Overview

A.-Gigli-Savaré: *Calculus and heat flows in metric measure spaces and applications to spaces with Ricci bounds from below.* http://cvgmt.sns.it, submitted.
(some results and proofs)

A.-Gigli-Savaré: *Riemannian Ricci curvature bounds in metric measure spaces.*
In preparation.
(just statements, no proofs)

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Some by now “classical” results

Let us consider in $\mathbb{R}^n$ the heat equation ($u_t(x) = u(t,x)$)

$$\partial_t u_t = \Delta u_t$$

Classically it can be viewed as the gradient flow of the energy

$$\mathrm{Dir}(u) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx \qquad (+\infty \text{ if } u \notin H^1(\mathbb{R}^n))$$

in the Hilbert space $H = L^2(\mathbb{R}^n)$.

Formally, $t \mapsto u_t$ solves the ODE $u' = -\nabla\mathrm{Dir}(u)$ in H because

$$\mathrm{Dir} \text{ “differentiable” at } u \qquad \Longleftrightarrow \qquad -\Delta u \in L^2,\ \nabla\mathrm{Dir}(u) = -\Delta u$$

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In 1998, Jordan-Kinderlehrer-Otto proved that the same equation arises as gradient flow of the *entropy* functional

$$\mathrm{Ent}(\rho\mathscr{L}^n) := \int_{\mathbb{R}^n} \rho \log \rho \, dx \qquad (+\infty \text{ if } \mu \text{ is not a.c. w.r.t. } \mathscr{L}^n)$$

in the space $\mathscr{P}_2(\mathbb{R}^n)$ of probability measures with finite quadratic moments, with respect to Wasserstein distance W_2.

$$W_2^2(\mu,\nu) := \min\left\{\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\, d\gamma(x,y) :\ (\pi_1)_\sharp\gamma = \mu,\ (\pi_2)_\sharp\gamma = \nu\right\}.$$

Push forward notation. $f : X \to Y$ Borel induces a map $f_\# : \mathscr{P}(X) \to \mathscr{P}(Y)$:

$$f_\#\mu(B) := \mu(f^{-1}(B)) \qquad \forall B \in \mathscr{B}(X).$$

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Proofs of this equivalence

1. By the so-called Otto calculus (formal);

2. Prove that the implicit time discretization scheme (Euler scheme), traditionally used for the approximation of gradient flows, when done with energy $\mathrm{Ent}(\mu)$ and distance W_2, does converge to the heat equation.

3. Give a meaning to what “gradient flow of Ent w.r.t. W_2 means”, and check that solutions of this gradient flow are solutions to the heat equation. Then, apply uniqueness for $\partial_t u_t = \Delta u_t$.

The last strategy is more abstract, but still uses the differentiable structure of $\mathbf{R}^n$. The question is to understand deeper reasons for this equivalence, in particular on which structural properties of the space it depends (Riemannian manifolds, Finsler spaces, Wiener spaces, sub-Riemannian spaces, etc.)

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Metric measure spaces

Let us consider a *metric measure space* $(X, d, \boldsymbol{m})$, with $\boldsymbol{m} \in \mathscr{P}(X)$. In this framework it is still possible to define a "Dirichlet energy", that we call Cheeger functional:

$$\mathsf{C}h(f) := \inf\left\{\liminf_{n\to\infty} \int_X |\nabla f_n|^2 \, d\boldsymbol{m} : \; f_n \in \mathrm{Lip}(X), \; \int_X |f_n - f|^2 \, d\boldsymbol{m} \to 0\right\},$$

where

$$|\nabla g|(x) := \limsup_{y\to x} \frac{|g(y) - g(x)|}{d(y,x)}$$

is the *slope* (also called local Lipschitz constant).
Also, one can consider the so-called *relative entropy functional* $\mathrm{Ent}_{\boldsymbol{m}} : \mathscr{P}(X) \to [0, +\infty]$

$$\mathrm{Ent}_{\boldsymbol{m}}(\rho\boldsymbol{m}) := \int_X \rho \log \rho \, d\boldsymbol{m} \qquad (+\infty \text{ if } \mu \text{ is not a.c. w.r.t. } \boldsymbol{m}).$$

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The basic result is that the equivalence between L^2-gradient flow of Ch and W_2-gradient flow of $\mathrm{Ent}_{\boldsymbol{m}}$ *always* holds, if the latter is properly understood. But, without additional assumptions on the space, both objects can be trivial.

Example. Let $X = [0,1]$, d the Euclidean distance, $\boldsymbol{m} = \sum_{n\geq 1} 2^{-n}\delta_{q_n}$, where $\{q_n\}_{n\geq 1}$ is an enumeration of $[0,1]\cap\mathbb{Q}$. If $A_n \supset \mathbb{Q}\cap X$ are open sets with $\mathscr{L}^1(A_n)\to 0$ and

$$\chi_n(t) := \int_0^t (1-\chi_{A_n}(s))\,ds \qquad t\in[0,1].$$

Then $f\circ\chi_n \to f$ in $L^2(X,\boldsymbol{m})$ for all $f\in\mathrm{Lip}(X)$ and $f\circ\chi_n$ is locally constant in $\mathbb{Q}\cap X$ hence

$$\mathrm{C}h(f) = 0 \qquad \forall f\in\mathrm{Lip}(X).$$

It follows that C$h \equiv 0$ in $L^2(X,\boldsymbol{m})$.

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Identification of weak gradients

A closely related question, relevant in particular for the second paper, is the identification of weak gradients. The first one, that we call *relaxed* gradient $|\nabla f|_*$, is the object that provides integral representation to $\mathrm{C}h$:

$$\mathrm{C}h(f) = \frac{1}{2}\int_X |\nabla f|_*^2 \, d\boldsymbol{m} \qquad \forall f \in D(\mathrm{C}h).$$

It has all the natural properties (locality, chain rules, etc.) a weak gradient should have.

This gradient is useful when doing "vertical" variations $\epsilon \mapsto f + \epsilon g$ (i.e. in the *dependent* variable).

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Identification of weak gradients

But, when computing variations of the entropy, the "horizontal" variations $\epsilon \to f(\gamma(\epsilon))$ (i.e. in the *independent* variable) are necessary. These are related to another weak gradient $|\nabla f|_w$, defined as follows.

We require the so-called upper gradient property

$$|f(\gamma_1) - f(\gamma_0)| \le \int_\gamma G$$

along "almost all" curves γ in $AC^2([0,1];X)$ and then we define $|\nabla f|_w$ as the element with smallest $L^2(X, \boldsymbol{m})$ norm.

The remarkable fact is that these two gradients *always* coincide (and, of course, maybe both trivial without extra assumptions). The proof of this identification uses ideas from optimal transportation, as lifting of solutions to the heat flow to probability measures in $AC^2([0,1];X)$ and the energy dissipation rate of $\mathrm{Ent}_{\boldsymbol{m}}$.

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The remarkable fact is that these two gradients *always* coincide (and, of course, maybe both trivial without extra assumptions). The proof of this identification uses ideas from optimal transportation, as lifting of solutions to the heat flow to probability measures in $AC^2([0,1];X)$ and the energy dissipation rate of $\mathrm{Ent}_{\boldsymbol{m}}$.

Identification of weak gradients

But, when computing variations of the entropy, the "horizontal" variations $\epsilon \to f(\gamma(\epsilon))$ (i.e. in the *independent* variable) are necessary. These are related to another weak gradient $|\nabla f|_w$, defined as follows.

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SCUOLA NORMALE SUPERIORE

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SCUOLA NORMALE SUPERIORE

Why gradients are not trivial in Lott-Sturm-Villani spaces

In these spaces one imposed convexity of W_2 geodesics of $\mathrm{Ent}_{\boldsymbol{m}}$ (the so-called $CD(0,\infty)$ condition) or of functionals

$$\rho\boldsymbol{m} \mapsto -\int_X \rho^{1-1/N}\,d\boldsymbol{m}$$

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In this case the gradient flow of $\mathrm{Ent}_{\boldsymbol{m}}$ is not trivial, and since it coincides with the heat flow, also this is not trivial.

The energy dissipation rate is

$$\begin{aligned}\frac{d}{dt}\int_X \rho_t\log\rho_t\,d\boldsymbol{m} &= \int_X \log\rho_t\Delta\rho_t\,d\boldsymbol{m} = -\int_{\{\rho_t>0\}}\frac{|\nabla\rho_t|^2}{\rho_t}\,d\boldsymbol{m}\\ &= -4\int_X |\nabla\sqrt{\rho_t}|^2\,d\boldsymbol{m}.\end{aligned}$$

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Standing assumptions (for the lectures).

(X, d) compact metric space, $\boldsymbol{m} \in \mathscr{P}(X)$

Prerequisites.

Basic facts of Optimal Transportation and Measure Theory

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