Viscosity Solutions and Aubry-Mather theory

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Chapter 1

Optimal Control

1.1 Optimal Control and Viscosity Solutions

The terminal cost problem in optimal control consists in minimizing the functional

$$J[t, x; u] = \int_t^{t_1} L(x, \dot{x}) ds + \psi(x(t_1)),$$

where $L : \mathbb{R}^{2n} \to \mathbb{R}$, and $\psi : \mathbb{R}^n \to \mathbb{R}$ are continuous functions, among all Lipschitz paths $x(\cdot)$, with initial condition $x(t) = x$ and satisfying the differential equation $\dot{x} = u$.

The infimum of J over all bounded Lipschitz controls $u \in L^\infty[t, t_1]$ is *value function* V

$$V(x, t) = \inf_u J(x, t; u). \tag{1.1}$$

Suppose $L(x, v)$ is convex in v, and satisfies the coercivity condition

$$\lim_{|v| \to \infty} \frac{L(x, v)}{|v|} = \infty.$$

The *Legendre transform* [1] of L, denoted by L^*, is the function

$$L^*(p, x) = \sup_v \left[-v \cdot p - L(x, v) \right].$$

$L^*(p, x)$ is the *Hamiltonian* and is frequently denoted by $H(p, x)$.

Next we list some important properties of the Legendre transform.

Proposition 1. *Suppose that $L(x, v)$ is convex and coercive in v. Let $H = L^*$. Then*

[1] This definition simplifies the treatment of the terminal value problem and is the usual in optimal control problems [FS93]; however, it is different from the customary in classical mechanics. The latter one is $L^\sharp(p, x) = \sup_v v \cdot p - L(x, v)$, as defined, for instance, in [Arn99]. The relation between them is $L^*(p, x) = L^\sharp(-p, x)$.

1. *$H(p,x)$ is convex in p;*

2. *$H^* = L$;*

3. *For each x, $\lim_{|p|\to\infty} \frac{H(x,p)}{|p|} = \infty$;*

4. *Define v^* by the equation $p = -D_v L(x, v^*)$; Then*

$$H(p,x) = -v^* \cdot p - L(x, v^*);$$

5. *Similarly define p^* by the equation $v = -D_p H(x, p^*)$; Then*

$$L(x,v) = -v \cdot p^* - H(x, v^*);$$

6. *If $p = -D_v L(x,v)$ or $v = -D_p H(x,p)$ then $D_x L(x,v) = -D_x H(p,x)$.*

Let ψ be a continuous function. The *superdifferential* $D_x^+\psi(x)$ of ψ at the point x is the set of values $p \in \mathbb{R}^n$ such that

$$\limsup_{|v|\to 0} \frac{\psi(x+v) - \psi(x) - p \cdot v}{|v|} \leq 0.$$

Consequently, $p \in D_x^+\psi(x)$ if and only if $\psi(x+v) \leq \psi(x) + p \cdot v + o(v)$, as $|v| \to 0$. Similarly, the *subdifferential* $D_x^-\psi(x)$ of ψ at the point x is the set of values p such that

$$\liminf_{|v|\to 0} \frac{\psi(x+v) - \psi(x) - p \cdot v}{|v|} \geq 0.$$

These sets are one-sided analog of derivatives. Indeed, if ψ is differentiable

$$D_x^-\psi(x) = D_x^+\psi(x) = \{D_x\psi(x)\}.$$

More precisely,

Proposition 2. *If $D_x^-\psi(x), D_x^+\psi(x) \neq \emptyset$ then $D_x^-\psi(x) = D_x^+\psi(x) = \{p\}$ and ψ is differentiable at x with $D_x\psi = p$. Conversely, if ψ is differentiable at x then*

$$D_x^-\psi(x) = D_x^+\psi(x) = \{D_x\psi(x)\}.$$

A point (x,t) is called *regular* if there exists a unique trajectory $x^*(s)$ such that $x^*(t) = x$ and

$$V(x,t) = \int_t^{t_1} L(x^*(s), \dot{x}^*(s))ds + \psi(x^*(t_1)).$$

We will see that regularity is equivalent to differentiability of the value function.

The next theorem collects the main results about the optimal control problem. Namely, whether V is finite, what are the optimal controls (if they exist), how the value function relates to the optimal trajectory, the regularity of V and uniqueness of optimal trajectory.

Theorem 1. *Suppose $x \in \mathbb{R}^n$ and $t_0 \leq t \leq t_1$. Assume $L(x,v)$ is a smooth function, strictly convex in v (i.e., $D^2_{vv}L$ positive definite), and satisfying the coercivity condition $\lim_{|v|\to\infty} \frac{L(x,v)}{|v|} = \infty$, for each x. Furthermore suppose L bounded below (without loss of generality, we may take $L(x,v) \geq 0$); assume also $L(x,0) \leq c_1$, $|D_xL| \leq c_2L + c_3$ for suitable constants c_1, c_2,and c_3; finally suppose that there exists a positive functions $C_0(R), C_1(R)$ such that $|D_vL| \leq C_0(R)$ and $|D^2_{xx}L| \leq C_1(R)$ whenever $|v| \leq R$. Then for any bounded Lipschitz function ψ:*

1. *V satisfies*
$$-\|\psi\|_\infty \leq V \leq c_1|t_1 - t| + \|\psi\|_\infty.$$
2. *Suppose $t_0 \leq t \leq t' \leq t_1$. Then*
$$V(x,t) = \inf_{y\in\mathbb{R}^n} \inf_{x(\cdot)} \left[\int_t^{t'} L(x(s),\dot{x}(s))ds + V(y,t') \right],$$
where $x(t) = x$ and $x(t') = y$.
3. *Suppose $\psi_1(x)$ and $\psi_2(x)$ are bounded Lipschitz functions with $\psi_1 \leq \psi_2$. Let $V_1(x,t)$ and $V_2(x,t)$ be the corresponding value functions. Then $V_1(x,t) \leq V_2(x,t)$. In particular this implies that for any $\psi_1(x)$ and $\psi_2(x)$*
$$\sup_x |V_1(x,t) - V_2(x,t)| \leq \sup_x |\psi_1(x) - \psi_2(x)|.$$
4. *There exists a control $u^* \in L^\infty[t,t_1]$ such that the corresponding path x^*, defined by the initial value ODE*
$$\dot{x}^*(s) = u(s) \qquad x^*(t) = x,$$
satisfies
$$V(x,t) = \int_t^{t_1} L(x^*(s),\dot{x}^*(s))ds + \psi(x^*(t_1)).$$
5. *There exists a constant C, which depends only on L, ψ and $t_1 - t_0$ but not on x or t such that $|u(s)| < C$ for $t \leq s \leq t_1$. The optimal trajectory $x^*(\cdot)$ is a $C^2[t,t_1]$ solution of the Euler-Lagrange equation*
$$\frac{d}{dt}D_vL - D_xL = 0$$
with initial condition $x(t) = x$.
6. *The adjoint variable p, defined by*
$$p(t) = -D_vL(x^*,\dot{x}^*),$$
satisfies the differential equation
$$\dot{p}(s) = D_xH(p(s),x^*(s)) \qquad \dot{x}^*(s) = -D_pH(p(s),x^*(s))$$

with terminal condition $p(t_1) \in D_x^- \psi(x^*(t_1))$*. Additionally*

$$(p(s), H(p(s), x^*(s))) \in D^- V(x^*(s), s)$$

for $t < s \leq t_1$*.*

7. *The value function* V *is Lipschitz continuous, thus differentiable almost everywhere.*

8. $(p(s), H(p(s), x^*(s))) \in D^+ V(x^*(s), s)$ *for* $t \leq s < t_1$ *and so* $D_x V(x^*(s), s)$ *exists for* $t < s < t_1$*.*

9. V *is differentiable at* (x, t) *if and only if* (x, t) *is a regular point.*

1.2 Viscosity Solutions

When the value function V is smooth it satisfies the Hamilton-Jacobi equation

$$-V_t + H(D_x V, x) = 0, \tag{1.2}$$

as corollary to theorem 1. However, it is not true that V is differentiable at any point (x, t). It satisfies (1.2) in a weaker sense - it is a viscosity solution. More precisely, a bounded uniformly continuous function V is a *viscosity subsolution* (resp. *supersolution*) of the Hamilton-Jacobi-Belmann PDE (1.2) if for any smooth function ϕ such that $V - \phi$ has a local maximum (resp. minimum) at (x, t) then $-D_t\phi + H(D_x\phi, x) \leq 0$ (resp. ≥ 0) at (x, t). A bounded uniformly continuous function V is a *viscosity solution* of the HJB equation provided it is both a subsolution and a supersolution.

Another useful characterization of viscosity solutions is given in the next proposition.

Proposition 3. *Suppose* V *is a bounded uniformly continuous function. Then* V *is a viscosity subsolution of (1.2) if and only if for any* $(p, q) \in D^+V(x, t)$, $-q + H(p, x; t) \leq 0$*. Similarly* V *is a viscosity supersolution if and only if for any* $(p, q) \in D^-V(x, t)$, $-q + H(p, x; t) \geq 0$*.*

A corollary of this proposition is that any smooth viscosity solution is, in fact, a classical solution.

1.3 Time independent problems

The separation of variables method applied to (1.2) motivates us to look for solutions of

$$H(P + D_x u, x) = \overline{H}(P), \tag{1.3}$$

here the parameter P is introduced artificially, but will be extremely useful in the following sections.

Any viscosity solution of (1.3) satisfies the fixed point property

$$u(x) = \inf_{x(\cdot)} \int_t^{t_1} L(x(s), \dot{x}(s)) + P \cdot \dot{x}(s) + \overline{H}(P) ds + u(x(t_1)). \tag{1.4}$$

The existence of such fixed points requires additional hypothesis on L (or H). For instance, if H is $\mathbb{Z}^n$ periodic in x, i.e., $H(x + k, p) = H(x, p)$ for $k \in \mathbb{Z}^n$ then there exists a periodic viscosity solution of (1.3). More precisely [LPV88],

Theorem 2 (Lions, Papanicolaou, Varadhan). *Suppose H is $\mathbb{Z}^n$ periodic in x. Then there exists a unique number $\overline{H}(P)$ for which the equation*

$$H(P + D_x u, x) = \overline{H}(P)$$

has a periodic viscosity solution u. Furthermore $\overline{H}(P)$ is convex in P.

Chapter 2

Hamiltonian Systems

2.1 Hamiltonian Systems

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ (we write $H(p, x)$ with $x, p \in \mathbb{R}^n$) be a smooth function. The *Hamiltonian Ordinary Differential Equation* (Hamiltonian ODE) associated with the *Hamiltonian H* and *canonical coordinates* (p, x) is

$$\dot{x} = (D_p H)^T \qquad \dot{p} = -(D_x H)^T. \tag{2.1}$$

2.2 Hamilton-Jacobi Theory

When changing coordinates in a Hamiltonian system one must be careful because the special structure of the Hamiltonian ODE is not preserved under general change of coordinates. To overcame this problem we study the theory of generating functions.

Proposition 4. *Let (p, x) be the original canonical coordinates and (P, X) be another coordinate system. Suppose $S(x, P)$ is a smooth function such that*

$$p = (D_x S(x, P))^T \qquad X = (D_P S(x, P))^T$$

defines a change of coordinates. Furthermore assume that $D^2_{xP} S$ is non-singular. Let $\overline{H}(P, X) = H(p, x)$. Then, in the new coordinate system, the equations of motion are

$$\dot{X} = (D_P \overline{H})^T \qquad \dot{P} = -(D_X \overline{H})^T, \tag{2.2}$$

i.e., (P, X) are canonical coordinates. In particular, if $\overline{H}$ does not depend on X, these equations simplify to

$$\dot{X} = (D_P \overline{H})^T \qquad \dot{P} = 0.$$

Proof. Observe that

$$-(D_x H)^T = \dot{p} = D^2_{xx} S (D_p H)^T + D^2_{Px} S \dot{P},$$

and so

$$D^2_{Px}S\dot{P} = -\left[D^2_{xx}S(D_pH)^T + (D_xH)^T\right] \tag{2.3}$$

Since $\overline{H}(P, D_PS) = H(D_xS, x)$,

$$D_X\overline{H}D^2_{xP}S = D_pHD^2_{xx}S + D_xH.$$

Transposing the previous equation and comparing with (2.3), using the fact that $D^2_{xP}S = (D^2_{Px}S)^T$ is non-singular and $D^2_{xx}S$ is symmetric,

$$\dot{P} = -(D_X\overline{H})^T.$$

We also have

$$\dot{X} = D_xX\dot{x} + D_PX\dot{P} = D^2_{xP}S(D_pH)^T + D^2_{PP}S\dot{P}.$$

Again using the identity $\overline{H}(P, (D_PS)^T) = H((D_xS)^T, x)$, we get

$$D_P\overline{H} + D_X\overline{H}D^2_{PP}S = D_pHD^2_{Px}S.$$

Again by transposition,we get

$$\dot{X} = (D_P\overline{H})^T + (D^2_{PP}S)^T\left(\dot{P} + (D_X\overline{H})^T\right),$$

which implies $\dot{X} = (D_P\overline{H})^T$. ■

The function S in the previous proposition is called a *generating function* (see, for instance, [Arn99] for details)

Proposition 5. *Suppose $S(x, P)$ is a smooth generating function such that in the new coordinates (X, P), $\overline{H}(X, P) \equiv \overline{H}(P)$. Then S is a solution of the PDE*

$$H(D_xS, x) = \overline{H}(P). \tag{2.4}$$

PROOF. If $p = D_xS$ then $H(D_xS, x) = \overline{H}(P)$. ■

When such a generating function is found, we say that the Hamiltonian ODE is *completely integrable*. However, in general, the PDE (2.4) does not have global smooth solutions.

Note that in the last proposition we have, in general, two unknowns, S and $\overline{H}(P)$. Finding $\overline{H}(P)$ is as important as finding S!

Suppose for each P we can find $\overline{H}(P)$ such that there exists a periodic smooth solution u of the PDE $H(P + D_xu, x) = \overline{H}(P)$. Then the generating function $S = P \cdot x + u$ yields a periodic (in x) change of coordinates. Assume further that

$$p = P + D_xu \qquad Q = x + D_Pu$$

defines a smooth change of coordinates. In the new coordinates

$$\dot{P} = 0 \qquad \dot{Q} = D_P\overline{H}.$$

The *rotation vector* $\omega \equiv \lim_{t\to\infty}\frac{x(t)}{t}$ of the orbits $x(t)$ exists and is

$$\omega = \lim_{t\to\infty}\frac{x(t)}{t} = \lim_{t\to\infty}\frac{Q(t)}{t} = D_P\overline{H},$$

since D_Pu is bounded (under smoothness and periodicity assumptions).

Chapter 3

Aubry-Mather Theory

3.1 Invariant sets

In the previous chapter we proved that, given a smooth periodic solution of the time independent Hamilton-Jacobi equation

$$H(P + D_x u, x) = \overline{H}(P), \tag{3.1}$$

it is possible construct an invariant set: the graph $(x, P + D_x u)$. Usually this set is identified with a n dimensional torus. Since, in general, there are no smooth solutions of (3.1), we would like to be able to prove an analogous result using viscosity solutions.

Suppose that u is a periodic viscosity solution of (3.1). Then u Lipschitz in x, and so, by Rademacher theorem, it is differentiable a.e.. Let $\mathcal{G}$ be the set

$$\mathcal{G} = \left\{(x, P + D_x u) : u \, \text{is differentiable at} \, x\right\}.$$

This set is not invariant, at least in general, but we will see that it is backwards invariant. Let Ξ_t be the *flow* associated with the backwards Hamiltonian ODE

$$\dot{p} = D_x H(p, x) \qquad \dot{x} = -D_p H(p, x). \tag{3.2}$$

Proposition 6. *$\mathcal{G}$ is backwards invariant under Ξ_t - more precisely, for all $t > 0$, we have $\Xi_t(\mathcal{G}) \subset \mathcal{G}$.*

Proof. Let u be a viscosity solution of (3.1). Consider the time dependent problem

$$-V_t + H(P + D_x V, x) = 0,$$

with terminal condition $V(t_1, x) = u(P, x)$. The (unique) viscosity solution is

$$V(x, t) = u(x) + \overline{H}(P)(t - t_1).$$

If u is differentiable at a point x_0 then, by theorem 1, $(t, x) = (0, x_0)$ is a regular point. Thus there exists a unique trajectory $x^*(s)$ such that $x^*(0) = x_0$ and

$$V(x_0, 0) = \int_0^{t_1} L(x^*(s), \dot{x}^*(s)) + P \cdot \dot{x}^*(s) ds + u(x^*(t_1)).$$

Along this trajectory the value function V is differentiable. The adjoint variable is defined by

$$p^*(s) = P + D_x V(x^*(s), s).$$

We know that the pair (x^*, p^*) solves the backwards Hamilton ODE (3.2). Therefore

$$(x^*(s), P + D_x V(x^*(s), s)) = (x^*(s), p(s)) = \Xi_s(x, p(0)) = \Xi_s(x, P + D_x V(x, 0)).$$

This implies

$$\Xi_s(x, P + D_x u) \in \mathcal{G},$$

for all $0 < s < t_1$. Since t_1 is arbitrary the previous inclusion holds for any $s \geq 0$. ■

Lemma 1. *If $\mathcal{G}$ is an invariant set then its closure $\overline{\mathcal{G}}$ is also invariant.*

Proof. Take a sequence $(x_n, p_n) \in \mathcal{G}$ and suppose this sequence converges to $(x, p) \in \overline{\mathcal{G}}$. Then, for any t, $\Xi_t(x_n, p_n) \to \Xi_t(x, p)$. This implies $\Xi_t(x, p) \in \overline{\mathcal{G}}$. ■

Define $\mathcal{G}_t = \Xi_t(\overline{\mathcal{G}})$. Note that $\mathcal{G}_t$ is, in general, a proper closed subset of $\overline{\mathcal{G}}$. Let

$$\mathcal{I} = \cap_{t>0} \mathcal{G}_t.$$

Theorem 3. *$\mathcal{I}$ is a nonempty closed invariant set for the Hamiltonian flow.*

Proof. Since $\mathcal{G}_t$ is a family of compact sets with the finite intersection property, its intersection is nonempty. Invariance follows from its definition. ■

This theorem generalizes the original one dimensional case by Moser et al. [JKM99] and W. E [E99]. A. Fathi has a different characterization of the invariant set using backward and forward viscosity solutions [Fat97a], [Fat97b], [Fat98a], and [Fat98b].

In the proof of theorem 3 we do not need to use the closure of $\mathcal{G}$. Even if $z \in \overline{\mathcal{G}} \backslash \mathcal{G}$ we have $\Xi_t(z) \in \mathcal{G}$, for all $t > 0$. Indeed, by theorem 1, the only points in an optimal trajectory that may fail to be regular are the end points.

3.2 Rotation vector

It turns out, as we explain next, that the dynamics in the invariant set $\mathcal{I}$ is particularly simple. Suppose there is a smooth (both in P and x) periodic solution of the time independent Hamilton-Jacobi equation (3.1). Define $X =$

$x + D_P u$. Then, for trajectories with initial conditions on the set $p = P + D_x u$ we have

$$\dot{X} = D_P \overline{H}(P),$$

or, equivalently, $X(t) = X(0) + D_P \overline{H}(P)t$. Therefore the dynamics of the original Hamiltonian system can be completely determined (assuming that one can invert $X = x + D_P u$).

We would like to prove an analog of this fact for orbits in the invariant set $\mathcal{I}$. A simple observation is that, in the smooth case,

$$\lim_{t \to \infty} \frac{x(t)}{t} = D_P \overline{H}(P) \equiv \omega, \tag{3.3}$$

the vector ω is called the rotation vector. The next theorem shows that (3.3) holds, under more general conditions, for all trajectories with initial conditions in the invariant set $\mathcal{I}$, provided $D_P \overline{H}$ exists.

Theorem 4. *Suppose $\overline{H}(P)$ is differentiable for some P. Then, the trajectories $x(t)$ of the Hamiltonian flow with initial conditions on the invariant set $\mathcal{I}(P)$ satisfy*

$$\lim_{t \to \infty} \frac{x(t)}{t} = D_P \overline{H}(P).$$

Proof. Fix P and P' and choose any $(x, p) \in \mathcal{I}$. By (1.4)

$$\overline{H}(P) = -\lim_{t \to \infty} \frac{\int_0^t L(x^*(s), \dot{x}^*(s)) + P \cdot \dot{x}^*(s) ds + u(x^*(t), P)}{t},$$

for some optimal trajectory x^*. Furthermore

$$\overline{H}(P') = -\lim_{t \to \infty} \inf_{x(\cdot): x(0)=x} \frac{\int_0^t L(x(s), \dot{x}(s)) + P' \cdot \dot{x}(s) ds + u(x(t), P)}{t}. \tag{3.4}$$

Thus

$$\overline{H}(P') \geq -\liminf_{t \to \infty} \frac{\int_0^t L(x^*(s), \dot{x}^*(s)) + P' \cdot \dot{x}^*(s) ds + u(P, x^*(t))}{t}.$$

The right hand side is equal to

$$-\liminf_{t \to \infty} \frac{\int_0^t (P' - P) \cdot \dot{x}^*(s) ds}{t} + \overline{H}(P).$$

Therefore

$$\overline{H}(P') - \overline{H}(P) \geq \limsup_{t \to \infty} \frac{\int_0^t (P - P') \cdot \dot{x}^*(s) ds}{t} = \limsup_{t \to \infty} \frac{(P - P') \cdot x^*(t)}{t}.$$

This implies immediately that for any vector Ω

$$-D_P \overline{H}(P) \cdot \Omega \geq \limsup_{t \to \infty} \frac{\Omega \cdot x^*(t)}{t}.$$

Replacing Ω by $-\Omega$ yields

$$-D_P\overline{H}(P)\cdot\Omega \le \liminf_{t\to\infty}\frac{\Omega\cdot x^*(t)}{t}.$$

Consequently

$$-D_P\overline{H}(P) = \lim_{t\to\infty}\frac{x^*(t)}{t}.$$

Now note that the optimal trajectory $x^*(s)$ with initial conditions $(x^*(0), p^*(0)) \in \mathcal{I}$ solves the backwards Hamilton ODE. So, any solution $x(t)$ of the Hamilton ODE with initial conditions on $\mathcal{I}$ satisfies

$$D_P\overline{H}(P) = \lim_{t\to\infty}\frac{x^*(t)}{t},$$

as required. ■

Corollary 1. *Suppose $x(t)$ is a optimal trajectory with initial conditions in $\mathcal{I}$. Then, for any subsequence t_j such that*

$$\omega \equiv \lim_{j\to\infty}\frac{x(t_j)}{t_j}$$

exists,

$$\overline{H}(P') \ge \overline{H}(P) + (P - P')\cdot\omega,$$

i.e. $\omega \in D_P^-\overline{H}(P)$.

Proof. By taking $t_j \to +\infty$ instead of $t \to +\infty$ in (3.4) we get

$$\overline{H}(P') - \overline{H}(P) \ge (P - P')\cdot\omega,$$

which proves the result. ■

3.3 Invariant Measures

J. Mather [Mat91] considered the problem of minimizing the functional

$$A[\mu] = \int L d\mu,$$

over the set of probability measures μ supported on $\mathbb{T}^n \times \mathbb{R}^n$ that are invariant under the flow associated with the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = 0.$$

Here $L = L(x, v)$ is the Legendre transform of H, and $\mathbb{T}^n$ the n-dimensional torus, identified with $\mathbb{R}^n/\mathbb{Z}^n$ whenever convenient.

One can add also the additional constraint

$$\int v d\mu = \omega,$$

restricting the class of admissible measures to the ones with an average rotation number ω. It turns out [Mn91] that this constrained minimization problem can be solved by adding a Lagrange multiplier term:

$$A_P[\mu] = \int L(x, v) + P v d\mu.$$

The main idea is that instead of studying invariant sets one should consider invariant probability measures. The supports of such measures correspond to the invariant sets (tori) defined by $P = \textit{constant}$ given by the classical theory. We show next how these measures appear naturally when using viscosity solutions.

Let $V(x, t)$ be a periodic viscosity solution (periodic both in x and t) of the Hamilton-Jacobi equation

$$-D_t V + H(P + D_x V, x, t) = \overline{H}(P).$$

For each ϵ, let $x^\epsilon(\cdot)$ be a minimizing trajectory for the optimal control problem and $p^\epsilon(\cdot)$ the corresponding adjoint variable. Then, for any s and t

$$V(x^\epsilon(s), s) = \int_s^t \left[L(x^\epsilon(r), \dot{x}^\epsilon(r), r) - P \cdot \dot{x}^\epsilon(r) - \overline{H}(P) \right] dr + V(x^\epsilon(t), t).$$

Theorem 5 (Mather measures). *For almost every $0 \leq t \leq 1$ there exists a measure (Mather measure) ν_t such that for any, smooth and periodic in y and τ, function $\Phi(p, y, \tau, t)$*

$$\Phi(p^\epsilon, \frac{x^\epsilon}{\epsilon}, \frac{t}{\epsilon}, t) \rightharpoonup \overline{\Phi}(t),$$

with $\overline{\Phi}(t) = \int \Phi(p, y, \tau, t) d\nu_t(p, y, \tau)$. More precisely, for any smooth function $\varphi(t)$

$$\int_0^1 \varphi(t) \Phi(p^\epsilon, \frac{x^\epsilon}{\epsilon}, \frac{t}{\epsilon}, t) dt \to \int_0^1 \varphi(t) \overline{\Phi}(t) dt,$$

as $\epsilon \to 0$ (through some subsequence, if necessary).

Proof. In general, the sequence $(\frac{x^\epsilon}{\epsilon}, \frac{t}{\epsilon})$ is not bounded. However if we consider $\frac{x^\epsilon}{\epsilon} \mod \mathbb{Z}^n$ and $\frac{t}{\epsilon} \mod 1$, this sequence is clearly bounded, and since, by hypothesis, Φ is periodic this does not change the result. Also p^ϵ can be uniformly bounded independently of ϵ. Thus, by the results of the previous section, we can find Young measures ν_t with the required properties. ■

We now prove that these measures are supported on the invariant set.

Proposition 7. *Let V be a periodic (in x and t) solution of $-D_t V + H(P + D_x V, x, t) = \overline{H}(P)$ and ν_t an associated Mather measure. Then $p = P + D_x V$ ν_t a.e..*

Proof. The measure ν_t was obtained as a weak limit of measures supported on the closure of $p = P + D_x V$, for some fixed V. Thus the support of the limiting measure should also be contained on the closure of $p = P + D_x V$. ■

Theorem 6. *Suppose μ a Mather measure, associated with a periodic viscosity solution of*

$$H(P + D_x u, x) = \overline{H}(P).$$

Then μ minimizes

$$\int L + P \cdot v d\eta,$$

over all invariant probability measures η.

Proof. If the claim were false, there would be an invariant probability measure ν such that

$$-\overline{H} = \int L + Pv d\mu > \int L + Pv d\nu = -\lambda.$$

We may assume that ν is ergodic, otherwise choose an ergodic component of ν for which the previous inequality holds. Take a generic point (x, v) in the support of ν and consider the projection $x(s)$ of its orbit. Then

$$u(x(0)) - \overline{H}(P)t \leq \int_0^t L(x(s), \dot{x}(s)) + P \cdot \dot{x}(s) ds + u(x(t)).$$

As $t \to \infty$

$$\frac{1}{t}\int_0^t L(x(s), \dot{x}(s)) + P \cdot \dot{x}(s) ds \to -\lambda,$$

by the ergodic theorem. Hence

$$-\overline{H} \leq -\lambda,$$

which is a contradiction. ■

Next we prove that any Mather measures (as defined originally by Mather) is "embedded" in a viscosity solution of a Hamilton-Jacobi equation. To do so we quote a theorem from [Mn96].

Theorem 7. *Suppose $\mu(P)$ is a ergodic minimizing measure. Then there exists a Lipschitz function $W : \operatorname{supp}(\mu) \to \mathbb{R}$ and a constant $\overline{H}(P) > 0$ such that*

$$-L - Pv = \overline{H}(P) + D_x W v + D_p W D_x H.$$

By taking W as initial condition (interpreting W as a function of x alone instead of (x, p) - which is possible because $\operatorname{supp}\mu$ is a Lipschitz graph) we can embed this minimizing measure in a viscosity solution. More precisely we have:

Theorem 8. *Suppose $\mu(P)$ is a ergodic minimizing measure. Then there exists a viscosity solution u of the cell problem*

$$H(P + D_x u, x) = \overline{H}(P)$$

such that $u = W$ on $\operatorname{supp}(\mu)$. Furthermore, for almost every $x \in \operatorname{supp}(\mu)$ the measures ν_t obtained by taking minimizing trajectories that pass trough x coincides with μ.

PROOF. Consider the terminal value problem $V(x, 0) = W(x)$ if $x \in \text{supp}(\mu)$ and $V(x, 0) = +\infty$ elsewhere, with

$$-D_t V + H(P + D_x V, x) = \overline{H}(P).$$

Then, for $x \in \text{supp}(\mu)$ and $t > 0$

$$V(x, -t) = W(x).$$

Also if $x \notin \text{supp}(\mu)$ then

$$V(x, -t) \leq V(x, -s),$$

if $s < t$. Hence, as $t \to \infty$ the function $V(x, -t)$ decreases pointwise. Since V is bounded and uniformly Lipschitz in x it must converge uniformly (because V is periodic) to some function u. Then u will be a viscosity solution of

$$H(P + D_x u, x) = \overline{H}(P).$$

Since $u = W$ on the support of μ, the second part of the theorem is a consequence of the ergodic theorem. ■

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