$

$$\int_X |\nabla\varphi|^2\,d\mu \leq W_2^2(\mu,\nu)$$

false.