26th Internet Seminar on Evolution Equations

Graphs and Discrete Dirichlet Spaces

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Lecture Notes

Contents

Preface

Most of the material presented here follows closely the monograph

"Graphs and Discrete Dirichlet Spaces"

by Keller, Lenz and Wojciechowski. This monograph appeared within the Springer series *Grundlehren der mathematischen Wissenschaften* and a version can be found at Matthias Keller's webpage

`https://www.math.uni-potsdam.de/professuren/graphentheorie/team/prof-dr-matthias-keller/`.

Specifically, each chapter of this monograph ends with a section called "Notes". Within these sections historical notes and references to the literature can be found there. For the lectures with the ISem 26 we therefore refer to these notes sections and provide a tabular below referencing the corresponding chapters in the lecture to the chapters in the book.

ISem 26	Monograph
Chapter 1	Chapter 0
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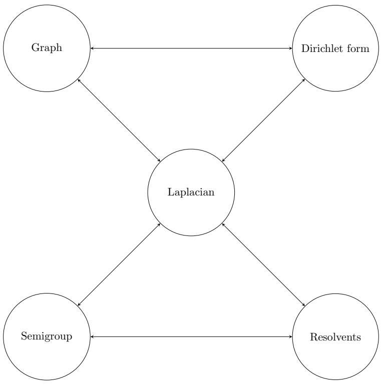
CHAPTER 1

Finite Graphs – The Theory in a Sandbox

Our topic deals with graphs, Dirichlet forms, Laplacians and Markovian semigroups and Markovian resolvents. It turns out that these five types of objects are in one-to-one correspondence to each other. This is the core of the theory.



In this chapter we present the theory in the situation of finite gaphs, i.e. graphs with finitely many vertices. The relevant vector spaces then become finite dimensional and the necessary operator theory is provided by linear algebra. This makes the considerations particularly accessible. The purpose of this chapter is twofold:

- The chapter introduces and discusses key topics of the course in a particularly simple situation.
- Results of this chapter are of direct use later in the context of approximation of the infinite dimensional situation by finite dimensional situations.

We also note in passing that finite graphs and their Laplacians, which are discussed in this chapter, are a topic of interest in itself.

1.1. Linear Algebra or Forms, Matrices Operators, Resolvents and Semigroups

In this section we present the necessary background from linear algebra for our considerations. At heart our theory is concerned with real valued functions (as these model diffusion processes and the like) and therefore all our vector spaces will be over the reals and all functions are real valued.

We consider a finite set X. We think of X as equipped with the discrete topology. Hence all functions on X are continuous. We denote by $C(X)$ the vector space of all (continuous) functions $f\colon X \longrightarrow \mathbb{R}$. For $x \in X$ we denote by 1_x the characteristic function of $\{x\}$. Thus, 1_x takes the value 1 at x and is 0 otherwise.

A *form* on $C(X)$ is a map

$$\mathcal{Q}\colon C(X) \times C(X) \longrightarrow \mathbb{R}$$

which is bilinear, i.e., satisfies

$$\mathcal{Q}(\alpha f + g, h) = \alpha \mathcal{Q}(f, h) + \mathcal{Q}(g, h)$$

and

$$\mathcal{Q}(f, \alpha g + h) = \alpha \mathcal{Q}(f, g) + \mathcal{Q}(f, h)$$

for all $f, g, h \in C(X)$ and all $\alpha \in \mathbb{R}$. A form $\mathcal{Q}$ is called *symmetric* if $\mathcal{Q}$ satisfies $\mathcal{Q}(f, g) = \mathcal{Q}(g, f)$ for all $f, g \in C(X)$. For the values of $\mathcal{Q}$ on the diagonal $\{(f, f) \mid f \in C(X)\}$ of $C(X) \times C(X)$ we will use the notation

$$\mathcal{Q}(f) := \mathcal{Q}(f, f).$$

In particular, when $\mathcal{Q}$ is symmetric, we get

$$\mathcal{Q}(f + g) = \mathcal{Q}(f) + 2\mathcal{Q}(f, g) + \mathcal{Q}(g).$$

To each form $\mathcal{Q}$ there exists a unique function $l\colon X \times X \longrightarrow \mathbb{R}$ with

$$\mathcal{Q}(f, g) = \sum_{x,y \in X} l(x, y) f(x) g(y)$$

for all $f, g \in C(X)$. We call $\mathcal{Q}$ the form *induced by* the matrix l and l the *matrix associated* to $\mathcal{Q}$. In particular, $\mathcal{Q}$ is symmetric if and only if the associated matrix l is symmetric. We note

$$\mathcal{Q}(1_x, 1_y) = l(x, y) \text{ and } \mathcal{Q}(1_x, 1) = \sum_{z \in X} l(x, z)$$

for all $x, y \in X$. Here, 1 denotes the function which is constant equal to 1 on X.

In order to make use of inner products and the powerful methods coming with them we will make $C(X)$ into an ℓ^2 space. So, we introduce next measures on discrete sets X. If $m\colon X \longrightarrow (0, \infty)$ is a strictly positive function on X, then we can extend m to a measure on X via

$$m(A) := \sum_{x \in A} m(x)$$

for all subsets $A \subseteq X$. Therefore, the pair (X, m) can be seen as a measure space. Clearly, m has *full support*, i.e. $m(A) > 0$ for all $A \neq \emptyset$. We refer to (X, m) consisting of a finite set X together with a measure of full support as a *finite set measure space.*

Example 1.1 (Counting measure). Let $m = 1$. Then m is called the *counting measure* on X. In this case, the measure of a set $A \subseteq X$ is the number of elements in the set, i.e.,

$$m(A) = \sum_{x \in A} 1.$$

The vector space $C(X)$ equipped with the inner product

$$\langle f, g\rangle := \sum_{x\in X} f(x)g(x)m(x)$$

and the norm

$$\|f\| := \langle f, f\rangle^{1/2}$$

is denoted by $\ell^2(X, m)$. It is finite dimensional and, hence, automatically complete and, therefore, a Hilbert space.

A linear map $L\colon \ell^2(X, m) \longrightarrow \ell^2(X, m)$ is called an *operator* on $\ell^2(X, m)$. The set of all operators is a vector space. It becomes a normed space with norm given by

$$\|L\| := \sup\{\|Lf\| \mid \|f\| \leq 1\}.$$

The characteristic feature of $\|L\|$ is that any $f \in \ell^2(X, m)$ satisfies

$$\|Lf\| \leq \|L\|\|f\|$$

and $\|L\|$ is the smallest number with this property. Convergence on the space of operators is always understood to refer to convergence with respect to this norm.

Any operator L on $\ell^2(X, m)$ comes with a form Q_L defined by

$$Q_L(f, g) := \langle f, Lg\rangle.$$

We call Q_L the form associated to L.

REMARK 1.2. Note that we write Q whenever we will interpret a form on $\ell^2(X, m)$ instead of $\mathcal{Q}$, the form on $C(X)$. In case X is finite we have $Q = \mathcal{Q}$ since $C(X) = \ell^2(X, m)$ as vector spaces. This will become different for infinite X, which we will see in the upcoming chapters.

An operator L on $\ell^2(X, m)$ is called *self-adjoint* if L satisfies

$$\langle Lf, g\rangle = \langle f, Lg\rangle$$

for all $f, g \in \ell^2(X, m)$. Clearly, L is self-adjoint if and only if its form is symmetric.

Whenever L is an operator and Q_L is the associated form we call the matrix l induced by Q_L the *matrix of the operator*. Thus,

$$\langle f, Lg\rangle = \sum_{x,y\in X} f(x)l(x, y)g(y)$$

holds for all $f, g \in \ell^2(X, m)$. In particular, l can be recovered from L by

$$l(x, y) = \langle 1_x, L1_y\rangle$$

and a direct calculation gives

$$(Lg)(x) = \frac{1}{m(x)} \sum_{y\in X} l(x, y)g(y)$$

for all $g \in \ell^2(X, m)$. So, L is uniquely determind by l and we call L the *operator induced by the matrix l*. Clearly, L is self-adjoint on $\ell^2(X, m)$ if and only if the matrix l associated to L is symmetric.

From these considerations we infer that there is a one-to-one correspondence between self-adjoint operators, symmetric forms and symmetric matrices.

Let now a self-adjoint L on $\ell^2(X,m)$ be given. An $e \in \ell^2(X,m)$ with $e \neq 0$ is called an *eigenvector* of L if

$$Le = \lambda e$$

holds for some $\lambda \in \mathbb{R}$ (note that self-adjointness implies that λ can only be real). The number λ is then called an *eigenvalue.* Eigenvectors to a fixed eigenvalue together with the null vector form a subspace of $\ell^2(X,m)$ called the *eigenspace* of this eigenvalue.

The set of all eigenvalues of L is called the *spectrum* of L and denoted by $\sigma(L)$. By basic results in linear algebra the operator L can be diagonalized, i.e. there exist pairwise orthogonal normalized eigenvectors $e_0,\ldots,e_{\#X-1}$ to eigenvalues $\lambda_0,\ldots,\lambda_{\#X-1}$ of L. Here, $\#X$ denotes the number of elements of X. The pairwise orthogonal normalized eigenvectors $e_0,\ldots,e_{\#X-1}$ form an orthonormal basis of $\ell^2(X,m)$, i.e. any $f \in \ell^2(X,m)$ can be represented as

$$f = \sum_{j=0}^{\#X-1} \langle e_j, f\rangle e_j,$$

and

$$Lf = \sum_{j=0}^{\#X-1} \lambda_j \langle e_j, f\rangle e_j$$

holds (as each e_j is an eigenvector to λ_j). If we take together all the $j \in \{0,\ldots,\#X-1\}$ whose λ_j agree we can exhibit these formulae in a more succinct way as follows: For $\lambda \in \sigma(A)$ let

$$E_\lambda := \sum_{j:\lambda_j=\lambda} \langle e_j, \cdot\rangle e_j$$

be the orthogonal projection onto the eigenspace of λ. Then, the mutual orthogonality of the e_j, the preceding formula for f and the preceding formula for Lf easily give

- $E_\lambda E_\mu = 0$ for $\lambda \neq \mu$.
- $I = \sum_{\lambda\in\sigma(L)} E_\lambda$.
- $L = \sum_{\lambda\in\sigma(L)} \lambda E_\lambda$.

We refer to these formulae as "spectral theorem" in the finite dimensional case. The formulae allow us to define for any function

$$\Phi : \sigma(L) \longrightarrow \mathbb{R}$$

the operator $\Phi(L)$ by

$$\Phi(L) := \sum_{\lambda\in\sigma(L)} \Phi(\lambda) E_\lambda.$$

The map

$$\text{functions on } \sigma(L) \longrightarrow \text{linear operators on } \ell^2(X,m), \quad \Phi \mapsto \Phi(L),$$

is called *spectral calculus (for L).* It has the following features:

- It is linear, i.e. $(\Phi + \lambda\Psi)(L) = \Phi(L) + \lambda\Psi(L)$ holds for all Φ, Ψ and $\lambda \in \mathbb{R}$.
- It is multiplicative, i.e. $(\Phi\Psi)(L) = \Phi(L)\Psi(L)$ holds for all Φ, Ψ.

- It is bounded, i.e. $\|\Phi(L)\| \leq \|\Phi\|_\infty$ holds (where we have $\|\Phi\|_\infty = \max\{|\Phi(\lambda)| \mid \lambda \in \sigma(L)\}$).

Indeed, these features follows easily from the definition and the mutual orthogonality of the E_λ, $\lambda \in \sigma(L)$. From these features, we easily infer that $\Phi_n(L) \to \Phi(L)$ whenever (Φ_n) is a sequence of functions on $\sigma(L)$ converging pointwise to Φ. Indeed, we have

$$\|\Phi(L) - \Phi_n(L)\| = \|(\Phi - \Phi_n)(L)\| \leq \|\Phi - \Phi_n\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Our definition of $\Phi(L)$ is consistent with natural definitions of $\Phi(L)$ for special Φ. We will discuss this in two special instances viz for the resolvents and the semigroup. Resolvents and semigroup are core objects of our further study.

We start by dealing with resolvents. Consider the linear equation — for specific L also known as *Poisson problem* — given by

$$(L + \alpha)u = f$$

for given $f \in \ell^2(X, m)$ and $\alpha \in \mathbb{R}$. By the very definition of $\sigma(L)$, for $-\alpha \notin \sigma(L)$ the operator $(L + \alpha)$ is bijective. Its inverse is a linear operator known as the *resolvent* (of L at $-\alpha$). The linear equation above is uniquely solved by

$$u = (L + \alpha)^{-1} f.$$

The resolvent can be computed by spectral calculus as follows. Define $\Phi_{(\alpha)}$ on $\sigma(-L)$ by $\Phi_{(\alpha)}(s) := \frac{1}{s+\alpha}$ for $\alpha \notin \sigma(-L)$. A short computation gives that

$$\Phi_{(\alpha)}(L) = \sum_{\lambda \in \sigma(L)} \frac{1}{\lambda + \alpha} E_\lambda$$

satisfies

$$\Phi_{(\alpha)}(L)(L + \alpha) = I = (L + \alpha)\Phi_{(\alpha)}(L).$$

This gives that $\Phi_{(\alpha)}(L)$ is just the inverse $(L + \alpha)^{-1}$ of $(L + \alpha)$.

Invoking

$$\alpha(1 - \frac{\alpha}{s + \alpha}) \to s \quad \text{as} \quad \alpha \to \infty,$$

for all $s \in \mathbb{R}$ we find from spectral calculus that we can recover L from the resolvents by

$$L = \lim_{\alpha \to \infty} \alpha(I - \alpha(L + \alpha)^{-1}).$$

We now turn to the semigroup. Consider the *Cauchy problem* — for specific L also referred to as *heat equation* — stated as:

$$-Lu = \partial_t u, \qquad u_0 = f.$$

Here, $u\colon [0, \infty) \longrightarrow \ell^2(X, m)$, $t \mapsto u_t$ is called a solution to the Cauchy problem if u is continuous with $u_0 = f$ and

$$\partial_t u_t := \lim_{s \to t} \frac{1}{t - s}(u(t) - u(s))$$

exists for all $t \in (0,\infty)$ and $-Lu_t = \partial_t u_t$ holds for all $t > 0$. Now, for $\Phi^{(t)}$ defined on $\sigma(L)$ by $\Phi^{(t)}(s) := e^{-ts}$ we can use spectral calculus to define

$$\Phi^{(t)}(L) = \sum_{\lambda\in\sigma(L)} e^{-t\lambda} E_\lambda.$$

We set

$$e^{-tL} := \Phi^{(t)}(L).$$

Then, a short computation shows that

$$u_t = e^{-tL} f$$

is a solution to the Cauchy problem (and by standard theory of ordinary differential equations this solution is unique).

A direct computation also gives that

$$e^{-tL} = \sum_{k=0}^{\infty} \frac{(-tL)^k}{k!}$$

is valid. Spectral calculus again gives that we can recover

$$L = \lim_{t\to 0} \frac{1}{t}(I - e^{-tL}).$$

Clearly, $e^{-(t+s)L} = e^{-tL}e^{-sL}$ holds for all $s,t \geq 0$ as well as $e^{-0L} = I$ and for this reason we refer to the family $(e^{-tL})_{t\geq 0}$, as the *semigroup* of L.

In the preceding discussion we have seen two families of operators associated to a self-adjoint L viz the semigroup and the resolvent. These families are — in some sense — equivalent objects. For example, we have already seen that the operator L can be obtained from either family by a limiting procedure. Moreover, one can actually obtain each of these families from the other as shown in the next lemma. The lemma can be understood to give a precise sense in which resolvent and semigroup are equivalent.

The lemma features the integral $\int_0^\infty e^{-t\alpha}e^{-tL}\mathrm{d}t$ over the operator-valued function $t \mapsto e^{-t\alpha}e^{-tL} =: A(t)$ (for $\alpha > 0$). There are various ways to make sense out of this integral and they all lead to the same result. One way is to think about the $A(t)$ as matrices (after taking a basis) and then the integral is just the matrix whose entries are the improper Riemann integrals over the corresponding entries of A. Another way is to first define $\int_0^N A(t)\mathrm{d}t$ by taking the limit of Riemann sums and then take the limit $N \to \infty$.

LEMMA 1.3 (Laplace transform, exponential formula). *Let (X,m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X,m)$ with non-negative eigenvalues.*

(a) *For all $\alpha > 0$,*

$$(L+\alpha)^{-1} = \int_0^\infty e^{-t\alpha}e^{-tL}\mathrm{d}t.$$

(*"Laplace transform"*)

(b) *For all $t > 0$,*

$$e^{-tL} = \lim_{n\to\infty} \left(\frac{n}{t}\left(L + \frac{n}{t}\right)^{-1}\right)^n.$$

(*"Exponential formula"*)

Proof. (a) Spectral calculus gives

$$e^{-t\alpha}e^{-tL} = \sum_{\lambda\in\sigma(L)} e^{-t(\alpha+\lambda)}E_\lambda \quad \text{and} \quad (L+\alpha)^{-1} = \sum_{\lambda\in\sigma(L)} \frac{1}{\lambda+\alpha}E_\lambda.$$

Now, the desired statement follows easily by integration.

(b) As follows from spectral calculus, for all natural numbers n we have

$$e^{-tL} = \sum_{\lambda\in\sigma(L)} e^{-t\lambda}E_\lambda \quad \text{and} \quad \left(\frac{n}{t}\left(\frac{n}{t}+L\right)^{-1}\right)^n = \sum_{\lambda\in\sigma(L)} \left(\frac{\frac{n}{t}}{\frac{n}{t}+\lambda}\right)^n E_\lambda.$$

Now, the desired statement follows easily from

$$\lim_{n\to\infty}\left(\frac{\frac{n}{t}}{\frac{n}{t}+\lambda}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{t\lambda}{n}}\right)^n = e^{-t\lambda}.$$

This completes the proof. □

Finally, we turn to the smallest eigenvalue of L, which can also be thought of as the minimum of the spectrum of L. We denote it by λ_0. One has

$$\lambda_0 = \inf_{\|f\|=1} Q_L(f).$$

Indeed, from $f = \sum_{\lambda\in\sigma(L)} E_\lambda f$ and $Lf = \sum_{\lambda\in\sigma(L)} \lambda E_\lambda f$ and $E_\lambda E_\mu = 0$ for $\lambda \neq \mu$ we easily find

$$Q_L(f) = \langle f, Lf\rangle = \sum_{\lambda\in\sigma(L)} \lambda\|E_\lambda f\|^2$$

as well as

$$\sum_{\lambda\in\sigma(L)} \|E_\lambda f\|^2 = \|f\|^2$$

for all $f \in \ell^2(X,m)$. So, for f with $\|f\| = 1$ we find that $Q_L(f)$ is a linear combination of the eigenvalues $\lambda \in \sigma(L)$ with coefficients $\|E_\lambda f\|^2$ adding up to 1. This shows the claim on λ_0.

1.2. Graphs and Matrices

In this section we introduce (finite) graphs and discuss some background. We then show that each such graph naturally comes with a matrix. This matrix in turn gives a form and a linear operator. The forms and linear operators associated to graphs share distinct features and are the topic of subsequent sections.

Definition 1.4 (Graph over finite X). Let X be a finite set. A *graph over* X or a *finite graph* is a pair (b,c) consisting of a function $b\colon X\times X \longrightarrow [0,\infty)$ satisfying

- $b(x,y) = b(y,x)$ for all $x,y\in X$
- $b(x,x) = 0$ for all $x\in X$

and a function $c\colon X \longrightarrow [0,\infty)$. If $c(x) = 0$ for all $x\in X$, then we speak of b as a graph over X (instead of $(b,0)$).

In the context of graphs we use the following notation: The elements of X are called the *vertices* of the graph. The map b is called the *edge weight.* The map c is called the *killing term.* Moreover, a pair (x, y) with $b(x, y) > 0$ is called an *edge* with *weight* $b(x, y)$ connecting x to y. The vertices x and y are called *neighbors* if they form an edge. We write $x \sim y$ in this case. For two vertices $x, y \in X$, a *path* from x to y is a finite number of vertices $x_0, \ldots, x_n$ with $x = x_0$, $x_n = y$ and $x_j \sim x_{j+1}$ for $j = 0, \ldots, n-1$. The graph (b, c) is called *connected* if for any $x, y \in X$ there exists a path from x to y. For any vertex x we define the *connected component* of x to be the set of $y \in X$ such that there exists a path from x to y.

Any graph comes with a matrix.

Definition 1.5 (Matrix associated to a graph). Let (b, c) be a graph over a finite set X. The matrix $l_{b,c}$ given by

$$l_{b,c}(x, y) = \begin{cases} -b(x, y) & \text{if} \quad x \neq y \\ \sum_{z \in X} b(x, z) + c(x) & \text{if} \quad x = y \end{cases}$$

is called the *matrix associated to the graph* (b, c). We say that (b, c) *induces* the matrix $l_{b,c}$.

Lemma 1.6 (Characterizing matrices arising from graphs). *Let X be a finite set. Let $l \colon X \times X \longrightarrow \mathbb{R}$ be a symmetric matrix. Then, the following statements are equivalent:*

(i) *There exists a graph (b, c) such that $l = l_{b,c}$.* (*“Graph”*)

(ii) *The matrix l satisfies*

$$l(x, y) \leq 0 \text{ and } \sum_{z \in X} l(x, z) \geq 0$$

for all $x, y \in X$ with $x \neq y$. (*“Matrix”*)

Moreover, if (i) *and* (ii) *hold, then the graph (b, c) and the matrix l are related by the equations*

$$l(x, y) = -b(x, y) \text{ and } c(x) = \sum_{z \in X} l(x, z)$$

for all $x, y \in X$ with $x \neq y$.

Proof. (i) $\Longrightarrow$ (ii): Let $l = l_{b,c}$ be the matrix associated to a graph (b, c). By the definition of $l_{b,c}$ we have $l(x, y) = -b(x, y) \leq 0$ for all $x \neq y$ as $b(x, y) \geq 0$. Furthermore,

$$\begin{aligned} \sum_{z \in X} l(x, z) &= l(x, x) + \sum_{z \neq x} l(x, z) = \sum_{z \in X} b(x, z) + c(x) - \sum_{z \neq x} b(x, z) \\ &= c(x) \geq 0 \end{aligned}$$

for all $x \in X$ as $b(x, x) = 0$. This gives (ii).

(ii) $\Longrightarrow$ (i): Define $b \colon X \times X \longrightarrow \mathbb{R}$ for $x \neq y$ by

$$b(x, y) = -l(x, y) \quad \text{and} \quad b(x, x) = 0.$$

Define $c \colon X \longrightarrow \mathbb{R}$ by

$$c(x) = \sum_{z \in X} l(x, z).$$

Then, (b, c) is a graph over X by (ii) and the symmetry of l.

Furthermore, by construction, $l_{b,c}(x, y) = -b(x, y) = l(x, y)$ for $x \neq y$ and

$$\begin{aligned} l_{b,c}(x, x) &= \sum_{z \in X} b(x, z) + c(x) = \sum_{z \neq x} b(x, z) + c(x) \\ &= -\sum_{z \neq x} l(x, z) + \sum_{z \in X} l(x, z) = l(x, x) \end{aligned}$$

for all $x \in X$. Therefore, l is the matrix associated to the graph (b, c). This gives (i).

The last statement is clear from the considerations above. $\square$

1.3. Graphs and Dirichlet Forms

Any graph gives rise to a form. This form has specific features. It is a Dirichlet form. Details are discussed in this section.

DEFINITION 1.7 (Form associated to a graph). Let (b, c) be a graph over a finite set X. The form $\mathcal{Q}_{b,c}$ acting on $C(X) \times C(X)$ by

$$\mathcal{Q}_{b,c}(f, g) := \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x)f(x)g(x)$$

is called the *form associated to the graph* (b, c) or the *energy form.*

Forms associated to graphs are particularly compatible with contractions. Specifically, let (b, c) be a graph over the finite set X and $\mathcal{Q}_{b,c}$ the associated form. Let $f, g \in C(X)$ be given and assume that g is a contraction of f in the sense that $|g(x)| \leq |f(x)|$ holds for all $x \in X$ and $|g(x) - g(y)| \leq |f(x) - f(y)|$ holds for all $x, y \in X$. Then, we immediately obtain from the definition that

$$\mathcal{Q}_{b,c}(g) \leq \mathcal{Q}_{b,c}(f).$$

To explore this more systematically, we make the following definition. A map $C \colon \mathbb{R} \longrightarrow \mathbb{R}$ is called a *normal contraction* if

$$C(0) = 0 \quad \text{and} \quad |C(s) - C(t)| \leq |s - t|$$

for all $s, t \in \mathbb{R}$. In particular, we note that $|C(s)| \leq |s|$ for all $s \in \mathbb{R}$ when C is a normal contraction.

In the context of normal contractions it is convenient to define

$$s \wedge t := \min\{s, t\} \quad \text{and} \quad s \vee t := \max\{s, t\}$$

for real numbers or for real-valued functions s and t. Examples of normal contractions include $C_+ : \mathbb{R} \longrightarrow \mathbb{R}, C_+(t) := t \vee 0$ and $C_- : \mathbb{R} \longrightarrow \mathbb{R}, C_-(t) := -(t \wedge 0)$ as well as

$$C_{[0,1]} : \mathbb{R} \longrightarrow \mathbb{R}, \quad C_{[0,1]}(t) := 0 \vee (t \wedge 1)$$

and the modulus $| \cdot | : \mathbb{R} \longrightarrow \mathbb{R}, t \mapsto |t|$.

PROPOSITION 1.8 (Compatibility of graph forms with normal contractions). *Let (b, c) be a graph over a finite set X and let $\mathcal{Q}_{b,c}$ be the form associated to (b, c). If $f \in C(X)$ is given and C is a normal contraction, then*

$$\mathcal{Q}_{b,c}(C \circ f) \leq \mathcal{Q}_{b,c}(f)$$

holds.

PROOF. Clearly, $C \circ f$ satisfies $|C \circ f(x)| \leq |f(x)|$ and

$$|C \circ f(x) - C \circ f(y)| \leq |f(x) - f(y)|$$

for all $x, y \in X$. Thus, the desired inequality follows directly from the definition of $\mathcal{Q}_{b,c}$. □

We are heading towards proving a converse to the proposition. We need the following general result.

PROPOSITION 1.9 (Representing forms via differences). *Let X be a finite set. Let $\mathcal{Q}$ be a symmetric form over X with associated matrix $l \colon X \times X \longrightarrow \mathbb{R}$. Define $b_{\mathcal{Q}} \colon X \times X \longrightarrow \mathbb{R}$ and $c_{\mathcal{Q}} \colon X \longrightarrow \mathbb{R}$ by*

$$b_{\mathcal{Q}}(x, y) := \begin{cases} -l(x, y) & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

and

$$c_{\mathcal{Q}}(x) := \sum_{y \in X} l(x, y).$$

Then, the form $\mathcal{Q}$ satisfies

$$\mathcal{Q}(f, g) = \frac{1}{2} \sum_{x,y \in X} b_{Q}(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c_{Q}(x) f(x) g(x)$$

for all $f, g \in C(X)$.

PROOF. This follows by a direct computation. Here are the details: By definition,

$$\mathcal{Q}(f, g) = \sum_{x,y \in X} \mathcal{Q}(1_x, 1_y) f(x) g(y).$$

Furthermore, by using the definitions of $c_{\mathcal{Q}}$ and $b_{\mathcal{Q}}$, we get

$$(*) \quad \mathcal{Q}(1_x, 1_x) = \mathcal{Q}(1, 1_x) - \sum_{y \neq x} \mathcal{Q}(1_y, 1_x) = c_{\mathcal{Q}}(x) + \sum_{y \in X} b_{\mathcal{Q}}(x, y).$$

Therefore,

$$\begin{aligned}
&\mathcal{Q}(f,g)\\
&= \sum_{x,y\in X} \mathcal{Q}(1_x,1_y)f(x)g(y)\\
&= \sum_{x\in X} \mathcal{Q}(1_x,1_x)f(x)g(x) + \sum_{x\in X}\sum_{y\neq x} \mathcal{Q}(1_x,1_y)f(x)g(y)\\
&\overset{(*)}{=} \sum_{x\in X}\left(c_{\mathcal{Q}}(x)+\sum_{y\in X} b_{\mathcal{Q}}(x,y)\right)f(x)g(x) - \sum_{x,y\in X} b_{\mathcal{Q}}(x,y)f(x)g(y)\\
&= \sum_{x,y\in X} b_{\mathcal{Q}}(x,y)f(x)(g(x)-g(y)) + \sum_{x\in X} c_{\mathcal{Q}}(x)f(x)g(x)\\
&= \frac{1}{2}\sum_{x,y\in X} b_{\mathcal{Q}}(x,y)(f(x)-f(y))(g(x)-g(y)) + \sum_{x\in X} c_{\mathcal{Q}}(x)f(x)g(x),
\end{aligned}$$

where in the last equality we use the symmetry of $b_{\mathcal{Q}}$ which follows from the symmetry of $\mathcal{Q}$. □

LEMMA 1.10 (Characterization of compatibility with normal contractions). *Let X be a finite set. Let $\mathcal{Q}$ be a symmetric form over X with associated matrix $l\colon X\times X \longrightarrow \mathbb{R}$.*

(a) *The following statements are equivalent:*
  (i) *The form $\mathcal{Q}$ satisfies, for all $f\in C(X)$,*

$$\mathcal{Q}(|f|)\leq \mathcal{Q}(f).$$

  (ii) *The matrix l satisfies, for all $x\neq y$,*

$$l(x,y)\leq 0.$$

(b) *The following statements are equivalent:*
  (i) *The form $\mathcal{Q}$ satisfies, for all $f\in C(X)$,*

$$\mathcal{Q}(C_{[0,1]}\circ f)\leq \mathcal{Q}(f).$$

  (ii) *The matrix l satisfies, for all $x\in X$ and $y\in X$ with $x\neq y$,*

$$l(x,y)\leq 0 \quad \textit{and} \quad \sum_{z\in X} l(x,z)\geq 0.$$

PROOF. As shown in Proposition 1.9, we have

$$\mathcal{Q}(f) = \frac{1}{2}\sum_{x\neq y} b_{\mathcal{Q}}(x,y)(f(x)-f(y))^2 + \sum_{x\in X} c_{\mathcal{Q}}(x)f(x)^2$$

with

$$b_{\mathcal{Q}}(x,y) = -l(x,y) \text{ for } x\neq y$$

and

$$c_{\mathcal{Q}}(x) = \sum_{z\in X} l(x,z).$$

This easily shows the implication (ii) $\Longrightarrow$ (i) in both (a) and (b).

(i) $\Longrightarrow$ (ii) in (a): Assume that $\mathcal{Q}$ satisfies $\mathcal{Q}(|f|) \le \mathcal{Q}(f)$ for all $f \in C(X)$. Let $x, y \in X$ with $x \neq y$ and consider $f := 1_x - 1_y$. Then, $|f| = 1_x + 1_y$. Hence, the assumption on $\mathcal{Q}$ gives

$$\mathcal{Q}(1_x + 1_y) \le \mathcal{Q}(1_x - 1_y).$$

Invoking the bilinearity and symmetry of $\mathcal{Q}$, we can easily infer

$$4\mathcal{Q}(1_x, 1_y) \le 0.$$

Since $l(x, y) = \mathcal{Q}(1_x, 1_y)$, the desired statement follows.

(i) $\Longrightarrow$ (ii) in (b): Assume that $\mathcal{Q}$ satisfies $\mathcal{Q}(C_{[0,1]} \circ f) \le \mathcal{Q}(f)$ for all $f \in C(X)$.

We start by showing $l(x, y) \le 0$ for all $x \neq y$. By part (a), which has already been proven, it suffices to show that $\mathcal{Q}(|f|) \le \mathcal{Q}(f)$ holds for all $f \in C(X)$.

Let $f \in C(X)$. After replacing f by αf with a suitable $\alpha > 0$, we can assume without loss of generality that $f \le 1$. Now, consider the decomposition of f into positive and negative parts $f = f_+ - f_-$ where $f_+(x) := f(x) \vee 0$ and $f_-(x) := (-f(x)) \vee 0$ for $x \in X$. Clearly, $|f| = f_+ + f_-$. For $s > 0$ set

$$f_s := f_+ - s f_-.$$

Then, $C_{[0,1]} \circ f_s = f_+$ for all $s > 0$. Thus, our assumption gives

$$\mathcal{Q}(f_+) = \mathcal{Q}(C_{[0,1]} \circ f_s) \le \mathcal{Q}(f_s) = \mathcal{Q}(f_+ - s f_-).$$

Invoking the bilinearity of $\mathcal{Q}$ and dividing by $s > 0$, we can then easily infer

$$0 \le -2\mathcal{Q}(f_+, f_-) + s\mathcal{Q}(f_-)$$

for all $s > 0$. Letting $s \to 0$, we obtain

$$0 \le -\mathcal{Q}(f_+, f_-).$$

Given this inequality, it follows that

$$\begin{aligned}\mathcal{Q}(|f|) = \mathcal{Q}(f_+ + f_-) &= \mathcal{Q}(f_+) + 2\mathcal{Q}(f_+, f_-) + \mathcal{Q}(f_-)\\ &\le \mathcal{Q}(f_+) - 2\mathcal{Q}(f_+, f_-) + \mathcal{Q}(f_-) = \mathcal{Q}(f).\end{aligned}$$

This gives the desired compatibility of $\mathcal{Q}$ with $|\cdot|$.

We now turn to proving $\sum_{z \in X} l(x, z) \ge 0$ for all $x \in X$. Let $x \in X$ and consider $f := 1 + s1_x$ with $s > 0$. Then, $C_{[0,1]} \circ f = 1$ for all $s > 0$ and we obtain by assumption

$$\mathcal{Q}(1) = \mathcal{Q}(C_{[0,1]} \circ f) \le \mathcal{Q}(f) = Q(1 + s1_x).$$

By the bilinearity of $\mathcal{Q}$ and after dividing by s, we find

$$0 \le 2\mathcal{Q}(1, 1_x) + s\mathcal{Q}(1_x).$$

Letting $s \to 0$, we obtain

$$0 \le \mathcal{Q}(1, 1_x) = \sum_{z \in X} l(x, z).$$

This gives the desired inequality for every $x \in X$. □

Theorem 1.11 (Characterization of forms associated to graphs). *Let $\mathcal{Q}$ be a symmetric form over a finite set X. Then, the following statements are equivalent:*

(i) *There exists a graph* (b, c) *over* X *with*

(*"Graph"*) $$\mathcal{Q} = \mathcal{Q}_{b,c}.$$

(ii) *The matrix* l *associated to* $\mathcal{Q}$ *satisfies, for* $x, y \in X$ *with* $x \neq y$,

(*"Matrix"*) $$l(x,y) \leq 0 \qquad \text{and} \qquad \sum_{z\in X} l(x,z) \geq 0.$$

(iii) *For all* $f \in C(X)$,

$$\mathcal{Q}(C_{[0,1]} \circ f) \leq \mathcal{Q}(f).$$

(*"Form compatible with one normal contraction"*)

(iv) *For all normal contractions* C *and* $f \in C(X)$,

$$\mathcal{Q}(C \circ f) \leq \mathcal{Q}(f).$$

(*"Form compatible with normal contractions"*)

(v) *If* $f, g \in C(X)$ *satisfy, for all* $x, y \in X$,

$$|f| \leq |g| \qquad \text{and} \qquad |f(x) - f(y)| \leq |g(x) - g(y)|,$$

then

$$\mathcal{Q}(f) \leq \mathcal{Q}(g).$$

Proof. This follows from the preceding considerations. Indeed, by Lemma 1.6, the equivalence between (i) and (ii) follows. The equivalence between (ii) and (iii) is the content of Lemma 1.10 (b). The implication (i) $\Longrightarrow$ (v) can be directly read off from the definition of $\mathcal{Q}_{b,c}$ (see also Proposition 1.8). The implication (v) $\Longrightarrow$ (iv) is clear from the definition of a normal contraction. Finally, (iv) $\Longrightarrow$ (iii) is obvious as $C_{[0,1]}$ is a normal contraction. □

Definition 1.12 (Dirichlet form). A form $\mathcal{Q}$ on $C(X)$ is called a *Dirichlet form* if

$$\mathcal{Q}(C_{[0,1]} \circ f) \leq \mathcal{Q}(f)$$

holds for all $f \in C(X)$.

With this definition the preceding theorem can be seen as a characterization of Dirichlet forms.

1.4. Graphs and Laplacians

From Section 1.3 we know that any graph comes with a form. Here, we discuss how it comes with an operator.

Definition 1.13 (Laplacian on $\ell^2(X, m)$). Let (b, c) be a graph over a finite set measure space (X, m). The operator $L_{b,c,m}$ acting on $\ell^2(X, m)$ via

$$L_{b,c,m} f(x) = \frac{1}{m(x)} \sum_{y\in X} b(x,y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x)$$

is called the *Laplacian on* $\ell^2(X, m)$ *associated to the graph* (b, c).

It is not hard to see that the Laplacian $L_{b,c,m}$ on $\ell^2(X,m)$ associated to (b,c) is self-adjoint. At this point we have associated to each graph (b,c) over (X,m) a symmetric form, viz $Q_{b,c} := \mathcal{Q}_{b,c}$ (recall Remark 1.2) and a self-adjoint operator, viz $L_{b,c,m}$. It turns out that form and operator are related. In fact, the form is exactly the form associated to the operator. This is the content of the next proposition.

Proposition 1.14 (Green's formula). *Let (b,c) be a graph over a finite set measure space (X,m). Let $Q_{b,c}$ and $L_{b,c,m}$ be the form and Laplacian associated to (b,c). Then, $Q_{b,c} = Q_{L_{b,c,m}}$ holds, i.e. the Green's formulae*

$$Q_{b,c}(f,g) = \langle L_{b,c,m}f, g\rangle = \langle f, L_{b,c,m}g\rangle$$

are valid for all $f, g \in \ell^2(X,m)$.

Proof. The proof is left as an exercise. □

In Section 1.3 we have seen that the forms associated to graphs are characterized within the set of all symmetric forms by their compatibility with normal contractions. It turns out that the Laplacians associated to graphs can be characterized within the set of self-adjoint operators by a distinctive feature. This feature is introduced next.

Definition 1.15 (Maximum principle and Laplacians). Let (X,m) be a finite set measure space and let L be a self-adjoint operator on $\ell^2(X,m)$. The operator L is said to satisfy the *maximum principle* if

$$Lf(x) \geq 0$$

whenever $f \in \ell^2(X,m)$ has a non-negative maximum at $x \in X$. An operator satisfying the maximum principle is called *Laplacian.*

Theorem 1.16 (Maximum principle and graphs). *Let (X,m) be a finite set measure space and let L be a self-adjoint operator on $\ell^2(X,m)$. Then, the following statements are equivalent:*

- (i) *The operator L satisfies the maximum principle.*
- (ii) *There exists a graph (b,c) over (X,m) such that $L = L_{b,c,m}$ is the Laplacian associated to (b,c).*

Proof. (i) $\Longrightarrow$ (ii): Let l be the matrix associated to L. By Lemma 1.6 it suffices to show that $l(x,y) \leq 0$ for all $x \neq y$ and $\sum_{z\in X} l(x,z) \geq 0$ for all $x \in X$. Applying the maximum principle to $f = 1$, we directly obtain $L1(x) = \frac{1}{m(x)}\sum_{z\in X} l(x,z) \geq 0$ for all $x \in X$. Applying the maximum principle at $x \in X$ to $f = -1_y$ for an arbitrary $y \in X$ with $y \neq x$ we infer $-L1_y(x) = -\frac{l(x,y)}{m(x)} \geq 0$ so that $l(x,y) \leq 0$ for all $x \neq y$.

(ii) $\Longrightarrow$ (i): As $L = L_{b,c,m}$ is the Laplacian associated to a graph (b,c) it follows that if f has a non-negative maximum at x, then

$$Lf(x) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)\underbrace{(f(x)-f(y))}_{\geq 0} + c(x)\underbrace{f(x)}_{\geq 0} \geq 0,$$

which completes the proof. □

1.5. Markov Resolvents and Semigroups

In Sections 1.3 and 1.4 we have seen that any graph comes with a form and an operator and we have characterized the form (via compatibility with normal contractions) and the operator (via a maximum principle). Here, we introduce two further objects coming with a graph. These are families of operators. Specifically, these are the semigroup and the resolvent associated to the Laplacian of the graph. We characterize them intrinsically by the Markov property.

Let L be a self-adjoint operator on $\ell^2(X,m)$ for a finite set measure space (X,m). We call $(e^{-tL})_{t\geq 0}$ the *semigroup associated to the operator* L. We say that the semigroup is *positivity preserving* if $f \in \ell^2(X,m)$, $f \geq 0$ implies

$$e^{-tL} f \geq 0$$

for all $t \geq 0$. Recall that a function f satisfying $f \geq 0$ is called *positive.* Therefore, the semigroup $(e^{-tL})_{t\geq 0}$ is positivity preserving if it maps positive functions to positive functions.

We say that the semigroup has the *Markov property* if $f \in \ell^2(X,m)$, $0 \leq f \leq 1$ implies

$$0 \leq e^{-tL} f \leq 1$$

for all $t \geq 0$. A semigroup with the Markov property is positivity preserving. Indeed, whenever $f \geq 0$ is given then sf with a suitable $s > 0$ will satisfy $0 \leq sf \leq 1$ and $e^{-tL} f = \frac{1}{s} e^{-tL}(sf) \geq 0$ follows.

In passing we note that a semigroup $(e^{-tL})_{t\geq 0}$ is Markov if and only if $e^{-tL} f \leq 1$ holds for all $f \leq 1$ and $t \geq 0$. (Indeed, if (e^{-tL}) is Markov then

$$e^{-tL} f = e^{-tL} f_+ - e^{-tL} f_- \leq e^{-tL} f_+ \leq 1$$

holds for any $f \leq 1$ and $t \geq 0$. Conversely, $e^{-tL} f \leq 1$ for $f \leq 1$ directly implies $e^{-tL} f \leq s$ for $f \leq s$ with some $s > 0$. This in turn gives $e^{-tL} f \leq 0$ for $f \leq 0$ as any such f satisfies $f \leq s$ for all $s > 0$. Hence, the semigroup is positivity preserving.)

We will characterize in terms of L and the associated form Q when a semigroup is positivity preserving and Markov. We will need an auxiliary lemma which does not involve graphs.

Lemma 1.17 (Lie–Trotter product formula on finite set measure spaces). *Let (X,m) be a finite set measure space. If A and B are operators on $\ell^2(X,m)$, then*

$$e^{A+B} = \lim_{n\to\infty} (e^{\frac{1}{n}A} e^{\frac{1}{n}B})^n.$$

Proof. Set $S_n := e^{\frac{1}{n}(A+B)}$ and $T_n := e^{\frac{1}{n}A} e^{\frac{1}{n}B}$ for $n \in \mathbb{N}$. We want to show that $\|S_n^n - T_n^n\| \to 0$ as $n \to \infty$.

We first note that for any operator L on $\ell^2(X,m)$ we have $\|e^L\| \leq e^{\|L\|}$. Consequently, it follows that

$$\|T_n\| \leq \|e^{\frac{1}{n}A}\| \|e^{\frac{1}{n}B}\| \leq e^{\frac{1}{n}\|A\|} e^{\frac{1}{n}\|B\|} = e^{\frac{1}{n}(\|A\|+\|B\|)}$$

and

$$\|S_n\| \leq e^{\frac{1}{n}\|A+B\|} \leq e^{\frac{1}{n}(\|A\|+\|B\|)}.$$

A telescoping argument gives

$$S_n^n - T_n^n = \sum_{j=0}^{n-1} S_n^j (S_n - T_n) T_n^{n-1-j}.$$

Therefore,

$$\|S_n^n - T_n^n\| \leq C_1 n \|S_n - T_n\|,$$

where $C_1 = e^{(\|A\|+\|B\|)}$. Moreover,

$$\begin{aligned} \|S_n - T_n\| &= \left\| \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{A+B}{n} \right)^j - \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{A}{n} \right)^k \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{B}{n} \right)^l \right\| \\ &= \left\| \sum_{j=2}^{\infty} \frac{1}{j!} \left(\frac{A+B}{n} \right)^j - \sum_{k+l \geq 2} \frac{1}{k! l!} \left(\frac{A}{n} \right)^k \left(\frac{B}{n} \right)^l \right\| \\ &\leq C \frac{1}{n^2} \end{aligned}$$

for some constant C. Therefore,

$$\|S_n^n - T_n^n\| \leq \frac{C_1 C}{n},$$

which yields the desired statement. □

We now characterize when a semigroup is positivity preserving in terms of the matrix and the form associated to a self-adjoint operator.

THEOREM 1.18 (First Beurling–Deny criterion). *Let (X, m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X, m)$ with associated matrix l and form $Q = Q_L$. Then, the following statements are equivalent:*

(i) *The matrix l of L satisfies, for all $x, y \in X$ with $x \neq y$,*

("Matrix") $$l(x,y) \leq 0.$$

(ii) *The form satisfies, for all $f \in \ell^2(X, m)$,*

("Form") $$Q(|f|) \leq Q(f).$$

(iii) *The semigroup satisfies, for all $f \in \ell^2(X, m)$, $f \geq 0$ and $t \geq 0$,*

("Semigroup") $$e^{-tL} f \geq 0.$$

PROOF. (i) $\Longrightarrow$ (iii): We first decompose L into a diagonal and an off-diagonal part. More specifically, we write

$$L = \widetilde{L} + \widetilde{D},$$

where $\widetilde{L}$ has matrix elements equal to those of L on the off-diagonal and matrix elements equal to zero on the diagonal and $\widetilde{D}$ has matrix elements equal to those of L on the diagonal and matrix elements equal to zero on the off-diagonal. The Lie–Trotter formula, Lemma 1.17, then gives

$$e^{-tL} = \lim_{n \to \infty} \left(e^{-\frac{t}{n}\widetilde{L}} e^{-\frac{t}{n}\widetilde{D}} \right)^n.$$

Now, by assumption, $-\widetilde{L}$ has only non-negative entries. This is then also true for $e^{-\frac{t}{n}\widetilde{L}}$. Also, $e^{-\frac{t}{n}\widetilde{D}}$ has only non-negative entries as it is a diagonal matrix with exponential functions on the diagonal. Putting this together, we infer that e^{-tL} has only non-negative matrix entries. This gives (iii).

(iii) $\Longrightarrow$ (ii): From (iii) we easily obtain

$$|e^{-tL}f| \leq e^{-tL}|f|.$$

Indeed, write $f = f_+ - f_-$ with $f_+ = f \vee 0$ and $f_- = -f \vee 0$. Note that $f_+ \geq 0$, $f_- \geq 0$ and $|f| = f_+ + f_-$. Now, a direct computation gives

$$\begin{aligned} |e^{-tL}f| &= |e^{-tL}f_+ - e^{-tL}f_-| \\ &\leq |e^{-tL}f_+| + |e^{-tL}f_-| \\ &= e^{-tL}f_+ + e^{-tL}f_- \\ &= e^{-tL}|f|. \end{aligned}$$

Here, we used assumption (iii) in the next to last step. From this preliminary consideration we infer

$$\langle e^{-tL}f, f\rangle \leq |\langle e^{-tL}f, f\rangle| \leq \langle e^{-tL}|f|, |f|\rangle.$$

Moreover, $\langle |f|, |f|\rangle = \langle f, f\rangle$. This gives

$$\langle (e^{-tL} - I)|f|, |f|\rangle \geq \langle (e^{-tL} - I)f, f\rangle.$$

Dividing by $t > 0$ we infer

$$\langle \frac{1}{t}(e^{-tL} - I)|f|, |f|\rangle \geq \langle \frac{1}{t}(e^{-tL} - I)f, f\rangle.$$

By the discussion of semigroups in Section 1.1 we know $\partial_t e^{-tL}f = -Le^{-tL}f$ so that $\partial_t e^{-tL}|_{t=0}f = -Lf$. Letting $t \to 0^+$ in the inequality above we then find

$$-Q(|f|) = \langle -L|f|, |f|\rangle \geq \langle -Lf, f\rangle = -Q(f).$$

This gives (ii).

(ii) $\Longrightarrow$ (i): This has already been shown in Lemma 1.10 (a). ☐

Having dealt with the positivity preserving part of the Markov property, we are now going to characterize the full Markov property.

THEOREM 1.19 (Second Beurling–Deny criterion). *Let (X, m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X, m)$ with associated matrix l and form $Q = Q_L$. Then, the following statements are equivalent:*

(i) *The matrix elements of the operator L satisfy, for all $x, y \in X$ with $x \neq y$,*

(*"Matrix"*) $$l(x, y) \leq 0 \quad \text{and} \quad \sum_{z\in X} l(x, z) \geq 0.$$

(ii) *The form satisfies, for all $f \in \ell^2(X, m)$,*

(*"Form"*) $$Q(C_{[0,1]} \circ f) \leq Q(f).$$

(iii) *The semigroup satisfies, for all $t \geq 0$ and $f \in \ell^2(X, m)$, $0 \leq f \leq 1$,*

(*"Semigroup"*) $$0 \leq e^{-tL}f \leq 1.$$

Proof. (i) $\iff$ (ii): This was already shown in Theorem 1.11.

(i) $\iff$ (iii): The equivalence of $l(x,y) \leq 0$ for $x \neq y$ and the semigroup being positivity preserving was already shown in Theorem 1.18. For the remaining part, we start with a preliminary consideration. Set $f := L1$ so that the second inequality of (i) is equivalent to $f \geq 0$. Consider now the function u definied by $u_t := e^{-tL}1$ for $t \geq 0$. This function satisfies $u_0 = 1$ and

$$\partial_t u_t = -Le^{-tL}1 = -e^{-tL}L1 = -e^{-tL}f$$

for all $t \geq 0$. In particular,

$$\lim_{t \to 0^+} \frac{1}{t}(u_t - u_0) = \partial_t u_t|_{t=0} = -f.$$

We now turn to proving the desired equivalence. If (i) holds, then u satisfies $u_0 = 1$ and $\partial_t u_t = -e^{-tL}f \leq 0$, where the last inequality follows as $(e^{-tL})_{t \geq 0}$ is positivity preserving and $f \geq 0$ due to (i). This shows that $t \mapsto u_t$ is non-increasing and gives

$$e^{-tL}1 \leq 1 \quad \text{for all} \quad t \geq 0.$$

Now, let $f \in \ell^2(X,m)$, $0 \leq f \leq 1$. Then the fact that $(e^{-tL})_{t \geq 0}$ is positivity preserving and the inequality above imply

$$0 \leq e^{-tL}f \leq e^{-tL}1 \leq 1$$

for all $t \geq 0$. This shows (iii).

Conversely, if (iii) holds, then we infer

$$-L1 = \partial_t e^{-tL}1|_{t=0} = \lim_{t \to 0^+} \frac{1}{t}\left(e^{-tL}1 - 1\right) \leq 0$$

from which $\sum_{z \in X} l(x,z) \geq 0$ follows. □

We now conclude this section with a characterization of the validity of the Markov property via graphs.

Theorem 1.20 (Characterization of the Markov property). *Let (X,m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X,m)$ with associated form $Q = Q_L$. Then, the following statements are equivalent:*

(i) *There exists a graph (b,c) over (X,m) with*

("Graph") $$Q = Q_{b,c} \quad \text{and} \quad L = L_{b,c,m}.$$

(ii) *The semigroup $(e^{-tL})_{t \geq 0}$, satisfies the Markov property, i.e.,*

("Semigroup") $$0 \leq e^{-tL}f \leq 1 \quad \text{for all} \quad 0 \leq f \leq 1,\ t \geq 0.$$

Proof. The statement follows by combining the second Beurling–Deny criterion in Theorem 1.19, with Lemma 1.6. □

As discussed in Section 1.1, semigroups and resolvents associated to self-adjoint operators share many features. One of these turns out to be validity of the Markov property. The resolvent has the Markov property if and only if the semigroup has the Markov property.

Corollary 1.21. *Let (X,m) be a finite set measure space. Let L be a self-adjoint operator on $\ell^2(X,m)$ with non-negative eigenvalues. Then, the following statements are equivalent:*

(i) *For all* $t \geq 0$ *and all* $f \in \ell^2(X,m)$ *with* $0 \leq f \leq 1$,

$$0 \leq e^{-tL} f \leq 1.$$

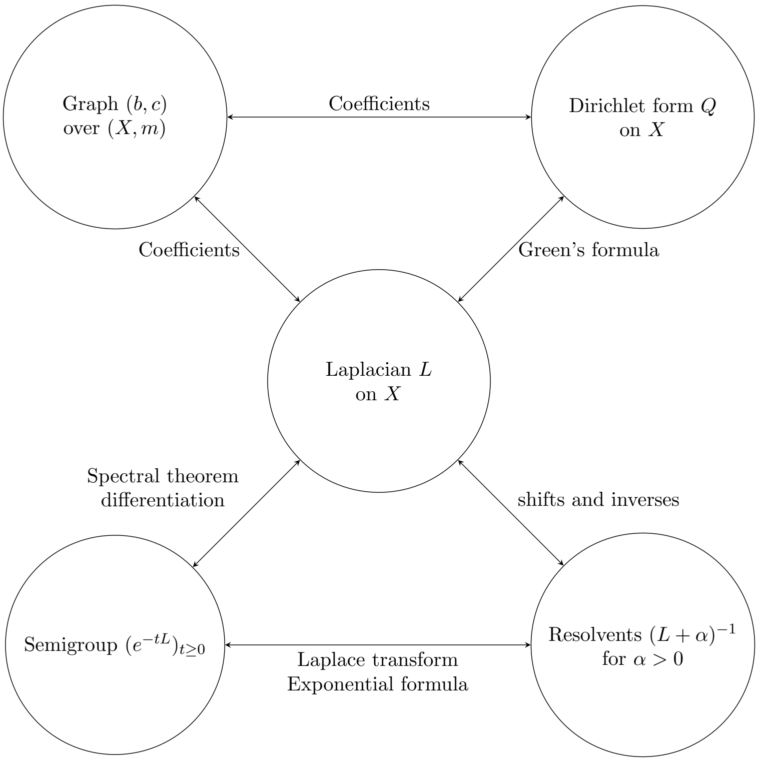
(ii) *For all* $\alpha > 0$ *and all* $f \in \ell^2(X,m)$ *with* $0 \leq f \leq 1$,

$$0 \leq \alpha(L+\alpha)^{-1} f \leq 1.$$

(iii) *There exists a graph* (b,c) *over* (X,m) *with* $L = L_{b,c,m}$.

Proof. The equivalence between (i) and (ii) follows easily from the formulae given in Lemma 1.3. The equivalence between (i) and (iii) was shown in Theorem 1.20. □

We can now visualize all the relations between the relevant objects in the following figure.



Remark 1.22 (Stochastic completeness). Let (b,c) be a graph over (X,m) and L the associated Laplacian. Then the Markov property implies $e^{-tL}1 \leq 1$ for all $t \geq 0$. The question whether actually $e^{-tL}1 = 1$ holds for all t is an interesting one. The graph is called *stochastically complete* if $e^{-tL}1 = 1$ holds for all $t \geq 0$. We will have a much closer look on this phenomenon in the case of infinite X. In the case considered here, where X is finite, it turns out that $e^{-tL}1 = 1$ holds if and only if $c = 0$. Indeed, $t \mapsto e^{-tL}1$ is the unique solution of the heat equation

$$\partial_t u_t = -Lu_t, \quad u_0 = 1.$$

Now, for $c = 0$ we have $L1 = 0$ and $u = 1$ is clearly a solution of that equation and from uniqueness $u_t = e^{-tL}1 = 1$ follows for all $t \geq 0$. Conversely, for $c \neq 0$ we have $L1 = c \neq 0$. In this case $u = 1$ is not a solution of the heat equation. Hence, $e^{-tL}1$ can not be identically to 1 for all t. A further analysis then shows that $e^{-tL}1$ is strictly less than 1 for all $t > 0$ on each connected component on which c does not vanish.

1.6. Connectedness and Large Time Behaviour

In this section we consider the behaviour of e^{-tL} for large t. This is intimately related to the behaviour of L at the bottom of the spectrum. Let (X, m) be a finite set measure space.

For a function $f\colon X \longrightarrow \mathbb{R}$ we write $f > 0$ provided $f(x) > 0$ for all $x \in X$.

Definition 1.23 (Positivity improving). Let $A\colon \ell^2(X, m) \longrightarrow \ell^2(X, m)$ be an operator on $\ell^2(X, m)$, i.e. A is linear. Then A is called *positivity improving* if $Af > 0$ holds for all $f \geq 0$ with $f \neq 0$.

Proposition 1.24 (Characterization of positivity improving semigroups and resolvents). *Let (b, c) be a graph over (X, m) with associated Laplacian $L = L_{b,c,m}$. Then, the following statements are equivalent:*

(i) *The semigroup operator e^{-tL} is positivity improving for one (all) $t > 0$.*
(ii) *The resolvent $(L + \alpha)^{-1}$ is positivity improving for one (all) $\alpha > 0$.*
(iii) *The graph (b, c) is connected.*

Proof. (i) $\Longrightarrow$ (ii): This follows immediately from the Laplace transform, i.e. from the fact that $(L + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL} \mathrm{d}t$, which is shown in Lemma 1.3 (b).

(ii) $\Longrightarrow$ (iii): Suppose that (b, c) is not connected so that there exists a non-empty connected component U of X with $U \neq X$. Write m_A for the restriction of m to the set $A \subseteq X$. We identify $\ell^2(X, m)$ with

$$\ell^2(U, m_U) \oplus \ell^2(X \setminus U, m_{X\setminus U})$$

and write elements of the latter space as (f, g).

As U is not connected to $X \setminus U$ (i.e., there is no path from any $x \in U$ to any $y \in X \setminus U$), the operator L can be decomposed as

$$L = L_U \oplus L_{X\setminus U},$$

where L_U is the restriction of L to $\ell^2(U, m_U)$ and $L_{X\setminus U}$ the restriction of L to $\ell^2(X \setminus U, m_{X\setminus U})$. It follows that

$$(L + \alpha)^{-1} = (L_U + \alpha)^{-1} \oplus (L_{X\setminus U} + \alpha)^{-1}.$$

Let $f \in \ell^2(U, m_U)$ be positive and non-trivial. Then we clearly have that $(f, 0) \in \ell^2(U, m_U) \oplus \ell^2(X \setminus U, m_{X\setminus U})$ is positive and non-trivial but

$$(L + \alpha)^{-1}(f, 0) = ((L_U + \alpha)^{-1} f, (L_{X\setminus U} + \alpha)^{-1} 0) = ((L_U + \alpha)^{-1} f, 0)$$

is not strictly positive. Hence $(L + \alpha)^{-1}$ is not positivity improving.

(iii) $\Longrightarrow$ (i): Let $f \geq 0$ with $f \neq 0$. Let $u\colon [0, \infty) \times X \longrightarrow [0, \infty)$ be defined by

$$u(t, x) := u_t(x) := e^{-tL} f(x).$$

By Corollary 1.21 we have $u_t(x) \geq 0$ for all $t \geq 0$ and $x \in X$. We wish to show that $u_t(x) > 0$ for all $t > 0$ and $x \in X$.

Assume that $u_{t_0}(x_0) = 0$ for some $t_0 > 0$ and some $x_0 \in X$. Then, $t \mapsto u_t(x_0)$ has a minimum at t_0. Thus,

$$\partial_t u_{t_0}(x_0) = 0.$$

As u_t solves $\partial_t u_t = -Lu_t$, this implies

$$\begin{aligned} 0 &= Lu_{t_0}(x_0) \\ &= \frac{1}{m(x_0)} \sum_{y\in X} b(x_0, y)(u_{t_0}(x_0) - u_{t_0}(y)) + \frac{c(x_0)}{m(x_0)} u_{t_0}(x_0) \\ &= -\frac{1}{m(x_0)} \sum_{y\in X} b(x_0, y) u_{t_0}(y). \end{aligned}$$

By $u \geq 0$ we conclude $u_{t_0}(y) = 0$ for all $y \sim x_0$. By connectedness of the graph, we obtain inductively that $u_{t_0} = 0$. This gives the contradiction $f = e^{t_0 L} u_{t_0} = 0$. □

Lemma 1.25 (Speed of convergence). *Let L be a self-adjoint operator on $\ell^2(X, m)$. Let λ_0, λ_1 be the smallest and second smallest eigenvalues of L, respectively, and let $\alpha := \lambda_1 - \lambda_0 > 0$. If E_0 is the orthogonal projection onto the eigenspace of λ_0, then*

$$\|e^{\lambda_0 t} e^{-tL} - E_0\| \leq e^{-\alpha t} \quad \textit{for all} \quad t \geq 0.$$

In particular,

$$\|e^{-tL} - E_0\| \leq e^{-\lambda_1 t} \quad \textit{for all} \quad t \geq 0$$

if $\lambda_0 = 0$.

Proof. We write $L = \sum_{j=0}^{n} \lambda_j E_j$ with pairwise different eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ of L and E_j the associated pairwise orthogonal spectral projections onto the eigenspaces. Then,

$$e^{\lambda_0 t} e^{-tL} = E_0 + \sum_{j=1}^{n} e^{-t(\lambda_j - \lambda_0)} E_j.$$

From this we derive

$$\|e^{\lambda_0 t} e^{-tL} - E_0\| \leq e^{-(\lambda_1 - \lambda_0)t}$$

as follows: Let $f \in \ell^2(X, m)$. We use the fact that the E_j are pairwise orthogonal twice to get

$$\begin{aligned} \|(e^{\lambda_0 t} e^{-tL} - E_0) f\|^2 &= \sum_{j,k=1}^{n} e^{-t(\lambda_j - \lambda_0)} e^{-t(\lambda_k - \lambda_0)} \langle E_j f, E_k f \rangle \\ &\overset{E_j \text{ pw. orth.}}{=} \sum_{j=1}^{n} e^{-2t(\lambda_j - \lambda_0)} \|E_j f\|^2 \\ &\leq e^{-2\alpha t} \sum_{j=0}^{n} \|E_j f\|^2 \\ &\overset{E_j \text{ pw. orth.}}{=} e^{-2\alpha t} \|\sum_{j=0}^{n} E_j f\|^2 \\ &= e^{-2\alpha t} \|f\|^2. \end{aligned}$$

Since this holds for all $f \in \ell^2(X,m)$, taking square roots yields the conclusion. □

The result above shows that $(e^{\lambda_0 t}e^{-tL})_{t\geq 0}$ converges exponentially to E_0, the orthogonal projection onto the eigenspace of λ_0. In particular, if $\lambda_0 = 0$, we get that the semigroup $(e^{-tL})_{t\geq 0}$ converges exponentially to E_0.

We will now investigate the properties of E_0 in the case when the graph is connected. The following result is known as the Perron–Frobenius theorem. It states that the eigenspace of λ_0 is of dimension one and consists of functions of a fixed sign.

We recall that by the variational characterization of the bottom of the spectrum we have

$$\lambda_0 = \inf Q(f),$$

where the infimum is taken over all $f \in \ell^2(X,m)$ with $\|f\| = 1$.

THEOREM 1.26 (Perron–Frobenius). *Let (b,c) be a connected graph over a finite set measure space (X,m). Let $L = L_{b,c,m}$ be the associated Laplacian with form $Q = Q_{b,c}$ and let λ_0 be the smallest eigenvalue of L with E_0 the associated orthogonal projection. Then, the eigenspace of λ_0 is one-dimensional and there exists a unique normalized strictly positive eigenfunction u corresponding to λ_0 with*

$$E_0 f = \langle u, f\rangle u$$

for all $f \in \ell^2(X,m)$.

PROOF. We first note the following general fact.

Claim. A normalized function u is an eigenfunction corresponding to λ_0 if and only if $Q(u) = \lambda_0$.

Proof of the claim. If $Lu = \lambda_0 u$ with $\|u\| = 1$, then $Q(u) = \langle Lu, u\rangle = \lambda_0\|u\|^2 = \lambda_0$.

Conversely, let u be normalized with $Q(u) = \lambda_0$. Let $\lambda_0 < \ldots < \lambda_n$ denote the eigenvalues of L. Writing $L = \sum_{j=0}^n \lambda_j E_j$, we note that

$$\lambda_0 = Q(u) = \langle u, Lu\rangle = \langle u, \sum_{j=0}^n \lambda_j E_j u\rangle = \sum_{j=0}^n \lambda_j \|E_j u\|^2$$

with $\sum_{j=0}^n \|E_j u\|^2 = \|u\|^2 = 1$. This shows $E_j u = 0$ for $j \geq 1$ and $E_0 u = u$, so that $Lu = \lambda_0 u$.

We now show that any eigenfunction corresponding to λ_0 is either strictly positive or strictly negative:

Let u be a normalized eigenfunction corresponding to λ_0. Then,

$$\lambda_0 \leq Q(|u|) \leq Q(u) = \lambda_0.$$

Here, we used the variational characterization of λ_0 in the first inequality and that Q is a Dirichlet form in the second inequality. Therefore,

$$\lambda_0 = Q(|u|).$$

As $|u|$ is normalized as well, we infer that $|u|$ is also an eigenfunction corresponding to λ_0 by the claim.

We now write $u = u_+ - u_-$, where $u_+ = u \vee 0$ and $u_- = -u \vee 0$, so that $|u| = u_+ + u_-$. Then

$$u_+ = \frac{1}{2}(|u| + u) \quad \text{and} \quad u_- = \frac{1}{2}(|u| - u)$$

are also eigenfunctions corresponding to λ_0 (or vanish identically). Assume, without loss of generality, that $u_+ \neq 0$. As e^{-tL} is positivity improving for all $t > 0$ by Proposition 1.24, we infer

$$0 < e^{-L}u_+ = e^{-\lambda_0}u_+.$$

This implies

$$u_+ > 0 \quad \text{and} \quad u_- = 0.$$

These considerations show that any eigenfunction corresponding to λ_0 has a strict sign. We conclude that the eigenspace of λ_0 is one-dimensional as eigenfunctions with a strict sign cannot be orthogonal to one another.

Now, as the eigenspace of λ_0 is one-dimensional, we then obtain

$$E_0 f = \langle u, f\rangle u$$

for any normalized eigenfunction u and $f \in \ell^2(X, m)$. Hence, any normalized strictly positive u has the desired properties and is uniquely determined by these properties. $\square$

DEFINITION 1.27 (Ground state and ground state energy). Let (b, c) be a connected graph over (X, m) with associated Laplacian $L = L_{b,c,m}$. The smallest eigenvalue λ_0 of L is called the *ground state energy* and the normalized positive eigenfunction u corresponding to λ_0 is called the *ground state.*

We also introduce the heat kernel, which arises from the heat semigroup $(e^{-tL})_{t\geq 0}$.

DEFINITION 1.28 (Heat kernel). Let (b, c) be a graph over (X, m) with associated Laplacian $L = L_{b,c,m}$. The map

$$p\colon [0, \infty) \times X \times X \longrightarrow [0, \infty)$$

defined by

$$e^{-tL}f(x) = \sum_{y\in X} p_t(x, y)f(y)m(y)$$

for all $t \geq 0$, $f \in \ell^2(X, m)$ and $x \in X$ is called the *heat kernel.*

THEOREM 1.29 (Convergence to the ground state and ground state energy). *Let (b, c) be a connected graph over (X, m). Let $L = L_{b,c,m}$ be the associated Laplacian with ground state energy λ_0, ground state u and heat kernel p. Let $\lambda_1 > \lambda_0$ be the second smallest eigenvalue of L and let $\alpha := \lambda_1 - \lambda_0$.*

(a) *For all $x, y \in X$,*

$$|e^{\lambda_0 t}p_t(x, y) - u(x)u(y)| \leq \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}}.$$

(*"Theorem of Chavel–Karp for finite graphs"*)

(b) *For all* $x, y \in X$,

$$\lim_{t\to\infty} \frac{1}{t} \log p_t(x,y) = -\lambda_0.$$

(*"Theorem of Li for finite graphs"*)

Proof. To prove (a), first observe that for any $f \in \ell^2(X,m)$ we have $|f(x)| \leq \|f\|/\sqrt{m(x)}$ for all $x \in X$. Now, the formula for E_0 in Theorem 1.26 gives, for all $x, y \in X$, that $E_0 1_y(x)/m(y) = u(x)u(y)$ while $p_t(x,y) = e^{-tL}1_y(x)/m(y)$ by definition. From Lemma 1.25 we then obtain

$$\begin{aligned} |e^{\lambda_0 t} p_t(x,y) - u(x)u(y)| &= \frac{|e^{\lambda_0 t} e^{-tL} 1_y(x) - E_0 1_y(x)|}{m(y)} \\ &\leq \frac{\|e^{\lambda_0 t} e^{-tL} - E_0\| \|1_y\|}{m(y)\sqrt{m(x)}} \\ &\leq \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}}. \end{aligned}$$

This gives (a).

To prove (b), note from the above that

$$u(x)u(y) - \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}} \leq e^{\lambda_0 t} p_t(x,y) \leq u(x)u(y) + \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}}$$

for all $x, y \in X$. As u is strictly positive by Theorem 1.26, (b) follows after taking logarithms for large t, dividing by t and letting $t \to \infty$. □

1.7. The Dirichlet Problem

In this section we discuss some further aspects of the theory. Throughout we assume that X is a finite set and the measure m is just the counting measure and we remove it from notation; so, e.g., we write $L_{b,c} := L_{b,c,1}$ for the Laplacian. We leave it as an exercise to include a measure in the considerations.

Lemma 1.30 (Non-vanishing c characterizes the bijectivity of $L_{b,c}$). *Let* (b,c) *be a graph over a finite set* X *and let* $L_{b,c}$ *be the associated Laplacian on* $\ell^2(X)$*. The operator* $L_{b,c}$ *is bijective if and only if* c *does not vanish identically on any connected component of* (b,c)*.*

Proof. As $L_{b,c}$ is a linear operator on a finite dimensional vector space, bijectivity is equivalent to injectivity. Furthermore, we can assume without loss of generality that the graph is connected.

If $c = 0$, then clearly $L_{b,c}1 = 0$. Therefore, $L_{b,c}$ is not injective in this case.

Now, suppose that c does not vanish at all $x \in X$. Let $u \in \ell^2(X)$ satisfy $L_{b,c}u = 0$. Green's formula, Proposition 1.14, gives

$$\begin{aligned} 0 &= \sum_{x\in X} u(x) L_{b,c} u(x) = Q_{b,c}(u) \\ &= \frac{1}{2} \sum_{x,y\in X} b(x,y)(u(x) - u(y))^2 + \sum_{x\in X} c(x)u(x)^2. \end{aligned}$$

As all terms appearing in the sums are non-negative, we infer $u(x) = u(y)$ whenever $b(x, y) > 0$ and $u(x) = 0$ whenever $c(x) \neq 0$. As the graph is connected, the first set of conditions implies u is constant and the second set of conditions implies $u = 0$ as c does not vanish identically. Therefore, $L_{b,c}$ is injective. $\square$

THEOREM 1.31 (The Dirichlet problem). *Let (b, c) be a connected graph over a finite set X. Let $B \subseteq X$ with $B \neq \emptyset$, $A := X \setminus B$ and $g\colon B \longrightarrow \mathbb{R}$. Then, the Dirichlet problem* (DP)*:*

- $L_{b,c}u = 0$ *on* A
- $u = g$ *on* B

has a unique solution. Moreover, for the set

$$\mathcal{A}_g := \{h \in C(X) \mid h = g \text{ on } B\}$$

and $f \in \mathcal{A}_g$ the following statements are equivalent:

(i) $Q_{b,c}(f) = \min\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}$.
(ii) *The function f solves the Dirichlet problem* (DP).

In particular, there exists a unique minimizer in (i)*. Moreover, if $0 \leq g \leq 1$, then $0 \leq f \leq 1$.*

PROOF. We will show a series of claims which will prove the theorem (and a bit more).

Claim 1. The solution of (DP) exists and is unique.

Proof of Claim 1. We transform the problem to an equivalent problem for which we will establish existence and uniqueness. Let f be a solution of $L_{b,c}f = 0$ on A with $f = g$ on B, that is, let f solve (DP). For any $x \in A$, we then have

$$\begin{aligned} 0 &= L_{b,c}f(x) \\ &= \sum_{y\in X} b(x,y)(f(x) - f(y)) + c(x)f(x) \\ &= \sum_{y\in A} b(x,y)(f(x) - f(y)) + \sum_{y\in B} b(x,y)(f(x) - f(y)) + c(x)f(x) \\ &= \sum_{y\in A} b(x,y)(f(x) - f(y)) + \Big(c(x) + \sum_{y\in B} b(x,y)\Big)f(x) - \sum_{y\in B} b(x,y)g(y) \\ &= \sum_{y\in A} b(x,y)(f(x) - f(y)) + d(x)f(x) - h(x) \end{aligned}$$

with

$$d(x) := c(x) + \sum_{y\in B} b(x,y) \quad \text{and} \quad h(x) := \sum_{y\in B} b(x,y)g(y).$$

Note that both d and h do not depend on f.

We let $L_A^{(D)} := L_{b_A,d}$, which we call the *Dirichlet Laplacian* associated to the graph (b_A, d) over A, given by $b_A(x, y) := b(x, y)$ for $x, y \in A$, d as above and the restriction f_A of f to A, we obtain from the above that

(P) $L_A^{(D)} f_A = h$ on A.

Now, if f is a solution of (DP), then f_A solves (P), as shown by the above calculation. Conversely, any solution $\widetilde{f}$ of (P) becomes a solution f to (DP) after extending $\widetilde{f}$ by g on B. This gives:

$$f \text{ solves (DP)} \quad \Longleftrightarrow \quad f_A \text{ solves (P)}.$$

Therefore, it suffices to show that (P) has a unique solution, that is, $L_A^{(D)}$ is bijective. By construction, $L_A^{(D)}$ is the Laplacian associated to the graph (b_A, d) over A. Thus, by Lemma 1.30, it suffices to show that d does not vanish on any connected component of A, where the connected components are defined with respect to b_A. Let Z be such a connected component. Invoking the definition of d, it suffices to find $x \in Z$ and $y \in B$ with $b(x, y) > 0$. First, we choose an arbitrary $y' \in B$ and $o \in Z$. As the graph is connected there exists a path $(x_0, x_1, \ldots, x_n)$ in (X, b) with $x_0 = o$ and $x_n = y'$. Let j be the smallest index such that x_j does not belong to Z. Then, letting $y := x_j$, y belongs to B as otherwise it would belong to Z since Z is a connected component. Thus, $x := x_{j-1} \in Z$ and $y = x_j \in B$ satisfy $b(x, y) > 0$. This finishes the proof of Claim 1.

Claim 2. Any minimizer of $Q_{b,c}$ on $\mathcal{A}_g$ solves (DP).

Proof of Claim 2. Suppose that there exists an $f \in \mathcal{A}_g$ with

$$Q_{b,c}(f) = \inf\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}.$$

Let φ be an arbitrary function supported on A. Then, $f + \lambda\varphi$ belongs to $\mathcal{A}_g$ for all $\lambda \in \mathbb{R}$. Thus, the function

$$\lambda \mapsto Q_{b,c}(f + \lambda\varphi) = Q_{b,c}(f) + 2\lambda Q_{b,c}(f, \varphi) + \lambda^2 Q_{b,c}(\varphi)$$

has a minimum at $\lambda = 0$. Taking the derivative at $\lambda = 0$ yields

$$0 = Q_{b,c}(f, \varphi) = \sum_{x \in X} L_{b,c} f(x) \varphi(x)$$

by Green's formula, Proposition 1.14. As φ supported in A was arbitrary, we conclude that $L_{b,c} f = 0$ on A.

Claim 3. There exists a minimizer of $Q_{b,c}$ on $\mathcal{A}_g$.

Proof of Claim 3. Let (f_n) be a sequence in $\mathcal{A}_g$ with

$$\lim_{n\to\infty} Q_{b,c}(f_n) = \inf\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}.$$

It follows that $(Q_{b,c}(f_n))$ is a bounded sequence. Let o be an arbitrary point in B. Then, $f_n(o) = g(o)$ for all $n \in \mathbb{N}$ as $f_n \in \mathcal{A}_g$. As we will show below, the boundedness of $(Q_{b,c}(f_n))$ together with the boundedness of $(f_n(o))$ implies that $(f_n(x))$ is bounded for any $x \in X$. By choosing a suitable subsequence we can, without of loss of generality, assume that (f_n) converges pointwise to a function f. Obviously, $f \in \mathcal{A}_g$ and

$$Q_{b,c}(f) = Q_{b,c}\left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} Q_{b,c}(f_n) = \inf\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}.$$

Thus, f is a minimizer of $Q_{b,c}$ on $\mathcal{A}_g$.

It remains to show the desired boundedness of $(f_n(x))$ for $x \in X$. Let $x \in X$ and let $\gamma := (x_0, \ldots, x_m)$ with $x_0 = o$ and $x_m = x$ be a path from o to x. Then, for any function u, we have by the Cauchy–Schwarz inequality

$$|u(x) - u(o)|$$

$$\begin{aligned}
&\leq \sum_{j=0}^{m-1} |u(x_j) - u(x_{j+1})| \\
&= \sum_{j=0}^{m-1} |u(x_j) - u(x_{j+1})| b(x_j, x_{j+1})^{1/2} \cdot \frac{1}{b(x_j, x_{j+1})^{1/2}} \\
&\leq \left(\sum_{j=0}^{m-1} (u(x_j) - u(x_{j+1}))^2 b(x_j, x_{j+1}) \right)^{1/2} \left(\sum_{j=0}^{m-1} b(x_j, x_{j+1})^{-1} \right)^{1/2} \\
&\leq Q_{b,c}(u)^{1/2} C(\gamma)
\end{aligned}$$

with $C(\gamma) := \left(\sum_{j=1}^{m} b(x_j, x_{j+1})^{-1} \right)^{1/2}$. Applying this to f_n and noting that $f_n(o) = g(o)$ for all n since $o \in B$, we get

$$|f_n(x) - g(o)| \leq C(\gamma) Q_{b,c}(f_n)^{1/2}.$$

As $(Q_{b,c}(f_n))_n$ is bounded and $C(\gamma)$ does not depend on n, it follows that $(f_n(x))_n$ is bounded.

Claim 4. If $0 \leq g \leq 1$, then $0 \leq f \leq 1$.

Proof of Claim 4. Recall that $C_{[0,1]} \circ f = 0 \vee f \wedge 1$. If $f \in \mathcal{A}_g$, then $C_{[0,1]} \circ f \in \mathcal{A}_g$ since $C_{[0,1]} \circ g = g$. Therefore, $C_{[0,1]} \circ f$ is also a minimizer of $Q_{b,c}$ as $Q_{b,c}$ is a Dirichlet form and thus $Q_{b,c}(C_{[0,1]} \circ f) \leq Q_{b,c}(f)$. The already proven uniqueness then gives $f = C_{[0,1]} \circ f$, which is equivalent to $0 \leq f \leq 1$.

By combining the preceding statements we now prove the theorem: Claim 1 yields the existence and uniqueness of solutions to (DP). Claim 2 shows the implication (i) $\Longrightarrow$ (ii). Furthermore, in Claim 3, we have shown the existence of a minimizer of $Q_{b,c}$ on $\mathcal{A}_g$. We next turn to (ii) $\Longrightarrow$ (i): The solution of (DP) and the minimizer of $Q_{b,c}$ on $\mathcal{A}_g$ both exist and are unique by the considerations above. As the minimizer of $Q_{b,c}$ on $\mathcal{A}_g$ solves (DP) by Claim 2, it coincides with the unique solution of (DP). Thus, this unique solution minimizes $Q_{b,c}$ on $\mathcal{A}_g$. Finally, the last statement of the theorem follows from Claim 4. □

CHAPTER 2

Infinite Graphs I – The Formal Objects

In this chapter we start the investigation of the general situation. Thus, we do not assume that the underlying set X is finite. We rather deal with (countably) infinite sets X. We will introduce the core objects and present some general theory.

Throughout we let X be a countable set. We think of X as being equipped with the discrete topology. We denote by $C(X)$ the set of all functions on X (which are automatically continuous) and by $C_c(X)$ the space of compactly, i.e., finitely, supported functions. For $x \in X$ we let 1_x denote the characteristic function of $\{x\}$. So, $1_x(y) = 1$ for $y = x$ and $1_x(y) = 0$ otherwise. Observe that the characteristic functions 1_x, $x \in X$, form a basis of $C_c(X)$. A function $m\colon X \longrightarrow (0,\infty)$ gives rise to a measure on (the σ-algebra consisting of all subsets of) X by

$$m(A) := \sum_{x\in A} m(x).$$

We will not distinguish between the measure and the function m in notation. Note that the measure has full support, i.e. $m(A) > 0$ holds for all $A \neq \emptyset$. The measure m gives naturally rise to the Hilbert space $\ell^2(X,m)$. By definition, the underlying vector space is given as

$$\ell^2(X,m) := \{f\colon X \longrightarrow \mathbb{R} \mid \sum_{x\in X} |f(x)|^2 m(x) < \infty\}.$$

The inner product is given by

$$\langle f, g\rangle := \sum_{x\in X} f(x)g(x)m(x).$$

Note that the sum in question is finite for $f, g \in \ell^2(X,m)$ since we have $|f(x)g(x)| \leq \frac{1}{2}(|f(x)|^2 + |g(x)|^2)$ for all $x \in X$. The inner product induces the norm $\|\cdot\|$ with

$$\|f\| := \langle f, f\rangle^{1/2} = \left(\sum_{x\in X} |f(x)|^2 m(x)\right)^{1/2}.$$

As is well-known from a basic course in functional analysis (and is also not hard to see) the space $\ell^2(X,m)$ is complete with respect to $\|\cdot\|$ and $C_c(X)$ is dense in $\ell^2(X,m)$.

2.1. Graphs

In this section we introduce graphs over countable X.

Definition 2.1 (Graph over X). A *graph over* X is a pair (b, c) consisting of a function $b \colon X \times X \longrightarrow [0, \infty)$ satisfying

- $b(x, y) = b(y, x)$ for all $x, y \in X$
- $b(x, x) = 0$ for all $x \in X$
- $\sum_{y \in X} b(x, y) < \infty$ for all $x \in X$

and a function $c \colon X \longrightarrow [0, \infty)$.

Whenever (b, c) is a graph over X we use the following pieces of notation (compare the Section 1.2): The elements of X are called *vertices.* A pair (x, y) with $b(x, y) > 0$ is called an *edge* with *weight* $b(x, y)$. Elements $x, y \in X$ forming an edge are also called *neighbors* and we write $x \sim y$ in this case.

The *degree* is the function

$$\deg \colon X \longrightarrow [0, \infty),\ \deg(x) := \sum_{y \in X} b(x, y) + c(x).$$

A tuple $(x_0, \ldots, x_n)$ of vertices is called a *path* from x_0 to x_n if $x_i \sim x_{i+1}$ holds for $i = 0, \ldots, n - 1$. As b is symmetric there exists a path from x to y if and only if there exists a path from y to x. Whenever $x \in X$ is given the set of vertices y such that there exists a path from x to y is called the *connected component of* x. If there exists a path between any two vertices the graph is called *connected.* Clearly, the graph is connected if and only if the connected component of one (each) $x \in X$ agrees with X.

We say that a graph (b, c) is *locally finite* if for every $x \in X$ the number of neighbors of x is finite, i.e.,

$$\#\{y \in X \mid y \sim x\} < \infty$$

for all $x \in X$. Here, as above, # denote the number of elements of a set. In general, we will not assume that graphs are locally finite.

Remark 2.2 (Uncountable graphs). In our discussion we have assumed that X is countable. This is not necessary in order to set up the theory. Indeed, all of the preceding definitions make sense also for uncountable X. However, the summability condition $\sum_{y \in X} b(x, y) < \infty$ for all $x \in X$ implies that any $x \in X$ can have at most countably many neighbors. Hence, the connected component of any x must be countable (even if X were uncountable at the beginning). So any graph over an uncountable X could be considered as an (uncountable) union of connected graphs with countably many vertices. So, all of the theory developed below will apply to each connected component of such a graph. For this reason we just assume countability of X from the very beginning.

2.2. The Energy Form

Any graph comes with a bilinear map on (a subspace of) $C(X)$. This map underlies all our subsequent considerations.

To a graph (b, c) over X, we associate the subspace $\mathcal{D} := \mathcal{D}_{b,c}$ of $C(X)$ given by

$$\mathcal{D} = \{f \in C(X) \mid \frac{1}{2} \sum_{x, y \in X} b(x, y)(f(x) - f(y))^2 + \sum_{x \in X} c(x) f(x)^2 < \infty\}.$$

Note that $\mathcal{D}$ is indeed a subspace of $C(X)$. To (b,c) we furthermore associate the map

$$\mathcal{Q} := \mathcal{Q}_{b,c} \colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$$

defined by

$$\mathcal{Q}(f,g) := \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x) f(x) g(x).$$

Clearly, $\mathcal{Q}$ is bilinear, i.e. linear in each argument. We call $\mathcal{Q}$ the *energy form* and refer to elements of $\mathcal{D}$ as *functions of finite energy.*

Clearly, $\mathcal{Q}$ is *symmetric*, i.e., satisfies

$$\mathcal{Q}(f,g) = \mathcal{Q}(g,f)$$

for all $f, g \in \mathcal{D}$. The form $\mathcal{Q}$ is also *positive*, i.e., satisfies

$$\mathcal{Q}(f,f) \geq 0$$

for all $f \in \mathcal{D}$.

We will often be interested in the values of $\mathcal{Q}$ on the diagonal only. In this case, we will use the notation

$$\mathcal{Q}(f) := \mathcal{Q}(f,f)$$

for $f \in \mathcal{D}$. We can then extend this restriction of $\mathcal{Q}$ to the diagonal to a map on the whole $C(X)$, again denoted by $\mathcal{Q}$, defined by $\mathcal{Q} \colon C(X) \longrightarrow [0,\infty]$ via

$$\mathcal{Q}(f) = \begin{cases} \mathcal{Q}(f,f) & \text{if } f \in \mathcal{D} \\ \infty & \text{else.} \end{cases}$$

Recall that a map $C \colon \mathbb{R} \longrightarrow \mathbb{R}$ is called a normal contraction if $C(0) = 0$ and $|C(s) - C(t)| \leq |t - s|$ for all $s, t \in \mathbb{R}$. The form $\mathcal{Q}$ is compatible with normal contractions in the following sense.

Proposition 2.3 (Compatibility with normal contractions). *Let (b,c) be a graph over X and let $C \colon \mathbb{R} \longrightarrow \mathbb{R}$ be a normal contraction. Then,*

$$\mathcal{Q}(C \circ f) \leq \mathcal{Q}(f)$$

holds for all $f \in C(X)$.

Proof. This is immediate from the definitions. □

The energy form has the following semi-continuity property.

Proposition 2.4 (Lower semi-continuity of $\mathcal{Q}$ on $C(X)$). *Let (b,c) be a graph over X. If a sequence (f_n) in $C(X)$ converges pointwise to $f \in C(X)$, i.e., $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$, then*

$$\mathcal{Q}(f) \leq \liminf_{n \to \infty} \mathcal{Q}(f_n).$$

Proof. This is a consequence of Fatou's lemma. Indeed, consider the measure space $X \times X$ with the measure B and X with the measure C given by

$$B(M) := \frac{1}{2} \sum_{(x,y) \in M} b(x,y) \quad \text{and} \quad C(N) := \sum_{x \in N} c(x)$$

for $M \subseteq X \times X$, $N \subseteq X$, and for $n \in \mathbb{N}$ the functions $F_n, F\colon X \times X \longrightarrow [0, \infty)$ defined by

$$F_n(x, y) := (f_n(x) - f_n(y))^2 \quad \text{and} \quad F(x, y) := (f(x) - f(y))^2.$$

Then, clearly $F_n(x, y) \to F(x, y)$ for all $x, y \in X$, $f_n(x)^2 \to f(x)^2$ for all $x \in X$ as $n \to \infty$ and

$$\int_{X\times X} F \mathrm{d}B + \int_X f^2 \mathrm{d}C = \mathcal{Q}(f), \qquad \int_{X\times X} F_n \mathrm{d}B + \int_X f_n^2 \mathrm{d}C = \mathcal{Q}(f_n).$$

Now, Fatou's lemma gives the desired statement. □

For $o \in X$, we define the map

$$\langle \cdot, \cdot \rangle_o \colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$$

by

$$\langle f, g \rangle_o := \mathcal{Q}(f, g) + f(o)g(o)$$

for $f, g \in \mathcal{D}$. Clearly, $\langle \cdot, \cdot \rangle_o$ is linear in each argument, symmetric and satisfies $\langle f, f \rangle_o \geq 0$ for all $f \in \mathcal{D}$. Hence, the map is a semi-scalar product. We let $\| \cdot \|_o$ be the corresponding semi-norm, i.e.

$$\|f\|_o := \langle f, f \rangle_o^{1/2}$$

for $f \in \mathcal{D}$.

For connected graphs the map $\langle \cdot, \cdot \rangle_o$ is a scalar-product and $\| \cdot \|_o$ is a norm on $\mathcal{D}$ and $\mathcal{D}$ becomes a Hilbert space. This (and more) is the content of the next lemma.

Lemma 2.5 (The Hilbert space $(\mathcal{D}, \langle \cdot, \cdot \rangle_o)$). *Let (b, c) be a connected graph over (X, m) and $o \in X$. Then, the following holds:*

(a) *The map $\langle \cdot, \cdot \rangle_o$ is a scalar-product and $\| \cdot \|_o$ is a norm.*
(b) *For any $x \in X$ the norms $\| \cdot \|_o$ and $\| \cdot \|_x$ are equivalent. In particular, for any $x \in X$ the pointwise evaluation map $\mathcal{D} \longrightarrow \mathbb{R}, f \mapsto f(x)$, is continuous.*
(c) *$(\mathcal{D}, \| \cdot \|_o)$ is a Hilbert space.*
(d) *A sequence (f_n) in $\mathcal{D}$ converges to f w.r.t. $\langle \cdot, \cdot \rangle_o$ if and only if $f_n \to f$ pointwise and*

$$\limsup_{n\to\infty} \mathcal{Q}(f_n) \leq \mathcal{Q}(f).$$

Proof. (a) We already know that $\langle \cdot, \cdot \rangle_o$ is a semi-inner product. Thus, it suffices to show that it is non-degenerate i.e. that $\langle f, f \rangle_o = 0$ is only possible for $f = 0$. Now, writing out the definition gives

$$\begin{aligned} 0 &= \langle f, f \rangle_o = \mathcal{Q}(f, f) + f(o)^2 \\ &= \frac{1}{2} \sum_{x,y\in X} b(x, y)(f(x) - f(y))^2 + \sum_{x\in X} c(x)f(x)^2 + f(o)^2. \end{aligned}$$

As b, c and all squares appearing are non-negative, we infer $f(o) = 0$ and $f(x) = f(y)$ for all $x \sim y$. As the graph is connected this implies $f = 0$.

(b) Since the graph is connected for any $x \in X$ there exists a path $(x_0, \ldots, x_n)$ with $x_0 = o$ and $x_n = x$. Without loss of generality we assume that the x_j are pairwise different. Then, for any $f \in \mathcal{D}$ the following estimate holds

$$\begin{aligned}
&|f(x) - f(o)| \\
&\leq \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})| \\
&\leq \left(\sum_{i=0}^{n-1} \frac{1}{b(x_i, x_{i+1})} \right)^{1/2} \left(\sum_{i=0}^{n-1} b(x_i, x_{i+1}) |f(x_i) - f(x_{i+1})|^2 \right)^{1/2} \\
&\leq C_{o,x} \mathcal{Q}(f)^{1/2}
\end{aligned}$$

with $C_{o,x} := \left(\sum_{i=0}^{n-1} \frac{1}{b(x_i,x_{i+1})} \right)^{1/2}$. From this the desired equivalence follows rather easily. Here are the details: From $|f(x)| \leq |f(o)| + |f(x) - f(o)|$ we obtain

$$f(x)^2 \leq 2f(o)^2 + 2|f(x) - f(o)|^2 \leq 2f(o)^2 + 2C_{o,x}^2 \mathcal{Q}(f).$$

This then implies

$$\|f\|_x^2 = \mathcal{Q}(f) + f(x)^2 \leq \max\{2, 2C_{o,x} + 1\} \|f\|_o^2$$

for all $f \in \mathcal{D}$. Reversing the roles of x and o we also find $C \geq 0$ such for all $f \in \mathcal{D}$ the inequality

$$\|f\|_o^2 \leq C \|f\|_x^2$$

holds. (Actually, $C = \max\{2, 2C_{o,x} + 1\}$ will do as well as we could just use the path above in the reverse order). This shows that $\| \cdot \|_o$ and $\| \cdot \|_x$ are equivalent.

Now, clearly the point evaluation $\mathcal{D} \longrightarrow \mathbb{R}, f \mapsto f(x)$ is continuous with respect to $\| \cdot \|_x$. As $\| \cdot \|_o$ is equivalent to $\| \cdot \|_x$ the point evaluation is continuous with respect to $\| \cdot \|_o$ as well.

(c) We have to show that $(\mathcal{D}, \langle \cdot, \cdot \rangle_o)$ is complete. Let (f_n) be a Cauchy sequence with respect to $\| \cdot \|_o$. This implies that

$$\mathcal{Q}(f_n - f_m) + |f_n(o) - f_m(o)|^2$$

becomes arbitrarily small for n, m sufficiently large. In particular, $(f_n(o))$ must be a Cauchy sequence. Now, by (b) the (f_n) must be a Cauchy sequence with respect to $\| \cdot \|_x$ for any $x \in X$. Hence, $(f_n(x))$ must be a Cauchy sequence for each x (by the same reasoning that we had applied just now for $x = o$). Altogether, we infer that (f_n) converges pointwise to some f. This f is now our candidate for a limit of (f_n) and we have to show that f is indeed the limit of (f_n) with respect to $\| \cdot \|_o$.

Since (f_n) is a Cauchy sequence, $(\|f_n\|_o)$ is bounded by some $C \geq 0$. Thus, by Fatou's lemma

$$\mathcal{Q}(f) \leq \liminf_{n \to \infty} \mathcal{Q}(f_n) \leq \liminf_{n \to \infty} \|f_n\|_o^2 \leq C^2.$$

Thus, f belongs to $\mathcal{D}$ and, again by Fatou's lemma,

$$\mathcal{Q}(f - f_n) \leq \liminf_{k \to \infty} \mathcal{Q}(f_k - f_n) \leq \liminf_{k \to \infty} \|f_k - f_n\|_o^2,$$

which becomes arbitrarily small since (f_n) is a Cauchy sequence. Hence, (f_n) converges indeed to f with respect to $\| \cdot \|_o$.

(d) Let (f_n) be a sequence in $\mathcal{D}$ that converges with respect to $\|\cdot\|_o$ to f. Then, the continuity of the point evaluation from (b) gives that (f_n) converges pointwise to f. Continuity of $\|\cdot\|_o$ then implies

$$\mathcal{Q}(f_n) = \|f_n\|_o^2 - f_n(o)^2 \to \|f\|_o^2 - f(o)^2 = \mathcal{Q}(f).$$

This shows one implication.

Conversely, assume pointwise convergence of $f_n \to f$ and the bound given in (d). This then implies

$$\limsup_{n\to\infty} \|f_n\|_o^2 = \limsup_{n\to\infty} \left(\mathcal{Q}(f_n) + f_n(o)^2\right) \leq \mathcal{Q}(f) + f(o)^2 = \|f\|_o^2.$$

As the sequence (f_n) is bounded in the Hilbert space $(\mathcal{D}, \|\cdot\|_o)$, the sequence as well as any of its subsequences must have weakly converging subsequences. By pointwise convergence the limit of any weakly converging subsequence coincides with f. Hence, (f_n) converges weakly to f itself, i.e., $\langle f_n, g\rangle_o \to \langle f, g\rangle_o$ for all $g \in \mathcal{D}$. This gives in particular

$$\langle f_n, f\rangle_o \to \|f\|_o^2, \quad n \to \infty.$$

From the preceding convergence statement and

$$0 \leq \|f - f_n\|_o^2 = \|f\|_o^2 + \|f_n\|_o^2 - 2\langle f, f_n\rangle_o$$

we obtain

$$0 \leq \limsup_{n\to\infty} \|f - f_n\|_o^2 \leq 0.$$

This gives the desired convergence and finishes the proof. □

Remark 2.6. Let (b, c) be a connected graph over (X, m) and $o \in X$. The inner product $\langle\cdot,\cdot\rangle_o$ can be rewritten as $\langle\cdot,\cdot\rangle_o = \mathcal{Q}_{b,\tilde{c}}$ with $\tilde{c} := c + 1_o$. In this sense, up to a scaling the previous lemma can be understood as a statement about connected graphs with nonvanishing c. In particular, if $c \neq 0$, then $(\mathcal{D}, \mathcal{Q})$ is a Hilbert space.

The preceding results make $\mathcal{D}$ a Hilbert space, whenever the underlying graph is connected. In particular, in this case a sequence of functions (f_n) converges to f if and only if (f_n) converges pointwise to f and $\mathcal{Q}(f - f_n) \to 0$ holds. The latter can be used to define convergence of sequences in $\mathcal{D}$ in the general case even if the graph is not connected. We will be interested in the 'closure' of $C_c(X)$ with respect to this notion of convergence.

Definition 2.7. Let (b, c) be a graph over X. We define $\mathcal{D}_0(X)$ to be the subspace of $\mathcal{D}$ consisting of those $f \in \mathcal{D}$ for which there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ pointwise and $\mathcal{Q}(f - \varphi_n) \to 0$ as $n \to \infty$.

Corollary 2.8. *Let (b, c) be a connected graph over X and $o \in X$. Then,*

$$\mathcal{D}_0(X) = \overline{C_c(X)}^{\|\cdot\|_o}$$

holds.

Proof. By (d) of Lemma 2.5, convergence of (φ_n) to f in $\mathcal{D}$ implies pointwise convergence of (φ_n) to f. Given this the equality is rather immediate. □

2.3. The Laplacian

Besides the energy form $\mathcal{Q}$ associated to a graph we will also consider the *formal Laplacian.* Details are discussed in this section.

Let m be a measure on X of full support and let (b, c) be a graph over X. We let $\mathcal{L} = \mathcal{L}_{b,c,m}$ be the operator acting on

$$\mathcal{F} = \mathcal{F}_b := \{f \in C(X) \mid \sum_{y \in X} b(x, y)|f(y)| < \infty \text{ for all } x \in X\}$$

by

$$\mathcal{L}f(x) := \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x).$$

We call $\mathcal{L}$ the *formal Laplacian* associated to (b, c) over (X, m). The word 'formal' appears as this is not an operator in an ℓ^2 space.

We note that the formal Laplacian $\mathcal{L}$ depends on b as well as c and m while the domain $\mathcal{F}$ depends only on b.

The operator $\mathcal{L}$ has a certain symmetry property and the form $\mathcal{Q}$ and operator $\mathcal{L}$ are related by an integration by parts formula which we refer to as Green's formula. This is the content of the next proposition.

PROPOSITION 2.9 (Green's formula). *Let (b, c) be a graph over (X, m).*

(a) *Every $\varphi \in C_c(X)$ belongs to $\mathcal{F}$ and for all $f \in \mathcal{F}$ and $\varphi \in C_c(X)$*

$$\sum_{x \in X} \varphi(x)\mathcal{L}f(x)m(x) = \sum_{x \in X} \mathcal{L}\varphi(x)f(x)m(x)$$
$$= \frac{1}{2} \sum_{x,y \in X} b(x, y)(\varphi(x) - \varphi(y))(f(x) - f(y)) + \sum_{x \in X} c(x)\varphi(x)f(x)$$

holds, where all of the sums are absolutely convergent.

(b) *We have*

$$\mathcal{D} \subseteq \mathcal{F}$$

and thus for all $f \in \mathcal{D}$ and $\varphi \in C_c(X)$

$$\mathcal{Q}(\varphi, f) = \sum_{x \in X} \varphi(x)\mathcal{L}f(x)m(x) = \sum_{x \in X} \mathcal{L}\varphi(x)f(x)m(x).$$

PROOF. (a) By the assumptions on f, φ and b we have

$$\sum_{x,y \in X} |b(x, y)f(y)\varphi(x)| = \sum_{x \in X} |\varphi(x)| \sum_{y \in X} b(x, y)|f(y)| < \infty$$

and

$$\sum_{x,y \in X} |b(x, y)f(x)\varphi(x)| = \sum_{x \in X} |f(x)\varphi(x)| \sum_{y \in X} b(x, y) < \infty.$$

Given this finiteness, the desired equalities follow easily by direct computations.

(b) Given (a), it suffices to show that every $f \in \mathcal{D}$ belongs to $\mathcal{F}$. To see this, we calculate

$$\sum_{y \in X} b(x, y)|f(y)| \leq \sum_{y \in X} b(x, y)|f(x) - f(y)| + \sum_{y \in X} b(x, y)|f(x)|.$$

Now, the first term can be seen to be finite via the Cauchy–Schwarz inequality as

$$\left(\sum_{y\in X} b(x,y)\right)^{1/2}\left(\sum_{y\in X} b(x,y)(f(x)-f(y))^2\right)^{1/2} \leq \deg(x)^{1/2}\mathcal{Q}(f)^{1/2}$$

and the second term is finite by the assumption on b. This gives the desired statement. $\square$

For $\alpha \in \mathbb{R}$ we say that a function u is *α-subharmonic* if $u \in \mathcal{F}$ and

$$(\mathcal{L}+\alpha)u \leq 0.$$

We say that u is *α-superharmonic* if $-u$ is α-subharmonic. We say that u is *α-harmonic* if u is both α-subharmonic and α-superharmonic, i.e., $u \in \mathcal{F}$ satisfies

$$(\mathcal{L}+\alpha)u = 0.$$

When $\alpha = 0$, we say that u is *(sub/super)harmonic*. We will see that various features of such functions are intimately related to the geometric, spectral and stochastic properties of graphs.

We next present three basic results concerning solutions of the equation

$$(\mathcal{L}+\alpha)u = f$$

which will be used in various later considerations. We refer to this equation as the *Poisson equation*.

As above, we will use the notation $u \wedge v := \min\{u,v\}$ and $u \vee v := \max\{u,v\}$ for the minimum and maximum of two functions u and v, respectively.

We start with a minimum principle for certain supersolutions of the Poisson equation.

THEOREM 2.10 (Minimum principle). *Let (b,c) be a graph over (X,m). Let $U \subseteq X$. Assume that the function $u \in \mathcal{F}$ satisfies*

- $(\mathcal{L}+\alpha)u \geq 0$ *on U for some $\alpha \geq 0$*
- $u \wedge 0$ *attains a minimum on every connected component of U*
- $u \geq 0$ *on $X \setminus U$.*

If $\alpha > 0$ or if every connected component of U is connected to $X \setminus U$, then $u \geq 0$ and, in fact, on each connected component of U either $u = 0$ or $u > 0$.

PROOF. Without loss of generality we can assume that U is connected. If $u > 0$ there is nothing to show. Therefore, assume there exists a vertex $x \in U$ with $u(x) \leq 0$. As $u \wedge 0$ attains a minimum on U, there exists a vertex $x_0 \in U$ with $u(x_0) \leq 0$ and $u(x_0) \leq u(y)$ for all $y \in U$. As $u(y) \geq 0$ for $y \in X \setminus U$, we obtain $u(x_0) - u(y) \leq 0$ for all $y \in X$. By the supersolution assumption we then find

$$\begin{aligned}0 &\leq (\mathcal{L}+\alpha)u(x_0)\\ &= \frac{1}{m(x_0)}\left(\sum_{y\in X} b(x_0,y)(u(x_0)-u(y)) + c(x_0)u(x_0)\right) + \alpha u(x_0) \leq 0.\end{aligned}$$

Therefore, if $\alpha > 0$, then $0 = u(x_0)$ and $u(y) = u(x_0) = 0$ for all $y \sim x_0$. As U is connected, iteration of this argument shows that $u = 0$ on U.

On the other hand, for $\alpha = 0$, we obtain by the same argument that u is constant on U. As U is connected to $X \setminus U$, namely there exist $x \in U$ and $z \in X \setminus U$ such that $x \sim z$, we conclude from the formula above for x that

$$0 = \frac{1}{m(x)} b(x,z)(u(x) - u(z)) + \frac{1}{m(x)} \left(\sum_{y \neq z} b(x,y)(u(x) - u(y)) + c(x)u(x) \right).$$

Since $u(x) = u(x_0)$, the second term is clearly smaller or equal to 0. Hence, the first term must be greater or equal to zero. This implies $0 \geq u(x_0) = u(x) \geq u(z) \geq 0$ and we conclude $u = 0$ on U. $\square$

For the following lemma, given a sequence of functions (u_n) and a function u we write

$$u_n \nearrow u \quad \text{as} \quad n \to \infty$$

if $u_n(x) \leq u_{n+1}(x)$ for all $x \in X$, $n \in \mathbb{N}$ and $u_n \to u$ pointwise as $n \to \infty$.

Lemma 2.11 (Monotone convergence of solutions). *Let (b, c) be a graph over (X, m). Let $\alpha \in \mathbb{R}$ and let $u, f \in C(X)$. Let (u_n) be a sequence of functions in $\mathcal{F}$ with $u_n \geq 0$ for all $n \in \mathbb{N}$. Assume that $u_n \nearrow u$ and $(\mathcal{L} + \alpha)u_n(x) \to f(x)$ for all $x \in X$ as $n \to \infty$. Then, $u \in \mathcal{F}$ and*

$$(\mathcal{L} + \alpha)u = f.$$

Proof. Without loss of generality, we assume that $m = 1$. By assumption

$$\sum_{y \in X} b(x,y)(u_n(x) - u_n(y)) + (c(x) + \alpha)u_n(x) = (\mathcal{L} + \alpha)u_n(x) \to f(x)$$

as $n \to \infty$ for any $x \in X$. As $\left(\sum_{y \in X} b(x,y)u_n(x)\right)_{n \in \mathbb{N}}$ converges increasingly to $u(x) \sum_{y \in X} b(x,y) < \infty$, the assumptions on (u_n) show that $\left(\sum_{y \in X} b(x,y)u_n(y)\right)_{n \in \mathbb{N}}$ must converge as well and, in fact, must converge to $\sum_{y \in X} b(x,y)u(y)$ by the monotone convergence theorem. From this, we easily obtain the conclusion. $\square$

We let

$$u_+ := u \vee 0 \qquad \text{and} \qquad u_- := -u \vee 0$$

denote the positive and negative part of u so that $u = u_+ - u_-$ and $|u| = u_+ + u_-$. The next lemma then shows that the positive and negative part of an α-harmonic function are α-subharmonic.

Lemma 2.12 (α-subharmonic and α-superharmonic functions). *Let (b, c) be a graph over (X, m). Let $\alpha \in \mathbb{R}$. If $u, v \in \mathcal{F}$ are α-subharmonic (α-superharmonic, respectively), then $u \vee v$ is α-subharmonic ($u \wedge v$ is α-superharmonic, respectively). In particular, if u is α-harmonic, then u_+, u_- and $|u|$ are all α-subharmonic.*

Proof. Let u, v be α-subharmonic for some $\alpha \in \mathbb{R}$ and let $w = u \vee v$. Let $x \in X$ and assume without loss of generality that $w(x) = u(x) \geq v(x)$.

Then,

$$w(x)-w(y)=\begin{cases}u(x)-u(y) & \text{if } u(y)\geq v(y)\\ u(x)-v(y) & \text{else}\end{cases}$$
$$\leq u(x)-u(y).$$

Thus,

$$(\mathcal{L}+\alpha)w(x)\leq(\mathcal{L}+\alpha)u(x)\leq 0$$

holds. As $x\in X$ was arbitrary, we infer that w is α-subharmonic.

Now, let u, v be α-superharmonic. We first observe that $u\wedge v=-((-u)\vee(-v))$. Hence, by what we have shown above, $(-u)\vee(-v)$ is α-subharmonic as $-u$ and $-v$ are α-subharmonic. Therefore, $u\wedge v$ is α-superharmonic. The "in particular" statement follows as $u_\pm=(\pm u)\vee 0$ and $|u|=u_++u_-$. ☐

We now introduce the *heat equation*

$$(\mathcal{L}+\partial_t)u=0.$$

More specifically, a function $u\colon[0,\infty)\times X\longrightarrow\mathbb{R}$ is called a *solution of the heat equation* if, for every $x\in X$, the mapping $t\mapsto u_t(x)$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, $u_t\in\mathcal{F}$ for all $t>0$ and

$$(\mathcal{L}+\partial_t)u_t(x)=0$$

for all $x\in X$ and $t>0$. If u has all of the properties above but instead of equality in the heat equation satisfies $(\mathcal{L}+\partial_t)u_t\geq 0$ for all $t>0$, then we call u a *supersolution* of the heat equation. If u is a solution of the heat equation and $u_0=f$ for $f\in C(X)$, then f is called the *initial condition* for u. We will say that u satisfies the heat equation with initial condition f in this case. We think of x as a space variable and t as time.

We now prove a minimum principle for the heat equation. In particular, for supersolutions of the heat equation on certain subsets, positivity on the boundary propagates to positivity on the subset. This will be used later to establish the minimality of certain solutions.

THEOREM 2.13 (Minimum principle for the heat equation). *Let (b,c) be a graph over (X,m). Let $U\subseteq X$ be a connected subset and suppose that U contains a vertex which is connected to a vertex outside of U. Let $T\geq 0$ and let $u\colon[0,T]\times X\longrightarrow\mathbb{R}$ be such that $t\mapsto u_t(x)$ is continuously differentiable on $(0,T]$ for every $x\in U$ and $u_t\in\mathcal{F}$ for all $t\in(0,T]$. Assume u satisfies*

- *$(\mathcal{L}+\partial_t)u_t\geq 0$ on U for $t\in(0,T]$,*
- *$u\wedge 0$ attains a minimum on $[0,T]\times U$,*
- *$u\geq 0$ on $((0,T]\times(X\setminus U))\cup(\{0\}\times U)$.*

Then, $u\geq 0$ on $[0,T]\times U$.

PROOF. By definition we have $u\wedge 0\leq 0$. Let (t,x) be a point where $u\wedge 0$ attains a minimum on $[0,T]\times U$. If $u_t(x)\geq 0$, the conclusion follows so we assume $u_t(x)<0$. Since u is positive on $\{0\}\times U$ we have $t>0$. Furthermore, since u attains a minimum at (t,x) with respect to t we obtain

$$\partial_t u_t(x)=0 \text{ if } t<T \text{ and } \partial_t u_t(x)\leq 0 \text{ if } t=T.$$

Since u also attains a negative minimum at (t, x) with respect to x, we infer from a direct computation

$$\mathcal{L}u_t(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y) \underbrace{(u_t(x) - u_t(y))}_{\leq 0} + \frac{c(x)}{m(x)} \underbrace{u_t(x)}_{<0} \leq 0.$$

Put together this gives $(\mathcal{L} + \partial_t)\, u_t(x) \leq 0$. As u also satisfies $(\mathcal{L} + \partial_t) u_t \geq 0$ by assumption, we obtain

$$(\mathcal{L} + \partial_t)\, u_t(x) = 0$$

and hence $\mathcal{L}u_t(x) \geq 0$ and therefore $\mathcal{L}u_t(x) = 0$. Looking at the formula for $\mathcal{L}u_t(x)$ again, we find

$$u_t(y) = u_t(x) < 0$$

for all $y \sim x$. Iterating this argument and using the assumption that U is connected we find that u_t is a negative constant on U. At the vertex $z \in U$ which has a neighbor not in U, the equation $\mathcal{L}u_t(z) = 0$ then contradicts the assumption $u \geq 0$ on $(0, T] \times (X \setminus U)$. □

2.4. Boundedness of Forms and Operators

We have associated to each graph a form and an operator. Now, in general form and operator will not be bounded and this will pose a major challenge right at the beginning of the investigation. We will deal with this challenge by using the theory of closed forms and unbounded self-adjoint operators. Basic elements of this theory will be discussed in the next chapter and this will allow us to proceed in the investigation of the general case. There is, however, a situation which can already be investigated right now without the theory of closed forms and unbounded operators. This is the situation that form and operator are bounded. Details will be discussed in this section.

For a graph (b, c) over (X, m) we define the *weighted degree*

$$\mathrm{Deg}\colon X \longrightarrow [0, \infty)$$

by

$$\mathrm{Deg}(x) := \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y) + c(x) \right)$$

We will characterize boundedness of the operator and the form in terms of the weighted degree. We first investigate the issue of whether the space of functions of compact support $C_c(X)$ is mapped into $\ell^2(X, m)$ by $\mathcal{L}$. We start by examples.

EXAMPLE 2.14 ($\mathcal{L}C_c(X)$ in $\ell^2(X, m)$). Let (b, c) be a locally finite graph over (X, m). Then,

$$\mathcal{L}C_c(X) \subseteq C_c(X) \subseteq \ell^2(X, m).$$

Indeed, for any $x \in X$, the function $\mathcal{L}1_x$ vanishes outside of $\{y \mid y \sim x\} \cup \{x\}$ and this set is finite by the local finiteness assumption. Hence, $\mathcal{L}1_x$ belongs to $C_c(X)$ for any $x \in X$. As the 1_x, $x \in X$, form a basis of $C_c(X)$ the inclusion $\mathcal{L}C_c(X) \subseteq C_c(X)$ follows and the inclusion $C_c(X) \subseteq \ell^2(X,m)$ is clear anyway.

Example 2.15 ($\mathcal{L}C_c(X)$ not contained in $\ell^2(X,m)$). We consider a star shaped graph. Specifically, let $X := \mathbb{N}_0$ and $b(0,k) = b(k,0) := k^{-2}$ for $k \geq 1$ and $b(k,l) := 0$ for $k,l \geq 1$ with $c := 0$. Furthermore, let m be given by $m(k) := k^{-3}$ for $k \geq 1$ with $m(0) := 1$. Then,

$$\sum_{k=0}^{\infty} (\mathcal{L}1_0(k))^2 m(k) \geq \sum_{k=1}^{\infty} \frac{b^2(0,k)}{m(k)} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Thus, $\mathcal{L}1_0$ does not belong to $\ell^2(X,m)$. Note that the set X has finite measure. So, finiteness of the measure is not relevant in this context.

We can now characterize when $C_c(X)$ is mapped into $\ell^2(X,m)$.

Theorem 2.16. *Let (b,c) be a graph over (X,m). Then, the following statements are equivalent:*

- (i) $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$.
- (ii) *The functions $X \longrightarrow [0,\infty)$, $y \mapsto b(x,y)/m(y)$, belong to $\ell^2(X,m)$ for all $x \in X$.*
- (iii) $\ell^2(X,m) \subseteq \mathcal{F}$.

Furthermore, the equivalent conditions above are satisfied if

$$\inf_{y \sim x} m(y) > 0$$

for all $x \in X$ which holds, in particular, if the graph is locally finite.

Proof. For any $x \in X$ we define φ_x on X by $\varphi_x(y) := b(x,y)/m(y)$. Clearly, φ_x vanishes outside of

$$N_x := \{y \in X \mid y \sim x\}.$$

(i) $\Longleftrightarrow$ (ii): Let $x \in X$ be arbitrary. We first observe

$$\mathcal{L}1_x = \operatorname{Deg}(x)1_x - \varphi_x.$$

Thus, $\mathcal{L}1_x$ belongs to $\ell^2(X,m)$ if and only if $\varphi_x \in \ell^2(X,m)$. As $x \in X$ is arbitrary this gives the desired equivalence.

(ii) $\Longleftrightarrow$ (iii): Assume that φ_x belongs to $\ell^2(X,m)$ for all $x \in X$. Then, for $f \in \ell^2(X,m)$, we get by the Cauchy–Schwarz inequality

$$\sum_{y \in X} b(x,y)|f(y)| = \sum_{y \in X} \varphi_x(y)|f(y)|m(y) \leq \|\varphi_x\|\|f\|.$$

Hence, f belongs to $\mathcal{F}$.

Conversely, assume $\ell^2(X,m) \subseteq \mathcal{F}$. Let $x \in X$. Then, $\ell^1(N_x, b(x,\cdot)) \supseteq \{f \in \mathcal{F} \mid \operatorname{supp} f \subseteq N_x\}$ (via restrictions) and we find

$$\ell^2(N_x, m_{N_x}) \subseteq \ell^1(N_x, b(x,\cdot)).$$

Thus, the map

$$j \colon \ell^2(N_x, m_{N_x}) \longrightarrow \ell^1(N_x, b(x,\cdot)), \quad f \mapsto f$$

is well-defined and linear. As convergence in an ℓ^p-space always implies pointwise convergence, we easily infer that the map j is closed (i.e. $f_n \to f$

and $j(f_n) \to g$ implies $j(f) = g$). Hence, by the closed graph theorem we infer the existence of a constant $C \geq 0$ such that for all $f \in \ell^2(X,m)$

$$\sum_{y\in X} b(x,y)|f(y)| = \|f 1_{N_x}\|_{\ell^1(N_x, b(x,\cdot))} \leq C\|f 1_{N_x}\| \leq C\|f\|.$$

Therefore,

$$\sum_{y\in X} \varphi_x(y)|f(y)|m(y) = \sum_{y\in X} b(x,y)|f(y)| \leq C\|f\|.$$

Hence, $\varphi_x \in \ell^2(X,m)$ by the Riesz representation theorem.

Finally, the condition $\inf_{y\sim x} m(y) = C_x > 0$ implies that $\varphi_x \in \ell^2(X,m)$ for $x \in X$, since

$$\|\varphi_x\|^2 = \sum_{y\in X} \frac{b^2(x,y)}{m(y)} \leq \frac{1}{C_x} \sum_{y\in X} b^2(x,y) < \infty.$$

This shows the "in particular" statement. □

We now turn to characterizing boundedness of $\mathcal{L}$ and $\mathcal{Q}$ on $\ell^2(X,m)$. The crucial feature will be a bound on the Deg. We start with a general lemma.

Lemma 2.17 (The bound lemma). *Let X be a countable set and m a measure on X of full support. Let $a\colon X \times X \longrightarrow [0,\infty)$ be symmetric and assume that for some $\kappa > 0$ we have*

$$\sum_{y\in X} a(x,y)m(y) \leq \kappa$$

for all $x \in X$. Then:

(a) *The map $A\colon \ell^2(X,m) \longrightarrow \ell^2(X,m)$ given by*

$$Af(x) := \sum_{y\in X} a(x,y)f(y)m(y)$$

for $f \in \ell^2(X,m)$ and $x \in X$ is well-defined and linear with $\|Af\| \leq \kappa\|f\|$.

(b) *The map $q\colon \ell^2(X,m) \times \ell^2(X,m) \longrightarrow \mathbb{R}$ with*

$$q(f,g) = \frac{1}{2}\sum_{x,y\in X} a(x,y)(f(x)-f(y))(g(x)-g(y))m(x)m(y) + \sum_{x\in X} a(x,x)f(x)g(x)m(x)m(x).$$

is well-defined and bilinear with $|q(f,g)| \leq 2\kappa\|f\|\|g\|$ for all $f,g \in \ell^2(X,m)$.

Proof. The proofs of (a) and (b) is very similar.

(a) We have to show that $\sum_{y\in X} a(x,y)f(y)m(y)$ is defined for all $x \in X$ and Af belongs to $\ell^2(X,m)$ with the given bound. It suffices to show

$$\sum_{x\in X}\left(\sum_{y\in X} a(x,y)|f(y)|m(y)\right)^2 m(x) \leq \kappa^2\|f\|^2.$$

This in turn follows by a direct computation using the Cauchy–Schwarz inequality (CSI):

$$\begin{aligned}
&\sum_{x\in X}\left(\sum_{y\in X}\underbrace{a(x,y)|f(y)|m(y)}_{=a(x,\cdot)^{1/2}m^{1/2}\cdot a(x,\cdot)^{1/2}fm^{1/2}}\right)^2 m(x)\\
&\overset{\text{CSI}}{\leq}\sum_{x\in X}\left(\underbrace{\sum_{z\in X}a(x,z)m(z)}_{\leq\kappa}\right)\left(\sum_{z\in X}a(x,z)f(z)^2m(z)\right)m(x)\\
&\leq\kappa\sum_{x\in X}\left(\sum_{z\in X}a(x,z)f(z)^2m(z)\right)m(x)\\
&\overset{\text{Fubini}}{=}\kappa\sum_{z\in X}f(z)^2m(z)\left(\underbrace{\sum_{x\in X}a(x,z)m(x)}_{\leq\kappa}\right)\\
&\leq\kappa^2\sum_{z\in X}f(z)^2m(z)=\kappa^2\|f\|^2.
\end{aligned}$$

Here, we used that a is symmetric to obtain

$$\sum_{x\in X}a(x,z)m(x)=\sum_{x\in X}a(z,x)m(x)\leq\kappa.$$

(b) We first consider the case $f=g$. Clearly, $(f(x)-f(y))^2\leq 2f(x)^2+2f(y)^2$ holds. Now, a direct computation invoking Fubini's theorem and the symmetry of a gives

$$\begin{aligned}
&\frac{1}{2}\sum_{x,y\in X,\,x\neq y}a(x,y)(2f(x)^2+2f(y)^2)m(x)m(y)+\sum_{x\in X}a(x,x)f(x)^2m(x)m(x)\\
&\leq\frac{1}{2}\sum_{x,y\in X,\,x\neq y}a(x,y)(2f(x)^2+2f(y)^2)m(x)m(y)\\
&\quad+\sum_{x\in X}a(x,x)f(x)^2m(x)m(x)+\sum_{y\in X}a(y,y)f(y)^2m(y)m(y)\\
&=\sum_{x\in X}f(x)^2m(x)\sum_{y\in X}a(x,y)m(y)+\sum_{y\in X}f(y)^2m(y)\sum_{x\in X}a(x,y)m(x)\\
&\leq\kappa\|f\|^2+\kappa\|f\|^2=2\kappa\|f\|^2.
\end{aligned}$$

This easily gives the claim for $f=g$. The general case now follows from $q(f,f)\geq 0$ and Cauchy–Schwarz inequality. □

THEOREM 2.18 (Characterization of boundedness). *Let (b,c) be a graph over (X,m). Then, the following statements are equivalent:*

(i) *The weighted degree* Deg *is a bounded function on X.*

(ii) *The form* $\mathcal{Q}$ *is bounded on* $\ell^2(X,m)$ *(i.e.* $\ell^2(X,m)$ *belongs to* $\mathcal{D}$ *and there exists a* $C \geq 0$ *with* $|\mathcal{Q}(f,g)| \leq C\|f\|\|g\|$ *for all* $f,g \in \ell^2(X,m)$*).*

(iii) *The operator* $\mathcal{L}$ *is bounded on* $\ell^2(X,m)$ *(i.e.* $\ell^2(X,m)$ *belongs to* $\mathcal{F}$ *and there exists* $D \geq 0$ *such that* $\|\mathcal{L}f\| \leq D\|f\|$ *for any* $f \in \ell^2(X,m)$*).*

Moreover, if Deg *is bounded by* D*, then*

$$|\mathcal{Q}(f,g)| \leq 2D\|f\|\|g\|$$

holds as well as

$$\|\mathcal{L}f\| \leq 2D\|f\|$$

for all $f,g \in \ell^2(X,m)$*.*

Proof. The implication (i) $\Longrightarrow$ (ii) follows from (b) of the preceding lemma with $a(x,y) := \frac{b(x,y)}{m(x)m(y)}$ for $x,y \in X$ with $x \neq y$ and $a(x,x) := \frac{c(x)}{m(x)m(x)}$ for $x \in X$. Moreover, the implication (i)$\Longrightarrow$(iii) follows from (a) of the preceding lemma with $a(x,y) := \frac{b(x,y)}{m(x)m(y)}$ for $x,y \in X$ with $x \neq y$ and $a(x,x) := 0$ for $x \in X$, since then $\mathcal{L}f(x) = \mathrm{Deg}(x)f(x) - Af(x)$ for all $f \in \ell^2(X,m)$ and $x \in X$. Indeed, the assumption of that lemma is satisfied with $\kappa := \sup_{x\in X} \mathrm{Deg}(x)$. The conclusion of the lemma then gives the bound $C = 2\kappa$ for the estimate of $\mathcal{Q}$ and the bound $D = 2\kappa$ for the estimate of $\mathcal{L}$.

(ii)$\Longrightarrow$(i): Assume $\mathcal{Q}(f) \leq C\|f\|^2$ for all $f \in \ell^2(X,m)$. Choosing $f := 1_x$ for $x \in X$ we find

$$\sum_{y\in X} b(x,y) + c(x) = \mathcal{Q}(1_x) \leq C\|1_x\|^2 = Cm(x).$$

This implies Deg $\leq C$.

(iii)$\Longrightarrow$(i): Assume that $\|\mathcal{L}f\| \leq D\|f\|$ holds for all $f \in \ell^2(X,m)$. Choosing $f := 1_x$ for $x \in X$ we find

$$\sum_{y\in X} b(x,y) + c(x) = m(x)\mathcal{L}1_x(x) = \langle 1_x, \mathcal{L}1_x\rangle \leq D\|1_x\|\|1_x\| = Dm(x)$$

for all $x \in X$. This implies Deg $\leq D$.

The last statement of the theorem has been proven along the way in (i)$\Longrightarrow$(ii) and (i)$\Longrightarrow$(iii) respectively. □

Remark 2.19 (Relationship between the theorems in this section). The boundedness of $\mathcal{L}$ obviously implies that $\mathcal{L}$ maps $C_c(X)$ in $\ell^2(X,m)$. Thus, the condition that Deg is bounded (appearing in Theorem 2.18) is stronger than the condition $\sum_{y\in X} \frac{b(x,y)^2}{m(y)} < \infty$ for all $x \in X$ (appearing in Theorem 2.16). Indeed, this can already be seen directly as boundedness of the degree gives

$$\sum_{z\in X} (b(y,z) + c(z)) \leq Cm(y)$$

which in turn implies

$$\sum_{y\in X}\frac{b(x,y)^2}{m(y)}\leq C\sum_{y\in X}\frac{b(x,y)^2}{\sum_z(b(y,z)+c(z))}\leq C\sum_{y\in X}b(x,y)<\infty.$$

2.5. Graphs with Standard Weights

Special b and c with rather restricted range have attracted particular attention in the literature. We present these cases here under the heading of standard weights.

Definition 2.20 (Graphs with standard weights). Let (b,c) be a graph over X. If b takes values in $\{0,1\}$ and $c=0$, we say that b is a graph with *standard weights.*

We denote the edges of the graph by

$$E=\{(x,y)\in X\times X\mid x\sim y\}.$$

For graphs with standard weights, the degree function deg given by $\deg(x):=\sum_{y\in X}b(x,y)$ is the *combinatorial degree*, i.e., if $x\in X$, then

$$\deg(x)=\#\{y\in X\mid x\sim y\}=\#\left(E\cap(\{x\}\times X)\right).$$

The assumption $\sum_{y\in X}b(x,y)<\infty$ for all $x\in X$ clearly implies that graphs with standard weights are locally finite.

We now explicitly write out the energy form and the Laplacian in the case of standard weights.

For a graph b with standard weights, the energy form $\mathcal{Q}$ is given by

$$\mathcal{Q}(f)=\frac{1}{2}\sum_{x,y\in X,\,x\sim y}(f(x)-f(y))^2$$

for $f\in C(X)$. Furthermore, by local finiteness, the domain $\mathcal{F}$ of the formal Laplacian consists of all functions, i.e.,

$$\mathcal{F}=C(X).$$

2.5.1. The Counting Measure.

The counting measure $m=1$ counts the number of vertices in a subset of X. In this case, the degree and the weighted degree satisfy

$$\deg=\mathrm{Deg}$$

and are equal to the combinatorial degree.

We denote the formal Laplacian $\mathcal{L}$ for graphs b with standard weights by Δ. This operator acts as

$$\Delta f(x)=\sum_{y\in X,\,y\sim x}(f(x)-f(y)).$$

We deduce the following corollaries from the results of the previous sections.

Corollary 2.21 (Characterization of boundedness). *Let b be a graph with standard weights and let $m=1$ be the counting measure. Then, the following statements are equivalent:*

(i) *The combinatorial degree* deg *is a bounded function on* X.
(ii) *The form* $\mathcal{Q}$ *is bounded on* $\ell^2(X)$.
(iii) *The operator* Δ *is bounded on* $\ell^2(X)$.

PROOF. This follows directly from Theorem 2.18 and the equality of the combinatorial and weighted degrees, deg = Deg, in this case. □

Furthermore, as a graph with standard weights is always locally finite we have

$$\Delta C_c(X) \subseteq C_c(X) \subseteq \ell^2(X)$$

2.5.2. The Normalizing Measure. We now introduce the normalizing measure and discuss how the resulting Laplacian is always bounded.

The normalizing measure n is given by deg which is the combinatorial degree in the case of standard weights. This measure counts the number of edges for a subset of vertices, more specifically,

$$n(A) = \#E_A + \frac{1}{2}\#\partial_E A$$

for $A \subseteq X$, where $E_A := E \cap (A \times A)$ and

$$\partial_E A := E \cap (((X \setminus A) \times A) \cup (A \times (X \setminus A)))$$

(Exercise). Letting $m := n$, the weighted degree Deg satisfies

$$\mathrm{Deg} = 1.$$

For the normalizing measure $n = \deg$, we denote the Laplacian by Δ_n refered to as the *normalized Laplacian.* We have

$$\Delta_n f(x) = \frac{1}{\deg(x)} \sum_{y \in X,\, y \sim x} (f(x) - f(y)).$$

COROLLARY 2.22 (Δ_n is bounded). *Let* b *be a graph with standard weights and let* $n = \deg$ *be the normalizing measure. Then, the normalized Laplacian* Δ_n *is a bounded operator on* $\ell^2(X, n)$. *In particular,* $C_c(X) \subseteq D(\Delta_n) = \ell^2(X, n)$.

PROOF. This follows directly from Theorem 2.18 and the equality Deg = 1 in this case. □

CHAPTER 3

Toolbox – The Spectral Theorem and Closed Forms

The basic objects of our study are graphs and the associated operators, forms, resolvents and semigroups. In the context of the general infinite graphs that we consider in this course the operators and forms are generally unbounded. To deal with this unboundedness requires some care. The necessary background for this careful dealing is provided by the spectral theorem and its consequences and the theory of closed forms. This is discussed in this chapter.

Throughout this chapter, we let H denote a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We consider complex Hilbert spaces and assume that the inner product is linear in the second argument. Real spaces can be complexified. Thus, all of our results below apply to the real case as well.

We will always assume that the Hilbert space is separable (i.e. contains a countable dense set). This is no restriction for us as the application we have in mind is the Hilbert space $\ell^2(X,m)$ with a countable set X and this Hilbert space is separable. The assumption of separability will allow us to restrict attention to measure spaces which are σ-finite (as separable Hilbert spaces are unitarily equivalent to $L^2(\Omega,\mu)$ with a σ-finite μ).

3.1. Basics on Operator Theory

In this section we present basic theory of unbounded operators. A key point to be taken care of is that such operators will not be defined on the entire Hilbert space but rather on a subspace.

An *operator* on H is a linear map

$$A\colon D(A) \longrightarrow H,$$

where $D(A)$ is a subspace of H which we call the *domain* of A. We say that A is *densely defined* if $D(A)$ is dense in H. We call an operator A *closed* if its graph $\{(f,Af)\in H\times H \mid f\in D(A)\}$ is closed in $H\times H$; put differently, if $f_n\to f$ for (f_n) in $D(A)$ along with $Af_n\to g$ imply $f\in D(A)$ and $Af=g$. We say that an operator A is *bounded* if there exists a constant $C\geq 0$ such that $\|Af\|\leq C\|f\|$ for all $f\in D(A)$. In this case, $\|A\|$, the *norm* of A, is the smallest such constant C.

If A is densely defined and bounded, then A can be uniquely extended to a bounded operator on the entire Hilbert space H and we denote this extension by A as well. We note that a bounded operator defined on the entire space is always closed. We denote the space of bounded operators defined on the entire Hilbert space H by $B(H)$.

For operators A and B on H we define the sum $A+B$ to be the linear map whose domain is

$$D(A+B) := D(A) \cap D(B)$$

and which acts by $(A+B)f := Af + Bf$.

We will also consider operators between different Hilbert spaces H_1 and H_2. In this case, the above definitions hold with the obvious modifications. In particular, an operator A from H_1 to H_2 is a linear map from a subspace $D(A)$ of H_1 into H_2. A most relevant instance is the product AB of operators B from H_1 into H_2 and A from H_2 into H_3. This product is defined on

$$D(AB) := \{f \in D(B) \mid Bf \in D(A)\}$$

and acts by $ABf := A(Bf)$.

Whenever A is an operator on H and $z \in \mathbb{C}$, we write $(A-z)$ for the operator $A - zI$ on $D(A)$, where I denotes the identity operator on H. We define the *resolvent set of* A to be

$$\varrho(A) := \{z \in \mathbb{C} \mid (A-z) \text{ is bijective and } (A-z)^{-1} \text{ is bounded}\}$$

and the *spectrum* of A as

$$\sigma(A) := \mathbb{C} \setminus \varrho(A).$$

We recall the standard fact that $\sigma(A)$ is always a closed set. For $z \in \varrho(A)$, we call the operator $(A-z)^{-1}$ the *resolvent of* A *at* z. For an operator A that is not closed, we have

$$\varrho(A) = \emptyset.$$

Indeed, if for some z the operator $(A-z)$ were bijective and $(A-z)^{-1}$ bounded, then $(A-z)^{-1}$ were bounded on the entire Hilbert space and, therefore, closed. But then $(A-z)$ and, hence, A would also be closed. This shows that the notion of a resolvent set is only relevant for closed operators.

On the other hand, for a closed operator A the definition of the resolvent set can be simplified to

$$\varrho(A) = \{z \in \mathbb{C} \mid (A-z) \text{ is bijective}\}.$$

This follows since if A is closed and $A-z$ is bijective, then $(A-z)^{-1}$ is bounded by the closed graph theorem.

An operator A is called *invertible* if $A\colon D(A) \longrightarrow H$ is bijective. If A and B are invertible operators and $D(B) \subseteq D(A)$, then

$$A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1},$$

as follows by a direct calculation. In particular, if $z_1, z_2 \in \varrho(A)$, then

$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1}.$$

We refer to these formulae as *resolvent identities*. As a particular consequence, we note that the second formula implies that resolvents commute. Moreover, the second formula implies that the resolvent map

$$\varrho(A) \longrightarrow B(H), \quad z \mapsto (A-z)^{-1},$$

is locally bounded and continuous, i.e., for a sequence (z_n) in $\varrho(A)$ with $z_n \to z \in \varrho(A)$ we have

$$\lim_{n\to\infty} \|(A-z_n)^{-1} - (A-z)^{-1}\| = 0.$$

If A is densely defined, then we define the *adjoint* A^* of A to be the operator with domain

$$D(A^*) := \left\{ f \in H \;\middle|\; \begin{array}{c} \text{there exists a } g \in H \text{ with } \langle f, Ah\rangle = \langle g, h\rangle \\ \text{for all } h \in D(A) \end{array} \right\}$$

acting as

$$A^* f := g,$$

where we note that this is well-defined. Specifically, we have

$$\langle Af, g\rangle = \langle f, A^* g\rangle$$

for all $f \in D(A)$ and $g \in D(A^*)$ and A^* has the maximal domain among all operators with this property. The operator A^* is always closed (as can be easily seen).

We note that $D((A - z)^*) = D(A^*)$ and

$$(A - z)^* = A^* - \overline{z}$$

for all $z \in \mathbb{C}$. Furthermore, for $z \in \varrho(A)$,

$$((A - z)^{-1})^* = (A^* - \overline{z})^{-1}.$$

If A is densely defined, we say that A is *symmetric* if A^* is an extension of A, that is, $D(A) \subseteq D(A^*)$ and $Af = A^* f$ for all $f \in D(A)$. Equivalently, A is symmetric if and only if A is densely defined and

$$\langle Af, g\rangle = \langle f, Ag\rangle$$

for all $f, g \in D(A)$. With these preparations, we now define the class of operators of primary interest.

Definition 3.1 (Self-adjoint operators). We call a densely defined operator A *self-adjoint* if $A = A^*$.

Clearly, a self-adjoint operator is symmetric. Moreover, as the adjoint is always a closed operator, all self-adjoint operators are closed.

Before we develop the general theory further, we present a key example, namely multiplication operators. These operators are self-adjoint if the underlying function is real-valued. The main result on self-adjoint operators (to be discussed in Section 3.2) states a converse to this observation.

Example 3.2 (Multiplication operators). Let (X, μ) be a measure space and let $u\colon X \longrightarrow \mathbb{C}$ be measurable. The operator M_u of *multiplication by* u has domain

$$D(M_u) := \{f \in L^2(X, \mu) \mid uf \in L^2(X, \mu)\}$$

and acts as

$$M_u f := uf$$

for all $f \in D(M_u)$. As discussed below, the operator M_u is densely defined. Furthermore, it can readily be seen that M_u is closed. The adjoint of M_u is given by $(M_u)^* = M_{\overline{u}}$. In particular, M_u is self-adjoint if u is real-valued. Finally, M_u is bounded if u is bounded.

The proofs of these statements are rather straightforward. We only sketch how to show that the domain of M_u is dense. For $n \in \mathbb{N}$, we define

$$X_n := \{x \in X \mid |u(x)| \leq n\}.$$

Then, the characteristic functions 1_{X_n} for $n \in \mathbb{N}$ tend pointwise increasingly towards the constant function with value 1. In particular, we have $1_{X_n} f \to f$ for any $f \in L^2(X,\mu)$ by Lebesgue's dominated convergence theorem. On the other hand, by the definition of X_n, the function $1_{X_n} f$ belongs to $D(M_u)$ for any $n \in \mathbb{N}$.

The *essential range* of a measurable function $u\colon X \longrightarrow \mathbb{C}$ over a measure space (X,μ) is defined as

$$\operatorname{ess\,ran} u := \{\lambda \in \mathbb{C} \mid \mu(u^{-1}(B_\varepsilon(\lambda))) > 0 \text{ for all } \varepsilon > 0\},$$

where $B_\varepsilon(\lambda)$ is the closed ball around λ of radius ε.

Lemma 3.3 (Spectrum of multiplication operators). *Let (X,μ) be a σ-finite measure space. Let $u\colon X \longrightarrow \mathbb{C}$ be measurable and M_u be the operator of multiplication by u. Then, $\sigma(M_u)$ equals the essential range of u.*

Proof. For λ not in the essential range, the operator $M_{1/(u-\lambda)}$ is obviously a bounded inverse for $M_u - \lambda = M_{u-\lambda}$. Conversely, consider λ belonging to the essential range of u. Using the assumption of σ-finiteness, for any $\varepsilon > 0$, we can construct $f \in L^2(X,\mu)$ with $\|f\| = 1$ and $\|(M_u - \lambda)f\| < \varepsilon$. This contradicts the existence of a bounded inverse to $M_u - \lambda$. □

From the definitions it is not hard to derive the following additional properties of multiplication operators. They will be used repeatedly in what follows.

Proposition 3.4 (Further features of multiplication operators). *Let (X,μ) be a σ-finite measure space and let $u\colon X \longrightarrow \mathbb{C}$ be measurable. Then, the following statements hold:*

(a) *The operator M_u is self-adjoint if and only if* $\operatorname{ess\,ran} u \subseteq \mathbb{R}$, *which, in turn, holds if and only if u is real-valued almost everywhere.*

(b) *The operator M_u is bounded if and only if* $\operatorname{ess\,ran} u$ *is bounded, which, in turn, holds if and only if $u \in L^\infty(X,\mu)$. In this case,*

$$\|M_u\| = \|u\|_\infty = \sup\{|\lambda| \mid \lambda \textit{ is in the essential range of } u\}.$$

(c) *$M_u = 0$ holds if and only if* $\operatorname{ess\,ran} u = \{0\}$ *if and only if $u = 0$ holds almost everywhere.*

Remark 3.5. In all three statements the if-part does not require the assumption of σ-finiteness of the measure space. This assumption is only used to obtain the only-if statement and the formula for the norm of the operator. Similarly, the assumption is used in obtaining the spectrum as the essential range in the preceding lemma.

3.2. Spectral Theorem and Spectral Calculus

The fundamental result about self-adjoint operators is known as spectral theorem. It states that they are (up to unitary equivalence) operators of multiplication. This can be seen as a (tremendous) generalization of the fact that symmetric matrices can be diagonalized. A consequence of the spectral theorem is the possibility to form functions of a self-adjoint operator. This is known as spectral calculus. In this section we discuss the details.

Without proof we present the spectral theorem.

THEOREM 3.6 (Spectral theorem). *Let A be a self-adjoint operator on the separable Hilbert space H. Then, there exists a measure space (X, μ), which is σ-finite, a measurable function $u\colon X \longrightarrow \mathbb{R}$ and a unitary map $U\colon L^2(X, \mu) \longrightarrow H$ with*

$$A = UM_uU^{-1}.$$

In particular, $\sigma(A)$ is equal to the essential range of u and, hence, contained in $\mathbb{R}$. Moreover, A is bounded if and only if u is essentially bounded.

The spectral theorem allows us to define functions of an operator.

DEFINITION 3.7 (Functional calculus - definition). If $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ is measurable, A is self-adjoint and (X, μ), u and U are as in Theorem 3.6, then we define the operator $\varphi(A)$ acting on the domain

$$D(\varphi(A)) := UD(M_{\varphi \circ u})$$

as

$$\varphi(A) := UM_{\varphi \circ u}U^{-1}.$$

REMARK 3.8. The spectral theorem does not state that (X, m), u and U are unique. Indeed, they are not. So, the preceding definition leaves open the possibility that $\varphi(A)$ is not well-defined but rather the definition depends on the choice of (X, m), u and U in the spectral theorem. This is not the case. We do not provide a full proof of independence here but rather sketch the idea: For $z \in \rho(A)$ we define $\varphi_z\colon \mathbb{R} \setminus \{z\} \longrightarrow \mathbb{C}, \varphi_z(t) := \frac{1}{t-z}$. Then, $\varphi_z(A)$ can (by direct computation with multipliciation operators) be seen to satisfy $\varphi_z(A)(A - z) = I_{D(A)}$ and $(A - z)\varphi_z(A) = I$ (where I denotes the identity on H and $I_{D(A)}$ denote the identity on $D(A)$). Hence, $\varphi_z(A) = (A - z)^{-1}$ is independent of the choice of (X, m), u and U and, hence, well-defined. This then applies to all sorts of functions φ, which can be approximated (in a suitable sense) by linear combinations of the φ_z, $z \in \rho(A)$. Ultimately, this then yields that $\varphi(A)$ is well-defined for all measurable φ on $\sigma(A)$.

PROPOSITION 3.9 (Basic properties of $\varphi(A)$). *Let A be a self-adjoint operator on H with spectrum $\sigma(A)$ and let $\varphi, \psi\colon \mathbb{R} \longrightarrow \mathbb{C}$ be measurable on $\sigma(A)$. Then, the following statements hold:*

(a) $\varphi(A)^* = \overline{\varphi}(A)$.

(b) *The operator $\varphi(A)$ is self-adjoint if and only if the essential range of $\varphi|_{\sigma(A)}$ is contained in $\mathbb{R}$.*

(c) *The operator $\varphi(A)$ is bounded if and only if $\varphi|_{\sigma(A)}$ is essentially bounded, in which case*

$$\|\varphi(A)\| = \|\varphi|_{\sigma(A)}\|_\infty.$$

(d) $D(\varphi(A)\psi(A)) \subseteq D((\varphi\psi)(A))$ *and on* $D(\varphi(A)\psi(A))$ *we have*

$$(\varphi\psi)(A) = \varphi(A)\psi(A).$$

(e) $D(\varphi(A) + \psi(A)) \subseteq D((\varphi + \psi)(A))$ *and on* $D(\varphi(A) + \psi(A))$ *we have*

$$(\varphi + \psi)(A) = \varphi(A) + \psi(A).$$

PROOF. This follows from the definition of $\varphi(A)$ and the corresponding properties of multiplication operators. In particular, (b) and (c) follow from Proposition 3.4. □

The spectral theorem makes it possible to introduce certain measures on the real line. This is done next.

Proposition 3.10 (Spectral measures). *Let A be a self-adjoint operator on H and $f \in H$ be given. Then, the map*

$$\mu_f : \textit{Measurable subsets of } \mathbb{R} \longrightarrow [0,\infty), \quad B \mapsto \langle f, 1_B(A) f\rangle,$$

is a measure on $\mathbb{R}$.

Proof. As 1_B is a bounded function, the operator $1_B(A)$ is a bounded operator. Moreover, from $1_B = 1_B^2 = \overline{1_B}$ and Proposition 3.9 we find

$$1_B(A) = 1_B(A) 1_B(A) = 1_B(A)^*.$$

This gives

$$\mu_f(B) = \langle f, 1_B(A) f\rangle = \langle f, 1_B(A) 1_B(A) f\rangle = \langle 1_B(A) f, 1_B(A) f\rangle \geq 0.$$

This shows that μ_f maps indeed into $[0,\infty)$. It remains to show that μ_f is σ-additive. Let B be a measurable subset of $\mathbb{R}$ and let B_n, $n \in \mathbb{N}$, be pairwise disjoint measurable subsets with $\bigcup_{n\in\mathbb{N}} B_n = B$. Then, we have

$$1_{\bigcup_{n=1}^N B_n}(A) f \to 1_B(A) f, \quad N \to \infty.$$

Indeed, this is clear for $A = M_u$ and then follows for general A by the spectral theorem. From this convergence we obtain

$$\begin{aligned}\sum_{n=1}^N \langle f, 1_{B_n}(A) f\rangle = \langle f, \sum_{n=1}^N 1_{B_n}(A) f\rangle &= \langle f, 1_{\bigcup_{n=1}^N B_n}(A) f\rangle \\ &\to \langle f, 1_B(A) f\rangle = \mu_f(B).\end{aligned}$$

This shows that μ_f is a measure. ☐

Definition 3.11 (Spectral measure). The measure μ_f appearing in the preceding proposition is called the *spectral measure* of f.

Proposition 3.12 (Computing the spectral measures via spectral theorem). *Let A be a self-adjoint operator on H and let (X,μ), u and U be as in Theorem 3.6. Let $f \in H$ be given and define $\psi = U^{-1} f$. Then, the formula*

$$\int_{\mathbb{R}} \varphi \mathrm{d}\mu_f = \int_X (\varphi \circ u) |\psi|^2 \mathrm{d}\mu$$

holds for all measurable $\varphi \colon \mathbb{R} \longrightarrow [0,\infty)$, where both sides may take the value ∞. For $\varphi \in L^1(\mathbb{R},\mu_f)$ both sides are finite.

Proof. For $\varphi = 1_B$ with a measurable subset B of $\mathbb{R}$ the formula is immediate from a direct computation, which uses that U is unitary:

$$\begin{aligned}\int_{\mathbb{R}} \varphi \mathrm{d}\mu_f = \langle f, 1_B(A) f\rangle = \langle f, U M_{1_B\circ u} U^{-1} f\rangle &= \langle \psi, M_{1_B \circ u} \psi\rangle \\ = \int_X (1_B \circ u) |\psi|^2 \mathrm{d}\mu.\end{aligned}$$

Now, the general case follows from taking linear combinations and limits. ☐

Given the preceding computation of the spectral measure, we now give some connections between the spectral calculus and the spectral measures. The arising formulas will be most useful for our subsequent considerations.

PROPOSITION 3.13 (Functional calculus and spectral measures). *Let A be a self-adjoint operator on H, let $f \in H$ and let $\varphi \colon \mathbb{R} \longrightarrow \mathbb{C}$ be measurable.*

(a) *$f \in D(\varphi(A))$ if and only if $\varphi \in L^2(\mathbb{R}, \mu_f) = L^2(\sigma(A), \mu_f)$, in which case*

$$\|\varphi(A)f\|^2 = \int |\varphi|^2 \mathrm{d}\mu_f.$$

In particular, $f \in D(A)$ if and only if $\int x^2 \mathrm{d}\mu_f(x) < \infty$.

(b) *If $f \in D(\varphi(A))$, then*

$$\langle f, \varphi(A)f \rangle = \int \varphi \mathrm{d}\mu_f \qquad \text{and} \qquad |\varphi|^2 \mu_f = \mu_{\varphi(A)f}.$$

PROOF. Let (X, μ), u and U be as in Theorem 3.6, i.e.,

$$A = U M_u U^{-1},$$

where $U \colon L^2(X, \mu) \longrightarrow H$ is unitary. Then, by definition,

$$\varphi(A) = U M_{\varphi \circ u} U^{-1}$$

holds for all measurable $\varphi \colon \mathbb{R} \longrightarrow \mathbb{C}$. We set

$$\psi := U^{-1} f.$$

(a) We first show the characterization of the domain of $\varphi(A)$. By Proposition 3.12, we have $\varphi \in L^2(\mathbb{R}, \mu_f)$ if and only if

$$\int |\varphi \circ u|^2 |\psi|^2 \mathrm{d}\mu < \infty$$

which, by the definition of the domain of a multiplication operator, is equivalent to

$$\psi \in D(M_{\varphi \circ u}).$$

As $\psi = U^{-1} f$, this holds if and only if $f \in D(\varphi(A))$ from the definition of the domain of $\varphi(A)$.

Now, if φ belongs to $L^2(\mathbb{R}, \mu_f)$, then, as U is unitary, Proposition 3.12 gives

$$\|\varphi(A)f\|^2 = \|M_{\varphi \circ u}\psi\|^2 = \int |\varphi \circ u|^2 |\psi|^2 \mathrm{d}\mu = \int |\varphi|^2 \mathrm{d}\mu_f,$$

which proves the formula given in (a). The last statement of (a) is immediate by taking $\varphi = \mathrm{id}$.

(b) As U is unitary and $U^{-1}\varphi(A) = M_{\varphi \circ u} U^{-1}$, we obtain

$$\langle f, \varphi(A)f \rangle = \langle U^{-1} f, U^{-1}\varphi(A) f \rangle = \langle \psi, M_{\varphi \circ u}\psi \rangle = \int (\varphi \circ u) |\psi|^2 \mathrm{d}\mu.$$

Since we assume $f \in D(\varphi(A))$, part (a) gives $\varphi \in L^2(\mathbb{R}, \mu_f)$. As μ_f is finite, $\varphi \in L^2(\mathbb{R}, \mu_f)$ implies $\varphi \in L^1(\mathbb{R}, \mu_f)$ and Proposition 3.12 yields

$$\int (\varphi \circ u) |\psi|^2 \mathrm{d}\mu = \int \varphi \mathrm{d}\mu_f.$$

Putting these equations together gives the first formula claimed in (b).

We now show

$$|\varphi|^2 \mu_f = \mu_{\varphi(A)f}.$$

It suffices to show

$$\int \chi|\varphi|^2 \mathrm{d}\mu_f = \int \chi \mathrm{d}\mu_{\varphi(A)f}$$

for all bounded measurable functions $\chi\colon \mathbb{R} \longrightarrow \mathbb{C}$. As χ is bounded and μ_f is finite for all $f \in H$, by part (a) the operator $\chi(A)$ is defined on the entire Hilbert space H. Hence, from the already established first formula of (b), the fact that U is unitary and the definitions of $\varphi(A)$ and $\chi(A)$, we find

$$\begin{aligned}\int \chi \mathrm{d}\mu_{\varphi(A)f} &= \langle \varphi(A)f, \chi(A)\varphi(A)f\rangle = \langle U^{-1}\varphi(A)f, U^{-1}\chi(A)UU^{-1}\varphi(A)f\rangle \\ &= \langle M_{\varphi\circ u}\psi, M_{\chi\circ u}M_{\varphi\circ u}\psi\rangle = \int (\chi\circ u)|\varphi\circ u|^2|\psi|^2 \mathrm{d}\mu \\ &= \int \chi|\varphi|^2 \mathrm{d}\mu_f,\end{aligned}$$

where we used Proposition 3.12 in the last equality. This is the desired statement. ☐

Corollary 3.14 (Bounded functional calculus). *Let A be a self-adjoint operator on H and $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ be measurable and bounded on $\sigma(A)$. Then, $D(\varphi(A)) = H$ and, for every $f \in H$,*

$$\|\varphi(A)f\|^2 = \int |\varphi|^2 \mathrm{d}\mu_f, \qquad \langle f, \varphi(A)f\rangle = \int \varphi \mathrm{d}\mu_f$$

and

$$|\varphi|^2\mu_f = \mu_{\varphi(A)f}.$$

Proof. As φ is bounded on $\sigma(A)$ and μ_f is a finite measure supported on $\sigma(A)$, we have $D(\varphi(A)) = H$ from (a) of Proposition 3.13, which also gives the first equality. The remaining equalities then follow from (b) of Proposition 3.13. ☐

Proposition 3.15. *Let A be a self-adjoint operator on H and let $f, g \in H$. Then, there exists a unique finite signed regular Borel measure μ on $\mathbb{R}$ with*

$$\langle f, (A-z)^{-1}g\rangle = \int \frac{1}{x-z} \mathrm{d}\mu(x)$$

for all $z \in \mathbb{C}\setminus\mathbb{R}$. If $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ is measurable and $f, g \in D(\varphi(A))$, then

$$\langle f, \varphi(A)g\rangle = \int \varphi \mathrm{d}\mu.$$

In particular, this holds for all $f, g \in H$ when φ is bounded and measurable.

Proof. The existence of such a signed measure is given by the existence of μ_f and μ_g and polarization. The remaining statements follow by Proposition 3.13 (b) and polarization as well. ☐

3.3. Spectral Projections

Of particular relevance for applications of the functional calculus are characteristic functions of measurable sets. They are discussed next. Recall that $B(H)$ denotes the space of bounded operators on H. Given a self-adjoint operator A we define

$$E\colon \text{measurable subsets of } \mathbb{R} \longrightarrow B(H)$$

via

$$E(B) := 1_B(A).$$

We call the operator $E(B)$ the *spectral projection* associated to B. Note that we have already encountered $E(B)$ in the proof of Proposition 3.10. In particular, we have used $1_B = 1_B 1_B = \overline{1_B}$ to conclude that $E(B)$ satisfies

$$E(B) = E(B)E(B) = E(B)^*$$

and hence is an orthogonal projection. Similarly, we infer from $1_{B_1}1_{B_2} = 1_{B_1 \cap B_2}$ that

$$E(B_1)E(B_2) = E(B_1 \cap B_2) = E(B_2)E(B_1)$$

whenever B_1, B_2 are measurable subsets of $\mathbb{R}$. Moreover, we obviously have $E(\emptyset) = 0$ as $1_\emptyset = 0$. These considerations give, in particular,

$$E(B_1)E(B_2) = E(\emptyset) = 0$$

whenever $B_1 \cap B_2 = \emptyset$. Moreover, as $1_{\cup_{n\in\mathbb{N}} B_n}$ is the monotone pointwise limit of $\left(\sum_{n=1}^N 1_{B_n}\right)_{N\in\mathbb{N}}$ whenever the sets B_n are mutually disjoint, we infer $E(\bigcup_{n\in\mathbb{N}} B_n) = \oplus_{n\in\mathbb{N}} E(B_n)$. Furthermore, $E(\mathbb{R}) = I$ is the identity operator.

To summarize, we note that E satisfies the following properties:

- $E(B)$ is an orthogonal projection for each measurable $B \subseteq \mathbb{R}$.
- $E(\bigcup_{n\in\mathbb{N}} B_n) = \oplus_{n\in\mathbb{N}} E(B_n)$ for mutually disjoint measurable sets.
- $E(\emptyset) = 0$.

In this sense, the map E resembles a measure. We refer to E as the *projection-valued measure associated to A* or the *spectral family*.

The map E is intimately linked to the spectral measures. This is discussed in the subsequent two propositions. The first proposition is a direct consequence of Proposition 3.10.

PROPOSITION 3.16 (Spectral measure via projection-valued measures). *Let A be a self-adjoint operator on H with associated projection-valued measure E. Then, for any $f \in H$, we have*

$$\mu_f(B) = \langle f, E(B)f\rangle = \|E(B)f\|^2$$

for any measurable set $B \subseteq \mathbb{R}$.

PROPOSITION 3.17. *Let A be a self-adjoint operator on H with associated projection-valued measure E. Let $B \subseteq \mathbb{R}$ be measurable. Then, for all $f \in H$,*

$$\mu_{E(B)f} = 1_B \mu_f.$$

In particular, for any $g \in E(B)H$, we have $\mu_g = 1_B\mu_g$,

$$\operatorname{supp}(\mu_g) \subseteq \overline{B},$$

where $\overline{B}$ denotes the closure of B, and $g \in D(\varphi(A))$ if and only if $\varphi \in L^2(\sigma(A), \mu_g)$ for any measurable $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$.

PROOF. The first statement follows from (b) of Proposition 3.13. Now, for $g \in E(B)H$, we have $g = E(B)g$ as $E(B)$ is an orthogonal projection and we find $\mu_g = 1_B\mu_g$. Clearly, $1_B\mu_g$ is supported on $\overline{B}$. Now, if $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ is measurable, then the statement on the domain of $\varphi(A)$ follows from Proposition 3.13 (a). This finishes the proof. $\square$

It is possible to characterize the spectrum of A via E. To do so we define the *support of* E as

$$\operatorname{supp}(E) := \{\lambda \in \mathbb{R} \mid E((\lambda-\varepsilon, \lambda+\varepsilon)) \neq 0 \text{ for all } \varepsilon > 0\}.$$

With this definition, we can show that the spectrum of A is equal to the support of E.

THEOREM 3.18. *Let A be a self-adjoint operator and E the associated projection-valued measure. Then,*

$$\sigma(A) = \operatorname{supp}(E).$$

PROOF. As the spectrum is preserved by unitary equivalence, we may assume that $A = M_u$, where (X, μ) is a σ-finite measure space and $u\colon X \longrightarrow \mathbb{R}$ is measurable by Theorem 3.6. In particular, the spectrum of A is given by the essential range of u. Hence, it remains to show that $\operatorname{supp}(E)$ equals the essential range of u. Now, for a measurable set $B \subseteq \mathbb{R}$, we have

$$E(B) = M_{1_B \circ u}$$

and, hence, $E(B)$ is not trivial if and only if

$$0 \neq 1_B \circ u = 1_{u^{-1}(B)}$$

if and only if $\mu(u^{-1}(B)) > 0$. This easily shows that $\operatorname{supp}(E)$ is equal to the essential range of u. ☐

3.4. Positive Operators

We will now restrict our attention further to those operators whose spectrum is contained in the non-negative real numbers.

LEMMA 3.19. *Let A be a self-adjoint operator on H with domain $D(A)$. Then, the following statements are equivalent:*

(i) $\sigma(A) \subseteq [0, \infty)$.
(ii) *A is unitarily equivalent to multiplication by an almost everywhere non-negative function.*
(iii) $\langle f, Af\rangle \geq 0$ *for all* $f \in D(A)$.
(iv) *There exists a self-adjoint operator S with $A = S^2$.*

PROOF. According to the spectral theorem, Theorem 3.6, we can assume without loss of generality that A is the operator M_u of multiplication by a measurable function $u\colon X \longrightarrow \mathbb{R}$, where (X, μ) is a σ-finite measure space and $D(M_u) = \{f \in L^2(X, \mu) \mid uf \in L^2(X, \mu)\}$. The spectrum of A is then the essential range of u by Lemma 3.3. Now, the essential range is contained in $[0, \infty)$ if and only if $u \geq 0$ almost everywhere and this in turn holds if and only if $\int u|f|^2 \mathrm{d}\mu = \langle f, M_u f\rangle \geq 0$ for all $f \in D(M_u)$. This shows the equivalence between (i), (ii) and (iii).

Now, if $u \geq 0$ almost everywhere, then $M_u = M_v^2$ with $v = \sqrt{u}$ and, thus, (ii) implies (iv). Finally, (iv) implies (iii) via

$$\langle f, Af\rangle = \langle f, S^2 f\rangle = \langle Sf, Sf\rangle = \|Sf\|^2 \geq 0.$$

This finishes the proof. ☐

We highlight the class of operators appearing in the previous statement by giving a definition.

Definition 3.20 (Positive operator). We say that a self-adjoint operator A is *positive* if A satisfies one of the equivalent conditions of Lemma 3.19. We write $A \geq 0$ in this case.

Remark 3.21. Note that positivity for self-adjoint operators means spectral positivity. If the Hilbert space has an additional lattice structure, which is the case in our typical situation of $\ell^2(X, m)$, this notion should not be confused with the property of being positivity preserving. An easy and instructive example is given by $A \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $Af := \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} f$. Then A is clearly self-adjoint and $\sigma(A) = \{2, 4\}$ implies that A is positive. However, A is not positivity preserving. Moreover, $A \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $Af := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} f$ is clearly positivity improving, but $\sigma(A) = \{-1, 3\}$. Here, note that $\mathbb{R}^2 = \ell^2(\{1, 2\})$.

We also remark that the notions of positivity and poritivitiy improving are used with different meanings in the literature (depending on the context).

Lemma 3.22 (Square root). *Let A be a positive operator on H. Then, the following statements hold:*

(a) *$\sqrt{A}$ is self-adjoint and positive.*

(b) *$(\sqrt{A})^2 = A$, i.e., $f \in D(A)$ if and only if $f \in D(\sqrt{A})$ and $\sqrt{A}f \in D(\sqrt{A})$ and in this case $Af = (\sqrt{A})^2 f$. In particular, $D(A) \subseteq D(\sqrt{A})$.*

Proof. By Lemma 3.19 we may assume that A is unitarily equivalent to multiplication by a function u which is positive almost everywhere on a σ-finite measure space (X, μ). It follows by the definition of the spectral calculus that $\sqrt{A}$ is then unitarily equivalent to multiplication by $\sqrt{u}$. As $\sqrt{u}$ is real-valued almost everywhere, the self-adjointness of $\sqrt{A}$ follows from Proposition 3.4. Positivity of $\sqrt{A}$ then follows by Lemma 3.19. This proves (a). Property (b) follows by a short argument involving the definition of the domain of a multiplication operator, see Example 3.2, and the definition of powers of unbounded operators. ☐

3.5. Semigroups and Resolvents

Any positive operator A on a Hilbert space comes with two families of operators. These are its semigroup and its resolvent. Each of these families can be seen as the solution to a certain equation. The two families are equivalent in a certain sense and in particular it is possible to compute one from the other.

We start with a discussion of the semigroup. For $t \geq 0$, we define

$$\varPhi^{(t)}\colon [0,\infty) \longrightarrow \mathbb{R}, \quad s \mapsto e^{-ts}.$$

Then, $\varPhi^{(t)}$ is a bounded function on $[0,\infty)$ (with bound 1) and $\varPhi^{(t)}\varPhi^{(r)} = \varPhi^{(t+r)}$ holds for all $t, r \geq 0$ as well as $\varPhi^{(0)} = 1$. Whenever A is a positive operator we define

$$e^{-tA} := \varPhi^{(t)}(A)$$

and call $(e^{-tA})_{t\geq 0}$ the *semigroup* associated to A.

Proposition 3.23 (Basic properties of the semigroup). *Let A be a positive operator on H. Then,*

(a) $e^{0A} = I$ *and for all* $s, t \geq 0$,

$$e^{-(s+t)A} = e^{-sA}e^{-tA}.$$

(b) *For all* $f \in H$,

$$\lim_{t\to 0^+} e^{-tA}f = f.$$

(c) *For all* $t \geq 0$,

$$\|e^{-tA}\| \leq 1.$$

Proof. (a) This follows immediately from Proposition 3.9 (d).

(b) By Corollary 3.14, for $f \in H$, we have

$$\|e^{-tA}f - f\|^2 = \int_0^\infty (e^{-tx} - 1)^2 \mathrm{d}\mu_f(x) \to 0$$

as $t \to 0^+$ by Lebesgue's dominated convergence theorem. This follows as the integrand is bounded above by 1, converges pointwise to 0 and each spectral measure is finite.

(c) By Corollary 3.14, for $f \in H$ we have

$$\|e^{-tA}f\|^2 = \int_{[0,\infty)} e^{-2tx}\mathrm{d}\mu_f(x) \leq \int_{[0,\infty)} \mathrm{d}\mu_f = \|f\|^2.$$

This gives the desired conclusion. □

We will now show that the semigroup generates solutions of the parabolic equation involving A. In order to make this precise, we recall that a function $u\colon (0,\infty) \longrightarrow H$ is called *differentiable* if for any $t > 0$ the limit

$$\lim_{h\to 0} \frac{1}{h}(u(t+h) - u(t))$$

exists. In this case, we denote this limit as $\partial_t u(t)$ and call it the *derivative* of u.

THEOREM 3.24 (Solution of the parabolic equation). *Let A be a positive operator on H and let $f \in H$. Then, $u\colon [0,\infty) \longrightarrow H$ given by*

$$u_t := e^{-tA} f$$

is continuous on $[0,\infty)$, differentiable on $(0,\infty)$, satisfies $u_t \in D(A)$ for $t > 0$ and

$$\partial_t u_t = -Au_t$$

for all $t > 0$ as well as $u(t) \to f$ for $t \to 0^+$.

PROOF. We prove the theorem through a series of claims.

Claim. The function u is continuous on $[0,\infty)$.

Proof of the claim. Let $t \geq 0$. Then, for all $h \in \mathbb{R}$ with $t + h \geq 0$, the operator $e^{-(t+h)A}$ is bounded and Corollary 3.14 gives

$$\left\| e^{-(t+h)A} f - e^{-tA} f \right\|^2 = \int_{[0,\infty)} \left| e^{-(t+h)x} - e^{-tx} \right|^2 \mathrm{d}\mu_f(x).$$

Now, $[0,\infty) \ni x \mapsto |e^{-(t+h)x} - e^{-tx}|^2$ is bounded by 4 and converges to 0 pointwise as $h \to 0$. Thus, we obtain from Lebesgue's dominated convergence theorem

$$\lim_{h \to 0} \int_{[0,\infty)} \left| e^{-(t+h)x} - e^{-tx} \right|^2 \mathrm{d}\mu_f(x) = 0.$$

This proves the continuity of u at t.

Claim. For any $t > 0$, $u_t \in D(A)$.

Proof of the claim. By Proposition 3.13 (a), we have to show that $\int x^2 \mathrm{d}\mu_{u_t}(x) < \infty$ for $t > 0$. Now, by Corollary 3.14, as $f \in H = D(e^{-tA})$ we have $\mu_{u_t} = \mu_{e^{-tA}f} = |e^{-t(\cdot)}|^2 \mu_f$. This easily gives

$$\int x^2 \mathrm{d}\mu_{u_t}(x) = \int_{[0,\infty)} x^2 e^{-2tx} \mathrm{d}\mu_f(x) < \infty,$$

where we used that μ_f is supported on $\sigma(A) \subseteq [0,\infty)$ and $x \mapsto x^2 e^{-2tx}$ is bounded on $[0,\infty)$.

Claim. For any $t > 0$, the function u is differentiable in t and satisfies

$$\partial_t u_t = -Au_t.$$

Proof of the claim. For $h \in \mathbb{R}$ with $|h| \leq t$, we define the function $\psi_h\colon [0,\infty) \longrightarrow \mathbb{R}$ by

$$\psi_h(x) := \frac{e^{-(t+h)x} - e^{-tx}}{h} + xe^{-tx}.$$

Then, (d) and (e) of Proposition 3.9, give

$$\frac{1}{h}(e^{-(t+h)A} f - e^{-tA} f) + Ae^{-tA} f = \psi_h(A) f,$$

where we use $e^{-tA} f \in D(A)$ for $t > 0$, which was established in the preceding claim to write down the expression on the left-hand side. Hence, Proposition 3.13 (a) yields

$$\left\| \frac{1}{h}(e^{-(t+h)A} f - e^{-tA} f) + Ae^{-tA} f \right\|^2 = \int_{[0,\infty)} |\psi_h(x)|^2 \mathrm{d}\mu_f(x).$$

Now, ψ_h can easily be seen to converge pointwise to 0 as $h \to 0$ and to be bounded by $x \mapsto 2xe^{-(t/2)x}$, which is bounded on $[0,\infty)$. Hence, by Lebesgue's dominated convergence theorem, we see that $\int |\psi_h|^2 \mathrm{d}\mu_f \to 0$ as $h \to 0$ and this gives the desired claim.

Claim. $u_t \to f$ as $t \to 0^+$.

Proof of the claim. This is immediate from the already established continuity of u on $[0,\infty)$ and $u_0 = f$. □

We now gather some basic properties of resolvents.

Proposition 3.25 (Basic properties of resolvents). *Let A be a positive operator on H. Then,*

(a) *For all $\alpha, \beta > 0$,*

$$(A+\alpha)^{-1} - (A+\beta)^{-1} = -(\alpha-\beta)(A+\alpha)^{-1}(A+\beta)^{-1}.$$

(b) *For all $f \in H$,*

$$\lim_{\alpha\to\infty} \alpha(A+\alpha)^{-1} f = f.$$

(c) *For all $\alpha > 0$,*

$$\|\alpha(A+\alpha)^{-1}\| \le 1.$$

Proof. (a) This follows directly from the identity

$$A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$$

for invertible operators A and B with $D(B) \subseteq D(A)$.

(b) For every $f \in H$ we have by Corollary 3.14

$$\|\alpha(A+\alpha)^{-1} f - f\|^2 = \int_{[0,\infty)} \left| \frac{\alpha}{x+\alpha} - 1 \right|^2 \mathrm{d}\mu_f(x) \to 0$$

as $\alpha \to \infty$ by Lebesgue's dominated convergence theorem.

(c) By Corollary 3.14, for every $f \in H$ we have

$$\|\alpha(A+\alpha)^{-1} f\|^2 = \int_{[0,\infty)} \left| \frac{\alpha}{x+\alpha} \right|^2 \mathrm{d}\mu_f(x) \le \int_{[0,\infty)} \mathrm{d}\mu_f = \|f\|^2.$$

This completes the proof. □

Theorem 3.26 (Semigroups and resolvents). *Let A be a positive operator on H. Let $f \in H$ be given.*

(a) *For every $\alpha > 0$,*

$$(A+\alpha)^{-1} f = \int_0^\infty e^{-t\alpha} e^{-tA} f \mathrm{d}t.$$

(Here the integral can be understood as limit of Riemannian sums.)

("Laplace transform")

(b) *For every $t > 0$,*

$$e^{-tA} f = \lim_{n\to\infty} \left(\frac{n}{t} \left(A + \frac{n}{t} \right)^{-1} \right)^n f.$$

("Exponential formula")

Proof. (a) From the formula

$$(x+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tx} \mathrm{d}t,$$

which holds for all $x \geq 0$ and $\alpha > 0$, we obtain by applying the functional calculus

$$(A+\alpha)^{-1} = \int_{[0,\infty)} e^{-t\alpha} e^{-tA} \mathrm{d}t.$$

This gives the conclusion.

(b) We note that

$$\varphi_n(x) := \left(\frac{n}{t}\left(x + \frac{n}{t}\right)^{-1}\right)^n = \left(1 + \frac{tx}{n}\right)^{-n} \to e^{-tx}$$

as $n \to \infty$ for $x, t \geq 0$. Hence, by Corollary 3.14 and Lebesgue's dominated convergence theorem, we obtain, for every $f \in H$,

$$\left\| e^{-tA} f - \left(\frac{n}{t}\left(A + \frac{n}{t}\right)^{-1}\right)^n f \right\|^2 = \int_{[0,\infty)} \left| e^{-tx} - \varphi_n(x) \right|^2 \mathrm{d}\mu_f(x) \to 0$$

as $n \to \infty$. This completes the proof. □

3.6. Forms

Forms and positive operators can be seen as two faces of the same medal. The advantage of forms is that they have a larger domain of definition and are generally much more easily written down than the underlying operator.

We start with the definition of the basic object.

Definition 3.27 (Symmetric positive form). A *symmetric positive form* Q on H consists of a dense subspace $D(Q) \subseteq H$ called the *domain of* Q together with a map

$$Q \colon D(Q) \times D(Q) \longrightarrow \mathbb{C}$$

satisfying

- $Q(f,g) = \overline{Q(g,f)}$ (*"Symmetry"*)
- $Q(f, \alpha g + \beta h) = \alpha Q(f,g) + \beta Q(f,h)$ (*"Linearity"*)
- $Q(f,f) \geq 0$ (*"Positivity"*)

for all $f, g, h \in D(Q)$ and $\alpha, \beta \in \mathbb{C}$.

We will often refer to positive symmetric forms as just forms.

Whenever such a form Q is given we define for $f \in H$ the value

$$Q'(f) := \begin{cases} Q(f,f) & \text{if } f \in D(Q) \\ \infty & \text{otherwise.} \end{cases}$$

We note that we can recover the form Q from the values $Q'(f)$ for $f \in H$ as the domain of Q is given by

$$D(Q) = \{f \in H \mid Q'(f) < \infty\}$$

and $Q(f,g)$ can be obtained by using the polarization identity, i.e.,

$$Q(f,g) = \frac{1}{4} \sum_{k=0}^{3} i^k Q(g + i^k f)$$

for $f, g \in D(Q)$. We will often write $Q(f)$ instead of $Q'(f)$.

Any form Q comes with an inner product $\langle \cdot, \cdot \rangle_Q$ on $D(Q)$ given by

$$\langle \cdot, \cdot \rangle_Q : D(Q) \times D(Q) \longrightarrow \mathbb{C}, \quad \langle f, g \rangle_Q := Q(f, g) + \langle f, g \rangle.$$

This inner product induces the norm $\| \cdot \|_Q$ given by

$$\|f\|_Q := \langle f, f \rangle_Q^{1/2}.$$

In the next example we begin to establish the connection between forms and positive operators. In particular, we show how to define a form from a positive operator.

Example 3.28 (Form associated to a positive operator). Let A be a positive operator on H. We define the form Q_A by letting $D(Q_A) := D(\sqrt{A})$ and

$$Q_A(f, g) := \langle \sqrt{A} f, \sqrt{A} g \rangle$$

for all $f, g \in D(\sqrt{A})$. We call Q_A the *form associated to* A.

In particular, if (X, μ) is a measure space, $u \colon X \longrightarrow [0, \infty)$ is measurable and M_u the operator of multiplication by u, then one easily seea that Q_{M_u} has domain

$$D(Q_{M_u}) = D(M_{\sqrt{u}}) = \{f \in L^2(X, \mu) \mid \int u|f|^2 \mathrm{d}\mu < \infty\}$$

and acts by

$$Q_{M_u}(f, g) = \int u \overline{f} g \mathrm{d}\mu$$

for $f, g \in D(Q_{M_u})$. We note that the integral defining $Q_{M_u}(f, g)$ exists as $u|fg| \leq \frac{1}{2}\big(u|f|^2 + u|g|^2\big)$.

We will show that the converse of the preceding example holds under some additional assumptions. For forms with suitable boundedness properties, this is not hard to see by using the Riesz representation theorem. This is the content of the next proposition.

Proposition 3.29 (Bounded forms and operators). *Let Q be a positive form with $D(Q) = H$ such that there exists a constant $C \geq 0$ with*

$$Q(f, g) \leq C \|f\| \|g\|$$

for all $f, g \in H$. Then, there exists a unique positive operator A with $D(A) = H$, $\|A\| \leq C$ and

$$Q(f, g) = \langle f, Ag \rangle = \langle Af, g \rangle = \langle \sqrt{A} f, \sqrt{A} g \rangle$$

for all $f, g \in H$.

Proof. For a fixed $f \in H$, we consider the map from H to $\mathbb{C}$ given by

$$g \mapsto Q(f, g).$$

This map is linear and bounded by the assumptions on Q. Hence, by the Riesz representation theorem, there exists a unique $f' \in H$ with

$$Q(f, g) = \langle f', g \rangle$$

for all $g \in H$. We define $A \colon H \longrightarrow H$ by

$$Af = f'.$$

It follows that A is linear and

$$Q(f,g) = \langle Af, g\rangle$$

for all $f, g \in H$. In particular, we infer

$$\|Af\| = \sup\{\langle Af, g\rangle \mid \|g\| \leq 1\} \leq C\|f\|$$

and, thus, $\|A\| \leq C$ follows. Moreover, by using the symmetry of Q, we have

$$\langle Af, g\rangle = Q(f,g) = \overline{Q(g,f)} = \overline{\langle Ag, f\rangle} = \langle f, Ag\rangle,$$

so that A is symmetric. As A is bounded, it follows that A is self-adjoint.

Finally, using the positivity of Q, we obtain

$$\langle f, Af\rangle = Q(f,f) \geq 0$$

and, thus, A is positive. The uniqueness of A is clear.

It remains to show the formula $\langle f, Ag\rangle = \langle \sqrt{A}f, \sqrt{A}g\rangle$. This, however, is clear as due to positivity of A the operator $\sqrt{A}$ is self-adjoint with $\sqrt{A}\sqrt{A} = A$. □

Any form Q with $D(Q) = H$ which satisfies $Q(f,g) \leq C\|f\|\|g\|$ for all $f, g \in H$ and some constant $C \geq 0$ is called *bounded*. Hence, we see from the proposition above that any positive bounded form gives rise to a unique positive bounded operator. Conversely, if A is a bounded positive operator, then $\sqrt{A}$ is bounded and, thus, Q_A as defined in Example 3.28 is a bounded form. Hence, from the considerations above, we see that there is a one-to-one correspondence between bounded positive operators and bounded positive forms. We will extend this result to a larger class of forms in what follows.

We first show that we can weaken the boundedness assumption on the form to a completeness assumption and still obtain the existence of an operator. This is the content of the next lemma.

LEMMA 3.30 (Associated operator). *Let Q be a positive form on H. If $(D(Q), \langle\cdot,\cdot\rangle_Q)$ is a Hilbert space, then there exists a positive operator A with $D(Q) = D(\sqrt{A})$ and*

$$Q(f,g) = \langle \sqrt{A}f, \sqrt{A}g\rangle$$

for all $f, g \in D(Q)$, i.e., $Q = Q_A$ is the form associated to A.

REMARK 3.31. As $\langle\cdot,\cdot\rangle_Q$ is an inner product on $D(Q)$, the assumption that $(D(Q), \langle\cdot,\cdot\rangle_Q)$ is a Hilbert space just means that $D(Q)$ is complete with respect to the norm $\|\cdot\|_Q$.

PROOF. We write H_Q to denote the Hilbert space $(D(Q), \langle\cdot,\cdot\rangle_Q)$. Consider

$$\langle\cdot,\cdot\rangle\colon H_Q \times H_Q \longrightarrow \mathbb{C},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on H. Then, as Q is positive,

$$|\langle f, g\rangle| \leq \|f\|\|g\| \leq \|f\|_Q\|g\|_Q,$$

so that $\langle\cdot,\cdot\rangle$ is a bounded form on H_Q. Hence, by Proposition 3.29, there exists a unique positive operator T with $D(T) = H_Q$ and

$$\langle f, g\rangle = \langle f, Tg\rangle_Q = Q(f, Tg) + \langle f, Tg\rangle$$

for all $f, g \in H_Q$.

We will ultimately show that

$$A = T^{-1} - I$$

has the desired properties. Indeed, assuming the definition of A as $T^{-1} - I$ makes sense, letting $g' = Tg$ and noting that $g - g' = Ag'$ we see from the above that

$$\langle f, Ag' \rangle = Q(f, g')$$

for all $f, g \in H_Q$. Using $A = \sqrt{A}\sqrt{A} = (\sqrt{A})^*(\sqrt{A})$ then gives

$$Q(f, g') = \langle \sqrt{A} f, \sqrt{A} g' \rangle$$

for all $f, g \in H_Q$.

To turn this into a rigorous argument, we have to show that T is injective and that $T^{-1} - I$ can be seen as a positive operator on H. One obstacle to overcome is that T and A are only defined on H_Q and we have to extend them to subspaces of H.

After this sketch, we now proceed to give the proof. As

$$\langle f, Tf \rangle_Q = \langle f, f \rangle = \|f\|^2$$

for all $f \in H_Q$, the operator T is positive and bounded on H_Q with $\|T\| \leq 1$. By the spectral theorem, Theorem 3.6, applied to T on H_Q, there exists a σ-finite measure space (X, μ_Q), a measurable function $u\colon X \longrightarrow [0, 1]$ and a unitary map $V\colon H_Q \longrightarrow L^2(X, \mu_Q)$ such that

$$T = V^{-1} M_u V.$$

Here, $0 \leq u \leq 1$ follows from the fact that T is positive and bounded with $\|T\| \leq 1$. Furthermore, as $\langle f, Tf \rangle_Q = \|f\|^2$, the operator T is injective and thus $u > 0$ almost everywhere so that $0 < u \leq 1$ almost everywhere.

We now define $a\colon X \longrightarrow [0, \infty)$ by

$$a := \frac{1}{u} - 1.$$

For all $f, g \in H_Q$, from

$$\langle f, g \rangle = \langle f, Tg \rangle_Q = \int u(\overline{Vf})(Vg) \mathrm{d}\mu_Q$$

we infer

$$\begin{aligned} Q(f, g) = \langle f, g \rangle_Q - \langle f, g \rangle &= \int (\overline{Vf})(Vg) \mathrm{d}\mu_Q - \int u(\overline{Vf})(Vg) \mathrm{d}\mu_Q \\ &= \int (1 - u)(\overline{Vf})(Vg) \mathrm{d}\mu_Q = \int a(\overline{Vf})(Vg) u \mathrm{d}\mu_Q \\ &= \int a(\overline{Vf})(Vg) \mathrm{d}\mu, \end{aligned}$$

where we define the measure $\mu := u\mu_Q$ and use that $1 - u = au$.

This is almost the desired formula for Q. It just remains to show that we can use $V\colon H_Q \longrightarrow L^2(X, \mu_Q)$ to define a unitary map

$$U\colon H \longrightarrow L^2(X, \mu)$$

which satisfies $Q(f,g) = \int a(\overline{Uf})(Ug)\mathrm{d}\mu$ for all $f, g \in D(Q)$ and

$$UD(Q) = \{f \in L^2(X,\mu) \mid \int a|f|^2 \mathrm{d}\mu < \infty\}.$$

If so, then we can define $M_{\sqrt{a}}$ on $D(M_{\sqrt{a}}) = UD(Q)$ and

$$\begin{aligned} Q(f,g) &= \int a(\overline{Uf})(Ug)\mathrm{d}\mu = \langle M_{\sqrt{a}}Uf, M_{\sqrt{a}}Ug\rangle_{L^2(X,\mu)} \\ &= \langle U^{-1}M_{\sqrt{a}}Uf, U^{-1}M_{\sqrt{a}}Ug\rangle \end{aligned}$$

for all $f, g \in D(Q)$. We then let

$$\sqrt{A} = U^{-1}M_{\sqrt{a}}U \quad \text{with} \quad D(\sqrt{A}) = U^{-1}D(M_{\sqrt{a}}) = D(Q),$$

which will complete the proof.

To this end, we note that V is isometric as a map from $D(Q) \subseteq H$ to $L^2(X,\mu)$ as

$$\begin{aligned} \langle f,g\rangle = \langle f, Tg\rangle_Q &= \int (\overline{Vf})(Vg)u\mathrm{d}\mu_Q = \int (\overline{Vf})(Vg)\mathrm{d}\mu \\ &= \langle Vf, Vg\rangle_{L^2(X,\mu)}. \end{aligned}$$

Furthermore, as $L^2(X,\mu_Q)$ is dense in $L^2(X,\mu)$, the image of V is dense. As $D(Q)$ is dense in H, we can extend V to an isometric operator $U\colon H \longrightarrow L^2(X,\mu)$ which is onto. As U is also one-to-one, U is unitary.

Moreover, the images of H_Q under U and V are equal. This image, by definition, is $L^2(X,\mu_Q)$ and clearly agrees with

$$\{f \in L^2(X,\mu) \mid \int a|f|^2 \mathrm{d}\mu < \infty\}.$$

Hence, we obtain the asserted formula for $UD(Q)$, which completes the proof. □

We now give the operator constructed above a name.

Definition 3.32 (Associated operator). Let Q be a positive form on H such that $(D(Q), \langle\cdot,\cdot\rangle_Q)$ is a Hilbert space. The positive operator A such that

$$D(\sqrt{A}) = D(Q) \quad \text{and} \quad Q(f,g) = \langle \sqrt{A}f, \sqrt{A}g\rangle$$

is called the operator *associated* to Q.

From Lemma 3.30 we see that every form which induces a Hilbert space structure on its domain gives rise to an associated operator. We will now show that all such forms come from positive operators. Along the way, we also characterize the completeness assumption in terms of lower semicontinuity. Recall that for a positive form Q we have defined Q' on the whole Hilbert space by $Q'(f) = Q(f)$ for $f \in D(Q)$ and $Q'(f) = \infty$ for $f \notin D(Q)$.

Theorem 3.33 (Characterization of closed forms). *Let Q be a positive form on H. Then, the following statements are equivalent:*

(i) *There exists a positive operator* A *with* $Q = Q_A$*, i.e.,* $D(Q) = D(\sqrt{A})$ *and*
$$Q(f,g) = \langle \sqrt{A}f, \sqrt{A}g\rangle$$
for all $f, g \in D(Q)$.

(ii) Q' *is lower semi-continuous, i.e.,*
$$Q'(f) \leq \liminf_{n\to\infty} Q'(f_n)$$
whenever $f_n \to f$ *as* $n \to \infty$ *in* H.

(iii) $(D(Q), \langle\cdot,\cdot\rangle_Q)$ *is a Hilbert space.*

Proof of Theorem 3.33. (i) $\Longrightarrow$ (ii): We will show that Q is the supremum of continuous functions Q_n, from which (ii) follows easily.

Since $A \geq 0$, the operator $(A+n)^{-1}$ exists and is bounded on H for all $n \in \mathbb{N}$. For $f \in H$, we denote by μ_f the spectral measure associated to f and note by Proposition 3.13 and Lemma 3.19 that $\operatorname{supp}(\mu_f) \subseteq \sigma(A) \subseteq [0,\infty)$. We let $\varphi_n\colon [0,\infty) \longrightarrow \mathbb{R}$ be given by $\varphi_n(x) := nx/(x+n)$ and note that φ_n is bounded for every $n \in \mathbb{N}$. Thus, by the bounded functional calculus, Corollary 3.14, we may define a continuous map $Q_n\colon H \longrightarrow [0,\infty)$ via
$$Q_n(f) := \int_{[0,\infty)} \frac{nx}{x+n} \mathrm{d}\mu_f(x) = \langle f, nA(A+n)^{-1}f\rangle.$$

We now claim that
$$Q_n(f) \nearrow \int_{[0,\infty)} x \mathrm{d}\mu_f(x) = Q(f)$$
as $n \to \infty$ for every $f \in H$. Here, the convergence follows easily by the monotone convergence theorem as $\varphi_n(x) \nearrow x$ as $n \to \infty$. The equality follows from Proposition 3.13 (a), which gives $f \in D(\sqrt{A}) = D(Q)$ if and only if $\int x \mathrm{d}\mu_f < \infty$, in which case
$$Q(f) = \|\sqrt{A}f\|^2 = \int_{[0,\infty)} x \mathrm{d}\mu_f(x).$$
This completes the proof.

(ii) $\Longrightarrow$ (iii): Let (f_n) be a Cauchy sequence in $(D(Q), \langle\cdot,\cdot\rangle_Q)$. Then, (f_n) is a Cauchy sequence in H. In particular, there exists an $f \in H$ with $f_n \to f$ with respect to $\|\cdot\|$.

Let $\varepsilon > 0$. As (f_n) is a Cauchy sequence in $(D(Q), \langle\cdot,\cdot\rangle_Q)$, there exists an $N \in \mathbb{N}$ with
$$\|f_n - f_m\|_Q < \varepsilon$$
for all $n, m \geq N$. Consider now $m \geq N$. Then, using (ii), we get
$$Q(f - f_m) \leq \liminf_{n\to\infty} Q(f_n - f_m) \leq \varepsilon.$$
This implies $f \in D(Q)$ and $Q(f - f_m) \leq \varepsilon$ for all $m \geq N$. Therefore, $f_n \to f$ with respect to $\|\cdot\|_Q$.

(iii) $\Longrightarrow$ (i): This is shown in Lemma 3.30. □

We highlight the class of forms appearing in the previous statement by giving a definition.

Definition 3.34 (Closed form). We say that a positive form Q on H is *closed* if Q satisfies one of the equivalent conditions of Theorem 3.33.

For application the following immediate consequence of the theorem is often useful.

Proposition 3.35. *Let Q be a closed form on H. Assume that (f_n) is a sequence in $D(Q)$ converging to $f \in H$. If the sequence $(Q(f_n))$ is bounded, then f belongs to $D(Q)$ and*

$$Q(f) \leq \liminf_n Q(f_n) < \infty$$

holds.

The preceding considerations show that all positive closed forms come from positive operators. We now discuss how to further describe the domain of the operator associated to such a form.

Theorem 3.36 (Domain and action of the operator). *Let Q be a positive closed form on H. Then, the associated operator A has domain*

$$D(A) = \left\{ f \in D(Q) \;\middle|\; \begin{array}{c} \text{there exists a } g \in H \text{ with } Q(h,f) = \langle h, g \rangle \\ \text{for all } h \in D(Q) \end{array} \right\}$$

and acts on $D(A)$ via

$$Af = g.$$

Proof. This follows from the definitions of the associated operator and the adjoint of the square root, the fact that $\sqrt{A}$ is self-adjoint, so that $D(\sqrt{A}) = D(\sqrt{A}^*)$, and the fact that $f \in D(A)$ if and only if $\sqrt{A}f \in D(\sqrt{A}^*) = D(\sqrt{A})$, see Lemma 3.22.

More specifically, for $f \in D(\sqrt{A}) = D(Q)$ we have $\sqrt{A}f \in D(\sqrt{A}^*)$ if and only if there exists an element $g \in H$ such that

$$\langle h, \sqrt{A}\sqrt{A}f \rangle = \langle h, g \rangle$$

for all $h \in D(\sqrt{A}) = D(Q)$, which is equivalent to

$$\langle \sqrt{A}h, \sqrt{A}f \rangle = Q(h,f) = \langle h, g \rangle$$

for all $h \in D(Q)$. This completes the proof. □

The following consequence of the previous theorem is a convenient way to think about the operator associated to a closed form. As a further fact, we also show that the operator domain is dense in the form domain with respect to the inner product arising from the form.

Corollary 3.37. *Let Q be a positive closed form on H. Then, there exists a unique self-adjoint operator A with*

$$Q(f,g) = \langle f, Ag \rangle$$

for all $f \in D(Q)$ and $g \in D(A)$. The operator A is positive and the form Q satisfies

$$D(Q) = D(\sqrt{A}) \quad \text{and} \quad Q(f,g) = \langle \sqrt{A}f, \sqrt{A}g \rangle$$

for all $f, g \in D(Q)$. Furthermore, $D(A) \subseteq D(Q)$ is dense with respect to $\|\cdot\|_Q$.

Proof. We first show uniqueness. Let A be such an operator. Then, by Theorem 3.36 the operator A is a restriction of the operator associated to Q. As both are self-adjoint, they must agree.

The existence of such an operator as well as the connection to the form follow from Theorem 3.33 and Lemma 3.22. Finally, to show that $D(A)$ is dense in $D(Q)$ with respect to $\|\cdot\|_Q$ we suppose not. Then there exists an $f \in D(Q)$, $f \neq 0$, which is in the orthogonal complement of $D(A)$ with respect to $\langle \cdot, \cdot \rangle_Q$, that is,

$$\langle f, g\rangle_Q = \langle f, g\rangle + Q(f,g) = 0$$

for all $g \in D(A)$. By the connection between the operator and form we then obtain

$$\langle f, Ag\rangle = -\langle f, g\rangle$$

for all $g \in D(A)$. This implies $f \in D(A^*)$ with $A^* f = -f$. As A is self-adjoint, it follows that $f \in D(A)$ with $Af = -f$. As A is positive, -1 can not be an eigenvalue. Hence, we conclude $f = 0$. This contradiction yields the claim. □

A form $\widetilde{Q}$ is called an *extension* of the form Q, written as $Q \subseteq \widetilde{Q}$ if $D(Q) \subseteq D(\widetilde{Q})$ holds and $\widetilde{Q}$ agrees with Q on $D(Q)$.

Sometimes a form Q is not closed but possesses closed extensions. Then, Q is called *closable.* In this case, the form can uniquely be extended to the intersection of the domains of all closed extensions of Q and this extension is a closed form. It is called the *closure* of the form.

3.7. Resolvents as Minimizers

In this section we prove a characterization of the resolvent of an operator. More specifically, given a closed form, we show that the resolvent of the associated operator gives the unique minimizer of an equation involving the form.

Theorem 3.38 (Characterization of the resolvent as a minimizer). *Let Q be a positive closed form on H with associated operator A. For $f \in H$ and $\alpha > 0$, define $j\colon D(Q) \longrightarrow [0,\infty)$ by*

$$j(v) := Q(v) + \alpha \left\| v - \frac{1}{\alpha} f \right\|^2 .$$

Then, j satisfies the formula

$$j(v) = j\left((A+\alpha)^{-1} f\right) + Q\left((A+\alpha)^{-1} f - v\right) + \alpha \left\|(A+\alpha)^{-1} f - v\right\|^2 .$$

In particular, $(A+\alpha)^{-1} f$ is the unique minimizer of j on $D(Q)$.

Proof. It suffices to show the formula for j. The statement on the minimizer is then immediate. For ease of notation we set

$$G_\alpha := (A+\alpha)^{-1}$$

and

$$Q_\alpha(u,v) := Q(u,v) + \alpha\langle u, v\rangle, \qquad u, v \in D(Q)$$

for $\alpha > 0$. Given this, the right-hand side of the formula for j can be written as

$$\mathrm{RHS} = j(G_\alpha f) + Q_\alpha(G_\alpha f - v).$$

We will compute the two terms appearing in RHS. In order to do so, we need a little bit of preparation. We obviously have

$$Q_\alpha(G_\alpha f, v) = \langle f, v\rangle$$

for all $f \in H$ and $v \in D(Q)$, which directly yields

$$Q_\alpha(G_\alpha f) = \langle f, G_\alpha f\rangle = \langle G_\alpha f, f\rangle,$$

where we use the self-adjointness of G_α. Furthermore, a direct computation gives

$$j(v) = Q_\alpha(v) - \langle v, f\rangle - \langle f, v\rangle + \frac{1}{\alpha}\|f\|^2.$$

Now, we turn to computing the two terms in RHS: By the last two equalities we obtain for the first term

$$\begin{aligned} j(G_\alpha f) &= Q_\alpha(G_\alpha f) - \langle G_\alpha f, f\rangle - \langle f, G_\alpha f\rangle + \frac{1}{\alpha}\|f\|^2 \\ &= \langle f, G_\alpha f\rangle - \langle G_\alpha f, f\rangle - \langle f, G_\alpha f\rangle + \frac{1}{\alpha}\|f\|^2 \\ &= -\langle f, G_\alpha f\rangle + \frac{1}{\alpha}\|f\|^2. \end{aligned}$$

For the second term, using $Q_\alpha(G_\alpha f, v) = \langle f, v\rangle$ repeatedly we obtain

$$\begin{aligned} Q_\alpha(G_\alpha f - v) &= Q_\alpha(G_\alpha f) - Q_\alpha(G_\alpha f, v) - Q_\alpha(v, G_\alpha f) + Q_\alpha(v) \\ &= \langle f, G_\alpha f\rangle - \langle f, v\rangle - \langle v, f\rangle + Q_\alpha(v). \end{aligned}$$

Putting the two terms together we can now compute

$$\begin{aligned} \mathrm{RHS} &= j(G_\alpha f) + Q_\alpha(G_\alpha f - v) \\ &= Q_\alpha(v) - \langle f, v\rangle - \langle v, f\rangle + \frac{1}{\alpha}\|f\|^2 = j(v), \end{aligned}$$

which finishes the proof. □

Remark 3.39 (Geometric interpretation). It is possible to interpret the previous result in terms of Hilbert space geometry on a suitably chosen Hilbert space. First, $v = G_\alpha f$ is equivalent to $(A + \alpha)v = f$, which in turn is equivalent to the fact that $v \in D(Q)$ with $Q_\alpha(v, w) = \langle f, w\rangle$ for all $w \in D(Q)$. We can write this as

$$Q(v, w) + \alpha\langle v - \frac{1}{\alpha}f, w\rangle = 0$$

for $w \in D(Q)$. Rewriting this with the (semi)-inner product

$$\langle (a, b), (c, d)\rangle_* := Q(a, c) + \alpha\langle b, d\rangle$$

on $D(Q) \times D(Q)$ we infer that $v = G_\alpha f$ if and only if $(v, v - \alpha^{-1}f)$ is perpendicular to the diagonal, i.e.,

$$(v, v - \frac{1}{\alpha}f) = (v, v) - (0, \frac{1}{\alpha}f) \perp U,$$

where U is the subspace

$$U := \{(w, w) \mid w \in D(Q)\}.$$

So, if $x = -(0, \alpha^{-1} f)$, then we want to find an element $\widetilde{v} \in U$ such that $x + \widetilde{v}$ is perpendicular to U.

By standard theory this problem has a unique solution, which is given by the minimizer of $\| \cdot \|_*$ on $x + U$ whenever $\langle \cdot, \cdot \rangle_*$ is an inner product inducing a Hilbert space structure. Now, in general, $\langle \cdot, \cdot \rangle_*$ is not an inner product and completeness may fail on $D(Q) \times D(Q)$. So, the basic theory does not apply directly. However, it is not necessary for $\langle \cdot, \cdot \rangle_*$ to be an inner product giving a Hilbert space structure on the entire space, it suffices that $\langle \cdot, \cdot \rangle_*$ is an inner product on U making U into a Hilbert space. This is indeed the case in our situation and we infer that $v = G_\alpha f$ holds if and only if v minimizes $\| \cdot \|_*$ on $x + U$. As, $j(\cdot) = \| \cdot \|_*^2$ on $x + U$ we obtain the statement of the theorem.

CHAPTER 4

Infinite Graphs II – Graphs and Regular Dirichlet Forms

In this chapter we are going to combine our setup of infinite graphs from Chapter 2 and the general theory of operators and forms and resolvents and semigroups from Chapter 3 in order to set up a Hilbert space theory of operators, forms, resolvents and semigroups associated to graphs. This theory will run in parallel to what has been developed in Chapter 1 for finite graphs.

The relevant Hilbert space will be $\ell^2(X,m)$, where X is a countable set and m is a measure on X with full support. Recall that a graph (b,c) over (X,m) consists of a symmetric map $b\colon X\times X\longrightarrow[0,\infty)$ with zero diagonal satisfying

$$\sum_{y\in X} b(x,y)<\infty, \qquad x\in X,$$

and a non-negative function $c\colon X\longrightarrow[0,\infty)$. Here m is a strictly positive function $X\longrightarrow(0,\infty)$ which extends to a measure via $m(A):=\sum_{x\in A}m(x)$, $A\subseteq X$.

4.1. Forms on Hilbert Spaces

In this section we introduce two closed forms arising from a graph.

Let a graph (b,c) over (X,m) be given. We have already met the space of functions of finite energy

$$\mathcal{D}:=\{f\in C(X)\mid \sum_{x,y\in X} b(x,y)(f(x)-f(y))^2+\sum_{x\in X}c(x)f(x)^2<\infty\}$$

and the energy form $\mathcal{Q}:=\mathcal{Q}_{b,c}\colon\mathcal{D}\times\mathcal{D}\longrightarrow\mathbb{R}$ defined by

$$\mathcal{Q}(f,g):=\frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))(g(x)-g(y))+\sum_{x\in X}c(x)f(x)g(x)$$

and extended on the diagonal to functions not in $\mathcal{D}$ by ∞. We will also need the norm $\|\cdot\|_{\mathcal{Q}}\colon\mathcal{D}\cap\ell^2(X,m)\longrightarrow[0,\infty)$ given by

$$\|f\|_{\mathcal{Q}}:=\left(\mathcal{Q}(f)+\|f\|^2\right)^{1/2},$$

where $\|f\|$ is the $\ell^2(X,m)$ norm of f.

We define the form $Q^{(N)}:=Q^{(N)}_{b,c,m}$ as the — so to speak — maximal restriction of $\mathcal{Q}$ to $\ell^2(X,m)$. Specifically, we define

$$D(Q^{(N)}):=\mathcal{D}\cap\ell^2(X,m)$$

and

$$Q^{(N)}(f,g) := \mathcal{Q}(f,g)$$

for $f,g \in D(Q^{(N)})$. Then, clearly $Q^{(N)}$ is symmetric and positive as $\mathcal{Q}$ has these properties. As above, we set

$$Q^{(N)}(f) := Q^{(N)}(f,f)$$

and extend $Q^{(N)}$ to all of $\ell^2(X,m)$ by setting it to be ∞ outside of $\mathcal{D} \cap \ell^2(X,m)$. We think of $Q^{(N)}$ as arising from some sort of Neumann boundary conditions and this is the reason for the superscript (N). We will refer to $Q^{(N)}$ as the *Neumann form.*

If a sequence (f_n) from $\ell^2(X,m)$ converges to f in $\ell^2(X,m)$, then it clearly converges pointwise and from Proposition 2.4 we obtain

$$Q^{(N)}(f) = \mathcal{Q}(f) \leq \liminf_n \mathcal{Q}(f_n) = \liminf_{n\to\infty} Q^{(N)}(f_n).$$

Thus, $Q^{(N)}$ is a lower semi-continuous map on a subspace of $\ell^2(X,m)$. By standard theory, see Theorem 3.33, $Q^{(N)}$ is then closed, i.e., $D(Q^{(N)})$ is complete with respect to $\|\cdot\|_{\mathcal{Q}}$.

In some sense, $Q^{(N)}$ is the "maximal" form associated to a graph. We will be even more concerned with the "minimal" form. This form comes about by considering all symmetric closed forms which are restrictions of $Q^{(N)}$ (or $\mathcal{Q}$) and whose domain contains $C_c(X)$. The intersection over the domains of all such forms will be a closed subspace of $D(Q^{(N)})$. Hence, the restriction of $\mathcal{Q}$ to this domain will yield a positive closed form. We denote this form by $Q^{(D)} := Q^{(D)}_{b,c,m}$ and its domain by $D(Q^{(D)}) := D(Q^{(D)}_{b,c,m})$.

By construction $Q^{(D)}$ is the smallest closed form extending the restriction of $\mathcal{Q}$ to $C_c(X) \times C_c(X)$. Thus, we can also obtain $D(Q^{(D)}_{b,c,m})$ by taking the closure of $C_c(X)$ with respect to the norm $\|\cdot\|_{\mathcal{Q}}$, that is,

$$D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}.$$

We think of $Q^{(D)}$ as arising from some sort of Dirichlet boundary conditions and this is the reason for the superscript (D).

We furthermore provide a structural characterization of the domain of $Q^{(D)}$. This structural characterization will involve the space $\mathcal{D}_0$. This space comes about as some form of closure of $C_c(X)$ in $\mathcal{D}$ (see Definition 2.7 and its corollary). More specifically, it is the subspace of $f \in \mathcal{D}$ for which there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ pointwise and $\mathcal{Q}(f - \varphi_n) \to 0$ as $n \to \infty$.

THEOREM 4.1 (Domain of $D(Q^{(D)})$). *Let (b,c) be a graph over (X,m) with associated energy form $\mathcal{Q}_{b,c}$. Then, $Q^{(D)}$ is the restriction of $\mathcal{Q}_{b,c}$ to*

$$D(Q^{(D)}) = \mathcal{D}_0 \cap \ell^2(X,m).$$

REMARK 4.2. To put this result into perspective, we compare it with the corresponding statement for the Neumann form $Q^{(N)}$. By definition, $Q^{(N)}$ arises as the restriction of $\mathcal{Q}_{b,c}$ to

$$D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m).$$

So, we see that the difference between the Dirichlet and Neumann boundary conditions comes from a corresponding difference between $\mathcal{D}$ and $\mathcal{D}_0$.

PROOF. To show

$$D(Q^{(D)}) = \mathcal{D}_0 \cap \ell^2(X,m)$$

we will prove two inclusions.

$D(Q^{(D)}) \subseteq \mathcal{D}_0 \cap \ell^2(X,m)$: By definition, $D(Q^{(D)})$ is the closure of $C_c(X)$ with respect to $\| \cdot \|_{\mathcal{Q}}$. This immediately gives the statement as $\ell^2(X,m)$ convergence implies pointwise convergence.

$\mathcal{D}_0 \cap \ell^2(X,m) \subseteq D(Q^{(D)})$: Let $f \in \mathcal{D}_0 \cap \ell^2(X,m)$. As $Q^{(D)}$ is a closed form, the restriction of $Q^{(D)}$ to the diagonal is lower semi-continuous, it suffices to find a sequence (χ_n) in $C_c(X)$ with $\chi_n \to f$ in $\ell^2(X,m)$ as $n \to \infty$ and $(Q^{(D)}(\chi_n))$ bounded.

Since $f \in \mathcal{D}_0$ we can find a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ pointwise and $\mathcal{Q}(f - \varphi_n) \to 0$ as $n \to \infty$. This implies, in particular, that the sequence $(Q^{(D)}(\varphi_n)) = (\mathcal{Q}(\varphi_n))$ is bounded. We will modify the sequence (φ_n) in order to obtain a sequence (χ_n) converging to f in $\ell^2(X,m)$. Consider

$$\psi_n := \varphi_n \wedge |f|, \quad n \in \mathbb{N}.$$

Claim. We have:

- $\psi_n \in C_c(X)$ for all $n \in \mathbb{N}$.
- $\psi_n \to f$ pointwise as $n \to \infty$.
- The sequence $(Q^{(D)}(\psi_n))_n$ is bounded.

Proof of the claim. The first two statements are straightforward. The last statement follows from

$$\begin{aligned} |\psi_n(x) - \psi_n(y)| &\leq |\varphi_n(x) - \varphi_n(y)| + |\, |f(x)| - |f(y)|\, | \\ &\leq |\varphi_n(x) - \varphi_n(y)| + |f(x) - f(y)|. \end{aligned}$$

Consider now $\chi_n := \psi_n \vee -|f|$ for $n \in \mathbb{N}$. Then, we clearly have

$$\chi_n = -(-\psi_n \wedge |f|), \quad n \in \mathbb{N}.$$

Thus, we can apply the reasoning of the previous claim to obtain:

- $\chi_n \in C_c(X)$ for all $n \in \mathbb{N}$.
- $\chi_n \to f$ pointwise as $n \to \infty$.
- The sequence $(Q^{(D)}(\chi_n))_n$ is bounded.

Moreover, by construction the sequence (χ_n) satisfies

$$-|f| \leq \chi_n \leq |f|$$

for $n \in \mathbb{N}$. Thus, by Lebesgue's dominated convergence theorem, the sequence (χ_n) converges to f in $\ell^2(X,m)$. Hence, the sequence (χ_n) has all of the desired properties. This finishes the proof. □

4.2. Graphs and (Regular) Dirichlet Forms

Let X be a countable set and let m be a measure on X with full support. We will be concerned with forms on $\ell^2(X,m)$. Indeed, we have already seen that any graph (b,c) over (X,m) gives rise to two forms. For short we will refer to forms on $\ell^2(X,m)$ also as forms over (X,m). Throughout we will freely use the theory and notation developed in Section 3.6.

Let $C\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a normal contraction, i.e., a map with $C(0)=0$ and $|C(s)-C(t)| \leq |s-t|$ for $s,t \in \mathbb{R}$. If a closed form Q over (X,m) has the property that $C \circ f$ belongs to $D(Q)$ with

$$Q(C \circ f) \leq Q(f)$$

for all $f \in D(Q)$ it is said to be *compatible* with the normal contraction C.

Definition 4.3. A closed form Q on $\ell^2(X,m)$ is called a *Dirichlet form* if it is compatible with all normal contractions.

For a graph (b,c) over (X,m), we show next that $Q^{(N)} = Q^{(N)}_{b,c,m}$ is a Dirichlet form. This form was introduced in Section 4.1 as the restriction of $\mathcal{Q} = \mathcal{Q}_{b,c}$ to $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$.

Proposition 4.4 ($Q^{(N)}$ is a Dirichlet form). *Let (b,c) be graph over (X,m). Then, $Q^{(N)}_{b,c,m}$ is a Dirichlet form.*

Proof. As $Q^{(N)}$ is a restriction of $\mathcal{Q}$, it is lower semi-continuous by Proposition 2.4. By Theorem 3.33 this implies that $Q^{(N)}$ is closed. Clearly, for all normal contractions C and $f \in \ell^2(X,m)$, it follows that $C \circ f \in \ell^2(X,m)$. Furthermore, for $f \in D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$, we find from the compatibility of $\mathcal{Q}_{b,c}$ with normal contractions

$$Q^{(N)}(C \circ f) = \mathcal{Q}_{b,c}(C \circ f) \leq \mathcal{Q}_{b,c}(f) = Q^{(N)}(f).$$

Thus, $Q^{(N)}$ is closed and compatible with normal contractions. Therefore, $Q^{(N)}$ is a Dirichlet form. □

Let $\|\cdot\|_\infty$ denote the supremum norm on $C_c(X)$. A Dirichlet form Q over (X,m) is called *regular* if $D(Q) \cap C_c(X)$ is dense in both $C_c(X)$ with respect to $\|\cdot\|_\infty$ and in $D(Q)$ with respect to the form norm $\|\cdot\|_Q$.

It turns out that a Dirichlet form Q on (X,m) is regular if and only if $C_c(X) \subseteq D(Q)$ and Q is the closure of the restriction of Q to the subspace $C_c(X)$. The "if" direction is immediate from the definition of a regular Dirichlet form. The "only if" direction is shown next.

Lemma 4.5. *Let Q be a regular Dirichlet form over (X,m). Then, $C_c(X)$ is contained in $D(Q)$. In particular, Q is the closure of the restriction of Q to $C_c(X) \times C_c(X)$.*

Proof. Let $x \in X$ be arbitrary and let $\varphi := 2 \cdot 1_x$ so that $\varphi \in C_c(X)$. We will show that $\varphi \in D(Q)$. As x is chosen arbitrarily, this will imply the first statement.

As Q is regular, $C_c(X) \cap D(Q)$ is dense in $C_c(X)$ with respect to the supremum norm, so there exists a $\psi \in D(Q)$ with $1 < \psi(x) < 3$ and $|\psi(y)| < 1$ for all $y \neq x$, i.e.,

$$\|\varphi - \psi\|_\infty < 1.$$

As Q is a Dirichlet form, $D(Q)$ is invariant under taking the modulus and we can assume $\psi \geq 0$. Furthermore, as taking the minimum with 1 is also a normal contraction, $\psi \wedge 1 \in D(Q)$. As $D(Q)$ is a vector space it contains $\psi - \psi \wedge 1$ and this is a nonzero multiple of φ by construction. Thus $\varphi \in D(Q)$ and as $x \in X$ is arbitrary, the first statement follows.

As Q was assumed to be regular, the space $C_c(X) = C_c(X) \cap D(Q)$ is dense in $D(Q)$ with respect to the form norm and the "in particular" statement follows. □

Next, we will show that the domain of $Q^{(D)} = Q^{(D)}_{b,c,m}$ is preserved by normal contractions.

Lemma 4.6 ($Q^{(D)}$ is a regular Dirichlet form). *Let (b,c) be a graph over (X,m). Then, $Q^{(D)}$ is a regular Dirichlet form.*

Proof. We first show that $Q^{(D)} = Q^{(D)}_{b,c,m}$ is a Dirichlet form. We denote the restriction of $Q^{(D)}$ to $C_c(X) \times C_c(X)$ by $\mathcal{Q}^{(\mathrm{comp})}_{b,c}$. Note that $\mathcal{Q}^{(\mathrm{comp})}_{b,c}$ is a restriction of $\mathcal{Q}_{b,c}$ by the very definition of $Q^{(D)}$. Note also that $C \circ \varphi$ belongs to $C_c(X)$ whenever φ belongs to $C_c(X)$ and C is a normal contraction. From the compatibility of $\mathcal{Q}_{b,c}$ with normal contractions we then infer for a normal contraction C that

$$\begin{aligned} Q^{(D)}(C \circ \varphi) &= \mathcal{Q}^{(\mathrm{comp})}_{b,c}(C \circ \varphi) = \mathcal{Q}_{b,c}(C \circ \varphi) \\ &\leq \mathcal{Q}_{b,c}(\varphi) = \mathcal{Q}^{(\mathrm{comp})}_{b,c}(\varphi) = Q^{(D)}(\varphi) \end{aligned}$$

for all $\varphi \in C_c(X)$.

We will next extend this inequality to the whole $D(Q^{(D)})$. In particular, we will show that $C \circ f \in D(Q^{(D)})$ for $f \in D(Q^{(D)})$.

As $Q^{(D)}$ is the closure of its restriction $\mathcal{Q}^{(\mathrm{comp})}_{b,c}$ to $C_c(X) \times C_c(X)$, there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ with respect to $\| \cdot \|_Q$. In particular, $\varphi_n \to f$ in $\ell^2(X,m)$. Then, clearly, the sequence $(C \circ \varphi_n)$ belongs to $C_c(X)$ and converges to $C \circ f$ in $\ell^2(X,m)$. Moreover, the sequence $(Q^{(D)}(C \circ \varphi_n))$ is bounded as

$$Q^{(D)}(C \circ \varphi_n) = \mathcal{Q}^{(\mathrm{comp})}_{b,c}(C \circ \varphi_n) \leq \mathcal{Q}^{(\mathrm{comp})}_{b,c}(\varphi_n) = Q^{(D)}(\varphi_n) \to Q^{(D)}(f)$$

as $n \to \infty$. From Proposition 3.35, we then infer $C \circ f \in D(Q^{(D)})$ and

$$Q^{(D)}(C \circ f) \leq Q^{(D)}(f).$$

Therefore, $Q^{(D)}$ is a Dirichlet form.

By construction, $Q^{(D)}$ is the closure of $\mathcal{Q}^{(\mathrm{comp})}_{b,c}$. Hence, $Q^{(D)}$ is regular. This finishes the proof. □

It turns out that the converse to the previous lemma holds as well.

LEMMA 4.7 (Regular Dirichlet forms arise from graphs). *Let Q be a regular Dirichlet form over (X, m). Then, there exists a graph (b, c) over (X, m) with $Q = Q^{(D)}_{b,c,m}$.*

PROOF. By Lemma 4.5, $C_c(X)$ is contained in $D(Q)$. Define $b\colon X \times X \longrightarrow \mathbb{R}$ by

$$b(x, y) := -Q(1_x, 1_y)$$

for $x \neq y$ and $b(x, x) := 0$ for $x \in X$, and define $c\colon X \longrightarrow \mathbb{R}$ by

$$c(x) := Q(1_x) - \sum_{y\in X} b(x, y).$$

We will show that (b, c) is a graph with $Q^{(D)}_{b,c,m} = Q$. This will also show that the sum appearing in the definition of c is absolutely convergent.

For a finite subset K of X we let

$$i_K\colon C(K) \longrightarrow C_c(X)$$

be the canonical inclusion (i.e. $i_K(f)(x) = f(x)$ for $x \in K$ and $i_K(f)(x) = 0$ for $x \notin K$). Note that — due to the finiteness of K — the map i_K maps indeed into $C_c(X)$. Due to regularity $C_c(X)$ belongs to the domain of Q (see Lemma 4.5). Thus, Q induces a form Q_K on $C(K)$ defined by

$$Q_K(f, g) := Q(i_K(f), i_K(g)).$$

Clearly, $C \circ i_K(f) = i_K(C \circ f)$ holds for all normal contractions C and all $f \in C(K)$. As Q is a Dirichlet form we therefore find

$$Q_K(Cf) = Q(i_K(Cf)) = Q(Ci_K(f)) \leq Q(i_K(f)) = Q_K(f).$$

Hence, Q_K is a Dirichlet form. By the results of the first chapter we then find

(a) For any $x, y \in X$ with $x \neq y$, we have

$$b(x, y) = -Q(1_x, 1_y) = -Q_{\{x,y\}}(1_x, 1_y) \geq 0.$$

(b) For any finite $K \subseteq X$ and $x \in K$, we have

$$Q(1_K, 1_x) = Q_K(1, 1_x) \geq 0.$$

From (a) the function b is positive. Moreover, for any $K \subseteq X$ finite and any $x \in K$ we compute

$$\begin{aligned} Q(1_x) &= Q(1_K, 1_x) - \sum_{y\in K, y\neq x} Q(1_y, 1_x) = Q(1_K, 1_x) + \sum_{y\in K, y\neq x} b(x, y) \\ &= Q(1_K, 1_x) + \sum_{y\in K} b(x, y). \end{aligned}$$

As, by (a) and (b) both $Q(1_K, 1_x)$ and b are positive, we can now conclude

$$\sum_{y\in K} b(x, y) \leq Q(1_x)$$

for any $K \subseteq X$ finite and this gives

$$\sum_{y\in X} b(x, y) \leq Q(1_x) < \infty.$$

From this we infer

$$Q(1_x) - \sum_{y \in X} b(x,y) \geq 0$$

for all $x \in X$. Thus, c defined at the beginning of the proof exists and is non-negative. Hence, (b,c) is indeed a graph.

Moreover, from the very definitions of b and c we conclude for $x, y \in X$ with $x \neq y$

$$Q(1_x, 1_y) = -b(x,y) = Q^{(D)}_{b,c,m}(1_x, 1_y)$$

and for $x \in X$

$$Q(1_x) = c(x) + \sum_{y \in X} b(x,y) = Q^{(D)}_{b,c,m}(1_x).$$

By bilinearity, Q and $Q^{(D)}_{b,c,m}$ agree on $C_c(X)$. As both are regular Dirichlet forms, they must then be equal. □

Combining the preceding lemmas we infer that regular Dirichlet forms on discrete sets are exactly the forms arising from graphs with Dirichlet boundary conditions. Specifically, the following holds.

THEOREM 4.8 (Regular Dirichlet forms and graphs). *The map*

$$(b,c) \mapsto Q^{(D)}_{b,c,m}$$

is a bijective correspondence between graphs (b,c) *over* (X,m) *and regular Dirichlet forms over* (X,m).

PROOF. This is a direct consequence of Lemmas 4.6 and 4.7. In particular, injectivity of the map follows directly from the first lines of the proof of Lemma 4.7. □

4.3. Laplacians on Hilbert Spaces

In this section we investigate the Laplacian associated to $Q^{(D)}_{b,c,m}$.

Let (b,c) be a graph over (X,m) and $Q^{(D)} := Q^{(D)}_{b,c,m}$ the associated regular form. By the theory of closed forms, see Lemma 3.30 and Corollary 3.37, there exists a unique self-adjoint operator $L^{(D)} := L^{(D)}_{b,c,m}$ on $\ell^2(X,m)$ whose domain $D(L^{(D)})$ is contained in $D(Q^{(D)})$ and which satisfies

$$\langle g, L^{(D)} f \rangle = Q^{(D)}(g,f)$$

for all $f \in D(L^{(D)})$ and $g \in D(Q^{(D)})$. We call $L^{(D)}$ the *Dirichlet Laplacian* or just *the Laplacian* associated to a graph. We denote the *spectrum* of $L^{(D)}$ by $\sigma(L^{(D)})$ and the bottom of the spectrum of $L^{(D)}$ by $\lambda_0(L^{(D)}) := \inf \sigma(L^{(D)})$. We note that $L^{(D)}$ is positive and thus $\sigma(L^{(D)}) \subseteq [0,\infty)$ and $\lambda_0(L^{(D)}) \geq 0$.

In general, it is rather hard to describe explicitly the domain of $L^{(D)}$. Still, the action of this operator is easy to describe.

To do so, we recall the definition of the formal operator $\mathcal{L} := \mathcal{L}_{b,c,m}$ associated to a graph (b,c) over the measure space (X,m). This operator has domain $\mathcal{F} = \{f \in C(X) \mid \sum_{y\in X} b(x,y)|f(y)| < \infty \text{ for all } x \in X\}$ and acts via

$$\mathcal{L}f(x) = \frac{1}{m(x)}\mathcal{L}_{b,c}f(x) = \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(f(x)-f(y)) + c(x)f(x)\right).$$

This operator is intimately linked to the form $\mathcal{Q} := \mathcal{Q}_{b,c}$. Indeed, Green's formula in Proposition 2.9 gives that any $f \in \mathcal{D}$ satisfies $f \in \mathcal{F}$ and

$$\mathcal{Q}(\varphi, f) = \sum_{x\in X} \varphi(x)\mathcal{L}f(x)m(x)$$

holds for all $\varphi \in C_c(X)$. This is the essential step in the proof of the following result.

Theorem 4.9 (Action of the Dirichlet Laplacian). *Let (b,c) be a graph over (X,m) and let $L^{(D)}$ be the Dirichlet Laplacian. Then $D(L^{(D)}) \subseteq \mathcal{F}$ and*

$$L^{(D)}f(x) = \mathcal{L}f(x)$$

for all $f \in D(L^{(D)})$ and $x \in X$.

Proof. By definition, $L^{(D)}$ is the unique self-adjoint operator with $D(L^{(D)}) \subseteq D(Q^{(D)})$ which satisfies $\langle g, L^{(D)}f\rangle = Q^{(D)}(g,f)$ for all $f \in D(L^{(D)})$ and $g \in D(Q^{(D)})$. Furthermore, as $Q^{(D)}$ is a restriction of $\mathcal{Q}$ and $C_c(X) \subseteq D(Q^{(D)}) \subseteq \mathcal{D} \subseteq \mathcal{F}$, from Green's formula, Proposition 2.9, we have

$$\langle \varphi, L^{(D)}f\rangle = Q^{(D)}(\varphi, f) = \mathcal{Q}(\varphi, f) = \sum_{x\in X} \varphi(x)\mathcal{L}f(x)m(x)$$

for all $\varphi \in C_c(X)$ and $f \in D(Q^{(D)})$. The conclusion follows by choosing $\varphi := 1_x/m$ for arbitrary x. □

4.4. Semigroups and Resolvents

In Section 3.5 we have discussed how positive operators come with a semigroup and a resolvent and how semigroups and resolvents give rise to solutions of the heat equation and the Poisson equation, respectively. In this section we apply this to the graph Laplacians.

Consider a graph (b,c) over (X,m) and let $\mathcal{L}$ be the associated formal Laplacian. A *solution of the heat equation* (for $\mathcal{L}$)) is a function $u\colon [0,\infty) \times X \longrightarrow \mathbb{R}$, $(t,x) \longmapsto u_t(x)$ such that $t \mapsto u_t(x)$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$ for all x and satisfies $u_t \in \mathcal{F}$ for all $t > 0$ such that

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

holds for all $x \in X$ and $t > 0$. We note that if $f \in \ell^2(X,m)$, then the function u given by

$$u_t(x) := e^{-tL^{(D)}}f(x), \quad t \geq 0, x \in X$$

is a solution of the heat equation with initial condition f, see Theorem 3.24. Similarly, for $f \in \ell^2(X,m)$ and $\alpha > 0$ the Poisson equation (associated to $\mathcal{L} + \alpha$)

$$(\mathcal{L} + \alpha)u = f$$

has the solution

$$u := (L^{(D)} + \alpha)^{-1} f.$$

By Theorem 3.26 the Laplace transform gives

$$(L^{(D)} + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL^{(D)}} \mathrm{d}t.$$

and the exponential formula gives

$$e^{-tL^{(D)}} = \lim_{n\to\infty} \left(\frac{n}{t} \left(L^{(D)} + \frac{n}{t} \right)^{-1} \right)^n .$$

CHAPTER 5

Regular Dirichlet forms: Approximation, Domain monotonicity and Markov Property

The crucial feature of regular Dirichlet forms is that they are determined by what happens on finite sets. This has various consequences, which are discussed in this section. In particular, we discuss domain monotonicity, the Markov property for semigroups and resolvents and a minimality property of solutions for the Dirichlet Laplacian $L = L^{(D)}$. Recall that the Markov property of the resolvents and semigroups mean that

$$0 \leq \alpha(L+\alpha)^{-1} f \leq 1, \quad \alpha > 0 \qquad \text{and} \qquad 0 \leq e^{-tL} f \leq 1, \quad t \geq 0$$

for $f \in \ell^2(X,m)$, $0 \leq f \leq 1$. While it can easily be derived by approximation it is not a consequence of regularity but rather a general feature of Dirichlet forms. This is elaborated upon in the second part of the section.

5.1. Approximation

We have already seen in the proof of Lemma 4.7 how a regular Dirichlet form is approximated by its restriction to finite set. Here, we are going to elaborate on this approximation and derive various consequences.

Let X be a countable set and m a measure on X with full support. For a graph (b,c) over (X,m) denote the associated energy form by $\mathcal{Q} = \mathcal{Q}_{b,c}$ and the formal Laplacian $\mathcal{L} = \mathcal{L}_{b,c,m}$.

For any finite set $K \subseteq X$, we denote the restriction of m to K by m_K and let $Q_K^{(D)}$ be the form defined on $\ell^2(K,m_K)$ by

$$Q_K^{(D)}(f) := \mathcal{Q}(i_K f)$$

for $f \in \ell^2(K,m_K)$. Here,

$$i_K \colon C(K) \longrightarrow C_c(X)$$

is the canonical embedding, i.e., $i_K f$ is the extension of $f \in C(K)$ to X by setting $i_K f$ to be identically zero outside of K. We call $Q_K^{(D)}$ the *restriction of* $\mathcal{Q}$ to K.

Clearly, $Q_K^{(D)}$ is a closed form on $\ell^2(K,m_K)$ since the domain of $Q_K^{(D)}$ is the entire (finite dimensional) Hilbert space $\ell^2(K,m_K)$. Also, $C \circ (i_K f) = i_K(C \circ f)$ obviously holds for all $f \in C(K)$ and all normal contractions C and this easily gives that $Q_K^{(D)}$ is a Dirichlet form (compare reasoning in the proof of Lemma 4.7 as well).

As $Q_K^{(D)}$ is a Dirichlet form on the finite set K, it must come from a graph (b_K, c_K) over K. It is not hard to determine this graph. Indeed, a short calculation gives

$$Q_K^{(D)}(f) = \mathcal{Q}(i_K f) = \mathcal{Q}_{b_K,c_K}(f) + \sum_{x\in K} d_K(x)f(x)^2 = \mathcal{Q}_{b_K,c_K+d_K}(f),$$

where b_K is the restriction of b to $K \times K$, c_K is the restriction of c to K and

$$d_K(x) := \sum_{y\in X\setminus K} b(x,y)$$

describes the edge deficiency of a vertex $x \in K$ compared to the same vertex in X. Thus, $Q_K^{(D)}$ is the Dirichlet form associated to the graph $(b_K, c_K + d_K)$ over (K, m_K).

We denote the self-adjoint operator associated to $Q_K^{(D)}$ by $L_K^{(D)}$ and call it the *Dirichlet Laplacian with respect to* K. As $\ell^2(K, m_K)$ is finite dimensional this is a bounded operator. Quite remarkably this operator can be computed just from $\mathcal{L}$ independent of K. More specifically, we have the following proposition.

Proposition 5.1 (Computing $L_K^{(D)}$ from $\mathcal{L}$). *Let (b, c) be a graph over (X, m) and $\mathcal{L}$ the associated formal Laplacian. Let $K \subseteq X$ be a finite set and $L_K^{(D)}$ the operator associated to the restriction of $\mathcal{Q}_{b,c}$ to K. Then,*

$$(L_K^{(D)} f)(x) = (\mathcal{L} \circ i_K)(f)(x)$$

holds for any $f \in \ell^2(K, m_K)$ and any $x \in K$.

Proof. From the definition of $L_K^{(D)}$ and Green's formula for $\mathcal{Q}_{b,c}$ we find by a direct computation

$$\begin{aligned}\sum_{x\in K} \varphi(x)L_K^{(D)} f(x)m(x) &= \langle \varphi, L_K^{(D)} f\rangle_{\ell^2(K,m_K)} = Q_K^{(D)}(\varphi, f)\\ &= \mathcal{Q}_{b,c}(i_K(\varphi), i_K(f)) = \sum_{x\in X} \varphi(x)\mathcal{L}i_K(f)(x)m(x)\end{aligned}$$

for all $\varphi, f \in C(K)$. This then implies the formula for $L_K^{(D)}$. □

Since K is assumed to be finite, we can apply the results from Chapter 1 about finite graphs. This gives that the eigenvalues of $L_K^{(D)}$ are non-negative. Moreover, it gives that the resolvents $(L_K^{(D)} + \alpha)^{-1}$ satisfy the Markov property for all $\alpha > 0$. Specifically, for a function $f \in \ell^2(K, m_K)$ with $0 \le f \le 1$ and $\alpha > 0$, we have

$$0 \le \alpha(L_K^{(D)} + \alpha)^{-1} f \le 1.$$

After these preparations we now turn to the first main result of this section. It says that the resolvent of a restriction to a set becomes larger if the set is increased. This is known as domain monotonicity. A precise version is contained in the next lemma.

Lemma 5.2 (Domain monotonicity). *Let (b, c) be a graph over (X, m). If K, H are finite subsets of X with $K \subseteq H$ and $\alpha > 0$, then*

$$(L_K^{(D)} + \alpha)^{-1} f \le (L_H^{(D)} + \alpha)^{-1} f$$

on K for all $f \in \ell^2(K, m_K)$ with $f \geq 0$, where f is extended by zero on $H \setminus K$.

Proof. Set $u_K := i_K(L_K^{(D)} + \alpha)^{-1} f$, $u_H := i_H(L_H^{(D)} + \alpha)^{-1} f$ and $v := u_H - u_K$. On K we have by Proposition 5.1 that

$$\begin{aligned}(\mathcal{L} + \alpha)v &= (\mathcal{L} + \alpha)i_H(L_H^{(D)} + \alpha)^{-1} f - (\mathcal{L} + \alpha)i_K(L_K^{(D)} + \alpha)^{-1} f \\ &= (L_H^{(D)} + \alpha)(L_H^{(D)} + \alpha)^{-1} f - (L_K^{(D)} + \alpha)(L_K^{(D)} + \alpha)^{-1} f \\ &= f - f \\ &= 0.\end{aligned}$$

We use this to show $v \geq 0$: Clearly, $v = 0$ holds on $X \setminus H$. Moreover, $v \geq 0$ holds on $H \setminus K$ as u_K vanishes outside of K and $u_H \geq 0$ holds by the Markov property of the resolvents (of Dirichlet forms on finite dimensional spaces). It remains to show $v \geq 0$ on K. As K is finite, the restriction of v to K must attain a minimum. Consider now $x_0 \in K$ such that $v(x_0)$ is the minimum of v on K. Assume $v(x_0) < 0$. As we have already established $v \geq 0$ outside of K we conclude that the value $v(x_0)$ is the minimal value of v on the whole of X. Using this and the already established vanishing of $(\mathcal{L} + \alpha)v$ on K we then obtain the contradiction

$$\begin{aligned}0 &= (\mathcal{L} + \alpha)v(x_0) \\ &= \frac{1}{m(x_0)}\left(\sum_{y \in X} b(x_0, y)\underbrace{(v(x_0) - v(y))}_{\leq 0} + \underbrace{c(x_0)}_{\geq 0}\underbrace{v(x_0)}_{<0}\right) + \alpha\underbrace{v(x_0)}_{<0} < 0.\end{aligned}$$

This contradiction shows $v(x_0) \geq 0$ and thus $v \geq 0$. □

Our next result will show convergence of the restrictions to finite subsets for both the resolvent and the semigroup. In order to be able to state the result conveniently we will use the following notation.

Notation. Let (b, c) be a graph over (X, m), let $Q = Q_{b,c,m}^{(D)}$ be the associated regular Dirichlet form and $Q_K^{(D)}$ be the restriction of Q to the finite set $K \subseteq X$ with associated Dirichlet Laplacian $L_K^{(D)}$ acting on $\ell^2(K, m_K)$ as defined above. We extend $L_K^{(D)}$ by zero on the orthogonal complement of $\ell^2(K, m_K)$ in $\ell^2(X, m)$. We will extend functions Φ of $L_K^{(D)}$ accordingly, that is, for $f \in \ell^2(X, m)$, we write $\Phi(L_K^{(D)})f$ for $i_K\Phi(L_K^{(D)})(f|_K)$. This is, in particular, used for the function $\Phi(\lambda) := (\lambda + \alpha)^{-1}$, i.e.,

$$(L_K^{(D)} + \alpha)^{-1} f \quad \text{for} \quad i_K(L_K^{(D)} + \alpha)^{-1}(f|_K),$$

but also applies to $\Phi(\lambda) := (\lambda + \alpha)$ or $\Phi(\lambda) := e^{-t\lambda}$. The extended operators will be denoted by the same symbols as the original ones.

Lemma 5.3 (Convergence of finite approximations). *Let (b, c) be a graph over (X, m) and let Q be the associated regular Dirichlet form with Laplacian L. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_{n \in \mathbb{N}} K_n$.*

(a) *If $f \in \ell^2(X,m)$ and $\alpha > 0$, then*

$$\lim_{n\to\infty} (L_{K_n}^{(D)} + \alpha)^{-1} f = (L+\alpha)^{-1} f.$$

(b) *If $f \in \ell^2(X,m)$ and $t \geq 0$, then*

$$\lim_{n\to\infty} e^{-tL_{K_n}^{(D)}} f = e^{-tL} f.$$

Furthermore, if additionally $f \geq 0$, then the sequences in both statements converge not only in $\ell^2(X,m)$ but also pointwise monotonically increasingly, i.e.,

$$(L_{K_n}^{(D)} + \alpha)^{-1} f \nearrow (L+\alpha)^{-1} f \quad \text{ and } \quad e^{-tL_{K_n}^{(D)}} f \nearrow e^{-tL} f$$

pointwise as $n \to \infty$.

Remark 5.4. The proof of (b) will actually show

$$\lim_{n\to\infty} \Phi(L_{K_n}^{(D)}) f = \Phi(L) f$$

for any $f \in \ell^2(X,m)$ and any continuous function $\Phi\colon [0,\infty) \longrightarrow \mathbb{R}$ vanishing at infinity (i.e. satisfying $\lim_{s\to\infty} \Phi(s) = 0$).

Proof. (a) In the proof we will use the following characterization of resolvents: Whenever Q is a positive closed form with associated self-adjoint operator L, the function f is an arbitrary element of the underlying Hilbert space and $\alpha > 0$, then $u := (L+\alpha)^{-1} f$ is the unique minimizer of

$$J(v) := Q(v) + \alpha \left\| v - \frac{1}{\alpha} f \right\|^2$$

over $v \in D(Q)$. See Theorem 3.38 for a proof of this characterization.

After decomposing f into positive and negative parts, we can restrict attention to $f \geq 0$. Define

$$u_n := (L_{K_n}^{(D)} + \alpha)^{-1} f, \quad n \in \mathbb{N}.$$

Then, $u_n \geq 0$ by the Markov property, see the paragraph before Lemma 5.2.

By domain monotonicity, Lemma 5.2, the sequence $(u_n(x))$ is monotonically increasing for any $x \in X$. Moreover, we have $\|u_n\| \leq \alpha^{-1} \|f\|$ since the operators $(L_{K_n}^{(D)} + \alpha)^{-1}$ are bounded uniformly in norm by $1/\alpha$, as follows from the spectral theorem. This implies that $(u_n(x))$ is also bounded for any $x \in X$. Thus, the sequence (u_n) converges pointwise and in $\ell^2(X,m)$ to a function $u \in \ell^2(X,m)$ by Lebesgue's dominated convergence theorem.

Let $\varphi \in C_c(X)$. Assume without loss of generality that the support of φ is contained in K_1. Then, $Q(\varphi) = Q_{K_n}^{(D)}(\varphi)$ for all n sufficiently large. Since Q is closed and thus lower semi-continuous, convergence of (u_n) to u and the minimizing property of u_n then give

$$\begin{aligned}Q(u) + \alpha \left\|u - \tfrac{1}{\alpha}f\right\|^2 &\leq \liminf_{n\to\infty} \left(Q(u_n) + \alpha \left\|u - \tfrac{1}{\alpha}f\right\|^2\right)\\ &= \liminf_{n\to\infty} \left(Q(u_n) + \alpha \left\|u_n - \tfrac{1}{\alpha}f\right\|^2\right)\\ &= \liminf_{n\to\infty} \left(Q^{(D)}_{K_n}(u_n) + \alpha \left\|u_n - \tfrac{1}{\alpha}f\right\|^2\right)\\ &\leq \liminf_{n\to\infty} \left(Q^{(D)}_{K_n}(\varphi) + \alpha \left\|\varphi - \tfrac{1}{\alpha}f\right\|^2\right)\\ &= Q(\varphi) + \alpha \left\|\varphi - \tfrac{1}{\alpha}f\right\|^2.\end{aligned}$$

As $\varphi \in C_c(X)$ is arbitrary and Q is regular, this implies

$$Q(u) + \alpha \left\|u - \tfrac{1}{\alpha}f\right\|^2 \leq Q(v) + \alpha \left\|v - \tfrac{1}{\alpha}f\right\|^2$$

for any $v \in D(Q)$. Thus, u is a minimizer of

$$Q(v) + \alpha \left\|v - \tfrac{1}{\alpha}f\right\|^2,$$

so that u must then be equal to $(L+\alpha)^{-1}f$ by the characterization of the resolvent stated at the start of the proof.

(b) The statement is clear for $t = 0$. Let $C_0([0,\infty)) := \{\Phi \in C([0,\infty)) \mid \lim_{s\to\infty} \Phi(s) = 0\}$ be the vector space of continuous functions vanishing at infinity. Define for $\alpha > 0$ the function $\varPhi_{(\alpha)} \colon [0,\infty) \longrightarrow \mathbb{R}$ by

$$\varPhi_{(\alpha)}(s) := (s+\alpha)^{-1}.$$

Then, clearly $\varPhi_{(\alpha)} \in C_0([0,\infty))$ for any $\alpha > 0$ and $\varPhi_{(\alpha)}(L) = (L+\alpha)^{-1}$ by the functional calculus, see Definition 3.7.

Let $\mathcal{A}$ be the closure in the supremum norm of the linear span of $\varPhi_{(\alpha)}$ for $\alpha > 0$. Then, by (a) we have

$$\lim_{n\to\infty} \Phi(L^{(D)}_{K_n})f = \Phi(L)f$$

for all $\Phi \in \mathcal{A}$ and $f \in \ell^2(X,m)$. We will show that for every $t > 0$, the function $[0,\infty) \longrightarrow \mathbb{R}$ given by $x \mapsto e^{-tx}$ belongs to $\mathcal{A}$, which will complete the proof.

We note that it suffices to show that

$$\mathcal{A} = C_0([0,\infty)).$$

We will do so by proving the following claim and then applying the Stone–Weierstrass theorem.

Claim. The set $\mathcal{A}$ has the following properties:

- $\mathcal{A}$ separates the points of $[0,\infty)$ (i.e., for any $x, y \in [0,\infty)$ with $x \neq y$ there exists a $\Phi \in \mathcal{A}$ with $\Phi(x) \neq \Phi(y)$).
- $\mathcal{A}$ does not vanish identically at any point (i.e., for any $x \in [0,\infty)$ there exists a $\Phi \in \mathcal{A}$ with $\Phi(x) \neq 0$).
- $\mathcal{A}$ is an algebra.

Proof of the claim. The first two points follow directly by considering $\Phi := \varPhi_{(1)}$. As for the last point, by definition, $\mathcal{A}$ is a vector space. Thus, it suffices to show that $\mathcal{A}$ is closed under multiplication. To show this it suffices to show $\Phi_{(\alpha)}\Phi_{(\beta)} \in \mathcal{A}$ for any $\alpha, \beta > 0$. For $\alpha \neq \beta$ this is clear as

$$\Phi_{(\alpha)}\Phi_{(\beta)} = \frac{1}{\alpha - \beta}(\Phi_{(\beta)} - \Phi_{(\alpha)}).$$

For $\alpha = \beta$ we can consider a sequence (β_n) of positive numbers with $\beta_n \to \beta = \alpha$ and $\beta_n \neq \beta$ for all n. Then, by what we have just shown $\Phi_{(\alpha)}\Phi_{(\beta_n)}$ belongs to $\mathcal{A}$ as $\beta_n \neq \alpha$. Thus, $\Phi_{(\alpha)}\Phi_{(\beta)} \in \mathcal{A}$ as $\lim_{n\to\infty} \Phi_{(\alpha)}\Phi_{(\beta_n)} = \Phi_{(\alpha)}\Phi_{(\beta)}$ in the supremum norm. This finishes the proof of the claim.

Given the claim, the desired statement that $\mathcal{A} = C_0([0,\infty))$ follows directly from the Stone–Weierstrass theorem. This concludes the proof of (b).

In the case of $f \geq 0$, the fact that the sequence (u_n) given by $u_n := (L_{K_n}^{(D)} + \alpha)^{-1} f$ is monotonically increasing pointwise follows from Lemma 5.2. The corresponding statement for $(e^{-tL_{K_n}^{(D)}} f)$ for $t > 0$ follows from the connection between resolvents and semigroups (the case $t = 0$ is clear). That is, from the formula

$$\left(\frac{k}{t}\left(x + \frac{k}{t}\right)^{-1}\right)^k = \left(1 + \frac{tx}{k}\right)^{-k} \to e^{-tx}$$

as $k \to \infty$ for any $t > 0$, it follows that

$$e^{-tL_{K_n}^{(D)}} f = \lim_{k\to\infty} \left(\frac{k}{t}\left(L_{K_n}^{(D)} + \frac{k}{t}\right)^{-1}\right)^k f$$

for any $f \in \ell^2(X, m)$ and $t > 0$, see Theorem 3.26 for more details. □

Remark 5.5. The convergence given in the previous lemma is a characterization of regularity (Exercise).

Combining the Markov property of the resolvents of restrictions to finite sets along with the convergence statements in Lemma 5.3 gives the Markov properties for the semigroups and resolvents associated to the regular form on the entire graph.

Corollary 5.6 (Markov property of resolvents and semigroups). *Let (b, c) be a graph over (X, m) with associated regular Dirichlet form Q and Laplacian L. Then, for any $f \in \ell^2(X, m)$ with $0 \leq f \leq 1$,*

$$0 \leq \alpha(L + \alpha)^{-1} f \leq 1 \qquad \textit{and} \qquad 0 \leq e^{-tL} f \leq 1$$

for all $\alpha > 0$ and $t \geq 0$.

Remark 5.7. It is not necessary for the function to be bounded in order for the positivity preserving property above to hold (Exercise).

Proof. After suitable approximation procedures, it suffices to consider $\varphi \in C_c(X)$ with $0 \leq \varphi \leq 1$. Consider now an increasing sequence (K_n) of finite subsets of X with $X = \bigcup_{n\in\mathbb{N}} K_n$. In particular, we may assume that the support of φ is contained in K_n for all $n \in \mathbb{N}$. By Lemma 5.3 we have

$$(L + \alpha)^{-1}\varphi = \lim_{n\to\infty} (L_{K_n}^{(D)} + \alpha)^{-1}\varphi.$$

By the Markov property for finite sets, we have $0 \leq \alpha(L_{K_n}^{(D)} + \alpha)^{-1}\varphi \leq 1$ for $n \in \mathbb{N}$. Combining these two observation we obtain the desired statement for the resolvents.

We now turn to proving the statement for the semigroups. The case $t = 0$ is clear so we restrict attention to the case $t > 0$. As above, the equality

$$e^{-tL} f = \lim_{k\to\infty} \left(\frac{k}{t}\left(L + \frac{k}{t}\right)^{-1}\right)^{k} f$$

for any $f \in \ell^2(X,m)$ given in Theorem 3.26 gives the statement from the already shown statement for the resolvents. □

LEMMA 5.8 (Resolvents as minimal solutions to $(\mathcal{L} + \alpha)u = f$). *Let (b,c) be a graph over (X,m) with associated regular Dirichlet form Q and Laplacian L. Let $\alpha > 0$ and $f \in \ell^2(X,m)$. Then $u := (L+\alpha)^{-1} f$ belongs to $\mathcal{F}$ and satisfies*

$$(\mathcal{L} + \alpha)u = f.$$

Furthermore, if additionally $f \geq 0$, then u is the smallest $v \in \mathcal{F}$ with $v \geq 0$ and $(\mathcal{L} + \alpha)v \geq f$.

PROOF. We first show that u is a solution as stated. For $\alpha > 0$, we note that the resolvent $(L + \alpha)^{-1}$ maps $\ell^2(X,m)$ into $D(L) \subseteq D(Q) \subseteq \mathcal{D} \subseteq \mathcal{F}$, where the last inclusion follows by Proposition 2.9 (b) and the other inclusions follow from the definitions. By Theorem 4.9, the operator L is a restriction of $\mathcal{L}$ so that $u = (L + \alpha)^{-1} f \in \mathcal{F}$ satisfies

$$(\mathcal{L} + \alpha)u = f,$$

as claimed.

We now establish the minimality of u when additionally $f \geq 0$. We first note that $u \geq 0$ whenever $f \geq 0$ as the resolvent is positivity preserving by Corollary 5.6. Now, let $v \geq 0$ be another function with $v \in \mathcal{F}$ and $(\mathcal{L}+\alpha)v \geq f$. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_{n\in\mathbb{N}} K_n$ and let $L_{K_n}^{(D)}$ be the Dirichlet Laplacian on $\ell^2(K_n, m_{K_n})$ for $n \in \mathbb{N}$. We recall that $L_{K_n}^{(D)}$ agrees with $\mathcal{L}$ on the set of functions supported in K_n. For $n \in \mathbb{N}$ let $f_n := f 1_{K_n}$,

$$u_n := (L_{K_n}^{(D)} + \alpha)^{-1} f_n$$

and extend u_n by 0 to $X \setminus K_n$. Then, letting $w_n := v - u_n$, for $n \in \mathbb{N}$ the function w_n satisfies

- $(\mathcal{L} + \alpha)w_n = (\mathcal{L} + \alpha)v - (L_{K_n}^{(D)} + \alpha)u_n \geq f - f_n = 0$ on K_n,
- $w_n \wedge 0$ attains a minimum on K_n since K_n is finite,
- $w_n = v \geq 0$ on $X \setminus K_n$.

Hence, we can apply the minimum principle, Theorem 2.10, and find $w_n = v - u_n \geq 0$ on X. Therefore, $v \geq u_n$ on X.

Finally, we show that (u_n) converges to u and thus $v \geq u$, which will complete the proof. Indeed, this can be seen by first fixing $k \in \mathbb{N}$ and considering $(L_{K_n}^{(D)} + \alpha)^{-1} f_k$ for $n \geq k$. Then, Lemma 5.3 (a) gives

$$\lim_{n\to\infty} (L_{K_n}^{(D)} + \alpha)^{-1} f_k = (L + \alpha)^{-1} f_k.$$

Furthermore, by the spectral theorem

$$\|\alpha(L+\alpha)^{-1}\| \leq 1 \quad \text{and} \quad \|\alpha(L_{K_n}^{(D)}+\alpha)^{-1}\| \leq 1$$

for all $n \in \mathbb{N}$ and all $\alpha > 0$. Therefore, as $f_k \to f$ in $\ell^2(X,m)$ we have

$$\lim_{k\to\infty} (L+\alpha)^{-1} f_k = (L+\alpha)^{-1} f$$

and

$$\|(L_{K_n}^{(D)}+\alpha)^{-1}(f_n - f_k)\| \leq \frac{1}{\alpha}\|f_n - f_k\| \to 0$$

as $k, n \to \infty$. Thus, the triangle inequality implies

$$\begin{aligned}\|u_n - u\| \leq \|(L_{K_n}^{(D)}+\alpha)^{-1}(f_n - f_k)\| &+ \|(L_{K_n}^{(D)}+\alpha)^{-1} f_k - (L+\alpha)^{-1} f_k\| \\ &+ \|(L+\alpha)^{-1}(f_k - f)\|,\end{aligned}$$

where we have shown that all three terms go to 0 as $k, n \to \infty$. □

As the resolvent associated to the operator coming from the regular Dirichlet form generates the minimal positive solution of the Poisson equation, so does the semigroup generate the minimal solution of the heat equation. This is discussed next.

We recall that a function

$$u \colon [0,\infty) \times X \longrightarrow \mathbb{R}$$

is called a solution of the heat equation with initial condition f if for all $x \in X$, the mapping $t \mapsto u_t(x)$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, $u_t \in \mathcal{F}$ for all $t > 0$ and

$$(\mathcal{L} + \partial_t) u_t(x) = 0$$

for all $x \in X$ and $t > 0$ with $u_0 = f$. We call u a supersolution of the heat equation with initial condition f if u satisfies all the assumptions above and instead of equality in the heat equation we have

$$(\mathcal{L} + \partial_t) u_t(x) \geq 0, \quad t > 0, x \in X.$$

We now show that if the initial condition is positive, then the semigroup of the associated Laplacian generates the minimal positive supersolution of the heat equation.

Lemma 5.9 (Semigroup as the minimal solution of the heat equation). *Let (b,c) be a graph over (X,m) with associated regular Dirichlet form Q and Laplacian L. Let $f \in \ell^2(X,m)$. If*

$$u_t(x) := e^{-tL} f(x)$$

for $t \geq 0$ and $x \in X$, then u is a solution of the heat equation with initial condition f.

Furthermore, if additionally $f \geq 0$, then u is the smallest positive supersolution of the heat equation with initial condition greater than or equal to f.

Proof. As L is a restriction of $\mathcal{L}$ by Theorem 4.9, the fact that u given by $u_t(x) := e^{-tL} f(x)$ for $t \geq 0$ and $x \in X$ is a solution of the heat equation with initial condition f for $f \in \ell^2(X,m)$ is a consequence of the spectral theorem and can be found as Theorem 3.24.

We now show minimality. Let f additionally satisfy $f \geq 0$. Then, by Corollary 5.6 we have $u_t(x) \geq 0$ for all $t \geq 0$ and $x \in X$ as the semigroup is positivity preserving. Thus, u is a positive solution of the heat equation with initial condition f. Now, suppose that w is a positive supersolution of the heat equation with initial condition $w_0 \geq f$. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_{n\in\mathbb{N}} K_n$ and let $L_{K_n}^{(D)}$ be the Dirichlet Laplacian on $\ell^2(K_n, m_{K_n})$. We recall that $L_{K_n}^{(D)}$ agrees with $\mathcal{L}$ on functions supported in K_n. For $n \in \mathbb{N}$ let $f_n := f1_{K_n}$ and

$$u_t^{(n)}(x) := e^{-tL_{K_n}^{(D)}} f_n(x)$$

for $x \in K_n$ and $t \geq 0$. We extend $u^{(n)}$ by 0 to $[0,\infty) \times X \setminus K_n$. If $w^{(n)} := w - u^{(n)}$, then for $n \in \mathbb{N}$ the function $w^{(n)}$ satisfies

- $(\mathcal{L} + \partial_t)w^{(n)} \geq 0$ on $(0,T) \times K_n$,
- $w^{(n)} \wedge 0$ attains a minimum on the compact set $[0,T] \times K_n$ since $w^{(n)}$ is continuous,
- $w^{(n)} \geq 0$ on $((0,T] \times (X \setminus K_n)) \cup (\{0\} \times K_n)$.

Hence, we can apply the minimum principle for the heat equation, Theorem 2.13, to obtain $w^{(n)} = w - u^{(n)} \geq 0$ on $[0,T] \times K_n$ for all $n \in \mathbb{N}$. Therefore, $w \geq u^{(n)}$ on $[0,T] \times X$ as $u^{(n)}$ vanishes outside of K_n and w is positive.

We now show that $(u^{(n)})$ converges to u from which it follows that $w \geq u$, thereby completing the proof. Indeed, this can be seen by first fixing $k \in \mathbb{N}$ and considering $e^{-tL_{K_n}^{(D)}} f_k$ for $n \geq k$. Then, Lemma 5.3 (b) gives

$$\lim_{n\to\infty} e^{-tL_{K_n}^{(D)}} f_k = e^{-tL} f_k.$$

Furthermore, by Proposition 3.23 we have

$$\|e^{-tL}\| \leq 1 \quad \text{and} \quad \|e^{-tL_{K_n}^{(D)}}\| \leq 1$$

for all $n \in \mathbb{N}$ and all $t \geq 0$. As $f_k \to f$ in $\ell^2(X,m)$ we have

$$\lim_{k\to\infty} e^{-tL} f_k = e^{-tL} f$$

and

$$\|e^{-tL_{K_n}^{(D)}}(f_n - f_k)\| \leq \|f_n - f_k\| \to 0$$

as $k, n \to \infty$. Thus, the triangle inequality implies

$$\|u_t^{(n)} - u_t\| \leq \|e^{-tL_{K_n}^{(D)}}(f_n - f_k)\| + \|e^{-tL_{K_n}^{(D)}} f_k - e^{-tL} f_k\| + \|e^{-tL}(f_k - f)\|,$$

where we have shown that all three terms go to 0 as $k, n \to \infty$. □

5.2. The Context of General Dirichlet Forms

Domain monotonicity, and minimality of solutions are special features of regular Dirichlet forms. The Markov property of the resolvents and semigroups, however, does not have to do with regularity. It is a general feature of Dirichlet forms. This is discussed in this section.

We first define the concept of a Dirichlet form. Let (X, μ) be a σ-finite measure space. Let H be the Hilbert space of square integrable real-valued functions on X, i.e., $H = L^2(X, \mu)$. We let Q be a positive closed form with domain $D(Q) \subseteq H$. In particular, Q is a positive symmetric form and $D(Q)$ is complete with respect to the form norm $\| \cdot \|_Q$ given by $\|f\|_Q = (Q(f) + \|f\|^2)^{1/2}$ for all $f \in D(Q)$, where $\| \cdot \|$ denotes the norm arising from the inner product on H. We recall that Q is extended on the diagonal to all of H via $Q(f) = \infty$ for $f \in H \setminus D(Q)$.

Recall that a map $C \colon \mathbb{R} \longrightarrow \mathbb{R}$ is a normal contraction if $C(0) = 0$ and $|C(s) - C(t)| \leq |s - t|$ for all $s, t \in \mathbb{R}$.

Definition 5.10 (Dirichlet form). A positive closed form Q with domain $D(Q)$ in $H = L^2(X, \mu)$ is called a *Dirichlet form* if $C \circ f \in D(Q)$ and

$$Q(C \circ f) \leq Q(f)$$

for all $f \in D(Q)$ and all normal contractions C.

This condition has a number of surprising consequences which we will discuss. We note that while the definition requires compatibility with all normal contractions, it actually suffices to check the condition for the normal contraction $C_{[0,1]}$ given by

$$C_{[0,1]}(s) := 0 \vee (s \wedge 1), \quad s \in \mathbb{R},$$

that is, cutting below by 0 and above by 1. This follows directly from the proof of Theorem 5.15 given below.

We recall that whenever Q is a positive closed form, the associated operator L is positive, that is, L is self-adjoint and $\sigma(L) \subseteq [0, \infty)$, see Theorem 3.33 for more details and Lemma 3.30 for the construction of L. As $\sigma(L) \subseteq [0, \infty)$, it follows that we can use the functional calculus to define both the resolvent $(L + \alpha)^{-1}$ for $\alpha > 0$ and the semigroup e^{-tL} for $t \geq 0$, which are bounded operators on H, see Propositions 3.13.

We now give some consequences for both semigroups and resolvents when the associated operator comes from a Dirichlet form. We say that an operator A with domain $D(A) \subseteq L^2(X, \mu)$ is *positivity preserving* if $Af \geq 0$ whenever $f \in D(A)$ satisfies $f \geq 0$. We say that A is *Markov* if $0 \leq Af \leq 1$ holds for all $f \in D(A)$ with $0 \leq f \leq 1$. It is not hard to see that any Markov operator is positivity preserving.

We start with a lemma which will be applied to the semigroup and resolvent in what follows.

Lemma 5.11. *Let A be a bounded self-adjoint positivity preserving operator on $H = L^2(X, \mu)$. Then, the quadratic form Q_{I-A} defined by*

$$Q_{I-A}(f, g) := \langle (I - A)f, g \rangle$$

satisfies

$$Q_{I-A}(|f|) \leq Q_{I-A}(f)$$

for all $f \in L^2(X, \mu)$. If, furthermore, A is Markov, then Q_{I-A} is a Dirichlet form and for $f, g \in L^2(X, \mu) \cap L^\infty(X, \mu)$ we have

$$Q_{I-A}(fg) \leq 2\|g\|_\infty^2 Q_{I-A}(f) + 2\|f\|_\infty^2 Q_{I-A}(g).$$

Proof. We show the first statement for simple functions. The statement for functions in $L^2(X,\mu)$ then follows by approximation. Let $f := \sum_{k=1}^n f_k 1_{U_k}$ for $f_1,\ldots,f_n \in \mathbb{R}$ and $U_1,\ldots,U_n \subseteq X$ which are measurable disjoint sets of finite measure. Then, by a direct calculation we find the following explicit formula for $Q_{I-A}(f)$

$$Q_{I-A}(f) = \frac{1}{2}\sum_{k,l=1}^n b_{k,l}(f_k - f_l)^2 + \sum_{k=1}^n c_k f_k^2,$$

where $b_{k,l} := \langle 1_{U_k}, A1_{U_l}\rangle$ and $c_k := \mu(U_k) - \sum_{l=1}^n b_{k,l}$.

If A is positivity preserving, then $b_{k,l} \geq 0$ and the explicit formula for $Q_{I-A}(f)$ above easily gives $Q_{I-A}(|f|) \leq Q_{I-A}(f)$.

If A is Markov, then for $U = \bigcup_{l=1}^n U_l$ we have $0 \leq A1_U \leq 1$ and thus

$$\sum_{l=1}^n b_{k,l} = \langle 1_{U_k}, A1_U\rangle \leq \mu(U_k).$$

Therefore, $c_k \geq 0$ by definition. Then, the explicit formula for $Q_{I-A}(f)$ above easily gives $Q_{I-A}(C \circ f) \leq Q_{I-A}(f)$ for any normal contraction C. As Q_{I-A} is clearly symmetric positive and closed, this shows that Q_{I-A} is a Dirichlet form.

For the last statement, we let $g := \sum_{k=1}^n g_k 1_{U_k}$, where we alter the sets $U_1,\ldots,U_n$ appearing in the definition of f if necessary. Then, using Young's inequality we get

$$(f_k g_k - f_l g_l)^2 = (g_k(f_k - f_l)) + f_l(g_k - g_l))^2 \leq 2g_k^2(f_k - f_l)^2 + 2f_l^2(g_k - g_l)^2,$$

which, along with the estimates

$$\sum_{k=1}^n c_k f_k^2 g_k^2 \leq \|f\|_\infty^2 \sum_{k=1}^n c_k g_k^2 \text{ and } \sum_{k=1}^n c_k f_k^2 g_k^2 \leq \|g\|_\infty^2 \sum_{k=1}^n c_k g_k^2,$$

yields

$$\begin{aligned} Q_{I-A}(fg) &= \frac{1}{2}\sum_{k,l=1}^n b_{k,l}(f_k g_k - f_l g_l)^2 + \sum_{k=1}^n c_k f_k^2 g_k^2 \\ &\leq 2\|g\|_\infty^2 Q(f) + 2\|f\|_\infty^2 Q(g). \end{aligned}$$

This concludes the proof. □

Define the quadratic forms $Q^{(t)}\colon H \longrightarrow \mathbb{R}$ associated to the semigroup by

$$Q^{(t)}(f) := \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$$

for $t > 0$. As the semigroup consists of bounded self-adjoint operators, see Propositions 3.23, we have $D(Q^{(t)}) = H$ as well as

$$Q^{(t)}(f,g) = \frac{1}{t}\langle (I - e^{-tL})f, g\rangle$$

for all $f, g \in H$ and $t > 0$ by polarization. Moreover, for $\alpha > 0$ we define the quadratic form $Q_{(\alpha)}\colon H \longrightarrow \mathbb{R}$ associated to the resolvent by

$$Q_{(\alpha)}(f) := \alpha\langle (I - \alpha(L + \alpha)^{-1})f, f\rangle.$$

We now show that the value of a closed form on the diagonal is the limit of the value of the quadratic forms associated to the resolvents and the semigroup.

LEMMA 5.12. *Let Q be a positive closed form on H and Q' the map on H with $Q'(f) := Q(f,f)$ for $f \in D(Q)$ and $Q'(f) := \infty$ otherwise. Then, for all $f \in H$,*

$$Q'(f) = \lim_{\alpha \to \infty} Q_{(\alpha)}(f) = \lim_{t \to 0^+} Q^{(t)}(f).$$

In particular, the limits are finite if and only if $f \in D(Q)$.

PROOF. The statement follows directly from the connection between the operator and form, properties of the functional calculus given in Proposition 3.13, and the monotone convergence theorem as

$$\alpha \left(1 - \alpha(s+\alpha)^{-1}\right) \nearrow s \qquad \text{and} \qquad \frac{1}{t}(1 - e^{-tx}) \nearrow x$$

as $\alpha \to \infty$ and $t \to 0^+$, respectively, for all $s \geq 0$ and $x \geq 0$, respectively. □

We now state and prove the Beurling–Deny criteria for positive closed forms. The first criterion shows that a form being compatible with the absolute value is equivalent to the fact that the heat semigroup and the resolvent are positivity preserving.

THEOREM 5.13 (First Beurling–Deny criterion). *Let Q be a positive closed form on $H = L^2(X,\mu)$ and let L be the associated positive operator. Then, the following statements are equivalent:*

(i) *$Q(|f|) \leq Q(f)$ for all $f \in H$.*
(ii) *$\alpha(L+\alpha)^{-1}$ is positivity preserving for every $\alpha > 0$.*
(iii) *e^{-tL} is positivity preserving for every $t \geq 0$.*

PROOF. (i) $\Longrightarrow$ (ii): Let $f \geq 0$ be given. By the characterization of the resolvent as a minimizer we know that $h := (L+\alpha)^{-1} f$ is the unique minimizer of ψ given by

$$\psi(v) := Q(v) + \alpha \|v - \frac{f}{\alpha}\|^2.$$

Now, by assumption (i) we have $Q(|h|) \leq Q(h)$. Moreover, for $f \geq 0$ clearly

$$\||h| - \frac{f}{\alpha}\|^2 \leq \|h - \frac{f}{\alpha}\|^2$$

holds. Thus, $|h|$ is a minimizer of ψ as well. This shows $h = |h| \geq 0$.

(ii) $\Longrightarrow$ (iii): This follows directly from Theorem 3.26 (b).

(iii) $\Longrightarrow$ (i): By Lemma 5.11 we have

$$\frac{1}{t}\langle (I - e^{-tL})|f|, |f|\rangle \leq \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$$

for all $t > 0$ and $f \in H$. Letting $Q^{(t)}(f) := \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$, Corollary 5.12 gives $\lim_{t \to 0^+} Q^{(t)}(f) = Q(f)$ for all $f \in H$. Thus, we conclude

$$Q(|f|) \leq Q(f).$$

This finishes the proof. □

REMARK 5.14. One can check that (i) in Theorem 5.13 is equivalent to:

(i.a) $Q(f_+) \leq Q(f)$ for all $f \in H$,

where $f_+ := f \vee 0$ denotes the positive part of f.

Indeed, as $f_+ = (f + |f|)/2$, it is clear that (i) implies (i.a). On the other hand, (i.a) implies $Q(f_-) \leq Q(f)$ and, by considering $f_s := f_+ - sf_-$ for $s > 0$, so that $(f_s)_+ = f_+$ and using bilinearity, $Q(f_+, f_-) \leq 0$, where $f_- := (-f) \vee 0$ is the negative part of f. Now, using the bilinearity of the form once more implies (i).

The second Beurling–Deny criterion deals with Dirichlet forms. In particular, being a Dirichlet form turns out to be equivalent to the Markov property for both the heat semigroup and the resolvent.

THEOREM 5.15 (Second Beurling–Deny criterion). *Let Q be a positive closed form on $H = L^2(X, \mu)$ and let L be the associated positive operator. Then, the following statements are equivalent:*

(i) Q is a Dirichlet form.

(i') $Q(C_{[0,1]} \circ f) \leq Q(f)$ holds for all $f \in D(Q)$.

(ii) $\alpha(L + \alpha)^{-1}$ is Markov for every $\alpha > 0$.

(iii) e^{-tL} is Markov for every $t \geq 0$.

PROOF. (i) $\Longrightarrow$ (i'): This is clear.

(i')$\Longrightarrow$(ii): Let $0 \leq g \leq 1$ be given and consider $h := (L + \alpha)^{-1}g$. By the characterization of the resolvent as a minimizer we infer that h is the unique minimizer of the functional

$$\psi : D(Q) \longrightarrow [0, \infty), \quad \psi(f) := Q(f) + \alpha \|f - \frac{g}{\alpha}\|^2.$$

Now, consider the map that cuts off at 0 and α, i.e. consider

$$C_{[0,\alpha]} : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \mapsto \alpha C_{[0,1]}(\frac{1}{\alpha}t).$$

Then, $C_{[0,\alpha]}$ is a normal contraction with

$$Q'(C_{[0,\alpha]} \circ f) = \alpha^2 Q'(C_{[0,1]}(\frac{1}{\alpha}f)) \leq \alpha^2 Q'(\frac{1}{\alpha}f) = Q'(f)$$

for all $f \in L^2(X, m)$ due to (i'). In particular,

$$Q(C_{[0,\alpha]} \circ h) \leq Q(h) < \infty$$

holds. Also, $\frac{g}{\alpha}$ is invariant under $C_{[0,\frac{1}{\alpha}]}$ due to $0 \leq g \leq 1$ and this implies

$$\|C_{[0,\frac{1}{\alpha}]} \circ h - \frac{g}{\alpha}\|^2 = \|C_{[0,\frac{1}{\alpha}]}(h) - C_{[0,\frac{1}{\alpha}]}(\frac{g}{\alpha})\|^2 \leq \|h - \frac{g}{\alpha}\|^2.$$

Thus, $C_{[0,\frac{1}{\alpha}]} \circ h$ is (another) minimizer of ψ. Uniqueness of the minimizer implies

$$C_{[0,\frac{1}{\alpha}]} \circ h = h$$

and this shows (ii).

(ii) $\Longrightarrow$ (iii): This follows directly from Theorem 3.26 (b).

(iii) $\Longrightarrow$ (i): As e^{-tL} is Markov for every $t \geq 0$, the form $Q^{(t)}$ defined by

$$Q^{(t)}(f) := \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$$

is a Dirichlet form for $t > 0$, by Lemma 5.11. Since $Q(f) = \lim_{t\to 0^+} Q^{(t)}(f)$ by Corollary 5.12, the statement follows. □

Remark 5.16. It is remarkable that compatibility with all normal contractions is equivalent to compatibility with the normal contraction $C_{[0,1]}$ with $C_{[0,1]}(s) := s \vee (s \wedge 1)$. In fact, compatibility with all normal contractions is also equivalent to compatibility with the normal contraction C_1 with $C_1(s) := s \wedge 1$. Indeed, whenever Q satisfies (i) it will also satisfy the following condition:

(i”) $Q(C_1 \circ f) \leq Q(f)$ for all $f \in L^2(X, \mu)$.

Conversely, using $s \vee (-\varepsilon) = -\varepsilon(-\varepsilon^{-1}s \wedge 1)$ for $s \in \mathbb{R}$ and $\varepsilon > 0$, one easily sees that (i”) also implies $Q(0 \vee f) \leq Q(f)$ for all $f \in L^2(X, \mu)$. Another application of (i”) then gives

$$Q(C_{[0,1]} \circ f) = Q(0 \vee (f \wedge 1)) \leq Q(f \wedge 1) \leq Q(f).$$

Hence, (i”) implies (i’).

Remark 5.17. By monotone convergence we see that (iii) in Theorem 5.15 is equivalent to

(iii.a) $0 \leq e^{-tL}f \leq 1$ for all $f \in L^\infty(X, \mu)$ with $0 \leq f \leq 1$.

By duality and the Riesz–Thorin interpolation theorem, one sees that (iii.a) is equivalent to

(iii.b) $0 \leq e^{-tL}f \leq 1$ for all $f \in L^p(X, \mu)$ with $0 \leq f \leq 1$ and $1 \leq p \leq \infty$.

CHAPTER 6

Large Time Behaviour

In this chapter we focus on the large time behaviour of the semigroup. We have seen this in Section 1.6 for finite graphs already, but now focus on infinite graphs.

6.1. Connectedness, irreducibility and positivity improving

In this section we show that connectedness of the graph, irreducibility of the form and the property that the semigroup/resolvent are positivity improving are equivalent.

Let X be a countable and discrete set and m a measure on X with full support, and let (b, c) be a graph over (X, m). In Theorem 5.13 we showed that the semigroup and resolvent associated to the Dirichlet Laplacian $L := L^{(D)}$ always map positive functions to positive functions. This property is called positivity preserving.

We recall that a subset of X is called connected if any two points in the subset can be connected by a path consisting of vertices in the subset. A maximal connected subset is called a connected component and (b, c) is called connected if it consists of one connected component.

We now introduce the necessary concepts of irreducible forms and positivity improving operators. A (quadratic) form Q on $\ell^2(X, m)$ with domain $D(Q)$ is called *irreducible* if the only subsets $U \subseteq X$ such that $1_U D(Q) \subseteq D(Q)$ (and therefore also $1_{X\setminus U} D(Q) \subseteq D(Q)$) and

$$Q(f) = Q(1_U f) + Q(1_{X\setminus U} f)$$

for all $f \in D(Q)$ are $U = \emptyset$ and $U = X$. Note that the equation above is equivalent to $Q(1_U f, 1_{X\setminus U} f) = 0$ and thus describes a version of a theorem of Pythagoras.

An operator A on $\ell^2(X, m)$ is called *positivity improving* if $Af > 0$ for all non-trivial $f \in D(A)$ with $f \geq 0$.

Theorem 6.1 (Characterization of connectedness and positivity improving). *Let (b, c) be a graph over (X, m) with associated regular Dirichlet form Q and Laplacian L. Then, the following statements are equivalent:*

(i) *(b, c) is connected.*
(ii) *Q is irreducible.*
(iii) *$(L + \alpha)^{-1}$ is positivity improving for all $\alpha > 0$.*
(iv) *e^{-tL} is positivity improving for all $t > 0$.*

Proof. (i) $\Longrightarrow$ (iv): Let $\varphi \in C_c(X)$ be positive and non-trivial. Let (K_n) be an increasing sequence of connected finite sets such that $X =$

$\bigcup_{n\in\mathbb{N}} K_n$. Denote by $L_{K_n}^{(D)}$ the operators corresponding to the restrictions of Q to $C_c(K_n)$. Then $(L_{K_n}^{(D)}+\alpha)^{-1}\varphi \geq 0$ for all $\alpha > 0$. Therefore,

$$e^{-tL_{K_n}^{(D)}}\varphi = \lim_{n\to\infty}\left(\frac{n}{t}\left(L_{K_n}^{(D)}+\frac{n}{t}\right)^{-1}\right)^{n}\varphi \geq 0, \quad t>0$$

so that the semigroup $(e^{-tL_{K_n}^{(D)}})_{t\geq 0}$ is positivity preserving.

Now, let $u(t,x) := e^{-tL_{K_n}^{(D)}}\varphi(x) \geq 0$ and assume that n is large enough so that the support of φ is included in K_n. We want to show that $u(t,x) > 0$ for all $x \in K_n$ and $t > 0$. If there exists $x_0 \in K_n$ and $t_0 > 0$ such that $u(t_0,x_0) = 0$, then (t_0,x_0) is a minimum for u in both variables. Having a minimum at t_0 gives

$$0 = \partial_t u(t_0,x_0) = -L_{K_n}^{(D)}u(t_0,x_0).$$

Now, having a minimum at x_0 yields $u(t_0,y) = 0$ for all $y \sim x_0$. As K_n is connected, this implies $u(t_0,x) = 0$ for all $x \in K_n$. However,

$$e^{t_0 L_{K_n}^{(D)}}u(t_0,x) = \varphi(x),$$

so that $\varphi = 0$ on K_n which gives a contradiction to the assumption on φ. Therefore, $e^{-tL_{K_n}^{(D)}}\varphi(x) > 0$ for all $t > 0$ and $x \in K_n$, so that $e^{-tL_{K_n}^{(D)}}$ is positivity improving for all $t > 0$.

As we assume that $\varphi \geq 0$, by Lemma 5.3, we get $e^{-tL_{K_n}^{(D)}}\varphi \to e^{-tL}\varphi$ as $n \to \infty$ where the convergence is pointwise monotonically increasing. Therefore, we infer that $e^{-tL}\varphi > 0$ for all $t > 0$.

For a non-trivial positive function $f \in \ell^2(X,m)$, let $f_n := 1_{K_n} f \in C_c(X)$ for $n \in \mathbb{N}$. Then, (f_n) converges monotonically increasingly to f in $\ell^2(X,m)$ as $n \to \infty$. By the above, applied to the functions $f_{n+1} - f_n$, we have that $0 < e^{-tL}f_n \to e^{-tL}f$ where the convergence is pointwise monotonically increasing. Therefore, $e^{-tL}f > 0$ for all $t > 0$.

(iv) $\Longrightarrow$ (iii): This follows directly from the Laplace transform formula in Theorem 3.26, that is,

$$(L+\alpha)^{-1} = \int_0^\infty e^{-\alpha t}e^{-tL}\mathrm{d}t.$$

(iii) $\Longrightarrow$ (ii): If the form Q is not irreducible, then there exists a proper non-trivial subset $U \subseteq X$ such that L decomposes into a direct sum of operators $L_U \oplus L_{X\setminus U}$ on $\ell^2(U,m_U) \oplus \ell^2(X\setminus U, m_{X\setminus U})$. Hence, the resolvent also decomposes into a direct sum $(L_U+\alpha)^{-1} \oplus (L_{X\setminus U}+\alpha)^{-1}$. In this case, taking a non-trivial function $f \in \ell^2(U,m_U)$ with $f \geq 0$ yields a non-trivial function $(f,0) \in \ell^2(U,m_U) \oplus \ell^2(X\setminus U, m_{X\setminus U})$ which is non-negative. However,

$$(L+\alpha)^{-1}(f,0) = ((L_U+\alpha)^{-1}f, (L_{X\setminus U}+\alpha)^{-1}0) = ((L_U+\alpha)^{-1}f, 0),$$

which is not strictly positive.

(ii) $\Longrightarrow$ (i): For any connected component U, we clearly have that $1_U\varphi \in D(Q)$ (see Lemma 4.5) and

$$Q(\varphi) = Q(1_U\varphi) + Q(1_{X\setminus U}\varphi)$$

for any $\varphi \in C_c(X)$. We want to show that the same holds for $f \in D(Q)$ so that we may apply irreducibility to conclude connectedness.

Let $f \in D(Q)$ and, by regularity of Q, let (φ_n) in $C_c(X)$ be such that $\|\varphi_n - f\|_Q \to 0$ as $n \to \infty$. Then, $(1_U\varphi_n)$ is a Cauchy sequence in $\|\cdot\|_Q$ since

$$\begin{aligned} Q(1_U\varphi_n - 1_U\varphi_m) &\leq Q(1_U(\varphi_n - \varphi_m)) + Q(1_{X\setminus U}(\varphi_n - \varphi_m)) \\ &= Q(\varphi_n - \varphi_m) \end{aligned}$$

and $\|1_U\varphi_n - 1_U\varphi_m\| \leq \|\varphi_n - \varphi_m\|$ for all $n, m \in \mathbb{N}_0$. Hence, $(1_U\varphi_n)$ converges in $D(Q)$ so that $1_U f \in D(Q)$. Furthermore, as $Q(1_U\varphi_n) \to Q(1_U f)$ and $Q(\varphi_n) \to Q(f)$ as $n \to \infty$, it follows that

$$Q(f) = Q(1_U f) + Q(1_{X\setminus U} f).$$

By irreducibility, we infer that either $U = \emptyset$ or $U = X$. This shows that (b, c) is connected. $\square$

6.2. Toolbox – Variational Characterization for the Bottom of the Spectrum

We prove a characterization of the bottom of the spectrum of a positive operator. To this end we need the following proposition.

Proposition 6.2 (Spectral parts and spectral family). *Let L be a self-adjoint operator on a Hilbert space H and let E be the associated spectral family. Let $\lambda \in \mathbb{R}$.*

(a) *$\lambda \in \sigma(L)$ if and only if $\lambda \in \operatorname{supp}(E)$, i.e.,*

$$E((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$$

for all $\varepsilon > 0$.

(b) *λ is an eigenvalue of L if and only if $E(\{\lambda\}) \neq 0$, in which case the range of $E(\{\lambda\})$ is the eigenspace of λ. Furthermore, $f \in H$ is an eigenfunction corresponding to λ if and only if μ_f is supported on $\{\lambda\}$.*

Proof. (a) This has already been shown in Theorem 3.18.

(b) From Proposition 3.13, as $(x - \lambda)^2 = 0$ for $x = \lambda$, we get

$$\|(L - \lambda)f\|^2 = \int (x - \lambda)^2 \mathrm{d}\mu_f(x) = \int_{\mathbb{R}\setminus\{\lambda\}} (x - \lambda)^2 \mathrm{d}\mu_f(x)$$

for any $f \in D(L)$. As $(x - \lambda)^2 > 0$ for all $x \neq \lambda$, we infer that $f \in D(L)$ with $f \neq 0$ is an eigenfunction corresponding to λ if and only if $\mu_f = 1_{\{\lambda\}}\mu_f$, i.e., if and only if $1_{\mathbb{R}\setminus\{\lambda\}}\mu_f = 0$. Now, $1_{\mathbb{R}\setminus\{\lambda\}}\mu_f = 0$ if and only if $\mu_f(\mathbb{R} \setminus \{\lambda\}) = 0$ and Proposition 3.16 gives

$$\|E(\mathbb{R} \setminus \{\lambda\})f\|^2 = \mu_f(\mathbb{R} \setminus \{\lambda\}) = 0.$$

Thus, we infer that $f \in D(L)$ is an eigenfunction corresponding to λ if and only if $E((\mathbb{R} \setminus \{\lambda\}))f = 0$. This, in turn, is equivalent to $f = E(\{\lambda\})f$

as $f = E(\{\lambda\})f + E(\mathbb{R} \setminus \{\lambda\})f$, where the summands are orthogonal. This shows the first statement of (b). The other statement has been shown along the way. $\square$

The equality we show now is also sometimes referred to as the Rayleigh–Ritz formula.

THEOREM 6.3 (Variational characterization of λ_0). *Let Q be a positive closed form and let L be the associated operator on a Hilbert space H. Let $\lambda_0(L) := \inf \sigma(L)$. Then,*

$$\lambda_0(L) = \inf_{f \in D(L), \|f\|=1} \langle f, Lf \rangle = \inf_{f \in D(Q), \|f\|=1} Q(f).$$

Furthermore, if $f \in D(Q)$ is normalized and satisfies $Q(f) = \lambda_0(L)$, then $f \in D(L)$ and $Lf = \lambda_0(L)f$, i.e., f is a normalized eigenfunction corresponding to the eigenvalue $\lambda_0(L)$.

PROOF. The second equality is clear from the connection between the form and the operator and the fact that $D(L)$ is dense in $D(Q)$ with respect to the form norm $\|\cdot\|_Q$ by Corollary 3.37. Hence, we focus on proving the first equality. In order to do so, we will show two inequalities.

To this end, we recall that by Proposition 3.13 (b) we have

$$\langle f, Lf \rangle = \int x \mathrm{d}\mu_f(x),$$

where the integral is taken over the support of the spectral measure μ_f for $f \in D(L)$ and $\mu_f(\sigma(L)) = \|f\|^2$.

Now, we let $\lambda_0 := \lambda_0(L)$ and let $f \in D(L)$ be normalized. As $\sigma(L) \subseteq [\lambda_0, \infty)$, we obtain

$$\langle f, Lf \rangle = \int_{[\lambda_0,\infty)} x \mathrm{d}\mu_f(x) \geq \lambda_0 \int_{[\lambda_0,\infty)} \mathrm{d}\mu_f(x) = \lambda_0 \|f\|^2 = \lambda_0.$$

This shows $\lambda_0 \leq \inf \langle f, Lf \rangle$ for all normalized $f \in D(L)$.

Conversely, since $\lambda_0 \in \sigma(L)$, we have $E([\lambda_0, \lambda_0 + \varepsilon)) \neq 0$ for all $\varepsilon > 0$ by Proposition 6.2 (a). Hence, for every $\varepsilon > 0$ there exists a normalized f with $f = E([\lambda_0, \lambda_0 + \varepsilon))f$. By Proposition 3.17, f has spectral measure supported on $[\lambda_0, \lambda_0 + \varepsilon]$ and $f \in D(L)$. We then find

$$\langle f, Lf \rangle = \int_{[\lambda_0,\lambda_0+\varepsilon]} x \mathrm{d}\mu_f(x) \leq (\lambda_0 + \varepsilon)\|f\|^2 = \lambda_0 + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this gives $\lambda_0 \geq \inf \langle f, Lf \rangle$, where the infimum is taken over all normalized $f \in D(L)$.

For the furthermore statement, suppose that $f \in D(Q) = D(\sqrt{L})$ is normalized and satisfies $Q(f) = \lambda_0$. By Proposition 3.13 (a), we now get

$$0 = Q(f) - \lambda_0 = \|\sqrt{L}f\|^2 - \lambda_0\|f\|^2 = \int_{[\lambda_0,\infty)} (x - \lambda_0) \mathrm{d}\mu_f(x).$$

As the integrand is non-negative, μ_f is supported on $\{\lambda_0\}$ which, by Proposition 6.2 (b), is equivalent to λ_0 being an eigenvalue with eigenfunction f. $\square$

6.3. Positivity Improving Semigroups and the Ground State

Let X be a countable set and m a measure on X with full support. For the self-adjoint operator L associated to a positive symmetric closed form Q on $\ell^2(X,m)$, the semigroup operators e^{-tL} for $t \geq 0$ are bounded self-adjoint positive operators. By the discreteness of the space X, the semigroup has a kernel, i.e., there exists a map

$$p\colon [0,\infty) \times X \times X \longrightarrow \mathbb{R}$$

such that

$$e^{-tL}f(x) = \sum_{y \in X} p_t(x,y)f(y)m(y)$$

for all $f \in \ell^2(X,m)$, $x \in X$ and $t \geq 0$. We call p the *heat kernel* associated to L. An easy calculation gives that

$$p_t(x,y) = \frac{1}{m(x)m(y)}\langle 1_x, e^{-tL}1_y\rangle$$

for all $x, y \in X$ and $t \geq 0$.

By the variational characterization of the bottom of the spectrum, Theorem 6.3, it follows that

$$\lambda_0(L) = \inf_{f \in D(Q), \|f\|=1} Q(f) = \inf_{f \in D(L), \|f\|=1} \langle f, Lf\rangle.$$

In what follows we denote by $Q := Q^{(D)}_{b,c,m}$ be the Dirichlet form associated to a graph (b,c) over (X,m) with operator $L := L^{(D)}_{b,c,m}$ and the bottom of the spectrum by $\lambda_0 := \inf \sigma(L)$.

The following lemma considers the bottom of the spectrum for connected graphs. Specifically, whenever the bottom of the spectrum is an eigenvalue, then there exists a unique strictly positive normalized eigenfunction.

Lemma 6.4 (Uniqueness of eigenfunctions to λ_0). *Let (b,c) be a connected graph over (X,m). Assume λ_0 is an eigenvalue of L. Then, there exists a unique strictly positive normalized eigenfunction corresponding to λ_0 and the eigenspace of λ_0 is one-dimensional.*

Proof. Let $u \in D(L)$ be a normalized eigenfunction corresponding to λ_0. We will show that u must be strictly positive or strictly negative. Without loss of generality, we may assume that $u(x) > 0$ for some $x \in X$. Let $u_+ := u \vee 0$ and $u_- := (-u) \vee 0$ so that $u = u_+ - u_-$ and $|u| = u_+ + u_-$. From the variational characterization of the bottom of the spectrum, Theorem 6.3, and the fact that Q is a Dirichlet form we get

$$\lambda_0 \leq Q(|u|) \leq Q(u) = \lambda_0$$

so that $Q(|u|) = Q(u)$. Therefore, $|u|$ is also a normalized eigenfunction corresponding to λ_0 by Theorem 6.3. As both u and $|u|$ are eigenfunctions corresponding to λ_0, we get that

$$u_+ = \frac{u + |u|}{2}$$

is also an eigenfunction corresponding to λ_0. We note that u_+ is non-zero as we assumed that $u(x) > 0$ for some $x \in X$.

The semigroup $(e^{-tL})_{t\geq 0}$ on a connected graph is positivity improving by Theorem 6.1. Therefore, as $u_+ \geq 0$ satisfies $Lu_+ = \lambda_0 u_+$ and is non-zero, by the functional calculus and the positivity improving property we obtain

$$0 < e^{-tL}u_+ = e^{-t\lambda_0}u_+$$

for any $t > 0$. Hence, $u_+ > 0$ so that $u = u_+ > 0$. Therefore, any eigenfunction corresponding to λ_0 which is positive at some vertex is strictly positive.

From the argument above, it follows that any eigenfunction corresponding to λ_0 has a strict sign, i.e., is strictly positive or strictly negative. It is clear that any two functions of strict sign are not orthogonal in $\ell^2(X,m)$. This gives that the dimension of the eigenspace is one-dimensional (note that the eigenspace is closed and thus has an orthonormal basis) and that u is unique. □

If L is a self-adjoint operator arising from a Dirichlet form associated to a connected graph and λ_0 is an eigenvalue, then we have a unique strictly positive eigenfunction which minimizes the energy by the lemma above. In this context, we will refer to this eigenfunction as the ground state and λ_0 as the ground state energy.

We now discuss the case when the ground state energy is zero.

Example 6.5 (When $\lambda_0 = 0$ is an eigenvalue). Suppose that (b,c) is a connected graph over (X,m) and L is an operator coming from a Dirichlet form Q associated to (b,c). If $\lambda_0 = 0$ is an eigenvalue for L, then $c = 0$ and $m(X) < \infty$.

Indeed, this follows as if $u > 0$ is a ground state for $\lambda_0 = 0$ given by the lemma above, then

$$0 = \lambda_0 = Q(u) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(u(x)-u(y))^2 + \sum_{x\in X} c(x)u(x)^2.$$

This shows that u is constant and $c = 0$. As $u \in D(L) \subseteq \ell^2(X,m)$, it follows that $m(X) < \infty$. In particular, as u is normalized, we obtain $u = 1/\sqrt{m(X)}$.

In the chapter about recurrence we will see that $\lambda_0 = 0$ is an eigenvalue for $L^{(D)}$ if and only if $c = 0$, $m(X) < \infty$ and the underlying graph is recurrent.

6.4. Theorems of Chavel–Karp and Li

In this section we prove two convergence results. We recall that the heat kernel of an operator L on $\ell^2(X,m)$ is given by

$$p_t(x,y) = \frac{\langle 1_x, e^{-tL}1_y\rangle}{m(x)m(y)}$$

for $t > 0$ and $x, y \in X$. The following result connects the heat kernel of the Dirichlet Laplacian $L := L^{(D)}$ of a graph (b,c) over (X,m), the bottom of the spectrum $\lambda_0 := \inf \sigma(L)$, and the ground state.

Theorem 6.6 (Theorem of Chavel–Karp). *Let (b,c) be a connected graph over (X,m). Then, there exists a function $u\colon X \longrightarrow [0,\infty)$ such* *that*

$$\lim_{t\to\infty} e^{\lambda_0 t} p_t(x,y) = u(x)u(y)$$

for all $x, y \in X$*. If* λ_0 *is not an eigenvalue of* L*, then* $u = 0$*. If* λ_0 *is an eigenvalue of* L*, then* u *is the ground state, i.e., the unique normalized positive eigenfunction corresponding to* λ_0*.*

PROOF. The proof is a direct application of the spectral theorem. Let $E := 1_{\{\lambda_0\}}(L)$ be the spectral projection onto the eigenspace of λ_0. By Proposition 6.2, $E = 0$ if λ_0 is not an eigenvalue and, if λ_0 is an eigenvalue, then $E = \langle u, \cdot\rangle u$, where u is the unique positive normalized eigenfunction corresponding to λ_0 given by Lemma 6.4.

Let μ be the signed spectral measure of L associated to $1_x, 1_y$ for $x, y \in X$. That is, μ is the unique signed measure which is characterized by

$$\langle 1_x, \psi(L) 1_y\rangle = \int_{[\lambda_0,\infty)} \psi(s) \mathrm{d}\mu(s)$$

for all bounded measurable functions ψ on $[\lambda_0, \infty)$, see Proposition 3.15. Assume that λ_0 is an eigenvalue so that $1_{\{\lambda_0\}}(L) = \langle u, \cdot\rangle u$. We then get

$$\begin{aligned} m(x)m(y)|e^{\lambda_0 t} p_t(x,y) - u(x)u(y)| &= |\langle 1_x, (e^{\lambda_0 t} e^{-tL} - 1_{\{\lambda_0\}}(L)) 1_y\rangle| \\ &= \left| \int_{[\lambda_0,\infty)} \left(e^{-t(s-\lambda_0)} - 1_{\{\lambda_0\}}(s) \right) \mathrm{d}\mu(s) \right| \\ &\to 0 \end{aligned}$$

as $t \to \infty$ by Lebesgue's dominated convergence theorem. Note that μ is a finite measure so that the bounding function can be chosen as 1. If λ_0 is not an eigenvalue, then a similar argument gives the conclusion.

Concerning uniqueness of u, if $\tilde{u}$ is another positive function determining the limit, then $\tilde{u}(x)^2 = u(x)^2$ for all $x \in X$, and therefore $\tilde{u} = u$. □

We highlight one immediate corollary of the theorem above which characterizes when there exists a ground state.

COROLLARY 6.7 (Characterization of existence of a ground state). *Let* (b, c) *be a connected graph over* (X, m)*. Then,* λ_0 *is an eigenvalue for* L *if and only if*

$$\lim_{t\to\infty} e^{\lambda_0 t} p_t(x,y) \neq 0$$

for some (all) $x, y \in X$*.*

We will now state and prove the second of our convergence statements, which gives that the logarithm of the heat kernel converges to the bottom of the spectrum.

THEOREM 6.8 (Theorem of Li). *Let* (b, c) *be a connected graph over* (X, m)*. Then,*

$$\lim_{t\to\infty} \frac{1}{t} \log p_t(x,y) = -\lambda_0$$

for all $x, y \in X$*.*

Proof. Let $e_x := 1_x/\sqrt{m(x)}$ for $x \in X$ and observe that $(e_x)_{x\in X}$ is an orthonormal basis for $\ell^2(X,m)$. Let

$$a_t(x,y) := \langle e_x, e^{-tL} e_y \rangle$$

for $x, y \in X$, $t \geq 0$ and write $a_t(x) := a_t(x,x)$. We will show that the function $t \mapsto \log a_t(x)$ on $[0,\infty)$ is superadditive for all $x \in X$.

Note that, as L is an operator coming from a Dirichlet form, e^{-tL} is positivity improving for $t > 0$ by Theorem 6.1 above and clearly positivity preserving for $t = 0$. Therefore, for all $x \in X$, $s, t \geq 0$, we obtain

$$\begin{aligned} a_{s+t}(x) &= \langle e_x, e^{-(s+t)L} e_x \rangle = \langle e^{-sL} e_x, e^{-tL} e_x \rangle \\ &= \sum_{y\in X} \langle e^{-sL} e_x, e_y \rangle \langle e_y, e^{-tL} e_x \rangle \\ &\geq \langle e^{-sL} e_x, e_x \rangle \langle e_x, e^{-tL} e_x \rangle = a_s(x) a_t(x). \end{aligned}$$

Theorem 6.1 implies $a_t(x) > 0$ for all $t \geq 0$ and $x \in X$, thus, we may take the logarithm of $a_t(x)$ for all $x \in X$ and $t \geq 0$. The estimate above then shows that $t \mapsto \log a_t(x)$ is superadditive, i.e., satisfies

$$\log a_s(x) + \log a_t(x) \leq \log a_{s+t}(x)$$

for $s, t \geq 0$. Furthermore, $a_t(x) \leq 1$ since $a_t(x) = e^{-tL} 1_x(x)$ and semigroups associated to operators coming from Dirichlet forms are contracting by Theorem 5.15. Therefore, $\log a_t(x) \leq 0$. Putting all of this together, by a version of Fekete's lemma for functions we get that the following limit exists for every $x \in X$

$$\lim_{t\to\infty} \frac{1}{t} \log a_t(x) = \sup_{t\in(0,\infty)} \frac{1}{t} \log a_t(x).$$

Now, for $t \geq 1$ and $x, y \in X$, by a similar reasoning as above we obtain

$$\begin{aligned} a_{t-1}(x) a_1(x,y) &= \langle e^{-(t-1)L} e_x, e_x \rangle \langle e_x, e^{-L} e_y \rangle \\ &\leq \sum_{z\in X} \langle e^{-(t-1)L} e_x, e_z \rangle \langle e_z, e^{-L} e_y \rangle \\ &= \langle e^{-(t-1)L} e_x, e^{-L} e_y \rangle = \langle e_x, e^{-tL} e_y \rangle = a_t(x,y). \end{aligned}$$

By the same arguments for $t \geq 0$,

$$a_1(x,y) a_t(x,y) \leq \sum_{z\in X} \langle e^{-L} e_y, e_z \rangle \langle e_z, e^{-tL} e_y \rangle = a_{t+1}(y).$$

Hence, as $a_1(x,y) > 0$, we get

$$a_{t-1}(x) a_1(x,y) \leq a_t(x,y) \leq \frac{1}{a_1(x,y)} a_{t+1}(y).$$

Combining this line of inequalities with the fact that $\lim_{t\to\infty} \frac{1}{t} \log a_t(x)$ exists and $a_t(x,y) = a_t(y,x)$ gives that $\lim_{t\to\infty} \frac{1}{t} \log a_t(x,y)$ exists and is independent of $x, y \in X$.

Let

$$\lim_{t\to\infty} \frac{1}{t} \log a_t(x,y) =: -\lambda.$$

Since

$$a_t(x,y) = \langle e_x, e^{-tL} e_y\rangle = \sqrt{m(x)m(y)}p_t(x,y),$$

we conclude that

$$-\lambda = \lim_{t\to\infty} \frac{1}{t} \log a_t(x,y) = \lim_{t\to\infty} \frac{1}{t} \log p_t(x,y).$$

We will now show that $\lambda = \lambda_0$, which will complete the proof. First, we note that

$$\lim_{t\to\infty} \frac{1}{t} \log \left(e^{\lambda_0 t} p_t(x,y) \right) = \lambda_0 - \lambda$$

for all $x, y \in X$. If λ_0 is an eigenvalue for L, it follows from Theorem 6.6 that $\lim_{t\to\infty} e^{\lambda_0 t} p_t(x,y) = u(x)u(y) > 0$ so that

$$\lim_{t\to\infty} \frac{1}{t} \log \left(e^{\lambda_0 t} p_t(x,y) \right) = 0$$

and, hence, $\lambda = \lambda_0$ in this case.

If λ_0 is not an eigenvalue for L, then Theorem 6.6 yields $e^{\lambda_0 t} p_t(x,y) \to 0$ as $t \to \infty$. Therefore, $\log \left(e^{\lambda_0 t} p_t(x,y) \right) < 0$ for all t large enough and since $\frac{1}{t} \log \left(e^{\lambda_0 t} p_t(x,y) \right) \to \lambda_0 - \lambda$ as $t \to \infty$, it follows that $\lambda_0 \leq \lambda$.

We will now show that $\lambda_0 \geq \lambda$. Let $\varepsilon > 0$. From Proposition 6.2 we get

$$1_{[\lambda_0, \lambda_0+\varepsilon]}(L) \neq 0$$

since $\lambda_0 \in \sigma(L)$. As the set of functions $\{1_x \mid x \in X\}$ is total (i.e. the span is dense) in $\ell^2(X,m)$, it follows that there exists an $x \in X$ such that

$$1_{[\lambda_0, \lambda_0+\varepsilon]}(L) 1_x \neq 0.$$

Let μ_x be the spectral measure of L associated to 1_x. Proposition 3.13 gives

$$\frac{p_t(x,x)}{m(x)^2} = \langle 1_x, e^{-tL} 1_x\rangle = \int_{[\lambda_0, \lambda_0+\varepsilon]} e^{-ts} \mathrm{d}\mu_x(s) \geq e^{-t(\lambda_0+\varepsilon)} \mu_x([\lambda_0, \lambda_0+\varepsilon])$$

as the spectral measure μ_x is supported on $[\lambda_0, \lambda_0+\varepsilon]$ by Proposition 3.17. Therefore,

$$-\lambda = \lim_{t\to\infty} \frac{1}{t} \log p_t(x,x) \geq -(\lambda_0+\varepsilon),$$

that is, $\lambda \leq \lambda_0 + \varepsilon$. As $\varepsilon > 0$ was arbitrary, it follows that $\lambda \leq \lambda_0$, which concludes the proof. □

From the theorem above we immediately obtain the following corollary which states that the existence of a positive eigenfunction implies that the eigenvalue is the bottom of the spectrum.

COROLLARY 6.9 (Positive eigenfunctions are multiples of ground states). *Let (b,c) be a connected graph over (X,m) and $\lambda \in \mathbb{R}$. If there exists a non-trivial $u \geq 0$ with $u \in D(L)$ such that*

$$Lu = \lambda u$$

i.e., u is a positive eigenfunction corresponding to λ, then $\lambda = \lambda_0$. Furthermore, u is a positive multiple of the ground state, in particular $u > 0$.

PROOF. As λ is an eigenvalue, $\lambda \in \sigma(L)$ so that $\lambda_0 \leq \lambda$ by definition. Now, if $u \in D(L)$, $u \geq 0$ is non-trivial and satisfies $Lu = \lambda u$, then the functional calculus gives

$$e^{-tL}u = e^{-t\lambda}u$$

for $t \geq 0$. Therefore, for an arbitrary $x \in X$, using the positivity of u we get

$$p_t(x,x)u(x)m(x) \leq \sum_{y \in X} p_t(x,y)u(y)m(y) = e^{-tL}u(x) = e^{-t\lambda}u(x).$$

Applying Theorem 6.8 and choosing $x \in X$ such that $u(x) \neq 0$, we get

$$-\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log\left(p_t(x,x)u(x)m(x)\right) \leq \lim_{t\to\infty} \frac{1}{t} \log\left(e^{-t\lambda}u(x)\right) = -\lambda$$

so that $\lambda_0 \geq \lambda$. Therefore, $\lambda_0 = \lambda$.

The strict positivity of u follows from the proof of Lemma 6.4, since the eigenspace of λ_0 is one-dimensional. ☐

CHAPTER 7

Intrinsic Metrics and Spectral Estimates

In this chapter we turn to spectral geometry of graphs. This means that we study the interplay between spectral theory of graph Laplacians L and the geometry of the graphs. Here, geometry is captured by intrinsic metrics. These are (pseudo)metrics which are adapted in some sense to the graph structure. A remarkable feature of graphs is that they will in general admit several intrinsic metrics which are not comparable. In particular, in general there is no natural canonical intrinsic metric.

The spectral theory we are interested in is the infimum $\lambda_0(L^{(D)})$ of the spectrum of $L^{(D)}$. Specifically, we will give lower and upper bounds of the form

$$\frac{h^2}{2} \leq \lambda_0(L^{(D)}) \leq \frac{\mu^2}{8},$$

where h is an isoperimetric or Cheeger constant and μ is an exponential volume growth rate.

7.1. Intrinsic metrics

We first discuss some generalities on pseudometrics and then turn to specific pseudometrics associated to graphs. Let X be a set. We will be interested in maps on X, which can be thought of as weak versions of metrics. A *pseudometric* on X is a map $\varrho\colon X \times X \longrightarrow [0, \infty]$ with

- $\varrho(x, x) = 0$,
- $\varrho(x, y) = \varrho(y, x)$, ("Symmetry")
- $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$ ("Triangle inequality")

for all $x, y, z \in X$.

Whenever ϱ is a pseudometric the inequality

$$|\varrho(o, x) - \varrho(o, y)| \leq \varrho(x, y)$$

holds for all $x, y, o \in X$. It is known as *reverse triangle inequality.*

Let now ϱ be a pseudometric on X. Let $C \geq 0$ be given. A function f on X is said to be *C-Lipschitz* (with respect to ϱ) if

$$|f(x) - f(y)| \leq C\varrho(x, y)$$

holds for all $x, y \in X$. A *Lipschitz function* is a function, that is C-Lipschitz for some $C \geq 0$. The set of 1-Lipschitz functions with respect to ϱ is denoted by $\mathrm{Lip}_{1,\varrho}(X)$. Clearly, f is C-Lipschitz (for some $C > 0$) if and only if $\frac{1}{C}f$ belongs to $\mathrm{Lip}_{1,\varrho}(X)$.

From the reverse triangle inequality we see that the function $\varrho_x := \varrho(x, \cdot)$, that gives the ($\varrho$-)distance to $x \in X$, is 1-Lipschitz for any $x \in X$.

In fact, this can be generalized as follows: Let $U \subseteq X$ be non-empty and define the *(ϱ-)distance* to U by

$$\varrho_U \colon X \longrightarrow [0,\infty], \quad \varrho_U(y) := \inf_{x \in U} \varrho(x,y).$$

Then, ϱ_U can easily be seen to belong to $\mathrm{Lip}_{1,\varrho}(X)$. So, the pseudometric ϱ gives rise to a wealth of 1-Lipschitz functions.

Conversely, any 1-Lipschitz function f gives rise to a pseudometric σ_f defined by $\sigma_f(x,y) := |f(x) - f(y)|$ with $\sigma_f \leq \varrho$. One can recover ϱ from $\mathrm{Lip}_{1,\varrho}(X)$ as

$$\varrho = \sup\{\sigma_f \mid f \in \mathrm{Lip}_{1,\varrho}(X)\}.$$

Here, the inequality $\geq$ is immediate from $\sigma_f \leq \varrho$. The converse inequality follows by considering $f = \varrho_x$, $x \in X$. So, $\mathrm{Lip}_{1,\varrho}(X)$ determines ϱ. In fact, this set has some special features. Specifically, the set of 1-Lipschitz functions is closed under taking maxima, minima, adding constants and multiplication with -1. Also it is closed under pointwise limits and suprema.

Let X be a countable set and m a measure on X with full support. Any graph (b,c) over (X,m) gives rise to a special class of pseudometrics.

Definition 7.1 (Intrinsic metric). A pseudometric ϱ is called an *intrinsic metric* for a graph b over (X,m) if

$$\sum_{y \in X} b(x,y)\varrho(x,y)^2 \leq m(x)$$

for all $x \in X$. We call a pseudometric ϱ an *intrinsic metric* for a graph (b,c) over (X,m) if ϱ is intrinsic for b over (X,m).

We will now put this definition in perspective. To do so, we note that a differentiable function f on the real line $\mathbb{R}$ satisfies

$$|f(s) - f(t)| \leq |s - t| \text{ for all } s,t \in \mathbb{R}$$

if and only if

$$|f'(s)| \leq 1 \text{ for all } s \in \mathbb{R}.$$

The basic idea is that similarly the 1-Lipschitz functions with respect to an intrinsic metric should be related to the functions whose derivative is bounded by 1. Of course, we first need a notion of derivative to make this precise. In this context, we define for $f \in C(X)$ the norm of the *gradient* by

$$|\nabla f| := |\nabla f|_{b,m} : X \longrightarrow [0,\infty]$$

$$|\nabla f|(x) := \left(\frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y))^2\right)^{1/2}$$

for $x \in X$ and

$$A_1(X) := \{f \in C(X) \mid |\nabla f|(x)^2 \leq 1 \text{ for all } x \in X\}.$$

Then, for $c = 0$,

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x \in X} |\nabla f|(x)^2 m(x)$$

holds for all $f \in C(X)$, and any $f \in A_1(X)$ satisfies the inequality

$$\frac{1}{2}\sum_{x\in K} |\nabla f|(x)^2 m(x) \leq \frac{1}{2} m(K)$$

for any finite $K \subseteq X$.

We now characterize intrinsic metrics as follows.

Lemma 7.2 (Characterization intrinsic metrics). *Let b be a graph over (X, m) and let ϱ be a pseudometric. Then, the following statements are equivalent:*

(i) *ϱ is an intrinsic metric.*
(ii) $\mathrm{Lip}_{1,\varrho}(X) \subseteq A_1(X)$.
(iii) $|\nabla \varrho(o,\cdot)|^2 \leq 1$, *i.e.,* $\varrho(o,\cdot) \in A_1(X)$ *for all* $o \in X$.
(iii)' $|\nabla \rho_U|^2 \leq 1$, *i.e.,* $\varrho_U \in A_1(X)$ *for all nonempty* $U \subseteq X$.

In particular, if $\eta \in C(X)$ is C-Lipschitz with respect to an intrinsic metric ϱ and $C \geq 0$, then

$$|\nabla \eta|^2 \leq C^2.$$

Proof. (i) $\Longrightarrow$ (ii): Let $f \in \mathrm{Lip}_{1,\varrho}(X)$ be given, where ϱ is an intrinsic metric. Then, the Lipschitz property of f and the defining feature of an intrinsic metric give

$$\begin{aligned}|\nabla f|(x)^2 &= \frac{1}{m(x)}\sum_{y\in X} b(x,y)(f(x)-f(y))^2 \leq \frac{1}{m(x)}\sum_{y\in X} b(x,y)\varrho(x,y)^2\\ &\leq 1\end{aligned}$$

for all $x \in X$. Hence, f belongs to $A_1(X)$.

(ii) $\Longrightarrow$ (iii)': As discussed above, we have $\varrho_U \in \mathrm{Lip}_{1,\varrho}(X)$. Since $\mathrm{Lip}_{1,\varrho}(X) \subseteq A_1(X)$ by (ii), we conclude $\varrho_U \in A_1(X)$ and (iii)' follows.

(iii)' $\Longrightarrow$ (iii): This follows immediately (as we can take $U = \{o\}$ for any $o \in X$).

(iii) $\Longrightarrow$ (i): The assumption $|\nabla \varrho(o,\cdot)|^2 \leq 1$ for all $o \in X$ gives

$$\begin{aligned}1 \geq |\nabla \varrho(o,\cdot)|(o)^2 &= \frac{1}{m(o)}\sum_{y\in X} b(o,y)(\varrho(o,o)-\varrho(o,y))^2\\ &= \frac{1}{m(o)}\sum_{y\in X} b(o,y)\varrho(o,y)^2.\end{aligned}$$

Hence,

$$\sum_{y\in X} b(o,y)\varrho(o,y)^2 \leq m(o)$$

holds for all $o \in X$. Thus, ϱ is an intrinsic metric.

The "in particular" statement follows immediately as C-Lipschitz means that

$$|\eta(x)-\eta(y)| \leq C\varrho(x,y)$$

for $x, y \in X$. □

Example 7.3. For a graph b over (X,m), ρ defined by

$$\rho(x,y) := \inf_{x=x_0\sim\ldots\sim x_n=y} \sum_{i=1}^{n} (\mathrm{Deg}(x_{i-1}) \vee \mathrm{Deg}(x_i))^{-1/2}$$

for $x,y \in X$ can be easily seen to be an intrinsic metric. We call it the *degree path metric.*

Remark 7.4 (Why there is no equality between $\mathrm{Lip}_{1,\varrho}(X)$ and $A_1(X)$). For graphs there will, in general, not exist a pseudometric ϱ with $\mathrm{Lip}_{1,\varrho}(X) = A_1(X)$. This is different from our motivating example of differentiable functions on the real line. To understand this better, we note that the set $\mathrm{Lip}_{1,\varrho}(X)$ is always closed under taking maxima. However, the set $A_1(X)$ is in general not closed under taking maxima (as can already be seen by simple examples of graphs with three vertices).

Remark 7.5 (What about the combinatorial distance?). Any graph (b,c) over (X,m) comes naturally with the *combinatorial distance* defined by

$$d_{\mathrm{comb}}(x,y) := \inf\{n \in \mathbb{N}_0 : \text{there exists a path of length } n \text{ from } x \text{ to } y\}.$$

Here, $x_0,\ldots,x_n$ from X are called a *path of length* n if $x_0 = x$, $x_n = y$ and $b(x_j,x_{j+1}) > 0$ whenever $j \in \{0,\ldots,n-1\}$. Here, as usual, the infimum over the empty set is defined to be ∞. Clearly, $d_{\mathrm{comb}}(x,y) = 1$ whenever $b(x,y) > 0$ holds. Thus,

$$\frac{1}{m(x)} \sum_{y\in X} b(x,y) d_{\mathrm{comb}}(x,y)^2 = \frac{1}{m(x)} \sum_{y\in X} b(x,y).$$

Now, clearly both combinatorial distance and the condition of being an intrinsic metric only depend on b (and not on c). Thus, we restrict now attention to graphs b over (X,m), i.e. $c = 0$. In this case, the expression featured in the last equality is just the weighted degree Deg. Thus, we infer that d_{comb} is an intrinsic metric for the graph b over (X,m) if and only if $\mathrm{Deg} \le 1$ holds. Now, we already know from Theorem 2.18 that boundedness of Deg is equivalent to boundedness of the associated Laplacian L. Hence, we can conclude that for a graph b over (X,m) the Laplacian is bounded if and only if a suitable multiple αd_{comb} is an intrinsic metric. For general graphs (b,c) still boundedness of the Laplacian implies that the combinatorial distance is an intrinsic metric (up to scaling). So, the advantage of intrinsic metrics over the combinatorial distance comes only about when situations with unbounded Laplacians are considered.

Remark 7.6 (Why intrinsic metrics?). The inequality

$$\frac{1}{2} \sum_{x\in K} |\nabla f|(x)^2 m(x) \le \frac{1}{2} m(K)$$

shown above for $f \in A_1(X)$ and any finite $K \subseteq X$ gives a first indication that $A_1(X)$ is a useful set of functions. A pseudometric is then intrinsic if its 1-Lipschitz functions satisfy this inequality. In Sections 7.2 and 7.3 we will see two instances of how this can be used to deal with general graphs without making any boundedness assumption. A particular feature is that intrinsic metrics allow one to exhibit cut-off functions. We will come back to this point below. Here, we already note that whenever ϱ is an intrinsic metric, for any $A \subseteq X$ and $R > 0$ the function

$$\eta := (1 - \frac{1}{R}\varrho_A)_+$$

satisfies the following properties:

- $1_A \leq \eta \leq 1_{B_R(A)}$. Here, 1_S denotes the characteristic function of the set $S \subseteq X$ and $B_R(A)$ denotes the set of points in X with ϱ-distance not exceeding R to A.
- η is $1/R$-Lipschitz with respect to ϱ. In particular,

$$|\nabla \eta|^2 \leq \frac{1}{R^2}.$$

7.2. Cheeger Inequality

In this section we provide a lower bound on the infimum of the spectrum of the Laplacian on a graph in terms of the Cheeger constant. Roughly speaking the Cheeger constant measures the ratio between the size of the boundary of a set and the set itself.

We first give the definition of the Cheeger constant. The *boundary* of a set $W \subseteq X$ is given by those pairs $(x, y) \in X \times X$ that have one element in W and one element in its complement, i.e.

$$\partial W := (W \times (X \setminus W)) \cup ((X \setminus W) \times W).$$

Moreover, analogously to the measures m on X, any $w\colon X \times X \longrightarrow [0, \infty)$ gives rise to a measure on $X \times X$ and therefore for $U \subseteq X \times X$ we write

$$w(U) := \sum_{(x,y)\in U} w(x, y) \in [0, \infty].$$

Definition 7.7 (Cheeger constant). Let b be a graph over (X, m) and let ϱ be an intrinsic metric. For a finite set $W \subseteq X$, we let the *area of the boundary* be given by

$$A_{b\varrho}(\partial W) := \frac{1}{2} \sum_{(x,y)\in\partial W} b(x, y)\varrho(x, y) = \frac{1}{2}(b\varrho)(\partial W).$$

We define the *Cheeger constant* $h := h_{b\varrho,m}$ (of the graph) by

$$h := \inf_{W\subseteq X, W \text{ finite}} \frac{A_{b\varrho}(\partial W)}{m(W)}.$$

We prove the following theorem.

Theorem 7.8 (Cheeger inequality). *Let b be a graph over (X, m) and $Q := Q^{(D)}_{b,0,m}$ the associated form and $L := L^{(D)}$ the induced Dirichlet Laplacian. Let ϱ be an intrinsic metric. Then,*

$$\lambda_0(L) \geq \frac{h^2}{2}.$$

The remaining part of this section is devoted to a proof of the theorem. We will first present two formulas which involve (strict) superlevel sets of functions. For a function $f \in C(X)$ and $t \in \mathbb{R}$, we define the *(strict) superlevel set*

$$\Omega_t(f) := \{x \in X \mid f(x) > t\}.$$

The first formula relates the differences of a function to an integral over the boundary of the superlevel sets. We refer to this as a *co-area formula.*

Lemma 7.9 (Co-area formula). *Let $w\colon X \times X \longrightarrow [0,\infty)$ and $f \in C(X)$ be given. Then,*

$$\sum_{x,y\in X} w(x,y)|f(x)-f(y)| = \int_{-\infty}^{\infty} w(\partial\Omega_t(f))\mathrm{d}t,$$

where both sides may take the value ∞.

Proof. For vertices $x, y \in X$ with $x \neq y$ we define the interval $I_{x,y}$ by

$$I_{x,y} := [f(x) \wedge f(y), f(x) \vee f(y)),$$

and let $|I_{x,y}| = |f(x) - f(y)|$ be the length of $I_{x,y}$. Denote by $1_{x,y}$ the characteristic function of $I_{x,y}$. Then, for $t \in \mathbb{R}$ we have $(x,y) \in \partial\Omega_t(f)$ if and only if $t \in I_{x,y}$ holds. Therefore,

$$w(\partial\Omega_t(f)) = \sum_{x,y\in X} w(x,y)1_{x,y}(t).$$

From these considerations and the monotone convergence theorem we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} w(\partial\Omega_t(f))\mathrm{d}t &= \int_{-\infty}^{\infty} \sum_{x,y\in X} w(x,y)1_{x,y}(t)\mathrm{d}t = \sum_{x,y\in X} w(x,y)\int_{-\infty}^{\infty} 1_{x,y}(t)\mathrm{d}t \\ &= \sum_{x,y\in X} w(x,y)|f(x)-f(y)|.\end{aligned}$$

This proves the statement. □

For the next formula, we assume that the function f is positive and that there exists a measure on the space. The formula then relates the values of the function to the measure of the superlevel sets associated to the function.

Lemma 7.10 (Area formula). *Let $m\colon X \longrightarrow [0,\infty)$ and $f\colon X \longrightarrow [0,\infty)$ be given. Then,*

$$\sum_{x\in X} f(x)m(x) = \int_0^{\infty} m(\Omega_t(f))\mathrm{d}t,$$

where both sides may take the value ∞.

Proof. We have $x \in \Omega_t(f)$ if and only if $1_{(t,\infty)}(f(x)) = 1$. From this and the monotone convergence theorem we obtain

$$\begin{aligned}\int_0^{\infty} m(\Omega_t(f))\mathrm{d}t &= \int_0^{\infty} \sum_{x\in\Omega_t(f)} m(x)\mathrm{d}t = \int_0^{\infty} \sum_{x\in X} m(x)1_{(t,\infty)}(f(x))\mathrm{d}t \\ &= \sum_{x\in X} m(x)\int_0^{\infty} 1_{(t,\infty)}(f(x))\mathrm{d}t = \sum_{x\in X} m(x)f(x).\end{aligned}$$

This finishes the proof. □

After these preparations we now turn to the proof of Theorem 7.8.

Proof of Theorem 7.8. Let $\varphi \in C_c(X)$ be given and denote the superlevel sets of φ^2 by

$$\Omega_t := \Omega_t(\varphi^2) = \{x \in X : \varphi(x)^2 > t\}.$$

Then, Ω_t is finite for any $t \geq 0$ as φ has finite support. Hence, by the definition of the Cheeger constant h we infer

$$hm(\Omega_t) \leq A_{b\varrho}(\partial\Omega_t)$$

for all $t \geq 0$. From this and the area and co-area formulae provided in Lemmas 7.10 and 7.9 we find

$$\begin{aligned} h\|\varphi\|^2 &= h\sum_{x\in X}\varphi(x)^2 m(x) \\ &\overset{\text{Area f.}}{=} h\int_0^\infty m(\Omega_t)\mathrm{d}t \leq \int_0^\infty A_{b\varrho}(\partial\Omega_t)\mathrm{d}t \\ &\overset{\text{Co-area f.}}{=} \frac{1}{2}\sum_{x,y\in X} b(x,y)\varrho(x,y)|\varphi(x)^2-\varphi(y)^2| \\ &= \frac{1}{2}\sum_{x,y\in X} b(x,y)\varrho(x,y)|\varphi(x)-\varphi(y)||\varphi(x)+\varphi(y)| \\ &= \frac{1}{2}\sum_{x,y\in X} b(x,y)|\varphi(x)-\varphi(y)|\varrho(x,y)|\varphi(x)+\varphi(y)|. \end{aligned}$$

Now, the Cauchy–Schwarz inequality on the space $\ell^2(X \times X, b)$ gives

$$\left|\sum_{x,y\in X} b(x,y)F(x,y)G(x,y)\right|^2 \leq \left(\sum_{x,y\in X} b(x,y)F(x,y)^2\right)\left(\sum_{x,y\in X} b(x,y)G(x,y)^2\right)$$

for all $F, G \in \ell^2(X \times X, b)$. Hence, we infer

$$h\|\varphi\|^2 \leq Q(\varphi)^{1/2}\left(\frac{1}{2}\sum_{x,y\in X} b(x,y)\varrho(x,y)^2(\varphi(x)+\varphi(y))^2\right)^{1/2}.$$

Now, Young's inequality $(\alpha+\beta)^2 \leq 2\alpha^2 + 2\beta^2$ for $\alpha, \beta \in \mathbb{R}$, symmetry of both b and ϱ and the intrinsic metric property give

$$\begin{aligned}
&\left(\frac{1}{2}\sum_{x,y\in X} b(x,y)\varrho(x,y)^2(\varphi(x)+\varphi(y))^2\right)^{1/2} \\
&\leq \left(2\sum_{x,y\in X} b(x,y)\varrho(x,y)^2\varphi(x)^2\right)^{1/2} \\
&= \left(2\sum_{x\in X}\varphi(x)^2\left(\sum_{y\in X} b(x,y)\varrho(x,y)^2\right)\right)^{1/2} \\
&\overset{\varrho \text{ intrinsic}}{\leq} \sqrt{2}\left(\sum_{x\in X}\varphi(x)^2 m(x)\right)^{1/2} = \sqrt{2}\|\varphi\|.
\end{aligned}$$

Putting everything together we arrive at

$$h\|\varphi\|^2 \leq \sqrt{2}Q(\varphi)^{1/2}\|\varphi\|$$

and this gives

$$\frac{h}{\sqrt{2}} \leq \frac{Q(\varphi)^{1/2}}{\|\varphi\|}.$$

By squaring both sides, we find

$$\frac{h^2}{2} \leq \frac{Q(\varphi)}{\|\varphi\|^2}$$

for all $\varphi \in C_c(X)$ with $\varphi \neq 0$. This gives the statement by the variational characterization of $\lambda_0(L)$, see Theorem 6.3. □

REMARK 7.11. Whenever (b,c) is a graph over (X,m) we clearly have $Q^{(D)}_{b,c,m}(\varphi) \geq Q^{(D)}_{b,0,m}(\varphi)$ for all $\varphi \in C_c(X)$. Hence, we find for the Laplacian $L^{(D)}$ associated to $Q^{(D)}_{b,c,m}$ as well the estimate $\frac{h^2}{2} \leq \lambda_0(L^{(D)})$.

7.3. Brooks–Sturm Inequality

In this section we show a version of Brooks–Sturm inequality. This inequality gives an upper bound on the infimum of the spectrum in terms of exponential volume growth of balls.

Let b be a graph over (X,m) and ϱ an intrinsic metric. The distance balls $B_r(x)$ of radius $r > 0$ about a vertex $x \in X$ with respect to ϱ are given by

$$B_r(x) := \{y \in X \mid \varrho(x,y) \leq r\}.$$

If the distance balls are finite, we define the *exponential volume growth rate with variable center*, which we call μ, by

$$\mu := \liminf_{r\to\infty} \inf_{x\in X} \frac{1}{r}\log\frac{m(B_r(x))}{m(B_1(x))}.$$

So, for any $\varepsilon > 0$ there exists an $R > 0$ with

$$m(B_r(x)) \geq m(B_1(x))e^{r(\mu-\varepsilon)}$$

for all $r \geq R$ and all $x \in X$.

THEOREM 7.12 (Theorem of Brooks–Sturm). *Let b be a connected graph over (X, m) and let ϱ be an intrinsic metric such that the distance balls are finite and $L := L^{(D)}$ the associated Dirichlet Laplacian. Then,*

$$\lambda_0(L) \leq \frac{\mu^2}{8}.$$

Brooks original result considered the bottom of the essential spectrum by a similar exponential volume growth constant.

The basic idea is to use test functions as follows.

LEMMA 7.13 (Test function). *Let b be a graph over (X, m) and ϱ and intrinsic metric. Assume that $f, g \in C(X)$ satisfy*

$$(f(x) - f(y))^2 \leq C(g(x)^2 + g(y)^2)\varrho(x, y)^2$$

for all $x, y \in X$, for some $C \geq 0$. Then,

$$\mathcal{Q}(f) \leq C \sum_{x \in X} g(x)^2 m(x)$$

holds (where the value ∞ is possible).

PROOF. This follows by a direct computation, invoking the symmetry of b and ϱ as follows

$$\begin{aligned}
\mathcal{Q}(f) &= \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2 \\
&\leq \frac{C}{2} \sum_{x,y \in X} b(x, y)\left(g(x)^2 + g(y)^2\right)\varrho(x, y)^2 \\
&= C \sum_{x \in X} g(x)^2 \left(\sum_{y \in X} b(x, y)\varrho(x, y)^2\right) \\
&\overset{\varrho \text{ intrinsic}}{\leq} C \sum_{x \in X} g(x)^2 m(x).
\end{aligned}$$

This finishes the proof. □

The test functions will be constructed via intrinsic metrics and exponential functions. To construct such functions we use the following proposition.

PROPOSITION 7.14 (Estimates for differences of exponentials). *For any $u, v \in \mathbb{R}$ the estimate*

$$|e^u - e^v| \leq \frac{|u - v|}{2}(e^u + e^v)$$

holds.

PROOF. For $w \geq 0$ we find by direct computation

$$\sum_{n=2}^{\infty} \frac{2}{n!} w^{n-1} = \sum_{n=1}^{\infty} \frac{2}{(n+1)!} w^n \leq \sum_{n=1}^{\infty} \frac{1}{n!} w^n = e^w - 1.$$

This gives

$$e^w - 1 = \sum_{n=0}^{\infty} \frac{w^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{w^n}{n!} = \frac{w}{2}\left(2 + \sum_{n=2}^{\infty} \frac{2}{n!} w^{n-1}\right) \leq \frac{w}{2}(1 + e^w).$$

Without loss of generality, we can now assume $u \leq v$. The preceding estimate for $w := v - u \geq 0$ gives

$$e^{v-u} - 1 \leq \frac{v-u}{2}(1 + e^{v-u}).$$

Multiplication with e^u then gives the desired statement. □

Let b be a graph over (X, m) and let ϱ be an intrinsic metric. Consider $\beta > 0$, $s \geq 0$ and $U \subseteq X$ non-empty, and define

$$h := h_{s,U,\beta} : X \longrightarrow (0, \infty), \quad h(x) := e^{\beta(s - \varrho_U(x))}.$$

By Proposition 7.14 the function h satisfies

$$\begin{aligned} |h(x) - h(y)| &\leq \frac{|\beta(s - \varrho_U(x)) - \beta(s - \varrho_U(y))|}{2}(h(x) + h(y)) \\ &\leq \frac{\beta\varrho(x,y)}{2}(h(x) + h(y)) \end{aligned}$$

for all $x, y \in X$, where we used ϱ_U is 1-Lipschitz. Taking squares and using $(a+b)^2 \leq 2a^2 + 2b^2$ we obtain for all $x, y \in X$

$$|h(x) - h(y)|^2 \leq \frac{\beta^2}{2}(h(x)^2 + h(y)^2)\varrho(x,y)^2.$$

The function h satisfies $h = e^{\beta s}$ on U and decays exponentially outside of U.

For our considerations we will need more than exponential decay viz vanishing outside of balls. Thus, we will construct modifications f and g from $h_{r,B_r(o),\beta}$, both of which can be seen as restrictions of h to balls $B_{2r}(o)$. Specifically, for the parameters $r \geq 0$, $o \in X$ and $\beta > 0$, we define the function $f := f_{r,o,\beta} \colon X \longrightarrow [0, \infty)$ by

$$f := (h_{r,B_r(o),\beta} - 1)_+,$$

i.e.

$$f(x) = (e^{\beta(r - \varrho_{B_r(o)}(x))} - 1)_+, \quad x \in X.$$

We observe the some basic properties of f. To this end let $B_r := B_r(o)$ and $B_r(B_r) = \{x \in X \mid \inf_{y \in B_r} \rho(x,y) \leq r\}$. Observe that on metric spaces one only has $B_r(B_r) \subseteq B_{2r}$ while one has equality on normed spaces. It can easily be seen that f satisfies

- $f|_{B_r} = e^{\beta r} - 1$
- $f|_{B_r(B_r) \setminus B_r} = e^{\beta(r - \varrho_{B_r}(\cdot))} - 1$
- $f|_{X \setminus B_r(B_r)} = 0$.

Furthermore, for the parameters $r \geq 0$, $o \in X$ and $\beta > 0$, we define the auxiliary functions $g := g_{r,o,\beta} \colon X \longrightarrow [0, \infty)$ by

$$g := f + 2 \cdot 1_{B_{2r}(o)} = (h_{r,B_r(o),\beta} + 1)1_{B_{2r}(o)}.$$

We observe the following basic properties for g

- $g|_{B_r} = e^{\beta r} + 1$

- $g|_{B_r(B_r)\setminus B_r} = e^{\beta(r-\varrho_{B_r}(\cdot))} + 1$
- $g|_{B_{2r}\setminus B_r(B_r)} = 2.$
- $g|_{X\setminus B_{2r}} = 0.$

Hence,

$$f \leq h \leq g$$

on $B_{2r}(o)$. The relevant features of f and g are provided in the next lemma.

LEMMA 7.15 (Properties of f and g). *Let b be a graph over (X, m) and let ϱ be an intrinsic metric. Let $o \in X$, $r \geq 0$ and $\beta > 0$.*

(a) For $f := f_{r,o,\beta}$ and $g := g_{r,o,\beta}$ the inequality

$$(f(x) - f(y))^2 \leq \frac{\beta^2}{2}\left(g(x)^2 + g(y)^2\right)\varrho(x,y)^2,$$

holds for all $x, y \in X$.

(b) Assume $\mu < \infty$ and choose $\beta > \frac{\mu}{2}$. Then, there exist sequences (o_k) in X and (r_k) in $[0,\infty)$ such that

$$\lim_{k\to\infty}\frac{\|g_{r_k,o_k,\beta}\|}{\|f_{r_k,o_k,\beta}\|} = 1.$$

PROOF. (a) We distinguish three cases:

Case 1: $x, y \in B_r(B_r(o))$. Then, $f = (h_{r,B_r(o),\beta} - 1)$ and we obtain from the calculation above

$$\begin{aligned} |f(x) - f(y)|^2 &= |h_{r,B_r(o),\beta}(x) - h_{r,B_r(o),\beta}(y)|^2 \\ &\leq \frac{\beta^2}{2}|h_{r,B_r(o),\beta)}(x)^2 + h_{r,B_r(o),\beta)}(y)^2|\varrho(x,y)^2 \\ &\leq \frac{\beta^2}{2}\left(g(x)^2 + g(y)^2\right)\varrho(x,y)^2, \end{aligned}$$

where we used $g = h_{r,B_r(o),\beta} + 1$ on $B_{2r}(o)$.

Case 2: $x, y \in X \setminus B_r(B_r(o))$. Then, $f(x) = 0 = f(y)$ and the estimate clearly follows.

Case 3: $x \in B_r(B_r(o))$ and $y \in X \setminus B_r(B_r(o))$ or $y \in B_r(B_r(o))$ and $x \in X \setminus B_r(B_r)$. By symmetry it suffices to consider the case $x \in B_r(B_r(o))$ and $y \in X \setminus B_r(B_r(o))$. By $y \in X \setminus B_r(B_r(o))$ we have $\varrho_{B_r(o)}(y) \geq r$. By $x \in B_r(B_r(o))$ we have

$$t := r - \varrho_{B_r(o)}(x) \geq 0.$$

Altogether we find

$$t = r - \varrho_{B_r(o)}(x) \leq \varrho_{B_r(o)}(y) - \varrho_{B_r(o)}(x) \leq \varrho(x,y).$$

Using this we find from Proposition 7.14

$$\begin{aligned} |f(x) - f(y)| = |e^{\beta t} - 1| = |e^{\beta t} - e^{\beta\cdot 0}| &\leq \frac{\beta}{2}(e^{\beta t} + 1)t \\ &\leq \frac{\beta}{2}(g(x) + g(y))\varrho(x,y). \end{aligned}$$

Now, the desired estimate follows after taking squares.

(b) Let $\beta > \frac{\mu}{2}$ and

$$0 < \varepsilon < \left(\beta - \frac{\mu}{2}\right) \wedge 1.$$

By definition of μ there exist a sequence (r_k) with $r_k \to \infty$ and a sequence (o_k) in X with

$$\frac{m(B_{2r_k}(o_k))}{m(B_1(o_k))} \leq e^{(2\mu+\varepsilon)r_k}$$

for $k \in \mathbb{N}$. With $f_k := f_{r_k,o_k,\beta}$ and $g_k := g_{r_k,o_k,\beta}$ for $k \in \mathbb{N}$ we have $g_k = (f_k + 2)1_{B_{2r_k}(o_k)}$ for all $k \in \mathbb{N}$, so we estimate using Cauchy–Schwarz and Young's inequality $(s+t)^2 \leq (1-\varepsilon)^{-1}s^2 + \varepsilon^{-1}t^2$,

$$\|g_k\|^2 \leq \left(\|f_k\| + 2\sqrt{m(B_{2r_k}(o_k))}\right)^2 \leq \frac{1}{1-\varepsilon}\|f_k\|^2 + \frac{4}{\varepsilon}m(B_{2r_k}(o_k)).$$

On the other hand we have

$$\|f_k\|^2 \geq m(B_{r_k}(o_k))(e^{\beta r_k} - 1)^2$$

for all $k \in \mathbb{N}$. Hence, for sufficiently large k, say $k \geq k_1$, we find

$$\|f_k\|^2 \geq \frac{1}{2}m(B_{r_k}(o_k))e^{2\beta r_k}.$$

Thus, for all $k \geq k_1$,

$$\frac{\|g_k\|^2}{\|f_k\|^2} \leq \frac{1}{1-\varepsilon} + \frac{8}{\varepsilon}e^{-2\beta r_k}\frac{m(B_{2r_k}(o_k))}{m(B_{r_k}(o_k))}.$$

Moreover, given ε as above, there exists $k_2 \geq k_1$ such that for all $k \geq k_2$,

$$\frac{m(B_{r_k}(o_k))}{m(B_1(o_k))} \geq \inf_{o\in X}\frac{m(B_{r_k}(o))}{m(B_1(o))} \geq e^{(\mu-\varepsilon)r_k}.$$

Therefore,

$$\frac{m(B_{2r_k}(o_k))}{m(B_{r_k}(o_k))} = \frac{m(B_{2r_k}(o_k))}{m(B_1(o_k))}\frac{m(B_1(o_k))}{m(B_{r_k}(o_k))} \leq e^{(\mu+2\varepsilon)r_k}.$$

Since $0 < \varepsilon < \beta - \frac{\mu}{2}$, we can combine this with the estimate above to conclude

$$\frac{\|g_k\|^2}{\|f_k\|^2} \leq \frac{1}{1-\varepsilon} + \frac{8}{\varepsilon}e^{(\mu-2\beta+2\varepsilon)r_k} \to \frac{1}{1-\varepsilon}$$

as $k \to \infty$. Since ε was chosen arbitrarily and $0 \leq f_k \leq g_k$, statement (b) follows. □

After these preparations we can now provide the proof of the main result of this section.

Proof of Theorem 7.12. Chose $\beta > \frac{\mu}{2}$. Let the sequences (o_k) and (r_k) be taken from Lemma 7.15 (b) and set $f_k := f_{r_k,o_k,\beta}$ and $g_k := g_{r_k,o_k,\beta}$ for $k \in \mathbb{N}$. By the finiteness of balls we have $f_k, g_k \in C_c(X) \subseteq D(Q^{(D)})$ for all $k \in \mathbb{N}$. By Lemma 7.13 combined with Lemma 7.15 (a) we find

$$Q^{(D)}(f_k) \leq \frac{\beta^2}{2}\|g_k\|^2$$

for all $k \in \mathbb{N}$. By the variational characterization of $\lambda_0(L)$, Theorem 6.3, we then get for any $k \in \mathbb{N}$ the estimate

$$\lambda_0(L) = \inf_{f\in C_c(X)}\frac{Q^{(D)}(f)}{\|f\|^2} \leq \inf_k \frac{Q^{(D)}(f_k)}{\|f_k\|^2} \leq \inf_k \frac{\beta^2}{2}\frac{\|g_k\|^2}{\|f_k\|^2} = \frac{\beta^2}{2},$$

where we used Lemma 7.15 (b) in the last equality. As this hold for any $\beta > \frac{\mu}{2}$ we arrive at the desired statement. $\square$

We finish this section by discussing a class of examples. A connected graph (b, c) over (X, m) is said to have *subexponential growth* with respect to the intrinsic metric ϱ if we find $o \in X$ such that for all $\varepsilon > 0$ there exists $C_\varepsilon \geq 0$ with

$$m(B_r(o)) \leq C_\varepsilon e^{r\varepsilon}$$

for all $r \geq 1$. It is not hard to see that this property does not depend on o. Specifically, if this property holds for o then for any other $o' \in X$ we clearly have $B_r(o') \subseteq B_{r+\varrho(o,o')}(o)$ for any $r > 0$ and this implies

$$m(B_r(o')) \leq m(B_{r+\varrho(o,o')}(o)) \leq C_\varepsilon e^{\varepsilon\varrho(o,o')} e^{\varepsilon r}$$

for any $r \geq 1$. For connected graphs with subexponential growth we clearly have $\mu = 0$. Thus, we arrive at the following corollary.

COROLLARY 7.16 (Subexponential growth implies $\lambda_0(L) = 0$). *Let (b, c) be a connected graph over (X, m) with subexponential growth with respect to the intrinsic metric ϱ and $L := L^{(D)}$ the Dirichlet Laplacian. Then, $\lambda_0(L) = 0$.*

CHAPTER 8

Agmon–Allegretto–Piepenbrink Theorem

In Chapter 7 we have seen a geometric approach to the infimum $\lambda_0(L)$ of the spectrum of a graph Laplacian $L = L^{(D)}$. Here, we consider a functional analytic characterization. Specifically, we show that $\lambda_0(L)$ can be characterized by the following property: There exists a strictly positive $u \in \mathcal{F}$ with

$$(\mathcal{L} - \lambda)u \geq 0$$

if and only if $\lambda \leq \lambda_0(L)$. This theorem is known as Agmon–Allgretto–Piepenbrink theorem.

Let X be a countable set and m a measure on X with full support.

8.1. Local Harnack inequality

Given a graph (b, c) over (X, m) and $\lambda \in \mathbb{R}$, our aim is to study $u \in \mathcal{F}$ with

$$(\mathcal{L} - \lambda)u \geq 0.$$

Such a u is known as a *supersolution* to $\mathcal{L} - \lambda$. Here, we first show that any such solution satisfies certain bounds, known as local Harnack inequality.

Theorem 8.1 (Strict positivity and local Harnack inequality). *Let (b, c) be a connected graph over (X, m), $\lambda \in \mathbb{R}$. Then, any positive supersolution $u \in \mathcal{F}$ to $(\mathcal{L} - \lambda)u \geq 0$ is strictly positive. Moreover, to any $x, x^* \in X$ there exists a decreasing function $C_{x,x^*} : \mathbb{R} \to [0, \infty)$ such that for every $\lambda \in \mathbb{R}$ and every $u \in \mathcal{F}$ with to $(\mathcal{L} - \lambda)u \geq 0$ the inequality*

$$u(x) \leq C_{x,x^*}(\lambda)u(x^*)$$

holds.

Proof. Let u be a positive supersolution to λ. Assume that u is nontrivial (as for trivial u there is nothing to show). By definition of $\mathcal{L}$ the inequality

$$(\mathcal{L} - \lambda)u \geq 0$$

reads as

$$\frac{1}{m(x)} \sum_{y \in X} b(x, y)(u(x) - u(y)) + \frac{c(x)}{m(x)} u(x) - \lambda u(x) \geq 0$$

for all $x \in X$. This implies

$$(\deg(x) - \lambda m(x))u(x) \geq \sum_{y \in X} b(x, y)u(y) \geq 0$$

with

$$\deg(x) = \sum_{y \in X} b(x,y) + c(x).$$

Since u is positive, so must be $(\deg - \lambda m)$. Furthermore, if $(\deg - \lambda m)(x) = 0$ or $u(x) = 0$ then the above inequality gives $u(y) = 0$ for all $y \sim x$. By connectedness of the graph we then deduce by induction $u = 0$ which contradicts non-triviality of u. Hence, $(\deg - \lambda m)(x) > 0$ and and $u(x) > 0$ and

$$u(x) \geq \frac{\sum_{y \in X} b(x,y)}{(\deg(x) - \lambda m(x))} u(y)$$

hold for all $x \in X$.

Now, chose an arbitrary path $x_0, \ldots, x_n$ connecting x and x^*. From what we have shown already we infer

$$\begin{aligned} u(x_0) &\geq \frac{\sum_{y \in X} b(x,y)}{(\deg(x) - \lambda m(x))} u(y) \geq \frac{b(x_0, x_1)}{(\deg(x_0) - \lambda m(x_0))} u(x_1) \\ &\geq \left(\prod_{j=0}^{n-1} \frac{b(x_j, x_{j+1})}{(\deg(x_j) - \lambda m(x_j))} \right) u(x_n). \end{aligned}$$

Letting

$$C_{x,x^*}(\lambda) := \prod_{j=0}^{n-1} \frac{(\deg(x_j) - \lambda m(x_j))}{b(x_j, x_{j+1})},$$

we find the desired inequality for any λ with $\lambda < \deg(x)/m(x)$ for all $x \in X$. For $\lambda \geq \inf_{x \in X} \frac{\deg(x)}{m(x)}$ there is no supersolution and we can choose $C_{x,x^*}(\lambda) := 0$. Clearly, the expression for $C_{x,x^*}(\lambda)$ is decreasing in λ. $\square$

Remark 8.2. As noted in the proof there are no positive non-trivial supersolutions for $\lambda \geq \inf_{x \in X} \deg(x)/m(x)$.

Remark 8.3. If there is a positive supersolution u for some λ then for all $\lambda' \leq \lambda$ we have

$$(\mathcal{L} - \lambda')u \geq (\mathcal{L} - \lambda)u \geq 0.$$

Hence, if there exists a non-trivial positive supersolution u for λ then u is a supersolution for every $\lambda' \in (-\infty, \lambda)$. In the next proposition we show that if there are supersolutions for all $\lambda' \in (-\infty, \lambda)$ then there is also a supersolution for λ.

Proposition 8.4 (Harnack principle). *Let (b,c) be a connected graph over (X,m). Let $\lambda \in \mathbb{R}$, (λ_n) in $\mathbb{R}$ such that $\lambda_n \to \lambda$ and assume there are positive non-trivial supersolutions u_n to λ_n with $u_n(o) = 1$ for some $o \in X$ and all $n \in \mathbb{N}$. Then, there is a subsequences (u_{n_k}) of (u_n) which converges pointwise to a positive non-trivial supersolution u for λ.*

Proof. Set $\widetilde{\lambda} := \inf_{n\in\mathbb{N}} \lambda_n$. Since $u_n(o) = 1$ for all $n \in \mathbb{N}$ we infer from the local Harnack inequality

$$0 \leq u_n(x) \leq C_{x,o}(\lambda_n) \leq C_{x,o}(\widetilde{\lambda})$$

for any $x \in X$ and all $n \in \mathbb{N}$. Thus, for any $x \in X$ the sequence $(u_n(x))_n$ is contained in $[0, C_{x,o}(\widetilde{\lambda})]$. As X is countable, (u_n) has a subsequence (u_{n_k}) that converges pointwise to some u. Furthermore, we have by Fatou's lemma applied to $(\sum_{y\in X} b(x,y)u_{n_k}(y))_k$

$$0 \leq \lim_{k\to\infty} (\mathcal{L} - \lambda_{n_k})u_{n_k}(x) \leq (\mathcal{L} - \lambda)u(x)$$

for all $x \in X$. Thus, u is a supersolution to λ. Since the u_{n_k} are positive and $u_{n_k}(o) = 1$ for all $k \in \mathbb{N}$, we have that u is positive and $u(o) = 1$. □

8.2. The Ground State Transform

On the intuitive level the ground state transform is a tool to convert a graph (b, c) into a new graph $(b_u, 0)$ (with $b_u(x, y) = b(x, y)u(x)u(y)$) provided u is a strictly positive (super)solution to $(\mathcal{L} - \lambda)u = 0$ for some $\lambda \in \mathbb{R}$. Precise versions (which give even more general statements) can be given both on the level of operators and on the level of forms. This is useful in order to obtain lower bounds.

We start with some notation. Let $u \in C(X)$ with $u > 0$ be given. Then, we denote $\mathcal{T}_u \colon C(X) \to C(X)$ by

$$\mathcal{T}_u f := uf.$$

Also, for $b \colon X \times X \longrightarrow [0, \infty)$ we define

$$b_u \colon X \times X \longrightarrow [0, \infty), \quad b_u(x, y) := b(x, y)u(x)u(y).$$

We next present the ground state transform on the level of operators.

Lemma 8.5 (Ground state transform – operator version). *Let (b, c) be a graph over (X, m). Let $u \in C(X)$ be strictly positive and $w \in C(X)$ such that $(\mathcal{L} - w)u = 0$. Then,*

$$\mathcal{L}_u := \mathcal{T}_u^{-1} \mathcal{L} T_u$$

acts on $f \in C(X)$ such that $uf \in \mathcal{F}$ as

$$\mathcal{L}_u f(x) = \frac{1}{u(x)^2 m(x)} \left(\sum_{y\in X} b(x, y)u(y)u(x)(f(x) - f(y)) \right) + w(x)f(x)$$

for all $x \in X$. In particular, if w is positive then $\mathcal{L}_u = \mathcal{L}_{b_u, u^2 w, u^2 m}$.

Proof. This follows by direct computation. First we note that $(\mathcal{L} - w)u = 0$ implies

$$w(x)u(x) = \frac{1}{m(x)}\Big(\sum_{y\in X} b(x, y)(u(x) - u(y)) + c(x)u(x)\Big)$$

for all $x \in X$ and that

$$u(x)f(x) - u(y)f(y) = u(y)(f(x) - f(y)) + (u(x) - u(y))f(x)$$

holds for all $f \in C(X)$ and $x, y \in X$. Given this we compute for $f \in C(X)$ and $x \in X$

$$\begin{aligned}
&\mathcal{T}_u\mathcal{L}_u f(x) = \mathcal{L}\mathcal{T}_u f(x)\\
&= \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(u(x)f(x) - u(y)f(y)) + c(x)u(x)f(x)\right)\\
&= \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)\left(u(y)(f(x) - f(y)) + (u(x) - u(y))f(x)\right) + c(x)u(x)f(x)\right)\\
&= \frac{1}{u(x)m(x)}\left(\sum_{y\in X} b(x,y)u(y)u(x)(f(x) - f(y))\right) + u(x)w(x)f(x).
\end{aligned}$$

Dividing by u then gives the desired statement. □

As a consequence we see how the operator $\mathcal{L} - w$, which can be seen as associated to the graph (b, c) perturbed by $-w$, is converted into the graph $(b_u, 0)$ over (X, u^2m):

Corollary 8.6. *Let (b, c) be a graph over (X, m). Let $u \in C(X)$ be strictly positive and $w \in C(X)$ with $w \geq 0$ such that $(\mathcal{L} - w)u = 0$. Then,*

$$\frac{1}{u}(\mathcal{L} - w)(u\varphi) = \mathcal{L}_{b_u,u^2m}\varphi$$

for all $\varphi \in C_c(X)$.

We define the quadratic form $\mathcal{Q}_u := \mathcal{Q}_{b_u,0}$. Hence, $\mathcal{Q}_u$ acts on $C(X)$ by

$$\mathcal{Q}_u(f) = \frac{1}{2}\sum_{x,y\in X} b(x,y)u(x)u(y)(f(x) - f(y))^2.$$

Now, we can state the ground state transform on the level of forms.

Theorem 8.7 (Ground state transform – form version). *Let (b, c) be a graph over (X, m). Let $u \in C(X)$ be strictly positive and $w \in C(X)$ such that $(\mathcal{L} - w)u = 0$. Then, for $\varphi \in C_c(X)$ we have both*

$$\mathcal{Q}(u\varphi) = \mathcal{Q}_u(\varphi) + \sum_{x\in X}\varphi(x)^2 w(x)u(x)^2 m(x).$$

and

$$\mathcal{Q}_u(\varphi/u) = \mathcal{Q}(\varphi) - \sum_{x\in X}\varphi(x)^2 w(x)m(x).$$

Proof. We calculate using Green's formula and the lemma on the ground state transform for operators

$$\begin{aligned}
\mathcal{Q}(u\varphi) &= \sum_{x\in X}(\mathcal{L}\mathcal{T}_u\varphi)(x)(\mathcal{T}_u\varphi)(x)m(x)\\
&= \sum_{x\in X}(\mathcal{T}_{u^{-1}}\mathcal{L}\mathcal{T}_u\varphi)(x)\varphi(x)u(x)^2m(x)\\
&= \mathcal{Q}_u(\varphi) + \sum_{x\in X}\varphi(x)^2w(x)u(x)^2m(x).
\end{aligned}$$

This shows the first part of the statement. The second part of the statement follows by replacing φ by φ/u. □

A main application of the ground state transform concerns the case where there is $u > 0$ with $(\mathcal{L} - \lambda)u \geq 0$. Such a u is sometimes known as *ground state* and this gives the name ground state transform. It is such u that provide a connection between the present section and Section 8.1. We will use this connection to derive a characterization of the infimum of the spectrum in Section 8.3. Here, we note the following corollary.

COROLLARY 8.8. *Let (b, c) be a graph over (X, m). Let $u \in C(X)$ be strictly positive and $\lambda \in \mathbb{R}$ with $(\mathcal{L} - \lambda)u \geq 0$. Then,*

$$\mathcal{Q}(\varphi) - \lambda\|\varphi\|^2 \geq \mathcal{Q}_u(\varphi/u) \geq 0$$

for all $\varphi \in C_c(X)$.

PROOF. Define

$$w \colon X \longrightarrow \mathbb{R}, \quad w(x) := \frac{\mathcal{L}u(x)}{u(x)}.$$

Then,

$$(\mathcal{L} - w)u = 0$$

by definition of w. Hence, the preceding theorem gives

$$\mathcal{Q}(\varphi) - \sum_{x \in X} w(x)\varphi(x)^2 m(x) = \mathcal{Q}_u(\varphi/u) \geq 0$$

for all $\varphi \in C_c(X)$.

Moreover, by assumption on u we have

$$w(x) - \lambda = \frac{1}{u(x)}(w(x)u(x) - \lambda u(x)) = \frac{1}{u(x)}((\mathcal{L}u)(x) - \lambda u(x)) \geq 0$$

for all $x \in X$. This implies

$$\begin{aligned}\mathcal{Q}(\varphi) - \lambda\|\varphi\|^2 &= \mathcal{Q}(\varphi) - \sum_{x \in X} w(x)\varphi(x)^2 m(x) + \sum_{x \in X}(w(x) - \lambda)\varphi(x)^2 m(x) \\ &\geq \mathcal{Q}(\varphi) - \sum_{x \in X} w(x)\varphi(x)^2 m(x).\end{aligned}$$

Combining these inequalities we arrive at the desired statement. □

REMARK 8.9. If the graph is connected any $u \geq 0$ with $(\mathcal{L} - \lambda)u = 0$ must satisfy $u > 0$ by the the Harnack inequality provided in Section 8.1.

8.3. Agmon–Allegretto–Piepenbrink Theorem

In this section we combine the results of Sections 8.1 and 8.2 in order to characterize the infimum of the spectrum.

THEOREM 8.10 (Agmon–Allegretto–Piepenbrink theorem). *Let (b, c) be a connected graph over (X, m), $L = L^{(D)}$ the Dirichlet Laplacian. For $\lambda \in \mathbb{R}$ the following are equivalent:*

(i) $\lambda \leq \lambda_0(L)$.
(ii) *There exists a positive non-trivial $u \in \mathcal{F}$ with $(\mathcal{L} - \lambda)u \geq 0$.*
(iii) *There exists a strictly positive $u \in \mathcal{F}$ with $(\mathcal{L} - \lambda)u \geq 0$.*

Proof. (ii)$\Longleftrightarrow$(iii): This follows by the local Harnack inequality.

(i)$\Longrightarrow$(ii): Consider first $\lambda < \lambda_0(L)$. Then, for any $o \in X$ the function $u_\lambda := (L-\lambda)^{-1}1_o$ is non-trivial and positive with

$$(\mathcal{L}-\lambda)u_\lambda = (\mathcal{L}-\lambda)(L-\lambda)^{-1}1_o = 1_o \geq 0.$$

This gives (ii) in this case. Note that u_λ is strictly positive by the local Harnack inequality.

To deal with the case $\lambda = \lambda_0(L)$ let (λ_n) converge to λ_0 from below. Consider

$$g_n := \frac{u_{\lambda_n}}{u_{\lambda_n}(o)}, \quad n \in \mathbb{N}.$$

(Here we use that the u_{λ_n} are strictly positive so that the denominator does not vanish.) Then, the g_n are positive non-trivial supersolutions by what we have just shown. Moreover, they satisfy $g_n(o) = 1$ by assumption. Hence, by the Harnack principle, there exists a subsequence of (g_n) which converges to some positive non-trivial supersolution g for $\lambda_0(L)$.

(iii)$\Longrightarrow$(i): Let $u \in \mathcal{F}$ be a strictly positive solution for λ. By Corollary 8.8, we have for all $\varphi \in C_c(X)$

$$\mathcal{Q}(\varphi) - \lambda\|\varphi\|^2 \geq \mathcal{Q}_u(\varphi/u) \geq 0.$$

Hence, we find

$$\mathcal{Q}(\varphi) \geq \lambda\|\varphi\|^2$$

for all $\varphi \in C_c(X)$. Since $\lambda_0(L) = \inf_{\|\varphi\|=1} Q(\varphi)$ we conclude $\lambda_0(L) \geq \lambda$. $\square$

The theorem begs the question to which extend one can find non-trivial solutions u of

$$(\mathcal{L}-\lambda)u = 0$$

for $\lambda \leq \lambda_0(L)$. For finite graphs the situation is clear. For $\lambda = \lambda_0(L)$ there exists a non-trivial solution (as $\lambda_0(L)$ is an eigenvalue) and to $\lambda < \lambda_0(L)$ there does not exist a solution (as there are no eigenvalues below the infimum of the spectrum). For infinite graphs the situation is more complicated. Still, for locally finite graphs we obtain a rather direct corollary of our considerations. The corollary is based on the following stability feature for pointwise convergent solutions on locally finite graphs.

Lemma 8.11. *Let (b,c) be a locally finite graph over (X,m). Let (u_n) in $\mathcal{F}$ and (f_n) in $C(X)$, $u, f \in C(X)$ with $u_n \to u$ pointwise and $f_n \to f$ pointwise and $\mathcal{L}u_n = f_n$ for all $n \in \mathbb{N}$. Then, $u \in \mathcal{F}$ and $\mathcal{L}u = f$.*

Proof. By Fatou's lemma we have $u \in \mathcal{F}$. For $n \in \mathbb{N}$ and $x \in X$ we have

$$\mathcal{L}u_n(x) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)(u_n(x) - u_n(y)) + \frac{c(x)}{m(x)}u_n(x) = f_n(x).$$

By local finiteness we can take the limit as $n \to \infty$ and directly obtain the statement. $\square$

Corollary 8.12. *Let (b,c) be an infinite connected locally finite graph over (X,m) and $L = L^{(D)}$ the Dirichlet Laplacian. Then, for any $\lambda \leq \lambda_0(L)$ there exists a non-trivial solution u of $(\mathcal{L}-\lambda)u = 0$.*

Proof. Fix $o \in X$. Choose a sequence (o_n) in X that leaves any finite set (i.e. that satisfies for each finite set $F \subseteq X$ that o_n does not belong to F for all large enough n).

We first deal with the case $\lambda < \lambda_0(L)$. Consider $(L-\lambda)^{-1}1_{o_n}$. This is a non-trivial solution to $(\mathcal{L}-\lambda)u \geq 0$. Hence, by the local Harnack inequality, it is strictly positive. We can therefore define

$$u_n := \frac{1}{(L-\lambda)^{-1}1_{o_n}(o)}(L-\lambda)^{-1}1_{o_n}, \quad n \in \mathbb{N}.$$

Then, each u_n satisfies

$$(\mathcal{L}-\lambda)u_n = \frac{1}{(L-\lambda)^{-1}1_{o_n}(o)}1_{o_n} \geq 0$$

as well as $u_n \geq 0$ and $u_n(o) = 1$. Hence, by the Harnack principle, there exists a pointwise convergent subsequence. Without loss of generality, we assume that (u_n) itself converges pointwise. Call the limit u. Then, u satisfies $u \geq 0$ as well as $u(o) = 1$. Moreover, clearly, (1_{o_n}) converges pointwise to 0 (as (o_n) leaves every finite set). Now, note that u_n solves

$$\mathcal{L}u_n = \lambda u_n + \frac{1}{(L-\lambda)^{-1}1_{o_n}(o)}1_{o_n},$$

for all $n \in \mathbb{N}$. Thus, Lemma 8.11 gives

$$\mathcal{L}u = \lambda u.$$

As u is positive and non-trivial, we infer $u > 0$ by the local Harnack inequality.

We now turn to the case $\lambda = \lambda_0(L)$. Let (λ_n) be a sequence converging from below to $\lambda_0(L)$. By what we have shown already, for all $n \in \mathbb{N}$ there exists a $u_n > 0$ with $(\mathcal{L}-\lambda_n)u_n = 0$. Without loss of generality, we can assume $u_n(o) = 1$ for all $n \in \mathbb{N}$. Moreover, invoking the Harnack principle we can assume without loss of generality that (u_n) converges pointwise to a limit. Call the limit u. Then, u is positive with $u(o) = 1$ and, by Lemma 8.11, $(\mathcal{L}-\lambda)u = 0$. □

CHAPTER 9

Recurrence and Transience

Recurrence and its counterpart transience refer to a property of the graph that can be phrased in many different ways. The term itself comes from a stochastic interpretation. In this interpretation the graph gives rise to a Markov process modeling a particle jumping between the vertices of the graph according to certain rules. Recurrence then describes the phenomenon that the particle returns again and again to any given vertex. Transience in turn describes the phenomenon that the particle leaves any vertex for good at one point of time. In the present lecture notes we are not concerned with stochastic processes but rather with an analytic description.

Let X be a countable set and m a measure on X with full support.

9.1. The Green Function

Let (b, c) be a graph over (X, m), $Q = Q^{(D)}_{b,c,m}$ the quadratic form and $L = L^{(D)}$ the Dirichlet Laplacian. The *Green function* of the graph is defined by $G\colon X \times X \to [0, \infty]$ with

$$G(x, y) := \int_0^\infty e^{-tL} 1_y(x)\mathrm{d}t.$$

The Green function can be thought of as modelling the (possibly non-existent) inverse L^{-1} of the Laplacian L associated to the graph. Indeed, this point of view is made precise (in two ways) in the first part of the subsequent theorem.

Recall from Section 5.1 that for any finite $K \subseteq X$ the restriction of Q to $C(K)$ gives rise to an operator $L^{(D)}_K$, which is associated to the graph (b_K, c_K) over (K, m_K) with $b_K(x, y) := b(x, y)$ for $x, y \in K$ and $c_K(x) := c(x) + \sum_{y \in X} b(x, y)$ for $x \in K$.

Theorem 9.1 (Basic features of the Green function). *Let (b, c) be a connected graph over (X, m). Then, the following holds.*

(a) *(Approximation property) For all $x, y \in X$*

$$G(x, y) = \lim_{\alpha \to 0+} (L + \alpha)^{-1} 1_y(x).$$

(b) *(Approximation property for infinite X) If X is infinite, the operator $L^{(D)}_K$ is invertible for any finite $K \subseteq X$, and for all sequences (K_n) of finite subsets of X with $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} K_n = X$ we have*

$$G(x, y) = \lim_{n\to\infty} (L^{(D)}_{K_n})^{-1} 1_y(x)$$

for all $x, y \in X$.

(c) *(Symmetry) For all* $x, y \in X$,

$$G(x,y)m(x) = G(y,x)m(y).$$

(d) *(Dichotomy) The inequality* $G(x,y) > 0$ *holds for all* $x, y \in X$ *and if* $G(x,y) = \infty$ *(respectively* $G(x,y) < \infty$*) for some* $x, y \in X$*, then* $G(x,y) = \infty$ *(respectively* $G(x,y) < \infty$*) for all* $x, y \in X$.

PROOF. (a) By positivity preservation, we see that

$$(L+\alpha)^{-1}1_y(x) = \int_0^\infty e^{-t\alpha}e^{-tL}1_y(x)\mathrm{d}t$$

is monotonically increasing as α is decreasing. Taking the limit $\alpha \to 0+$, we obtain the equality.

(b) If X is infinite and $K \subseteq X$ is finite the operator $L_K^{(D)}$ belongs to the graph (b_K, c_K) with $c_K(x) = c(x) + \sum_{y\in X} b(x,y)$ for all $x \in K$. As (b,c) is connected, c_K does not vanish. Hence, $L_K^{(D)}$ is invertible. Furthermore, for $x, y \in X$, $((L_{K_n}^{(D)})^{-1}1_y(x))$ is monotonically increasing by domain monotonicity and therefore we can interchange the limits

$$\lim_{\alpha\to 0}(L+\alpha)^{-1}1_y(x) = \lim_{\alpha\to 0}\lim_{n\to\infty}(L_{K_n}^{(D)}+\alpha)^{-1}1_y(x) = \lim_{n\to\infty}(L_{K_n}^{(D)})^{-1}1_y(x).$$

(c) The symmetry follows directly from the equality

$$e^{-tL}1_x(y)m(y) = \langle e^{-tL}1_x, 1_y\rangle = \langle 1_x, e^{-tL}1_y\rangle = e^{-tL}1_y(x)m(x)$$

for all $x, y \in X$ and $t \geq 0$.

(d) The strict positivity $G > 0$ follows directly from the definition of G and the fact that the semigroup operators e^{-tL} are positivity improving for $t > 0$ by Theorem 6.1 as we assume connectedness.

To show that $G(x,y) = \infty$ for all $x, y \in X$ if $G(x,y) = \infty$ for some $x, y \in X$ let $e_x := 1_x/\sqrt{m(x)}$ for $x \in X$. Let $x, y, x_0 \in X$. We calculate for $t > 1$ that

$$\begin{aligned} e^{-tL}1_y(x) &= e^{-L}e^{-(t-1)L}1_y(x) = \frac{1}{m(x)}\langle e^{-L}e^{-(t-1)L}1_y, 1_x\rangle \\ &= \frac{1}{m(x)}\sum_{z\in X}\langle e^{-(t-1)L}1_y, e_z\rangle\langle e^{-L}1_x, e_z\rangle \\ &\overset{(e^{-tL})\text{ positive}}{\geq} \frac{1}{m(x)}\langle e^{-(t-1)L}1_y, e_{x_0}\rangle\langle e^{-L}1_x, e_{x_0}\rangle \\ &= \frac{m(x_0)}{m(x)}e^{-(t-1)L}1_y(x_0)e^{-L}1_x(x_0). \end{aligned}$$

Since the semigroup is positivity improving on a connected graph by Theorem 6.1, we infer that $C := C_{x,x_0} := e^{-L}1_x(x_0)m(x_0)/m(x) > 0$. Then,

$$\begin{aligned} G(x,y) &= \int_0^\infty e^{-tL}1_y(x)\mathrm{d}t \geq \int_1^\infty e^{-tL}1_y(x)\mathrm{d}t \geq C\int_0^\infty e^{-tL}1_y(x_0)\mathrm{d}t \\ &= CG(x_0,y). \end{aligned}$$

As this holds for all $x_0, x, y \in X$ the we find, by symmetry in (c), that

$$G(x_0,y) \geq C'G(x_0,y_0),$$

where $C' = C_{y,y_0}\frac{m(y)}{m(y_0)}$, for any $y_0 \in X$. As $x, y, x_0, y_0 \in X$ were chosen arbitrarily, the statement follows. $\square$

We have already noted that the Green function models the inverse of L. Here is another precise version of this.

THEOREM 9.2 (Characterization of $G(\cdot, o)$). *Let (b, c) be a connected graph over (X, m) and $o \in X$. If $G(x, y) < \infty$ for some $x, y \in X$, then $G(\cdot, o)$ is superharmonic and satisfies*

$$\mathcal{L}G(\cdot, o) = 1_o.$$

Furthermore, $G(\cdot, o)$ is the smallest $u \in \mathcal{F}$ with $u \geq 0$ such that $\mathcal{L}u \geq 1_o$.

PROOF. Consider first the case that X is finite. Then, $G(x, y) < \infty$ yields that 0 is not an eigenvalue of $L = \mathcal{L}$ (note that X is finite). Indeed, note that the Dichotomy in Theoem 9.1 yields $G(x, y) < \infty$ for all $x, y \in X$. If 0 is an eigenvalue then 1 is a (strictly positive) eigenfuntion (see Example 6.5), and we observe

$$\infty > \sum_{z\in X} G(x, z) = \int_0^\infty e^{-tL}1(x)\mathrm{d}t = \int_0^\infty 1\mathrm{d}t = \infty,$$

a contradiction. Hence, $\inf \sigma(L) > 0$ and L is invertible. Spectral calculus then gives $G(\cdot, o) = L^{-1}1_o$ and the desired statement follows.

Consider now the case that X is infinite. We first show that $\mathcal{L}G(\cdot, o) = 1_o$ holds. This implies in particular that $G(\cdot, o)$ is superharmonic.

Let (K_n) be an arbitrary sequence of increasing finite subsets of X such that $\bigcup_{n\in\mathbb{N}} K_n = X$ and $o \in K_n$ for all $n \in \mathbb{N}$. Set

$$g_n := (L^{(D)}_{K_n})^{-1}1_o, \quad n \in \mathbb{N}.$$

By domain monotonicity we see that (g_n) is monotonically increasing in n and, by the previous theorem, it converges to $G(\cdot, o)$. By monotone convergence we get

$$1_o = \mathcal{L}g_n \to \mathcal{L}G(\cdot, o), \qquad n \to \infty.$$

This is the desired equality.

We now show that $G(\cdot, o)$ is the smallest function $u \in \mathcal{F}$ such that $u \geq 0$ and $\mathcal{L}u \geq 1_o$. So, let $u \in \mathcal{F}$ satisfy $u \geq 0$ and $\mathcal{L}u \geq 1_o$. Then, for $n \in \mathbb{N}$, $v_n := u - g_n$ is superharmonic on K_n, satisfies $v_n \geq 0$ outside of K_n and $v_n \wedge 0$ assumes its minimum on the finite set K_n. Hence, by the minimum principle, Theorem 2.10, we infer $v_n \geq 0$ and, therefore, $u \geq g_n$. Since (g_n) converges to $G(\cdot, o)$, it follows that $u \geq G(\cdot, o)$. $\square$

9.2. Null Sequences and $\mathcal{D}_0(X)$

In this section we use the space $\mathcal{D}_0(X)$ defined earlier. Let (b, c) be a connected graph over X. Note that in this section we do not need the measure m. For $o \in X$, let the semi-scalar product $\langle\cdot,\cdot\rangle_o$ be given by

$$\langle f, g\rangle_o := \mathcal{Q}(f, g) + f(o)g(o)$$

for $f, g \in \mathcal{D}$ and let $\|\cdot\|_o$ be the corresponding semi-norm. Then, $\langle\cdot,\cdot\rangle_o$ is a scalar-product and $\|\cdot\|_o$ is a norm on $\mathcal{D}$ whenever the graph is connected. Moreover, recall that

$$\mathcal{D}_0 := \mathcal{D}_0(X) = \overline{C_c(X)}^{\|\cdot\|_o}.$$

In Lemma 2.5 it was shown that for any $x \in X$ the norms $\|\cdot\|_o$ and $\|\cdot\|_x$ are equivalent and $(\mathcal{D}, \|\cdot\|_o)$ is a Hilbert space. Also, it was shown there that a sequence (f_n) in $\mathcal{D}$ converges to f w.r.t. $\|\cdot\|_o$, if and only if $f_n \to f$ pointwise and

$$\limsup_{n\to\infty} \mathcal{Q}(f_n) \leq \mathcal{Q}(f).$$

We now turn to the question whether the constant function 1 can be approximated by functions in $C_c(X)$ with respect to $\|\cdot\|_o$ for one (all) $o \in X$. In this context we provide the following definition.

Definition 9.3 (Null sequence). Let (b, c) be a connected graph over X. A sequence (e_n) in $C_c(X)$ with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ is called a *null sequence* if $e_n \to 1$ pointwise and $\mathcal{Q}(e_n) \to 0$ as $n \to \infty$.

Corollary 9.4 (Characterization of null sequences). *Let (b, c) be a connected graph over X.*

(a) *If (b, c) admits a null sequence then $c = 0$ holds.*
(b) *Assume $c = 0$. Then, a sequence (e_n) in $C_c(X)$ is a null sequence if and only if $(1 - e_n)$ converges to 0 with respect to $\|\cdot\|_o$ for one (all) $o \in X$.*
(c) *The graph $(b, 0)$ admits a null sequence if and only if $1 \in \mathcal{D}_0$.*

Proof. (a) Let (e_n) be a null sequence. Then, $e_n(x) \to 1$ for all $x \in X$. By

$$\sum_{x\in X} c(x) e_n(x)^2 \leq \mathcal{Q}(e_n) \to 0$$

the claim on c is immediate.

(b) Given that $c = 0$ holds, we directly see

$$\mathcal{Q}(e_n) = \mathcal{Q}(1 - e_n).$$

From this (b) follows.

(c) This is just a reformulation of (b). □

By definition, null sequences give a way to approximate the constant function $1 \in \mathcal{D}$ by functions with finite support. In fact, they offer the possibility to approximate any function in $\mathcal{D}$ by functions with finite support. Details are given in the subsequent lemma.

Lemma 9.5. *Let (b, c) be a connected graph over X and $o \in X$. If (e_n) is a null sequence, then for any $f \in \mathcal{D}$ we have $e_n f \to f$ with respect to $\|\cdot\|_o$ as $n \to \infty$.*

Proof. We first deal with bounded functions f in $\mathcal{D}$. So, let $f \in \mathcal{D} \cap \ell^\infty(X)$ be given. Now, $e_n f \in C_c(X)$ for all $n \in \mathbb{N}$. Moreover, simple algebraic manipulations give

$$\begin{aligned}
&(f(x)(1-e_n)(x) - f(y)(1-e_n)(y))^2 \\
&= (f(x)(1-e_n)(x) - f(y)(1-e_n)(x) + f(y)(1-e_n)(x) - f(y)(1-e_n)(y))^2 \\
&= ((1-e_n)(x)(f(x)-f(y)) + f(y)(e_n(x)-e_n(y)))^2 \\
&\leq 2(1-e_n(x))^2(f(x)-f(y))^2 + 2f(y)^2(e_n(x)-e_n(y))^2
\end{aligned}$$

for all $x, y \in X$. This yields

$$\begin{aligned}
\mathcal{Q}(f - e_n f) &= \mathcal{Q}(f(1-e_n)) \\
&= \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)(1-e_n)(x) - f(y)(1-e_n)(y))^2 \\
&\leq \sum_{x\in X}(1-e_n(x))^2 \sum_{y\in X} b(x,y)(f(x)-f(y))^2 \\
&\qquad + \sum_{y\in X} f(y)^2 \sum_{x\in X} b(x,y)(e_n(x)-e_n(y))^2 \\
&\leq \sum_{x\in X}(1-e_n(x))^2 \sum_{y\in X} b(x,y)(f(x)-f(y))^2 + 2\|f\|_\infty^2 \mathcal{Q}(e_n) \\
&\to 0.
\end{aligned}$$

Here, we used in the last step that (e_n) is a null sequence to treat the second term and the Lebesgue's dominated convergence theorem to treat the first term. Note that Lebesgue's theorem is applicable since f belongs to $\mathcal{D}$. Altogether, we have shown that $(e_n f)$ converges to f with respect to $\|\cdot\|_o$ for bounded f in $\mathcal{D}$.

Now, consider an arbitrary function $f \in \mathcal{D}$. Then for any $k \in \mathbb{N}$, the function $f_k := (-k) \vee f \wedge k$ is bounded and belongs to $\mathcal{D}$ (as they arise from f by taking a normal contraction). Hence, it suffices to show $f_k \to f$ w.r.t. $\|\cdot\|_o$ as $k \to \infty$. Clearly $f_k \to f$ pointwise as $k \to \infty$. Moreover, as $\mathcal{Q}$ is compatible with normal contractions we find

$$\limsup_{k\to\infty} \mathcal{Q}(f_k) \leq \mathcal{Q}(f).$$

Together with the characterization of convergence with respect to $\|\cdot\|_o$ discussed at the beginning of the section, we infer the desired statement. ☐

We get the following immediate consequence.

THEOREM 9.6. *Let b be a connected graph over X. The following statements are equivalent:*

(i) $\mathcal{D}_0 = \mathcal{D}$.
(ii) $1 \in \mathcal{D}_0$.
(iii) *There exists a null sequence.*
(iv) *For any $w \geq 0$ such that*

$$\mathcal{Q}(\varphi) \geq \sum_{x\in X} w(x)\varphi(x)^2, \qquad \varphi \in C_c(X)$$

we have $w = 0$.

PROOF. The equivalence between (ii) and (iii) is shown in Lemma 9.4.

(iv) $\Longrightarrow$ (ii): For $x \in X$, let

$$\mathrm{cap}(x) := \inf_{\varphi \in C_c(X), \varphi(x)=1} \mathcal{Q}(\varphi).$$

Then,

$$\mathcal{Q}(\varphi) \geq \mathrm{cap}(x)\varphi(x)^2$$

for any $\varphi \in C_c(X)$. In particular, (iv) implies

$$\mathrm{cap}(x) = 0$$

for all $x \in X$. Thus, for $x \in X$, there exists (φ_n) in $C_c(X)$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n(x) = 1$ for all $n \in \mathbb{N}$ such that $\mathcal{Q}(\varphi_n) \to 0$. Hence, (φ_n) is a Cauchy sequence with respect to $\|\cdot\|_x$. By completeness, we infer that there exists $f \in \mathcal{D}$ such that (φ_n) converges to f with respect to $\|\cdot\|_x$. Then, $f = 1$ must hold as otherwise $\mathcal{Q}(f) > 0$ must hold (by connectedness of the graph) and, by Fatou's lemma, we had the contradiction

$$0 < \mathcal{Q}(f) \leq \liminf_{n\to\infty} \mathcal{Q}(\varphi_n) = 0.$$

This shows the desired implication (iv) $\Longrightarrow$ (ii).

(iii) $\Longrightarrow$ (iv): This is clear.

(ii) $\Longrightarrow$ (i): We have already discussed that $1 \in \mathcal{D}_0$ is equivalent to existence of a null sequence. Hence, (i) follows from (ii) by Lemma 9.5.

(i) $\Longrightarrow$ (ii): From $\mathcal{D} = \mathcal{D}_0$ and $1 \in \mathcal{D}$ we infer $1 \in \mathcal{D}_0$. □

9.3. Characterization of Recurrence and Transience

In Section 9.2 we have been concerned with $\mathcal{D}_0$ and it being equal to $\mathcal{D}$. Here, we consider and characterize the non-equality.

We call a connected graph b over X *transient* if $\mathcal{D} \neq \mathcal{D}_0$, and *recurrent* otherwise.

THEOREM 9.7 (Characterization of transience). *Let b be a connected graph over X. The following statements are equivalent:*

(i) *$\mathcal{D}_0 \neq \mathcal{D}$, i.e. b is transient.*

(ii) *There exists a non-trivial $w \geq 0$ such that*

$$\mathcal{Q}(\varphi) \geq \sum_{x\in X} w(x)\varphi(x)^2, \qquad \varphi \in C_c(X).$$

(iii) *There exists a positive non-constant superharmonic function.*

(iv) *The set X is infinite and there exist infinitely many linearly independent positive superharmonic functions.*

(v) *$G(x,y) < \infty$ for some (all) $x, y \in X$.*

PROOF. The equivalence (i) $\Longleftrightarrow$ (ii) is already given in Theorem 9.6.

(ii) $\Longrightarrow$ (v): By the dichotomy given in Theorem 9.1 it suffices to find one $x \in X$ with $G(x,x) < \infty$. Let $x \in X$ with $w(x) > 0$ be given. Now, note that (ii) implies that X is infinite (as otherwise $\varphi = 1$ would give a contradiction to (ii)). Let (K_n) be an increasing sequence of finite subsets of X with $\bigcup_{n\in\mathbb{N}} K_n = X$, and let $g_n := (L_{K_n}^{(D)})^{-1}1_x$ for $n \in \mathbb{N}$. Then, $g_n \to G(\cdot, x)$ pointwise by Theorem 9.1. Moreover, a direct computation invoking the Green formula and (ii) gives

$$g_n(x)m(x) = \langle 1_x, g_n\rangle = \langle \mathcal{L}g_n, g_n\rangle = \mathcal{Q}(g_n) \geq \sum_{y\in X} w(y)g_n(y)^2 \geq w(x)g_n(x)^2.$$

Hence,

$$g_n(x) \leq \frac{m(x)}{w(x)}, \quad n \in \mathbb{N},$$

which implies $G(x,x) < \infty$ after taking the limit $n \to \infty$.

(v) $\Longrightarrow$ (iv): First we note that X must be infinite if $G(x,y) < \infty$ holds. (Otherwise, by $c = 0$ we had that 1 were an eigenfunction to the eigenvalue 0 of L and this would imply $G(x,y) = \infty$ by the very definition of G. Compare reasoning within the proof of Theorem 9.2.) As discussed in Theorem 9.2 the functions $G(\cdot,x)$ are positive and superharmonic with $\mathcal{L}G(\cdot,x) = 1_x$ for all $x \in X$. By $c = 0$ they can not be constant. We show that they are linearly independent. Assume

$$\sum_{x\in X} \lambda_x G(\cdot,x) = 0$$

for some $\lambda_x \in \mathbb{R}$ with $\lambda_x = 0$ for all but finitely many $x \in X$. Then,

$$\sum_{x\in C} \lambda_x 1_x = \sum_{x\in X} \lambda_x \mathcal{L}G(\cdot,x) = L(\sum_{x\in X} \lambda_x G(\cdot,x)) = L0 = 0.$$

This gives $\lambda_x = 0$ for all $x \in X$.

(iv) $\Longrightarrow$ (iii): This is clear.

(iii) $\Longrightarrow$ (ii): Let u be a non-constant postive superharmonic function. By the Harnack inequality we have $u > 0$. Then, $v := u^{1/2}$ satisfies

$$\begin{aligned}
\mathcal{L}v(x) &= \frac{1}{m(x)}\sum_{y\in X} b(x,y)(u(x)^{1/2} - u(y)^{1/2}) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)(u(x)^{1/2} - u(y)^{1/2}) \\
&= \frac{1}{m(x)u(x)^{1/2}}\sum_{y\in X} b(x,y)(\frac{1}{2}u(x) - \frac{1}{2}u(y) + \frac{1}{2}u(x) - u(x)^{1/2}u(y)^{1/2} + \frac{1}{2}u(y)) \\
&= \frac{1}{2m(x)u(x)^{1/2}}\mathcal{L}u(x) + \frac{1}{2m(x)u(x)^{1/2}}\sum_{x\in X} b(x,y)(u(x)^{1/2} - u(y)^{1/2})^2 \\
&> 0.
\end{aligned}$$

Hence, by the ground state transform, Theorem 8.7, we get with

$$w(x) := \frac{1}{v(x)}\mathcal{L}v(x)m(x), \quad x \in X,$$

that

$$\mathcal{Q}(\varphi) = Q_v(\varphi/v) + \sum_{x\in X} \frac{\varphi(x)^2}{v(x)}\mathcal{L}v(x)m(x) \geq \sum_{x\in X} w(x)\varphi(x)^2.$$

This finishes the proof. $\square$

CHAPTER 10

Stochastic Completeness

Graphs and their Laplacians can be used to model diffusion of heat on discrete sets. The corresponding equation is known as heat equation. The basic ingredient is the distribution of heat at a given point of time. This is modeled by a function (with values in $[0, \infty)$) on the set and the heat equation captures how this function develops in time. A particularly relevant aspect is whether the total amount of heat on the set is conserved over time. This is known as stochastic completeness or conservativeness or honesty. Details are discussed in this chapter. To put things in perspective we already provide an outline of the material here.

Let X be a countable set and m a measure on X with full support. Let (b, c) be a graph over (X, m) and $\mathcal{L}$ the associated formal Laplacian. Then the equation

$$-\mathcal{L}v = \partial_t v$$

is known as heat equation (induced from $\mathcal{L}$)). A function

$$u \colon [0, \infty) \times X \longrightarrow \mathbb{R}$$

is said to be a *solution of the heat equation* with *initial condition* $f \in C(X)$ if the following holds:

- For each $x \in X$ the function $[0, \infty) \longrightarrow \mathbb{R}$, $t \mapsto u_t(x)$ is continuous and differentiable on $(0, \infty)$ with $u_0 = f$.
- For each $t > 0$ the map $X \longrightarrow \mathbb{R}, x \mapsto u_t(x)$ belongs to $\mathcal{F}$.
- The equality $-\mathcal{L}u_t(x) = \partial_t u_t(x)$ holds for all $t > 0$ and all $x \in X$.

By Theorem 4.9 the action of the self-adjoint operator L associated to (b, c) is given by $Lf = \mathcal{L}f$ for $f \in D(L)$. Hence, by Theorem 3.24 for any $f \in \ell^2(X, m)$ the function

$$[0, \infty) \ni t \mapsto u_t := e^{-tL} f \in \ell^2(X, m)$$

is a solution to the heat equation with initial condition f (as had indeed been already discussed above). Now, the operators $S(t) := e^{-tL}$, $t \geq 0$, satisfy the Markov property and this can be used to extend them to operators $S^{(p)}(t)$ on $\ell^p(X, m)$ for any $1 \leq p \leq \infty$. Details are discussed in Section 10.1. This can be used to show that the heat equation can be solved for bounded initial condition f, i.e. for $f \in \ell^\infty(X, m)$. Specifically, as discussed in Section 10.2, for any $f \in \ell^\infty(X, m)$ the function

$$u \colon [0, \infty) \times X \longrightarrow \mathbb{R}, u_t(x) := S^{(\infty)}(t) f(x),$$

is a solution to the heat equation with initial condition f. Moreover,

$$\sum_{x \in X} g(x)(S^{(\infty)}(t)f)(x)m(x) = \sum_{x \in X} (S^{(1)}(t)g)(x)f(x)m(x)$$

holds for all $g \in \ell^1(X,m)$, $f \in \ell^\infty(X,m)$ and $t \geq 0$. Now, the question arises whether for any $g \in \ell^1(X,m)$ with $g \geq 0$ the total amount of heat at time t given by

$$A(t) := \sum_{x \in X} (S^{(1)}(t)g)(x)m(x)$$

remains constant (in t). This is what is known as stochastic completeness and the second part of this chapter, starting in Section 10.3, is devoted to various equivalent characterizations.

10.1. The Semigroup and Resolvent on ℓ^p

Here, we discuss how the operators e^{-tL}, $t \geq 0$, on $\ell^2(X,m)$ associated to the graph (b,c) with induced Laplacian $L = L^{(D)}$ can be extended to operators on each $\ell^p(X,m)$, $1 \leq p \leq \infty$. The crucial point here is that these operators satisfy the Markov property.

We start with the definition of a strongly continuous contraction Markov semigroup. For $p \in [1,\infty]$, we denote the bounded linear operators on $\ell^p(X,m)$ by

$$B(\ell^p(X,m)) := \{L \colon \ell^p(X,m) \longrightarrow \ell^p(X,m) \mid L \text{ is linear and bounded}\}.$$

Note that we have $\ell^\infty(X,m) = \ell^\infty(X)$.

Definition 10.1 (Semigroup). Let $p \in [1,\infty]$. A map $S\colon [0,\infty) \longrightarrow B(\ell^p(X,m))$ is called a *semigroup* on $\ell^p(X,m)$ if

$$S(0) = I, \quad \text{and} \quad S(s+t) = S(s)S(t)$$

for all $s, t \geq 0$. A semigroup S is called *strongly continuous* if

$$\lim_{t \to 0^+} S(t)f = f$$

for all $f \in \ell^p(X,m)$. A semigroup S is called a *contraction semigroup* if

$$\|S(t)\|_p \leq 1$$

for all $t \geq 0$. Finally, a semigroup is called a *Markov semigroup* if

$$0 \leq S(t)f \leq 1$$

for all $t \geq 0$ and $f \in \ell^p(X,m)$ with $0 \leq f \leq 1$.

Example 10.2. If $A \in B(\ell^p(X,m))$, then $(e^{-tA})_{t \geq 0}$ defined by the absolutely convergent series

$$e^{-tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

for $t \geq 0$ gives a strongly continuous semigroup which yields a solution of the parabolic equation involving A (Exercise). Furthermore, given an initial condition, this solution is unique (Exercise).

Any strongly continuous semigroup S defines an operator called the generator of the semigroup.

Definition 10.3 (Generator of a semigroup). Let $p \in [1, \infty]$. If S is a strongly continuous semigroup on $\ell^p(X, m)$, then the operator A with

$$D(A) := \{f \in \ell^p(X, m) \mid \lim_{t \to 0^+} \frac{1}{t}(f - S(t)f) \text{ exists in } \ell^p(X, m)\}$$

and

$$Af := \lim_{t \to 0^+} \frac{1}{t}(f - S(t)f)$$

for $f \in D(A)$ is called the *generator* of S.

It follows from general considerations (which we omit here as we do not need them) that for a strongly continuous semigroup S on $\ell^p(X, m)$ with generator A the domain $D(A)$ of A is dense in $\ell^p(X, m)$ and that A is a closed operator. If S is additionally a contraction semigroup, then it can be shown that $A + \alpha$ is a bijection for $\alpha > 0$ with inverse given by

$$(A + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} S(t) \mathrm{d}t.$$

We now introduce resolvents and point out their connections to semigroups.

Definition 10.4 (Resolvents). Let $p \in [1, \infty]$. A map $G \colon (0, \infty) \longrightarrow B(\ell^p(X, m))$ is called a *resolvent* on $\ell^p(X, m)$ if G satisfies the *resolvent identity*

$$G(\alpha) - G(\beta) = -(\alpha - \beta)G(\alpha)G(\beta)$$

for all $\alpha, \beta > 0$. A resolvent G is called *strongly continuous* if

$$\lim_{\alpha \to \infty} \alpha G(\alpha) f = f$$

for all $f \in \ell^p(X, m)$. A resolvent G is called a *contraction resolvent* if

$$\|\alpha G(\alpha)\|_p \leq 1$$

for all $\alpha > 0$. Finally, a resolvent G is called a *Markov resolvent* if

$$0 \leq \alpha G(\alpha) f \leq 1$$

for all $\alpha > 0$ and $f \in \ell^p(X, m)$ with $0 \leq f \leq 1$.

Of course, our main concern are semigroups and resolvents arising from the self-adjoint operator $L = L^{(D)}$ associated to a graph (b, c) over (X, m) and these have the Markov property.

The Markov property of a semigroup and a resolvent will allow us to extend $S = (e^{-tL})_{t \geq 0}$ to all $\ell^p(X, m)$ spaces for $p \in [1, \infty]$. A crucial ingredient in this extension process is the fact that a Markov matrix defines a bounded operator on $\ell^p(X, m)$ for $p \in [1, \infty]$.

Definition 10.5 (Markov matrix). A function $a \colon X \times X \longrightarrow \mathbb{R}$ is called a *Markov matrix* if a satisfies the following properties:

- $a(x, y) = a(y, x)$
- $a(x, y) \geq 0$
- $\sum_{z \in X} a(x, z) m(z) \leq 1$

for all $x, y \in X$.

With this notion we now show that a Markov matrix can be used to define a bounded operator on $\ell^p(X,m)$ for all $p \in [1,\infty]$.

LEMMA 10.6 (General bound for a Markov matrix). *Let a be a Markov matrix. Then, for all $f \in C(X)$,*

$$\sup_{x\in X} \sum_{y\in X} |a(x,y)f(y)|m(y) \leq \sup_{x\in X} |f(x)|,$$

and for $p \in [1,\infty)$,

$$\sum_{x\in X} \left(\sum_{y\in X} |a(x,y)f(y)|m(y) \right)^p m(x) \leq \sum_{x\in X} |f(x)|^p m(x).$$

Here, the value ∞ is allowed to occur.

In particular, for $p \in [1,\infty]$ the matrix a induces a bounded operator $A^{(p)}$ with norm not exceeding 1 on each $\ell^p(X,m)$ by

$$A^{(p)}f(x) := \sum_{y\in X} a(x,y)f(y)m(y).$$

PROOF. The "in particular" statement is a direct consequence of the inequalities. Thus, it suffices to show these inequalities. The case $p=\infty$ is clear and the case $p=1$ follows easily from Fubini's theorem.

Consider now $p \in (1,\infty)$. Let $q \in (1,\infty)$ satisfy $1/p + 1/q = 1$. Then, we can estimate

$$\begin{aligned}
&\sum_{x\in X} \left(\sum_{y\in X} a(x,y)|f(y)|m(y) \right)^p m(x) \\
&\quad = \sum_{x\in X} \left(\sum_{y\in X} |a(x,y)m(y)|^{1/q} |a(x,y)m(y)|^{1/p} |f(y)| \right)^p m(x) \\
&\quad \leq \sum_{x\in X} \left(\sum_{y\in X} a(x,y)m(y) \right)^{p/q} \left(\sum_{y\in X} a(x,y)|f(y)|^p m(y) \right) m(x) \\
&\quad \leq \sum_{x\in X} \sum_{y\in X} a(x,y)|f(y)|^p m(y)m(x) \\
&\quad = \sum_{y\in X} |f(y)|^p m(y) \sum_{x\in X} a(x,y)m(x) \\
&\quad \leq \sum_{y\in X} |f(y)|^p m(y),
\end{aligned}$$

where we used Hölder's inequality in the third line, $\sum_{y\in X} a(x,y)m(y) \leq 1$ in the fourth line, Fubini's theorem in the fifth line, and $a(x,y) = a(y,x)$ and $\sum_{x\in X} a(y,x)m(x) \leq 1$ in the last line. The case of $p=1$ follows in a similar manner by using Fubini's theorem. This finishes the proof. $\square$

We need a further piece of notation. Whenever $p, q \in [1,\infty]$ satisfy $1/p + 1/q = 1$ (where the cases $p=1$, $q=\infty$ and $p=\infty$, $q=1$ are allowed) we can appeal to the Hölder inequality to infer that

$$(f,g) := \sum_{x\in X} f(x)g(x)m(x)$$

exists as an absolutely convergent sum for $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$. Then, $(\cdot,\cdot)$ is called the *dual pairing* between $\ell^p(X,m)$ and $\ell^q(X,m)$. Of course, for $p = q = 2$ and $f, g \in \ell^2(X,m)$, we just have

$$(f,g) = \sum_{x\in X} f(x)g(x)m(x) = \langle f, g\rangle.$$

THEOREM 10.7 (Extension theorem – semigroups). *Let (b,c) be a graph over (X,m) and $L := L^{(D)}$ the induced self-adjoint operator. Let $S := (e^{-tL})_{t\geq 0}$ be the associated semigroup. Then, there exists a unique family of contraction Markov semigroups $S^{(p)}$ on $\ell^p(X,m)$ for $p \in [1,\infty]$ satisfying the following properties:*

- *$S^{(2)} = S$.* (*"Extension"*)
- *For all $t \geq 0$ and all $p,q \in [1,\infty]$ with $1/p + 1/q = 1$*
$$(S^{(p)}(t)f, g) = (f, S^{(q)}(t)g)$$
for $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$. (*"Symmetry"*)
- *For all $t \geq 0$ and all $p,q \in [1,\infty]$*
$$S^{(p)}(t)f = S^{(q)}(t)f$$
for all $f \in \ell^p(X,m) \cap \ell^q(X,m)$. (*"Consistency"*)

For $p \in [1,\infty)$, the semigroup $S^{(p)}$ is strongly continuous. For $p = \infty$, the semigroup $S^{(\infty)}$ is pointwise continuous, i.e. the map

$$t \mapsto S^{(\infty)}(t)f(x)$$

is continuous for all $f \in \ell^\infty(X,m)$ and $x \in X$.

REMARK 10.8. The pointwise continuity of $S^{(\infty)}$ can be seen to be equivalent to weak* continuity in the discrete setting (Exercise).

PROOF. We first deal with the *uniqueness statement*. By consistency and the extension property, the semigroups are defined on $C_c(X)$. As $C_c(X)$ is dense in $\ell^p(X,m)$ for $p \in [1,\infty)$ and all $S^{(p)}$ are bounded, this shows that the semigroups are uniquely determined on $\ell^p(X,m)$ for $p \in [1,\infty)$. For $p = \infty$, we note that the semigroup on $\ell^\infty(X,m)$ is uniquely determined by the semigroup on $\ell^1(X,m)$ by the symmetry condition.

We now turn to proving *existence*. By Corollary 5.6, $S = (e^{-tL})_{t\geq 0}$ is a Markov semigroup of self-adjoint operators on $\ell^2(X,m)$. Now, for every $t \geq 0$, there exists a $p_t\colon X \times X \longrightarrow \mathbb{R}$ with

$$S(t)f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y)$$

for all $f \in \ell^2(X,m)$. This p is called the *heat kernel* of the semigroup of S. We will now show that p_t is a Markov matrix for every $t \geq 0$.

First, note that by direct calculation $S(t)1_y(x) = p_t(x,y)m(y)$. As $S(t)$ is self-adjoint, we get

$$p_t(x,y)m(y)m(x) = \langle S(t)1_y, 1_x\rangle = \langle 1_y, S(t)1_x\rangle = p_t(y,x)m(x)m(y)$$

so that $p_t(x,y) = p_t(y,x)$ for all $x, y \in X$ and $t \geq 0$.

Since $S(t)$ is Markov, it follows that

$$0 \leq S(t)1_y(x) = p_t(x,y)m(y).$$

Therefore, $p_t(x,y) \geq 0$ for all $x, y \in X$ and $t \geq 0$.

Finally, for $n \in \mathbb{N}$ let $K_n \subseteq X$ be finite such that $K_n \subseteq K_{n+1}$ and $X = \bigcup_{n\in\mathbb{N}} K_n$. It follows by the Markov property that $0 \leq S(t)1_{K_n} \leq 1$ and thus

$$0 \leq \sum_{y\in X} p_t(x,y)1_{K_n}(y)m(y) = \sum_{y\in K_n} p_t(x,y)m(y) \leq 1$$

for all $n \in \mathbb{N}$, $x \in X$ and $t \geq 0$. By the monotone convergence theorem

$$\sum_{y\in X} p_t(x,y)m(y) \leq 1$$

for every $x \in X$ and $t \geq 0$.

Hence, for every $t \geq 0$, p_t is a Markov matrix and Lemma 10.6 gives for any $p \in [1,\infty]$ that the operator $S^{(p)}(t)\colon \ell^p(X,m) \longrightarrow \ell^p(X,m)$ given by

$$S^{(p)}(t)f(x) := \sum_{y\in X} p_t(x,y)f(y)m(y)$$

is bounded with norm not exceeding 1. We now show that these operators have the desired properties.

Markov property. As p_t is a Markov matrix for every $t \geq 0$, each operator $S^{(p)}(t)$ satisfies

$$0 \leq S^{(p)}(t)f \leq 1$$

whenever $0 \leq f \leq 1$ for $f \in \ell^p(X,m)$.

Consistency. By definition, we have

$$S^{(p)}(t)f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y) = S^{(q)}(t)f(x)$$

for all $t \geq 0$ whenever $f \in \ell^p(X,m) \cap \ell^q(X,m)$.

$S^{(2)} = S$. This is clear from the definition of $S^{(p)}$.

Semigroup property for $S^{(p)}$. By the consistency of the family the space

$$\mathcal{C} := \bigcap_{p\in[1,\infty]} \ell^p(X,m)$$

is invariant under any $S^{(p)}(t)$ for $t \geq 0$ and $p \in [1,\infty]$. Moreover, the action of $S^{(p)}$ on $\mathcal{C}$ agrees with the action of S. As S satisfies $S(0) = I$ and $S(s)S(t) = S(s+t)$ for all $s, t \geq 0$, the same will hold for $S^{(p)}$ on $\mathcal{C}$. As $\mathcal{C}$ contains $C_c(X)$, the space $\mathcal{C}$ is dense in $\ell^p(X,m)$ for $p \in [1,\infty)$. Then, the semigroup property follows on $\ell^p(X,m)$ for $p \in [1,\infty)$ as each $S^{(p)}(t)$ is a bounded operator. To deal with the case $p = \infty$, it suffices to consider $f \geq 0$. Any such function can be written as a monotone limit of functions in $C_c(X)$. By the Markov property, the operators $S^{(\infty)}(t)$ on $\ell^\infty(X,m)$ are compatible with monotone limits and the desired statement follows.

Symmetry. The symmetry property is clear for $f, g \in C_c(X)$. It then follows in the generality stated by approximating $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$, where $1/p + 1/q = 1$, by sequences (f_n) and (g_n) in $C_c(X)$.

Strong continuity for $p \in [1,\infty)$. From the strong continuity of S on $\ell^2(X,m)$ we infer pointwise continuity of p_t for $t \to 0^+$ in the sense that we have

$$p_t(x,y) = \frac{1}{m(y)} e^{-tL} 1_y(x) \to \frac{1}{m(y)} 1_y(x)$$

as $t \to 0^+$ for every $x, y \in X$.

We now treat the general case and let $f \in \ell^p(X,m)$ for $p \in [1,\infty)$. In order to simplify the notation we set

$$u_t(x) := S^{(p)}(t)f(x) = \sum_{y \in X} p_t(x,y)f(y)m(y).$$

Let $\varepsilon > 0$. Since $C_c(X)$ is dense in $\ell^p(X,m)$ for $p \in [1,\infty)$ we can choose as finite subset $K \subseteq X$ with

$$\|(1 - 1_K)f\|_p^p < \varepsilon$$

so that

$$\|1_K f\|_p^p = \|f\|_p^p - \|(1 - 1_K)f\|_p^p > \|f\|_p^p - \varepsilon.$$

For t sufficiently close to 0, we then infer from the pointwise continuity and the finiteness of K that

$$\|1_K(u_t - f)\|_p^p < \varepsilon \text{ and } \|1_K u_t\|_p^p \geq \|f\|_p^p - \varepsilon.$$

Therefore, combining with the above, we obtain

$$\|1_K u_t\|_p^p > \|f\|_p^p - 2\varepsilon.$$

Moreover, as each $S^{(p)}(t)$ has norm not exceeding 1 we also have

$$\|u_t\|_p^p \leq \|f\|_p^p.$$

Therefore, for small enough $t > 0$, we infer from the last two inequalities

$$\|f\|_p^p \geq \|u_t\|_p^p = \|1_K u_t\|_p^p + \|(1 - 1_K)u_t\|_p^p > \|f\|_p^p - 2\varepsilon + \|(1 - 1_K)u_t\|_p^p.$$

Hence,

$$\|(1 - 1_K)u_t\|_p^p < 2\varepsilon.$$

This gives the desired continuity at 0 as

$$\|u_t - f\|_p \leq \|(1 - 1_K)u_t\|_p + \|1_K(u_t - f)\|_p + \|(1 - 1_K)f\|_p.$$

Pointwise continuity for $p = \infty$. This follows from the strong continuity for $p = 1$ and the symmetry of the family. □

From the preceding discussion we infer the following: Whenever (b,c) is a graph over (X,m) and $L := L^{(D)}$ is the associated operator and p is the heat kernel then

$$S^{(p)}(t)f(x) = \sum_{y \in X} p_t(x,y)f(y)m(y)$$

holds for all $p \in [1,\infty]$ and $f \in \ell^p(X,m)$, $t \geq 0$ and $x \in X$. For this reason we will subsequently — by a slight abuse of notation — write

$$e^{-tL}f(x) := \sum_{y \in X} p_t(x,y)f(y)m(y) = S^{(p)}(t)f(x)$$

whenever f belongs to $\ell^p(X,m)$ for some $p \in [1,\infty]$, $t \geq 0$ and $x \in X$.

As one can extend the semigroups to all $\ell^p(X,m)$ one can also extend the resolvents to all $\ell^p(x,m)$. We omit the details but rather state the result and give some indication of the proof.

THEOREM 10.9 (Extension theorem – resolvents). *Let (b,c) be a graph over (X,m) and $L := L^{(D)}$ the associated self-adjoint operator. Let $G(\alpha) := (L+\alpha)^{-1}$ be the resolvent of L for $\alpha > 0$. Then, there exists a unique family of contraction Markov resolvents $G^{(p)}$ on $\ell^p(X,m)$ for $p \in [1,\infty]$ satisfying the following properties:*

- *$G^{(2)} = G$.* (*"Extension"*)
- *For all $\alpha > 0$ and all $p, q \in [1,\infty]$ with $1/p + 1/q = 1$*
$$(G^{(p)}(\alpha)f, g) = (f, G^{(q)}(\alpha)g)$$
for $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$. (*"Symmetry"*)
- *For all $\alpha > 0$ and all $p, q \in [1,\infty]$*
$$G^{(p)}(\alpha)f = G^{(q)}(\alpha)f$$
for $f \in \ell^p(X,m) \cap \ell^q(X,m)$. (*"Consistency"*)

For $p \in [1,\infty)$, the resolvent $G^{(p)}$ is strongly continous. For $p = \infty$, the resolvent $G^{(\infty)}$ ispointwise continuous, i.e., the map
$$\alpha \mapsto \alpha G^{(\infty)}(\alpha)f(x)$$
is continuous for all $f \in \ell^\infty(X,m)$ and $x \in X$.

If $S^{(p)}$ is the contraction Markov semigroup with generator $L^{(p)}$, then $G^{(p)}$ satisfies
$$G^{(p)}(\alpha)f(x) = \int_0^\infty e^{-t\alpha} S^{(p)}(t)f(x)\mathrm{d}t$$
for all $\alpha > 0$, $p \in [1,\infty]$, and, $f \in \ell^p(X,m)$ and $x \in X$.
("Laplace transform")

IDEA OF THE PROOF. Uniqueness is clear from the given properties. For existence we define the resolvent, for $\alpha > 0$, $p \in [1,\infty]$, $f \in \ell^p(X,m)$ and $x \in X$ by
$$G^{(p)}(\alpha)f(x) = \int_0^\infty e^{-t\alpha} S^{(p)}(t)f(x)\mathrm{d}t$$
and show the claimed properties. □

THEOREM 10.10 (Resolvents as minimal solutions to $(\mathcal{L}+\alpha)u = f$). *Let (b,c) be a graph over (X,m) and let $L = L^{(D)}$ be the associated self-adjoint operator. Let $p \in [1,\infty]$ and let $G^{(p)}$ be the resolvent on $\ell^p(X,m)$ associated to L. If $f \in \ell^p(X,m)$, $\alpha > 0$ and*
$$u := G^{(p)}(\alpha)f,$$
then u belongs to $\mathcal{F}$ and satisfies the Poisson equation

(*"Poisson equation"*)
$$(\mathcal{L} + \alpha)u = f.$$

Furthermore, if additionally $f \geq 0$, then $u \geq 0$ and for the resolvent $G_{b,c,m}$ associated to L we have that
$$u = G^{(p)}_{b,c,m}(\alpha)f$$

is the smallest $v \in \mathcal{F}$ *with* $v \geq 0$ *and* $(\mathcal{L} + \alpha)v \geq f$.

Proof. Let $f \in \ell^p(X, m)$ for $p \in [1, \infty]$. Without loss of generality, we assume $f \geq 0$. For a general $f \in \ell^p(X, m)$, we can decompose $f = f_+ - f_-$ into its positive and negative part.

Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_{n \in \mathbb{N}} K_n$ and let $f_n := f 1_{K_n}$ so that $f_n \in C_c(X)$ for all $n \in \mathbb{N}$. Let $\alpha > 0$ and

$$u_n := G^{(p)}(\alpha) f_n, \quad n \in \mathbb{N}.$$

As $G^{(p)}$ is Markov by Theorem 10.9, $u_n \geq 0$ for all $n \in \mathbb{N}$ and the sequence (u_n) is monotonically increasing and converges to $u := G^{(p)}(\alpha) f \in \ell^p(X, m)$ by the monotone convergence theorem.

As the resolvents agree on their common domain due to the consistency statement in Theorem 10.9 and $f_n \in C_c(X) \subseteq \ell^p(X, m)$ for all $p \in [1, \infty]$, we have

$$u_n = G^{(p)}(\alpha) f_n = G^{(2)}(\alpha) f_n$$

for all $n \in \mathbb{N}$. By definition we have $G^{(2)}(\alpha) = (L + \alpha)^{-1}$ and since $L = \mathcal{L}$ on $D(L) = G^{(2)}(\alpha)\ell^2(X, m) \supseteq G^{(2)} C_c(X)$ we get

$$(\mathcal{L} + \alpha) u_n = (L + \alpha) G^{(2)}(\alpha) f_n = f_n$$

for all $n \in \mathbb{N}$. We conclude that $u \in \mathcal{F}$ solves the Poisson equation by taking monotone limits, Lemma 2.11. Furthermore, $u = G^{(p)}(\alpha) f \geq 0$ for $f \geq 0$ since $G^{(p)}$ is Markov and, therefore, positivity preserving by Theorem 10.9.

Now, let $v \in \mathcal{F}$ with $v \geq 0$ satisfy $(\mathcal{L} + \alpha)v \geq f$. Then, $(\mathcal{L} + \alpha)v \geq f_n$ for all $n \in \mathbb{N}$ and as $u_n := G^{(2)}_{b,c,m}(\alpha) f_n$ is the minimum positive solution of this inequality by Lemma 5.8 we obtain $u_n \leq v$ for all $n \in \mathbb{N}$. Taking the limit gives $u \leq v$. ☐

10.2. The Heat Equation on ℓ^∞

In this section we show that $t \mapsto e^{-tL} f$ provides a solution to the heat equation for bounded f.

Let (b, c) be a graph over (X, m). A function $v \colon [0, \infty) \longrightarrow \mathbb{R}$ is called a *supersolution* to the heat equation with initial condition f if $t \mapsto v_t(x)$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ for all $x \in X$, and

$$-\mathcal{L} v_t \geq \partial_t v$$

holds for all $t > 0$ and $v_0 = f$ is valid.

Theorem 10.11 (Existence of bounded solutions of the heat equation). *Let* (b, c) *be a graph over* (X, m) *with associated self-adjoint operator* $L := L^{(D)}$ *and let* $f \in \ell^\infty(X)$ *be given. Then,*

$$u \colon [0, \infty) \times X \longrightarrow \mathbb{R}, \quad u_t(x) := e^{-tL} f(x),$$

is a bounded solution of the heat equation with initial condition f.

Furthermore, if additionally $f \geq 0$, *then* u *is the smallest positive supersolution of the heat equation with initial condition greater than or equal to* f.

Proof. We start by showing the continuity and boundedness of u. We denote the dual pairing between $\ell^1(X,m)$ and $\ell^\infty(X,m)$ by $(\cdot,\cdot)$ and for $x \in X$ let $\eta_x \in \ell^1(X,m)$ be given by $\eta_x := \frac{1}{m(x)} 1_x$. Since

$$u_t(x) = (\eta_x, e^{-tL} f)$$

for $x \in X$ and $t \geq 0$, continuity of the function $t \mapsto u_t(x)$ for $t \geq 0$ and $x \in X$ follows from the weak* continuity of the semigroup on $\ell^\infty(X,m)$ established in Theorem 10.7. Furthermore, as the semigroup on $\ell^1(X,m)$ is strongly continuous, we have $u_0 = f$. Finally, as $(e^{-tL})_{t\geq 0}$ is a contraction semigroup on $\ell^\infty(X,m)$ by Theorem 10.7, it follows that u_t is bounded by $\|f\|_\infty$ for every $t \geq 0$. In particular, $u_t \in \mathcal{F}$ for every $t \geq 0$.

As an intermediate step, we next show the continuity of

$$t \mapsto \mathcal{L}u_t(x) = \frac{1}{m(x)} \left(\sum_{y\in X} b(x,y)(u_t(x) - u_t(y)) + c(x)u_t(x) \right)$$

on $[0,\infty)$ for every $x \in X$. Indeed, this is immediate from the continuity of $t \mapsto u_t(y)$ for each $y \in X$, the uniform boundedness of u in both variables, and the summability of $b(x,\cdot)$ for every $x \in X$.

We will now show differentiability and the fact that u satisfies the heat equation. We will do so by approximating f by functions with finite support and using that the heat equation holds for functions in $\ell^2(X,m)$ and, hence, for functions with finite support.

We note that the preceding considerations hold for any bounded function f and, in particular, for the elements of the approximating sequence (f_n) and $u_t^{(n)} := e^{-tL} f_n$ which we introduce below. Hence, the functions $t \mapsto u_t(x), t \mapsto u_t^{(n)}(x)$, $t \mapsto \mathcal{L}u_t(x)$ and $t \mapsto \mathcal{L}u_t^{(n)}(x)$ will be continuous for all $x \in X$ and all $n \in \mathbb{N}$.

Let $t > 0$. By decomposing f into positive and negative parts, we can assume without loss of generality that f is positive. Let (K_n) be a sequence of finite increasing subsets of X such that $X = \bigcup_{n\in\mathbb{N}} K_n$. Furthermore, let $f_n := f 1_{K_n}$ and let

$$u_t^{(n)} := e^{-tL} f_n, \quad n \in \mathbb{N}.$$

Since $(e^{-tL})_{t\geq 0}$ on $\ell^\infty(X,m)$ is a bounded Markov semigroup by Theorem 10.7, $(e^{-tL})_{t\geq 0}$ admits a positive kernel p. That is,

$$e^{-tL} f(x) = \sum_{y\in X} p_t(x,y) f(y) m(y), \quad x, y \in X, t \geq 0,$$

where $p_t(x,y) \geq 0$ for all $x, y \in X$ and $t \geq 0$. Thus, $u_t^{(n)}(x) \nearrow u_t(x)$ as $n \to \infty$ for all $x \in X$ and $t > 0$. Moreover, the convergence is uniform on compact subintervals of $(0,\infty)$ by Dini's theorem as $(t \mapsto u_t^{(n)}(x))_n$ in $C([0,\infty))$ and $t \mapsto u_t(x)$ is continuous.

Since $f_n \in C_c(X) \subseteq \ell^2(X,m) \cap \ell^\infty(X,m)$ and the semigroup on $\ell^\infty(X,m)$ agrees with the semigroup on $\ell^2(X,m)$ for functions in $\ell^2(X,m) \cap \ell^\infty(X,m)$ by Theorem 10.7, it follows that $u_t^{(n)} \in \ell^2(X,m)$. Therefore, for all $x \in X$,

$t \geq 0$ and $n \in \mathbb{N}$, we infer by Lemma 5.9 that

$$\begin{aligned}\partial_t u_t^{(n)}(x) &= -Lu_t^{(n)}(x) = -\mathcal{L}u_t^{(n)}(x)\\ &= -\frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(u_t^{(n)}(x) - u_t^{(n)}(y)) + c(x)u_t^{(n)}(x)\right).\end{aligned}$$

Monotone convergence of $(u_t^{(n)}(y))_n$ to $u_t(y)$ for all $y \in X$ and $t \geq 0$, and the fact that $u_t \in \ell^\infty(X) \subseteq \mathcal{F}$, yields the convergence of the right-hand side to $-\mathcal{L}u_t(x)$ as $n \to \infty$ for each $x \in X$ and $t \geq 0$. Therefore, we obtain the convergence of $(\partial_t u_t^{(n)}(x))_n$ to $\mathcal{L}u_t(x)$ for each $x \in X$ and $t > 0$. In fact, this convergence is uniform in t on compact subintervals of $(0,\infty)$ as the convergence of $(t \mapsto u_t^{(n)}(y))_n$ to $t \mapsto u_t(y)$ is uniform on compact subintervals of $(0,\infty)$ for each $y \in X$ and $b(x,\cdot)$ is summable for each $x \in X$.

Altogether we have established that $(t \mapsto u_t^{(n)}(x))_n$ converges uniformly on compact subintervals of $(0,\infty)$ to $t \mapsto u_t(x)$ and $(t \mapsto \partial_t u_t^{(n)}(x))_n$ converges uniformly on compact subintervals of $(0,\infty)$ to $t \mapsto -\mathcal{L}u_t(x)$ for each $x \in X$. As discussed above, all involved functions are continuous. Thus, this gives that $t \mapsto u_t(x)$ is differentiable for all $x \in X$ with the desired derivative.

It remains to show the last statement of the theorem. That u is positive whenever f is positive follows immediately from the fact that the semigroup on $\ell^\infty(X,m)$ is Markov and, in particular, positivity preserving by Theorem 10.7. We now show the minimality statement. Let w be a supersolution of the heat equation with initial condition greater than or equal to f. From what we have shown above, $u^{(n)}$ satisfies

$$(\mathcal{L} + \partial_t)u_t^{(n)} = 0$$

for $t > 0$ and $u_0^{(n)} = f_n$ for $n \in \mathbb{N}$. Furthermore, the $u^{(n)}$ agree with the solution generated by the semigroup on $\ell^2(X,m)$ as the semigroups agree on their common domain by Theorem 10.7. As w is a positive supersolution with initial condition greater than or equal to f_n we obtain $u^{(n)} \leq w$ for all n by Lemma 5.9. Letting $n \to \infty$ gives $u \leq w$ which completes the proof. □

PROPOSITION 10.12 (Heat equation at $t = 0$). *Let (b,c) be a graph over (X,m) and let u be a bounded solution of the heat equation. Then, the function*

$$[0,\infty) \longrightarrow \mathbb{R}, t \mapsto \mathcal{L}u_t(x),$$

is continuous for all $x \in X$, the limit

$$\partial_t u_t(x)|_{t=0} = \lim_{h\to 0^+} \frac{u_h(x) - u_0(x)}{h}$$

exists for all $x \in X$ and

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

for all $x \in X$ and $t \geq 0$.

Proof. Let $x \in X$. We first show the continuity of $t \mapsto \mathcal{L}u_t(x)$ on $[0,\infty)$: As u is a solution of the heat equation the map $t \mapsto u_t(y)$ is continuous for each $y \in X$. As $b(x,\cdot)$ is summable and u is bounded, we then obtain that

$$t \mapsto \mathcal{L}u_t(x) = \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(u_t(x)-u_t(y)) + c(x)u_t(x)\right)$$

is continuous on $[0,\infty)$. Moreover, as u is a solution of the heat equation, we have

$$\partial_t u_t(x) = -\mathcal{L}u_t(x)$$

for all $t > 0$.

Altogether, for each $x \in X$, the function $t \mapsto u_t(x)$ is a continuous function on $[0,\infty)$ which is differentiable on $(0,\infty)$ and $t \mapsto -\mathcal{L}u_t(x)$ is a continuous function on $[0,\infty)$ which agrees with the derivative of $t \mapsto u_t(x)$ on $(0,\infty)$. This implies that $t \mapsto u_t(x)$ is differentiable at $t = 0$ with derivative given by $-\mathcal{L}u_0(x)$: Indeed, fix $x \in X$. By the mean value theorem, for every $h > 0$, there exists a $\zeta(h) \in (0,h)$ such that

$$\frac{u_h(x)-u_0(x)}{h} = \partial_t u_t|_{t=\zeta(h)} = -\mathcal{L}u_{\zeta(h)}(x).$$

This implies

$$\lim_{h\to 0^+} \frac{u_h(x)-u_0(x)}{h} = -\mathcal{L}u_0(x)$$

by continuity. ☐

The next proposition connects bounded solutions of an inhomogeneous heat equation with solutions of the Poisson equation.

Proposition 10.13 (Solutions of heat and Poisson equations). *Let (b,c) be a graph over (X,m) and $L := L^{(D)}$ the associated operator. Let $f,g \in \ell^\infty(X,m)$ and let $u\colon [0,\infty)\times X \longrightarrow \mathbb{R}$ be a bounded solution of*

$$(\mathcal{L}+\partial_t)\,u_t = f$$

with initial condition $u_0 = g$. Then, for $\alpha > 0$ the function

$$v := \int_0^\infty \alpha e^{-t\alpha} u_t \mathrm{d}t$$

is bounded and satisfies

$$(\mathcal{L}+\alpha)v = f + \alpha g.$$

Moreover, if additionally $f, g \geq 0$, then

$$w := \int_0^\infty e^{-t\alpha} e^{-tL}(f+\alpha g)\mathrm{d}t$$

is the smallest positive function $w \in \mathcal{F}$ with $(\mathcal{L}+\alpha)w \geq f+\alpha g$. In particular,

$$\int_0^\infty e^{-t\alpha}e^{-tL}(f+\alpha g)\mathrm{d}t \leq \int_0^\infty \alpha e^{-t\alpha}u_t\mathrm{d}t.$$

PROOF. The boundedness of v follows since we assume that u is bounded and since $[0,\infty) \ni t \mapsto \alpha e^{-t\alpha}$ is a probability density function.

Furthermore, by the boundedness of u and Fubini's theorem, we have for all $x \in X$

$$\mathcal{L}v(x) = \int_0^\infty \alpha e^{-t\alpha} \mathcal{L}u_t(x)\mathrm{d}t = \lim_{T\to\infty} \int_0^T \alpha e^{-t\alpha} \mathcal{L}u_t(x)\mathrm{d}t.$$

Since u satisfies $\mathcal{L}u_t = -\partial_t u_t + f$, we infer

$$\begin{aligned}\mathcal{L}v(x) &= \lim_{T\to\infty} \int_0^T \alpha e^{-t\alpha}(-\partial_t u_t(x))\mathrm{d}t + \int_0^\infty \alpha e^{-t\alpha} f(x)\mathrm{d}t \\ &= \lim_{T\to\infty} \left(-\alpha e^{-t\alpha} u_t(x)|_0^T - \int_0^T \alpha^2 e^{-t\alpha} u_t(x)\mathrm{d}t \right) + f(x),\end{aligned}$$

where we used integration by parts and again the fact that $[0,\infty) \ni t \mapsto \alpha e^{-t\alpha}$ is a probability density function. Next, we conclude from the boundedness of u and $u_0 = g$ that the first term tends to αg and, therefore,

$$\begin{aligned}\mathcal{L}v(x) &= \alpha g(x) - \alpha \int_0^\infty \alpha e^{-t\alpha} u_t(x)\mathrm{d}t + f(x) \\ &= \alpha g(x) - \alpha v(x) + f(x)\end{aligned}$$

by the definition of v. Therefore, v is bounded and satisfies $(\mathcal{L}+\alpha)v = f + \alpha g$.

If $f, g \in \ell^\infty(X,m)$ and $x \in X$, then

$$\int_0^\infty e^{-t\alpha} e^{-tL}(f + \alpha g)(x)\mathrm{d}t = (L+\alpha)^{-1}(f+\alpha g)(x)$$

by the Laplace transform formula, see Theorem 10.9. Furthermore, if f and g are positive, Theorem 10.10 gives that $(L+\alpha)^{-1}(f+\alpha g)$ is the minimal positive function $w \in \mathcal{F}$ with $(\mathcal{L}+\alpha)w \geq f + \alpha g$. As v is such a function by what we have shown above, $(L+\alpha)^{-1}(f+\alpha g) \leq v$ follows. □

COROLLARY 10.14 (Solutions of the heat equation and α-harmonic functions). *Let (b,c) be a graph over (X,m) and let u be a bounded solution of the heat equation with $u_0 = 0$. Then, for $\alpha > 0$ the function*

$$v := \int_0^\infty e^{-t\alpha} u_t \mathrm{d}t$$

is bounded and satisfies

$$(\mathcal{L}+\alpha)v = 0.$$

In particular, if there exists a positive non-trivial bounded solution of the heat equation with trivial initial conditions, then there exists a positive non-trivial bounded α-harmonic function for any $\alpha > 0$.

PROOF. This follows immediately from Proposition 10.13 by letting f and g be 0. □

10.3. The Heat Equation Perspective

In this section we turn to stochastic completeness and present the heat equation viewpoint on it.

We first define the notion.

Definition 10.15 (Stochastic completeness). A graph (b,c) over (X,m) is called *stochastically complete* or *conservative* if

$$e^{-tL}1 = 1$$

holds for all $t \geq 0$, where $L := L^{(D)}$ is the associated operator. Otherwise, (b,c) over (X,m) is called *stochastically incomplete.*

Remark 10.16. We have defined the notion of stochastic completeness via the semigroup. We will see that stochastic completeness can equivalently be characterized via resolvents by the condition that

$$\alpha(L+\alpha)^{-1}1 = 1$$

holds $\alpha > 0$.

Remark 10.17. Stochastic completeness is only possible for $c = 0$. Indeed, assume that there exist an $x \in X$ with $c(x) > 0$. Let $u_t := e^{-tL}1$ and assume $u_t = 1$ for all $t \geq 0$. Then, $t \mapsto u_t(x)$ is constant (to 1) and this gives (for any $t > 0$) the contradiction

$$0 = \partial_t u_t(x) = -\mathcal{L}u_t(x) = -\mathcal{L}1(x) = -\frac{c(x)}{m(x)} < 0.$$

To better understand the concept and its name we present the following lemma. Recall that $(\cdot,\cdot)$ given by

$$(g,f) := \sum_{y\in X} g(y)f(y)m(y)$$

denotes the dual pairing between $\ell^\infty(X,m)$ and $\ell^1(X,m)$.

Lemma 10.18 (Stochastic completeness as preservation of heat). *Let (b,c) be a graph over (X,m) and $L := L^{(D)}$ the associated operator. Define for $f \in \ell^1(X,m)$ the amount of heat at time $t \geq 0$ by*

$$A_f(t) := \sum_{y\in X} e^{-tL}f(y)m(y).$$

Then, the following assertions are equivalent:

- (i) *The graph is stochastically complete.*
- (ii) *For any $x \in X$ the function A_{1_x} is constant on $[0,\infty)$.*
- (iii) *For any $f \in \ell^1(X,m)$ the function A_f is constant on $[0,\infty)$.*

Proof. This is basically a direct consequence of

$$A_f(t) = (1, e^{-tL}f) = (e^{-tL}1, f),$$

where the first equality follows directly from the definition of the dual pairing and the second equality follows from the symmetry. For the convenience of the reader we provide the details.

(iii)$\Longrightarrow$ (ii): This is clear.

(ii)$\Longrightarrow$ (i): We have

$$m(x) = A_{1_x}(0) = A_{1_x}(t) = (1, e^{-tL}1_x) = (e^{-tL}1, 1_x) = m(x)e^{-tL}1(x)$$

for all $x \in X$ and this gives (i).

(i)$\Longrightarrow$(iii): For all $t > 0$ we can compute

$$A_f(t) = (1, e^{-tL}f) = (e^{-tL}1, f) = (1, f) = A_f(0).$$

This gives (iii). □

Here is the main characterization of stochastic completeness in terms of the heat equation.

THEOREM 10.19 (Stochastic completeness and the heat equation). *Let b be a connected graph over (X, m) and $L := L^{(D)}$ the associated operator. Then, the following statements are equivalent:*

(i) *For some (all) $t > 0$ and some (all) $x \in X$,*

$$e^{-tL}1(x) = 1.$$

(*"Stochastic completeness"*)

(i.a) *For some (all) $\alpha > 0$ and some (all) $x \in X$,*

$$(L + \alpha)^{-1}1(x) = 1.$$

(vi) *For every $f \in \ell^\infty(X)$ there exists a unique bounded solution u of the heat equation*

$$(\mathcal{L} + \partial_t)u = 0 \quad \text{with} \quad u_0 = f.$$

(*"Heat equation"*)

(vi.a) *Every bounded solution u of the heat equation $(\mathcal{L} + \partial_t)u = 0$ with $u_0 = 0$ is trivial.*

We start the proof of Theorem 10.19 by showing the equivalence of the "for some" and "for all" statements in (i). For this, the connectedness of the graph is essential. We need the following lemma. It shows that if the total amount of heat in the graph drops below 1 at some time, then it drops below 1 for all times.

LEMMA 10.20. *Let (b, c) be a graph over (X, m) and $L := L^{(D)}$ the associated operator. If $t \geq s \geq 0$, then*

$$e^{-tL}1 \leq e^{-sL}1.$$

PROOF. From Theorem 10.7, we get that the heat semigroup is both positivity preserving and contracting. Let $t = s + h$ with $s, h \geq 0$. As the semigroup is contracting we have

$$e^{-hL}1 \leq 1.$$

As the semigroup is positivity preserving, this gives, after we apply e^{-sL} to both sides,

$$e^{-tL}1 = e^{-sL}e^{-hL}1 \leq e^{-sL}1.$$

This is the desired statement. □

Lemma 10.21. *Let b be a connected graph over (X, m) and $L := L^{(D)}$ the associated operator. If $e^{-tL}1(x) < 1$ for some $t > 0$ and some $x \in X$, then $e^{-tL}1(x) < 1$ for all $t > 0$ and all $x \in X$.*

Proof. Assume that $e^{-tL}1(x) < 1$ for some $t > 0$ and some $x \in X$. By Lemma 10.20 we have $e^{-tL}1 \leq 1$. Thus, $(1 - e^{-tL}1)$ is positive and non-trivial. As the graph is connected and the semigroup is positivity improving we obtain that $e^{-sL}(1 - e^{-tL}1) > 0$. Hence, we find $e^{-(s+t)L}1 < 1$ for all $s > 0$. So, $e^{-rL}1 < 1$ for all $r > t$. It remains to show $e^{-rL}1 < 1$ also for $0 < r \leq t$. Assume not. Then, there exists a $y \in X$ and $0 < r \leq t$ such that $e^{-rL}1(y) = 1$. By what we have shown already this gives $e^{-uL}1 = 1$ for all $0 < u < r$. Fix any such $u > 0$. Let $n \in \mathbb{N}$ be such that $nu > t$. By the semigroup property, we get $e^{-nuL}1 = 1$. Thus, $e^{-tL}1 = 1$ as $nu > t$, which yields a contradiction to the assumption that $e^{-tL}1(x) < 1$. This completes the proof. $\square$

Proof of Theorem 10.19. The equivalence of the "for some" and "for all" statements found in (i) has already been shown in Lemma 10.21.

(i) $\iff$ (i.a): This follows readily from Theorem 10.11, the Laplace transform and symmetry, as well as Lemma 10.21. This also shows the equivalence of the "for some" and "for all" statements found in (i.a).

(vi) $\implies$ (vi.a): This is clear since if we let $f = 0$, then $u = 0$ is the unique bounded solution of the heat equation with initial condition 0.

(vi.a) $\implies$ (vi): Given $f \in \ell^\infty(X, m)$, the existence of a bounded solution of the heat equation with initial condition f has been shown in Theorem 10.11. Hence, we only need to establish uniqueness. Therefore, let u and v be two bounded solutions of the heat equation with initial condition f. Then, $w := u - v$ is a bounded solution of the heat equation with initial condition $w_0 = 0$. By (vi.a) we get that $w = 0$, so that $u = v$.

(vi.a) $\implies$ (i): We show this by contraposition. If $e^{-tL}1(x) < 1$ for some $x \in X$ and some $t > 0$, then $e^{-tL}1 < 1$ for all $t > 0$ by Lemma 10.21. Hence, suppose that $e^{-tL}1 < 1$ for $t > 0$. Moreover, $u_t := e^{-tL}1$ defines a bounded solution to the heat equation

$$(\mathcal{L} + \partial_t)u_t = 0$$

with $u_0 = 1$ by Theorem 10.11. Furthermore, it is clear that $(\mathcal{L} + \partial_t)1 = 0$. Therefore, it follows that

$$v_t := 1 - e^{-tL}1$$

is a bounded solution of the heat equation with initial condition 0, which is non-trivial since $e^{-tL}1 < 1$ for $t > 0$.

(i) $\implies$ (vi.a): We show this by contraposition as well. Let u be a non-zero bounded solution of the heat equation with $u_0 = 0$. Without loss of generality, we may assume that $u_{t_0}(x_0) > 0$ for some $t_0 > 0$ and some $x_0 \in X$ as otherwise we work with $-u$. Furthermore, by rescaling, we may assume $|u| \leq 1$.

Let $w := 1 - u$. Then, w is positive, bounded and $w_{t_0}(x_0) < 1$. Furthermore, since $u_0 = 0$, we get $w_0 = 1$ and since u solves the heat equation, we get

$$(\mathcal{L} + \partial_t)\, w = 0$$

for all $t > 0$. Since $v_t := e^{-tL}1$ defines the smallest positive function with these properties by Theorem 10.11, we infer $e^{t_0L}1(x_0) \leq w_{t_0}(x_0) < 1$. Therefore, $e^{-tL}1(x) < 1$ for all $t > 0$ and all $x \in X$ by Lemma 10.21. This completes the proof. $\square$

10.4. The Poisson Equation Perspective

It is possible to characterize stochastic completeness by means of Poisson equation. Details are discussed in this section.

Theorem 10.22 (Stochastic completeness and the Poisson equation). *Let b be a connected graph over (X, m) and $L = L^{(D)}$ the associated operator. Then, the following statements are equivalent:*

(i) *For some (all) $t > 0$ and some (all) $x \in X$,*

$$e^{-tL}1(x) = 1.$$

(*"Stochastic completeness"*)

(v) *For some (all) $\alpha > 0$ and every $f \in \ell^\infty(X)$ there exists a unique $u \in \ell^\infty(X)$ satisfying*

$$(\mathcal{L} + \alpha)u = f.$$

(*"Poisson equation"*)

(v.a) *For some (all) $\alpha > 0$ every positive $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u \leq 0$ is trivial.*

(v.b) *For some (all) $\alpha > 0$ every $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u = 0$ is trivial.*

(v.c) *For some (all) $\alpha > 0$ every positive $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u = 0$ is trivial.*

The following lemma is the key to proving Theorem 10.22. It connects $(e^{-tL}1)_{t\geq 0}$ with bounded α-harmonic functions for $\alpha > 0$.

Lemma 10.23 (Largest α-subharmonic function). *Let b be a graph over (X, m) and $L := L^{(D)}$ the associated operator. For $\alpha > 0$, the function*

$$w_\alpha := \int_0^\infty \alpha e^{-t\alpha}(1 - e^{-tL}1)\mathrm{d}t = 1 - \alpha(L + \alpha)^{-1}1$$

satisfies $0 \leq w_\alpha \leq 1$, solves $(\mathcal{L} + \alpha)w_\alpha = 0$ and is the largest function $u \in \mathcal{F}$ with $0 \leq u \leq 1$ such that $(\mathcal{L} + \alpha)u \leq 0$.

Proof. The Laplace transform formula proven in Theorem 10.9 gives that for every $\alpha > 0$ the function

$$v_\alpha := \int_0^\infty \alpha e^{-t\alpha} e^{-tL}1\mathrm{d}t = \alpha(L + \alpha)^{-1}1$$

satisfies $(\mathcal{L} + \alpha)v_\alpha = \alpha 1$ and is the minimal positive $v \in \mathcal{F}$ such that $(\mathcal{L} + \alpha)v \geq \alpha 1$ by Lemma 10.10. Furthermore, as $0 \leq e^{-tL}1 \leq 1$ for all $t \geq 0$

by Lemma 10.20, we get $0 \leq v_\alpha \leq 1$. Therefore,

$$\begin{aligned} w_\alpha = 1 - v_\alpha = 1 - \alpha(L+\alpha)^{-1}1 &= 1 - \int_0^\infty \alpha e^{-t\alpha} e^{-tL} 1 \mathrm{d}t \\ &= \int_0^\infty \alpha e^{-t\alpha} \mathrm{d}t - \int_0^\infty \alpha e^{-t\alpha} e^{-tL} 1 \mathrm{d}t \\ &= \int_0^\infty \alpha e^{-t\alpha} (1 - e^{-tL}1) \mathrm{d}t, \end{aligned}$$

and $0 \leq w_\alpha \leq 1$. Furthermore, as v_α satisfies $(\mathcal{L}+\alpha)v_\alpha = \alpha 1$ and since $(\mathcal{L}+\alpha)1 = \alpha 1$ by a direct calculation, we get

$$(\mathcal{L}+\alpha)\, w_\alpha = 0.$$

We now show the maximality of w_α. Hence, let u satisfy $(\mathcal{L}+\alpha)u \leq 0$ with $0 \leq u \leq 1$. Then, $1-u \geq 0$ satisfies $(\mathcal{L}+\alpha)(1-u) \geq \alpha 1$. As v_α is the minimal such positive function by Theorem 10.10, we get $v_\alpha \leq 1-u$. As $w_\alpha = 1 - v_\alpha$, $w_\alpha \geq u$ follows. This completes the proof. □

The equivalence of the "for some" and "for all" statements in (v.a), (v.b) and (v.c) is shown in the next lemma.

LEMMA 10.24. *Let (b,c) be a graph over (X,m). If there exists a bounded non-trivial $v \geq 0$ such that $(\mathcal{L}+\alpha)v \leq 0$ for some $\alpha > 0$, then for every $\alpha > 0$ there exists a bounded non-trivial $v \geq 0$ such that $(\mathcal{L}+\alpha)v = 0$.*

PROOF. Let $\alpha > 0$ and let v be a bounded non-trivial positive function on X satisfying $(\mathcal{L}+\alpha)v \leq 0$. By rescaling, we may assume that $0 \leq v \leq 1$. By Lemma 10.23, $w_\alpha := \int_0^\infty \alpha e^{-t\alpha}(1 - e^{-tL}1)\mathrm{d}t$ is the maximal function $u \in \mathcal{F}$ with $0 \leq u \leq 1$ such that $(\mathcal{L}+\alpha)u \leq 0$. Therefore, $v \leq w_\alpha$. As v is non-trivial, w_α is non-trivial and we conclude that $e^{-tL}1 < 1$ for some t. Therefore, $e^{-tL}1 < 1$ for all $t > 0$ by Lemma 10.21. Hence, for all $\beta > 0$, the function $w_\beta = \int_0^\infty \beta e^{-t\beta}(1 - e^{-tL}1)\mathrm{d}t$ is non-trivial. Furthermore, by Lemma 10.23 we have $0 \leq w_\beta \leq 1$ and $(\mathcal{L}+\beta)w_\beta = 0$ for $\beta > 0$. This completes the proof. □

PROOF OF THEOREM 10.22. The equivalence of the "for some" and "for all" statements in (v.a) and (v.c) follows from Lemma 10.24. The equivalence of the "for some" and "for all" statements in (v) and (v.b) will follow from the arguments given below.

For the rest of the proof recall that

$$w_\alpha := \int_0^\infty \alpha e^{-t\alpha}(1 - e^{-tL}1)\mathrm{d}t$$

solves $(\mathcal{L}+\alpha)w_\alpha = 0$ and is the largest function u with $0 \leq u \leq 1$ and $(\mathcal{L}+\alpha)u \leq 0$ by Lemma 10.23. Obviously, $w_\alpha = 0$ for some (all) $\alpha > 0$ if and only if $e^{-tL}1 = 1$ for some (all) $t > 0$, i.e., if and only if the graph is stochastically complete.

We first show (i) $\Longrightarrow$ (v.a) $\Longrightarrow$ (v.b) $\Longrightarrow$ (v.c) $\Longrightarrow$ (i) for fixed $\alpha > 0$.

(i) $\Longrightarrow$ (v.a): Let $u \geq 0$ be a bounded solution of $(\mathcal{L}+\alpha)u \leq 0$. By rescaling, we may assume that $u \leq 1$. Then, $0 \leq u \leq w_\alpha$ since w_α is the largest such solution. If $e^{-tL}1 = 1$, then $w_\alpha = 0$ and, therefore, $u = 0$.

(v.a) $\Longrightarrow$ (v.b): This follows immediately from Lemma 2.12, which states that if $u \in \mathcal{F}$ is α-harmonic, then $|u|$ is α-subharmonic.

(v.b) $\Longrightarrow$ (v.c): This is clear.

(v.c) $\Longrightarrow$ (i): If there do not exist non-trivial positive functions $u \leq 1$ such that $(\mathcal{L} + \alpha)u = 0$, then the largest such function w_α satisfies $w_\alpha = 0$. Therefore, $e^{-tL}1 = 1$ for all $t > 0$.

Next, we show the implications (v)(for fixed α) $\Longrightarrow$ (i) $\Longrightarrow$ (v.b) $\Longrightarrow$ (v) (for all α). This completes the proof as it gives in particular the equivalence of the "for some" and "for all" statements in (v) and (v.b).

(v) $\Longrightarrow$ (i): We show this by contraposition. So, suppose that $e^{-tL}1 < 1$. Let $\alpha > 0$ and let $f \in \ell^\infty(X)$. Then, $u := (L + \alpha)^{-1} f \in \ell^\infty(X)$ solves $(\mathcal{L}+\alpha)u = f$ by Theorem 10.10. As we assume that $e^{-tL}1 < 1$, $w_\alpha > 0$ and, therefore, $v := u + w_\alpha > u$ also solves $(L + \alpha)v = f$ since $(\mathcal{L} + \alpha)w_\alpha = 0$. Therefore, there is no uniqueness of solutions to the Poisson equation for any $\alpha > 0$.

(i) $\Longrightarrow$ (v.b): We have already shown this in the first round of equivalences above.

(v.b) $\Longrightarrow$ (v): Let $f \in \ell^\infty(X)$ and let $\alpha > 0$. The existence of solutions to the Poisson equation for $\alpha > 0$ is given by $u := (L + \alpha)^{-1} f$. So, we have to show uniqueness. Therefore, assume that there exists $f \in \ell^\infty(X)$ and two bounded solutions u_1, u_2 such that $(\mathcal{L} + \alpha)u_1 = f = (\mathcal{L} + \alpha)u_2$. Then, $u := u_1 - u_2$ is bounded and satisfies $(\mathcal{L} + \alpha)u = 0$. From (v.b) we infer $u = 0$ and, therefore, $u_1 = u_2$. □

10.5. What we have shown (and more)

In the subsequent theorem we present the results of Section 10.4 in a combined version. The theorem contains some further aspects of stochastic completeness that we have not discussed.

THEOREM 10.25 (Characterization of stochastic completeness). *Let b be a connected graph over (X, m) and $L := L^{(D)}$ the associated operator. Then, the following statements are equivalent:*

(i) *For some (all) $t > 0$ and some (all) $x \in X$,*

$$e^{-tL}1(x) = 1.$$

(i.a) *For some (all) $\alpha > 0$ and some (all) $x \in X$,*

$$\alpha(L + \alpha)^{-1}1(x) = 1$$

(ii) *There exists a sequence of functions (e_n) in $D(Q)$ (equivalently, (e_n) in $C_c(X)$) with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*

$$\lim_{n\to\infty} Q(e_n, v) = 0$$

for all $v \in D(Q) \cap \ell^1(X, m)$.

(ii.a) *There exists a sequence of functions (e_n) in $D(Q)$ (equivalently, (e_n) in $C_c(X)$) with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*

$$\lim_{n\to\infty} Q(e_n, (L + \alpha)^{-1} v) = 0$$

for one $v \in \ell^2(X,m) \cap \ell^1(X,m)$ *with* $v > 0$ *and some (all)* $\alpha > 0$*.*

(iii) *If* $v \in \mathcal{D} \cap \ell^1(X,m) \cap \ell^2(X,m)$ *satisfies* $\mathcal{L}v \in \ell^1(X,m)$*, then*

(*"Green's formula"*)
$$\sum_{x \in X} \mathcal{L}v(x)m(x) = 0.$$

(iii.a) *If* $v \in \mathcal{D} \cap \ell^1(X,m) \cap \ell^2(X,m)$ *satisfies* $\mathcal{L}v \in \ell^1(X,m) \cap \ell^2(X,m)$*, then*

$$\sum_{x \in X} \mathcal{L}v(x)m(x) = 0.$$

(iv) *If* $u \in \mathcal{F}$ *satisfies* $\sup u \in (0,\infty)$ *and* $\beta \in (0, \sup u)$*, then*

$$\sup_{x \in X_\beta} \mathcal{L}u(x) \geq 0,$$

where $X_\beta := \{x \in X \mid u(x) > \sup u - \beta\}$*.*

(*"Omori–Yau maximum principle"*)

(v) *For some (all)* $\alpha > 0$ *and every* $f \in \ell^\infty(X)$ *there exists a unique bounded solution* u *of the Poisson equation*

(*"Poisson equation"*)
$$(\mathcal{L} + \alpha)u = f.$$

(v.a) *For some (all)* $\alpha > 0$ *every positive* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L} + \alpha)u \leq 0$ *is trivial.*

(v.b) *For some (all)* $\alpha > 0$ *every* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L} + \alpha)u = 0$ *is trivial.*

(v.c) *For some (all)* $\alpha > 0$ *every positive* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L} + \alpha)u = 0$ *is trivial.*

(vi) *For every* $f \in \ell^\infty(X)$ *there exists a unique bounded solution* u *of the heat equation*

(*"Heat equation"*)
$$(\mathcal{L} + \partial_t)u = 0 \quad \text{with} \quad u_0 = f.$$

(vi.a) *Every bounded solution* u *of the heat equation* $(\mathcal{L} + \partial_t)u = 0$ *with* $u_0 = 0$ *is trivial.*

CHAPTER 11

Spherically Symmetric Graphs

In this chapter we apply the theory developed in the preceding chapters to a certain class of graphs defined by a symmetry property. This class contains both trees (satisfying a natural symmetry condition) and anti-trees. Hence, it covers a wealth of (counter)examples. We can characterize this class by an operator theoretic condition. Specifically, the Laplacian of graphs in these class commutes with a certain averaging operator. For graphs in this class we can give an explicit lower bound on the infimum of the spectrum and characterize both recurrence and stochastic completeness by finiteness of certain sums.

Let X be a countable set and m a measure on X with full support.

11.1. Weakly Spherically Symmetric Graphs

Let (b, c) be a connected graph over (X, m). Denote the combinatorial distance on X by d. Let $O \subseteq X$ be nonempty and define the combinatorial graph distance to O by

$$d(O, x) := \min_{o \in O} d(o, x)$$

for $x \in X$. For most of our results below we will assume that O is a finite set.

We denote the *distance sphere of radius* $r \in \mathbb{N}_0$ *about* O by

$$S_r(O) := \{x \in X \mid d(O, x) = r\}.$$

For convenience, we let $S_{-1}(O) := \varnothing$. Moreover, we denote the *distance ball of radius* $r \in \mathbb{N}_0$ *about* O by

$$B_r(O) := \bigcup_{n=0}^{r} S_n(O) = \{x \in X \mid d(O, x) \leq r\}.$$

In case $O = \{o\}$ for some $o \in X$ then we just write $S_r(o) := S_r(O)$ and $B_r(o) := B_r(O)$. If (b, c) is locally finite and O is finite, then the sets $S_r(O)$ and $B_r(O)$ are finite for all $r \in \mathbb{N}_0$. In any case, connectedness of the graph means that

$$X = \bigcup_{r \in \mathbb{N}_0} B_r(O)$$

holds. So, we can exhaust the graph by balls.

We call a function $f \in C(X)$ *spherically symmetric (with respect to* O*)* if there exists a function $g\colon \mathbb{N}_0 \longrightarrow \mathbb{R}$ such that $f(x) = g(r)$ for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$. With a slight abuse of notation, we then write

$$f(r) := f(x)$$

for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Although all of our notions involving symmetry depend on O, we will mostly omit this dependence in our notation and statements. Clearly, any spherically symmetric functions is bounded on each distance ball. Hence, it can easily be seen to belong to the domain $\mathcal{F}$ of the formal Laplacian.

We next define the functions for which we will assume spherical symmetry. Let $O \subseteq X$ nonempty be given. By connectedness of the graph, we can define $k_\pm \colon X \longrightarrow [0,\infty)$ via

$$k_\pm(x) := \frac{1}{m(x)} \sum_{y \in S_{r\pm1}(O)} b(x,y)$$

for $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Here, the sum over the empty set arising in the definition of $k_-(x)$ for $x \in S_0(O) = O$ is defined to be zero.

We call $k_\pm$ the *outer and inner degrees (with respect to O)*, which are functions. Furthermore, we define the *potential* $q \colon X \longrightarrow [0,\infty)$ by

$$q(x) := \frac{c(x)}{m(x)}$$

for $x \in X$.

With these preparations we can now define the class of graphs which will be studied in this chapter.

Definition 11.1 (Weakly spherically symmetric graphs). We call a connected graph (b,c) over (X,m) *weakly spherically symmetric* with respect to a nonempty set $O \subseteq X$ if the outer and inner degrees $k_\pm$ and the potential q are spherically symmetric with respect to O.

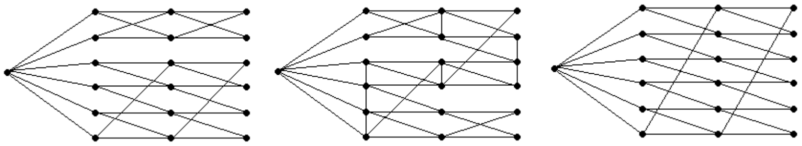


Figure 1. Examples of weakly spherically symmetric graphs.

We call a sequence of vertices $(x_0,\ldots,x_n)$ in X with $n \in \mathbb{N}$ a *cycle* if $x_j \sim x_{j+1}$ for all $j = 0,1,\ldots,n-1$, $x_j \neq x_k$ for $0 \leq j,k \leq n$ with $j \neq k$, and $x_0 = x_n$. Put differently, a cycle is a path whose start and end vertex agrees but all other vertices are pairwise different. A connected graph with no cycles is called a *tree*.

Example 11.2 (Spherically symmetric trees). Let (b,c) be a connected graph over (X,m) with standard weights and counting measure, i.e., $b \colon X \times X \longrightarrow \{0,1\}$, $c = 0$ and $m = 1$. Let $O = \{o\}$ for $o \in X$. We say that b is a *spherically symmetric tree with branching numbers k* if there exists a $k \colon \mathbb{N}_0 \longrightarrow \mathbb{N}$ such that

$$k_+(x) = k(r)$$

for every vertex $x \in S_r(o)$ and $r \in \mathbb{N}_0$ and

$$k_-(x) = 1$$

for all $x \in S_r(o)$ for $r \in \mathbb{N}$ and $b|_{S_r(o)\times S_r(o)} = 0$ for all $r \in \mathbb{N}_0$.

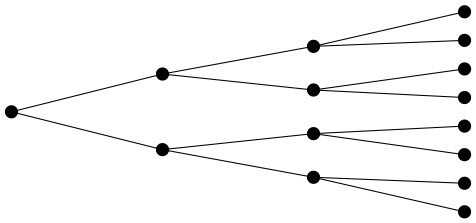


FIGURE 2. A tree with $k(r) = 2$.

We note that these graphs are indeed trees. Furthermore, we note that removing a single edge between spheres will disconnect any tree. This contrasts with anti-trees, which we now define.

EXAMPLE 11.3 (Anti-trees). Let (b, c) be a connected graph over (X, m) with standard weights and counting measure, i.e., $b\colon X \times X \longrightarrow \{0, 1\}$, $c = 0$ and $m = 1$. Let $O = \{o\}$ for $o \in X$. Let $s\colon \mathbb{N}_0 \longrightarrow \mathbb{N}$ be given by $s(r) := \#S_r(o)$ for all $r \in \mathbb{N}_0$. We then say that b is an *anti-tree with sphere size* s if

$$k_\pm(x) = s(r) \qquad \text{for all } x \in S_{r\mp1}(o) \text{ and } r \in \mathbb{N}_0.$$

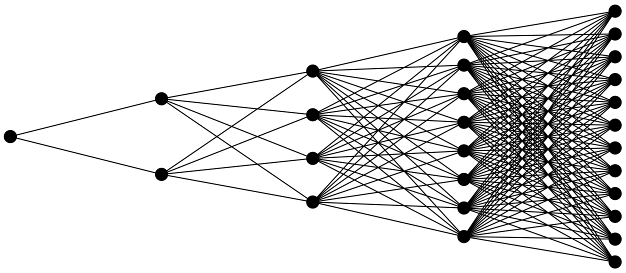


FIGURE 3. An anti-tree with $s(r) = 2^r$.

We note that the definition of an anti-tree implies

$$b|_{S_r(o)\times S_{r\pm1}(o)} = 1$$

for all $r \in \mathbb{N}_0$. In other words, every vertex in $S_r(o)$ is connected to all vertices in $S_{r+1}(o)$ for all $r \in \mathbb{N}_0$ (and every vertex in $S_{r+1}(o)$ is connected to every vertex in $S_r(o)$ for all $r \in \mathbb{N}_0$). Hence, to disconnect such a graph, we must remove all vertices between spheres. Furthermore, we note that we impose no restrictions on $b|_{S_r(o)\times S_r(o)}$.

For a given sequence $s\colon \mathbb{N}_0 \longrightarrow \mathbb{N}$ of natural numbers with $s(0) = 1$, we can easily construct an anti-tree with sphere size s. Indeed, we can just partition the (infinite) set X into disjoint subsets U_r with $\#U_r = s(r)$ and let $b|_{U_r\times U_{r\pm1}} = 1$ for $r \in \mathbb{N}_0$ with $b = 0$ otherwise, $m = 1$ and $c = 0$.

We will regularly revisit spherically symmetric trees and anti-trees to illustrate our results in this chapter.

The following formulas will play a crucial role in the proofs of several results below. Hence we gather them together into one statement.

Lemma 11.4. *Let (b, c) be a weakly spherically symmetric graph over (X, m) with respect to a nonempty $O \subseteq X$. Then,*

$$k_+(r)m(S_r(O)) = k_-(r+1)m(S_{r+1}(O))$$

for all $r \in \mathbb{N}_0$, where the value ∞ is allowed on both sides. In particular, $m(S_r(O)) < \infty$ holds for all $r \in \mathbb{N}_0$ if and only if $m(O) < \infty$.

If f is a spherically symmetric function, then $f \in \mathcal{F}$ and $\mathcal{L}f$ is spherically symmetric with

$$\mathcal{L}f(x) = k_+(r)\big(f(r) - f(r+1)\big) + k_-(r)\big(f(r) - f(r-1)\big) + q(r)f(r)$$

for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$.

Proof. The first formula follows by a simple computation using $k_+(r) = k_+(x)$ for all $x \in S_r(O)$, Fubini's theorem and the symmetry of b. Specifically, we have

$$\begin{aligned} k_+(r)m(S_r(O)) &= \sum_{x \in S_r(O)} k_+(x)m(x) = \sum_{x \in S_r(O)} \sum_{y \in S_{r+1}(O)} b(x, y) \\ &= \sum_{y \in S_{r+1}(O)} \sum_{x \in S_r(O)} b(y, x) = \sum_{y \in S_{r+1}(O)} k_-(y)m(y) \\ &= k_-(r+1)m(S_{r+1}(O)). \end{aligned}$$

The "in particular" statement now follows by the formula and induction.

A spherically symmetric function f is clearly a bounded function on the neighbors of any vertex and, therefore, $f \in \mathcal{F}$. The second formula follows immediately from the definition of $\mathcal{L}$ and the assumption that f is spherically symmetric. □

11.2. Symmetry of the Kernel and Green Function

In this section we establish the symmetry of the heat kernel and the Green function on a weakly spherically symmetric graph. Here and subsequently we will often make the additional assumption that the graph is locally finite. This will simplify our dealing with the averaging operator below.

Let (b, c) be a graph over (X, m). We first recall the definition of the heat kernel. The semigroup $(e^{-tL})_{t \geq 0}$ of the Laplacian $L := L^{(D)}$ on $\ell^2(X, m)$ for $t \geq 0$ gives rise to the heat kernel $p\colon [0, \infty) \times X \times X \longrightarrow \mathbb{R}$ with

$$e^{-tL}f(x) = \sum_{y \in X} p_t(x, y)f(y)m(y)$$

for all $f \in \ell^2(X, m)$, $x \in X$ and $t \geq 0$.

By the fact that the semigroup is positivity preserving, established in Corollary 5.6, we have $p_t(x, y) \geq 0$ for all $x, y \in X$ and $t \geq 0$ as $p_t(x, y) = e^{-tL}1_y(x)/m(y)$.

For a nonempty finite set $O \subseteq X$, we now define

$$\begin{aligned} p_t(x,O) &:= \frac{1}{m(x)m(O)}\langle 1_x, e^{-tL}1_O\rangle \\ &= \frac{1}{m(O)}e^{-tL}1_O(x) \\ &= \frac{1}{m(O)}\sum_{o\in O} p_t(x,o)m(o) \end{aligned}$$

for $x \in X$ and $t \geq 0$. Thus, whenever $O = \{o\}$ for $o \in X$, we recover the heat kernel

$$p_t(x,o) = p_t(x,\{o\})$$

for $x \in X$ and $t \geq 0$.

The first theorem of this section states that the functions $p_t(\cdot,O)$ for $t \geq 0$ are spherically symmetric whenever the graph is weakly spherically symmetric with respect to the subset O.

Theorem 11.5 (Spherical symmetry of the heat kernel). *Let (b,c) be a locally finite graph over (X,m). If (b,c) is weakly spherically with respect to a nonempty finite set $O \subseteq X$, then $p_t(\cdot,O)$ is a spherically symmetric function for all $t \geq 0$.*

To start the proof we introduce the *averaging operator*

$$\mathcal{A}\colon C(X) \longrightarrow C(X)$$

on a locally finite graph with respect to a nonempty finite set $O \subseteq X$ by

$$\mathcal{A}f(x) := \frac{1}{m(S_r(O))}\sum_{y\in S_r(O)} f(y)m(y)$$

for $f \in C(X)$ $x \in S_r(O)$ and $r \in \mathbb{N}_0$. As the graph is locally finite, the operator $\mathcal{A}$ is indeed defined on the whole of $C(X)$. Moreover, we note that $\mathcal{A}f$ is spherically symmetric for any $f \in C(X)$ and a function $f \in C(X)$ is spherically symmetric if and only if $\mathcal{A}f = f$. In particular, $\mathcal{A}$ maps $C(X)$ onto the spherically symmetric functions. We will denote the restriction of $\mathcal{A}$ to $\ell^2(X,m)$ by A, i.e.,

$$A = \mathcal{A}|_{\ell^2(X,m)}.$$

We now collect some basic facts about A.

Lemma 11.6 (Basic facts about A). *Let (b,c) be a locally finite connected graph over (X,m) and let $O \subseteq X$ be a nonempty finite set. Let $\mathcal{A}$ be the averaging operator with respect to O and let A be the restriction of $\mathcal{A}$ to $\ell^2(X,m)$. Then, A is a bounded self-adjoint operator on $\ell^2(X,m)$. More specifically, A is an orthogonal projection of $\ell^2(X,m)$ onto the subspace of spherically symmetric functions in $\ell^2(X,m)$.*

Proof. Let $f \in \ell^2(X,m)$. We note that $X = \bigcup_{r\in\mathbb{N}_0} S_r(O)$ from the assumption that (b,c) is connected. To show that A maps $\ell^2(X,m)$ to

$\ell^2(X,m)$ and that A is bounded, we use the Cauchy–Schwarz inequality to obtain for $f \in \ell^2(X,m)$

$$\begin{aligned}\sum_{x\in X}(Af)(x)^2 m(x) &= \sum_{r=0}^{\infty}\sum_{x\in S_r(O)}\left(\frac{1}{m(S_r(O))}\sum_{y\in S_r(O)} f(y)m(y)\right)^2 m(x)\\ &= \sum_{r=0}^{\infty}\frac{1}{m(S_r(O))}\left(\sum_{y\in S_r(O)} f(y)m(y)\right)^2\\ &\leq \sum_{r=0}^{\infty}\frac{1}{m(S_r(O))}\left(\sum_{y\in S_r(O)} m(y)\right)\left(\sum_{y\in S_r(O)} f(y)^2 m(y)\right)\\ &= \|f\|^2.\end{aligned}$$

Hence, A is a bounded operator of norm 1 since $Af = f$ for any spherically symmetric function $f \in \ell^2(X,m)$.

Moreover, A is symmetric, and thus self-adjoint, by a direct calculation. As the range of A is included in the spherically symmetric functions and A is the identity on the spherically symmetric functions, we have $A^2 = A$ and the range of A is just the set of spherically symmetric functions in $\ell^2(X,m)$. Hence, A is a bounded self-adjoint operator with $A^2 = A$. Hence, it is the projection on its range. ☐

REMARK 11.7. The operator A can easily be seen to be a Markovian operator, i.e. to satisfy $0 \leq Af \leq 1$ for all $0 \leq f \leq 1$. Hence, the considerations on Markovian operators apply and would also give boundedness of A.

The next lemma shows that weak spherical symmetry is equivalent to $\mathcal{A}$ and A commuting with the Laplacians $\mathcal{L}$ and $L := L^{(D)}$ on suitable spaces. Since $O \subseteq X$ is assumed to be finite and (b,c) locally finite below, $S_r(O)$ is finite for all $r \in \mathbb{N}_0$ and A and L both map $C_c(X)$ to $C_c(X)$, i.e., $C_c(X)$ is invariant under both A and L.

LEMMA 11.8 (Characterization of weak spherical symmetry). *Let (b,c) be a connected locally finite graph over (X,m) and let $O \subseteq X$ be nonempty and finite. Then, the following statements are equivalent:*

(i) *The graph (b,c) is weakly spherically symmetric.*

(ii) *The operator $\mathcal{A}$ commutes with $\mathcal{L}$ on $C(X)$, i.e.,*

$$\mathcal{A}\mathcal{L} = \mathcal{L}\mathcal{A} \qquad \text{on } C(X).$$

(iii) *The operator A commutes with L on $C_c(X)$, i.e.,*

$$AL = LA \qquad \text{on } C_c(X).$$

PROOF. We denote $S_r := S_r(O)$ for $r \in \mathbb{N}_0$.

(i) $\Longrightarrow$ (ii): Obviously, multiplication by the spherically symmetric function q commutes with $\mathcal{A}$. Hence, we may assume that $q = 0$.

Since $\mathcal{A}f$ is spherically symmetric for $f \in C(X)$, by Lemma 11.4 we get, for $r \in \mathbb{N}_0$ and $x \in S_r$,

$$\mathcal{L}\mathcal{A}f(x) = k_+(r)(\mathcal{A}f(r) - \mathcal{A}f(r+1)) + k_-(r)(\mathcal{A}f(r) - \mathcal{A}f(r-1)).$$

On the other hand, a direct computation combined with the recursion of Lemma 11.4 and the formula for $\mathcal{L}$ given in Lemma 11.4 for symmetric functions gives for $r \in \mathbb{N}_0$ and $x \in S_r$,

$$\begin{aligned}
&\mathcal{A}\mathcal{L}f(x) = \frac{1}{m(S_r)} \sum_{y\in S_r} \mathcal{L}f(y)m(y) \\
&= \frac{1}{m(S_r)} \sum_{y\in S_r} \sum_{z\in S_{r-1}\cup S_{r+1}} b(y,z)(f(y)-f(z)) \\
&= \frac{1}{m(S_r)} \sum_{y\in S_r} f(y) \sum_{z\in S_{r-1}\cup S_{r+1}} b(y,z) - \frac{1}{m(S_r)} \sum_{z\in S_{r-1}\cup S_{r+1}} f(z) \sum_{y\in S_r} b(y,z) \\
&= (k_+(r)+k_-(r))\mathcal{A}f(r) \\
&\quad - \frac{k_+(r-1)}{m(S_r)} \sum_{z\in S_{r-1}} f(z)m(z) - \frac{k_-(r+1)}{m(S_r)} \sum_{z\in S_{r+1}} f(z)m(z) \\
&= (k_+(r)+k_-(r))\mathcal{A}f(r) \\
&\quad - \frac{k_-(r)}{m(S_{r-1})} \sum_{z\in S_{r-1}} f(z)m(z) - \frac{k_+(r)}{m(S_{r+1})} \sum_{z\in S_{r+1}} f(z)m(z) \\
&= k_+(r)(\mathcal{A}f(r) - \mathcal{A}f(r+1)) + k_-(r)(\mathcal{A}f(r) - \mathcal{A}f(r-1)) \\
&= \mathcal{L}\mathcal{A}f(x).
\end{aligned}$$

Thus we see $\mathcal{L}\mathcal{A}f = \mathcal{A}\mathcal{L}f$.

(ii) $\Longrightarrow$ (iii): This is clear as A and L are restrictions of $\mathcal{A}$ and $\mathcal{L}$.

(iii) $\Longrightarrow$ (i): Let $r \in \mathbb{N}_0$. Obviously, $A1_{S_r} = 1_{S_r}$ as 1_{S_r} is a spherically symmetric function. Furthermore, for $x \in S_{r\pm1}$, we have

$$L1_{S_r}(x) = -k_{\mp}(x)$$

by direct calculations. Thus, for $x \in S_{r\pm1}$, we have

$$\begin{aligned}
-k_{\mp}(x) = LA1_{S_r}(x) = AL1_{S_r}(x) &= -\frac{1}{m(S_{r\pm1})} \sum_{y\in S_{r\pm1}} k_{\mp}(y)m(y) \\
&= -Ak_{\mp}(r\pm1).
\end{aligned}$$

Therefore, $k_{\pm}$ are spherically symmetric.

Similarly, for $x \in S_r$, we calculate $L1_{S_r}(x) = k_+(x) + k_-(x) + q(x)$. Therefore, for $x \in S_r$, we have

$$\begin{aligned}
k_+(x) + k_-(x) + q(x) = L1_{S_r}(x) &= LA1_{S_r}(x) = AL1_{S_r}(x) \\
&= \frac{1}{m(S_r)} \sum_{y\in S_r} (k_+(y)+k_-(y)+q(y))m(y) \\
&= A(k_+ + k_- + q)(r).
\end{aligned}$$

As we have already shown that $k_{\pm}$ are spherically symmetric, this shows that the potential q and, thus, the graph is weakly spherically symmetric. □

LEMMA 11.9. *Let (b,c) be a connected locally finite graph over (X,m) and let $O \subseteq X$ be nonempty and finite. Then, the following statements are equivalent:*

(i) *$AL = LA$ on $C_c(X)$.*

(ii) *A maps $D(L)$ into $D(L)$ and $AL = LA$ on $D(L)$.*
(iii) *$Ae^{-tL} = e^{-tL}A$ on $\ell^2(X, m)$ for all $t \geq 0$.*

Proof. The equivalence between (ii) and (iii) is a general result on self-adjoint operators commuting with orthogonal projections. We only sketch the proof. Indeed, (ii) just says that L operates — so to speak —- independently on the range of A and on its orthogonal complement. This then easily gives that the same applies to functions of L and this gives (iii). Conversely, from (iii) and the fact that L arises from the semigroup by differentiation we can infer that (ii) holds.

Furthermore, (ii) implies (i) as $C_c(X) \subseteq D(L)$ for locally finite graphs. Finally, from (i) and local finiteness of the graph and finiteness of O we infer that both $\mathcal{L}$ and $\mathcal{A}$ are defined on the whole of $C(X)$. The symmetry properties of A and L combined with (i) then give $\mathcal{A}\mathcal{L} = \mathcal{L}\mathcal{A}$. From this we infer (ii) by restriction. □

Proof of Theorem 11.5. Let $t \geq 0$. From Lemmas 11.8 and 11.9 we see that if (b, c) is a locally finite and weakly spherically symmetric graph with respect to a nonempty finite set O, then A and e^{-tL} commute. Hence, as $p_t(x, O) = e^{-tL}1_O(x)/m(O)$ for $x \in X$, we get

$$Ap_t(x, O) = \frac{1}{m(O)} Ae^{-tL}1_O(x) = \frac{1}{m(O)} e^{-tL}A1_O(x) = p_t(x, O)$$

for $x \in X$. Hence, $p_t(\cdot, O)$ is spherically symmetric. □

We now extend the definition of the Green function from Section 9.1 by letting

$$G(x, O) := \frac{1}{m(O)} \sum_{o \in O} G(x, o)$$

whenever $O \subseteq X$ is nonempty and finite, and $x \in X$.

Corollary 11.10 (Spherical symmetry of the Green function). *Let b be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$. Assume that b is transient. Then, the Green function $G(\cdot, O)$ is spherically symmetric.*

Proof. We calculate by the definitions above and Fubini's theorem

$$\begin{aligned} G(x, O) &= \frac{1}{m(O)} \sum_{o \in O} G(x, o) = \frac{1}{m(O)} \sum_{o \in O} \int_0^\infty p_t(x, o) m(o) \mathrm{d}t \\ &= \int_0^\infty \frac{1}{m(O)} \sum_{o \in O} p_t(x, o) m(o) \mathrm{d}t = \int_0^\infty p_t(x, O) \mathrm{d}t \end{aligned}$$

for $x \in X$. By Theorem 11.5, $p_t(\cdot, O)$ is spherically symmetric for $t \geq 0$. This completes the proof. □

11.3. The Spectral Gap

Having introduced and studied spherically symmetric graphs, we begin in this section to apply the theory developed earlier. Here, we deal with a lower bound on the infimum of the spectrum.

Let (b, c) be a graph over (X, m). We recall the notion of the *boundary* ∂W of a set $W \subseteq X$ as

$$\partial W := (W \times (X \setminus W)) \cup ((X \setminus W) \times W).$$

We remark that there are many notions of boundaries when considering subsets of a graph. Here, we take a notion that is symmetric.

We will be particularly interested in the boundary of balls around a set and the total edge weight of this boundary. We therefore introduce the *area of the boundary of* W for a set $W \subseteq X$ as

$$b(\partial W) := \sum_{(x,y)\in\partial W} b(x, y)$$

for $r \in \mathbb{N}_0$. We will be interested in the situation where W has the form $B_r(O)$ for a set $O \subseteq X$ and $r \in \mathbb{N}_0$. We note that in this case

$$b(\partial B_r(O)) = 2 \sum_{x\in S_r(O)} k_+(x)m(x)$$

by a direct computation. So if the graph is weakly spherically symmetric, we obtain

$$b(\partial B_r(O)) = 2k_+(r)m(S_r(O))$$

for $r \in \mathbb{N}_0$.

With these notions, we will prove the following summability criterion for positivity of the bottom of the spectrum for weakly spherically symmetric graphs.

Theorem 11.11 (Area-volume ratio and spectrum). *Let b be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$. If*

$$a := \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty,$$

then

$$\lambda_0(L) \geq \frac{1}{2a}.$$

The proof uses the Agmon–Allegretto–Piepenbrink theorem in Theorem 8.10. In order to apply it we need the following lemma on solutions. The lemma basically gives a recursion for solutions on spherically symmetric graphs. It will be used in subsequent sections as well.

Lemma 11.12 (Recursion formula for spherically symmetric solutions). *Let (b, c) be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$ and let $f \in C(X)$ be spherically symmetric. Then, a spherically symmetric function $u \in C(X)$ satisfies $(\mathcal{L} + f)u = 0$ if and only if*

$$u(r + 1) - u(r) = \frac{2}{b(\partial B_r(O))} \sum_{n=0}^{r} (q(n) + f(n))m(S_n(O))u(n)$$

for all $r \in \mathbb{N}_0$. In particular, u is uniquely determined by the choice of $u(0)$. Furthermore, if $u(0) > 0$ and $f > 0$, then $u(r + 1) > u(r)$ for all $r \in \mathbb{N}_0$.

Proof. We will prove the recursion formula by induction. The uniqueness and monotonicity statements are then obvious from the recursion formula.

We will omit O from our notation below, writing $B_r := B_r(O)$ and $S_r := S_r(O)$. We recall that $b(\partial B_r) = 2k_+(r)m(S_r)$ for $r \in \mathbb{N}_0$.

For $r = 0$, from $(\mathcal{L} + f)u(0) = 0$ we obtain

$$\begin{aligned} 0 &= k_+(0)(u(0) - u(1)) + (q(0) + f(0))u(0) \\ &= \frac{b(\partial B_0)}{2m(S_0)}(u(0) - u(1)) + (q(0) + f(0))u(0), \end{aligned}$$

which yields the desired formula after rearranging the terms.

Now, we assume that the recursion formula holds for $r - 1$, where $r \in \mathbb{N}$. From $(\mathcal{L} + f)u(r) = 0$ we obtain

$$k_+(r)(u(r) - u(r+1)) + k_-(r)(u(r) - u(r-1)) + (q(r) + f(r))u(r) = 0.$$

Therefore, by the induction hypothesis, $b(\partial B_r) = 2k_+(r)m(S_r)$ and the formula $k_+(r-1)m(S_{r-1}) = k_-(r)m(S_r)$ proven in Lemma 11.4, we obtain

$$\begin{aligned} u(r+1) - u(r) &= \frac{k_-(r)}{k_+(r)}(u(r) - u(r-1)) + \frac{1}{k_+(r)}(q(r) + f(r))u(r) \\ &= \frac{k_-(r)}{k_+(r)}\left(\frac{2}{b(\partial B_{r-1})}\sum_{n=0}^{r-1}(q(n) + f(n))m(S_n)u(n)\right) \\ &\quad + \frac{2}{b(\partial B_r)}(q(r) + f(r))m(S_r)u(r) \\ &= \frac{k_-(r)}{k_+(r)}\left(\frac{1}{k_-(r)m(S_r)}\sum_{n=0}^{r-1}(q(n) + f(n))m(S_n)u(n)\right) \\ &\quad + \frac{2}{b(\partial B_r)}(q(r) + f(r))m(S_r)u(r) \\ &= \frac{2}{b(\partial B_r)}\sum_{n=0}^{r}(q(n) + f(n))m(S_n)u(n). \end{aligned}$$

This proves the recursion formula and thus completes the proof. □

We now use the recursion formula above to show that under the summability assumption found in Theorem 11.11, there exists a strictly positive α-harmonic function for $\alpha < 0$. This will prove the spectral gap via the Agmon–Allegretto–Piepenbrink theorem.

Lemma 11.13. *Let b be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$. If*

$$a := \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty,$$

then there exists a strictly positive monotonically decreasing spherically symmetric function u which satisfies $u(0) = 1$ and

$$\left(\mathcal{L} - \frac{1}{2a}\right)u = 0.$$

PROOF. We will define a spherically symmetric function u with the required properties. We start by letting $u(0) := 1$. Then, by Lemma 11.12, u will satisfy $\left(\mathcal{L} - \frac{1}{2a}\right) u = 0$ if and only if u satisfies the recursion formula

$$u(r+1) - u(r) = -\frac{1}{a \cdot b(\partial B_r(O))} \sum_{n=0}^{r} m(S_n(O))u(n)$$

for $r \in \mathbb{N}_0$, as we assume $c = 0$ and, thus, $q = 0$.

We will show that u is strictly monotonically decreasing and remains positive by using strong induction. More specifically, we will show that

$$0 < 1 - \frac{1}{a} \sum_{n=0}^{r} \frac{m(B_n(O))}{b(\partial B_n(O))} \leq u(r+1) < u(r)$$

for all $r \in \mathbb{N}_0$. The first inequality above is clear from the definition of a. For $r = 0$, the remaining inequalities follow directly from the recursion formula and $u(0) = 1$ as

$$u(1) - u(0) = -\frac{1}{a \cdot b(\partial B_0(O))} m(S_0(O)) = -\frac{m(O)}{a \cdot b(\partial O)} < 0$$

gives

$$1 - \frac{m(O)}{a \cdot b(\partial O)} = u(1) < 1 = u(0).$$

Now, assume that the inequalities hold up to $r - 1$, that is,

$$0 < 1 - \frac{1}{a} \sum_{n=0}^{k} \frac{m(B_n(O))}{b(\partial B_n(O))} \leq u(k+1) < u(k)$$

for $k = 0, 1, \ldots, r-1$. Therefore, $u(k) > 0$ for $k = 0, 1, \ldots, r$ and the recursion formula gives $u(r+1) - u(r) < 0$. Moreover, as u is then strictly decreasing up to r, we get $u(n) < u(0) = 1$ for all $n = 1, 2, \ldots, r$. Hence, from the recursion formula and the inductive hypotheses we obtain

$$\begin{aligned}
u(r+1) &= u(r) - \frac{1}{a \cdot b(\partial B_r(O))} \sum_{n=0}^{r} m(S_n(O))u(n) \\
&> u(r) - \frac{m(B_r(O))}{a \cdot b(\partial B_r(O))} \\
&\geq 1 - \frac{1}{a} \sum_{n=0}^{r-1} \frac{m(B_n(O))}{b(\partial B_n(O))} - \frac{m(B_r(O))}{a \cdot b(\partial B_r(O))} \\
&= 1 - \frac{1}{a} \sum_{n=0}^{r} \frac{m(B_n(O))}{b(\partial B_n(O))}.
\end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 11.11. By Lemma 11.13 for

$$a := \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty$$

there exists a strictly positive function u which satisfies

$$\left(\mathcal{L} - \frac{1}{2a}\right) u = 0.$$

Thus, $\lambda_0(L) \geq 1/2a$ follows from the Agmon–Allegretto–Piepenbrink theorem in Theorem 8.10. □

We next illustrate Theorem 11.11 for spherically symmetric trees and anti-trees, i.e., for Examples 11.2 and 11.3. For trees, we get the following criterion for the spectral gap and discreteness of the spectrum.

Example 11.14 (Spherically symmetric trees and spectrum). Let b be a spherically symmetric tree with branching number k. If

$$a := \sum_{r=0}^{\infty} \frac{1 + \sum_{n=1}^{r} \prod_{j=0}^{n-1} k(j)}{2 \prod_{j=0}^{r} k(j)} < \infty,$$

then $\lambda_0(L) \geq 1/2a$ (Exercise).

For anti-trees we obtain the following criterion. We note, in particular, that this can be used to construct examples of graphs with strictly positive bottom of the spectrum and whose distance balls grow polynomially.

Example 11.15 (Anti-trees and spectrum). Let b be an anti-tree with sphere size s. If

$$a := \sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{2s(r)s(r+1)} < \infty,$$

then $\lambda_0(L) \geq 1/2a$ (Exercise).

Remark 11.16 (Discrete spectrum for the graphs in question). The argument to bound the infimum of the spectrum above can be iterated after truncating the original graph by chopping of larger and larger balls around o. This shows that the infima of the spectrum of the truncated graphs goes to ∞. Combined with some standard spectral theory this then yields that the graphs in questions have pure point spectrum with eigenvalues that are of finite multiplicity and tend to ∞.

11.4. Recurrence

We continue our investigation of spherically symmetric graphs by characterizing recurrence.

Theorem 11.17 (Area ratio and recurrence). *Let b be a locally finite weakly spherically symmetric graph over X with respect to a nonempty finite set $O \subseteq X$. Then, b is recurrent if and only if*

$$\sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} = \infty$$

holds.

PROOF. As noted in Section 9.2, the measure plays no role in recurrence. Hence, we let $m = 1$ be the counting measure on X. We also let $a > 0$ be a constant. By Lemma 11.12, the unique spherically symmetric function u with $u(0) = 1$ and

$$\left(\mathcal{L} - \frac{1}{2a \cdot m(O)} 1_O\right) u = 0$$

satisfies

$$\begin{aligned} u(r+1) - u(r) &= \frac{2}{b(\partial B_r(O))} \sum_{n=0}^{r} \frac{-1}{2a \cdot m(O)} 1_O(n) m(S_n(O)) u(n) \\ &= \frac{-1}{a \cdot b(\partial B_r(O))} \end{aligned}$$

for all $r \in \mathbb{N}_0$. Iterating this and using $u(0) = 1$, we obtain

$$u(r+1) = 1 - \frac{1}{a} \sum_{k=0}^{r} \frac{1}{b(\partial B_k(O))}$$

for $r \in \mathbb{N}_0$. We will use this equality with different constants a for both implications in the proof.

First, if we assume that

$$a := \sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} < \infty,$$

then u is a non-constant strictly positive superharmonic function. Thus, b is transient by Theorem 9.7.

Conversely, assume that b is transient. Then, by Theorem 9.7, the Green function is finite, i.e., $G(x,y) < \infty$ for all $x, y \in X$. Furthermore, by Theorem 9.2, $G(\cdot, o)$ for $o \in O$ satisfies

$$\mathcal{L} G(\cdot, o) = 1_o.$$

Furthermore, $G(\cdot, o)$ is strictly positive as the graph is connected. By Corollary 11.10, the function g_O given by

$$g_O(x) := \frac{1}{m(O)} \sum_{o \in O} G(x, o)$$

for $x \in X$ is spherically symmetric and from the above satisfies

$$\mathcal{L} g_O = \frac{1}{m(O)} \sum_{o \in O} \mathcal{L} G(\cdot, o) = \frac{1}{m(O)} 1_O = \frac{1}{g_O(o') m(O)} 1_O g_O,$$

where the last equality holds for all $o' \in O$ since g_O is spherically symmetric. Hence, if we let $u := g_O / g_O(o')$ for $o' \in O$, then u is strictly positive spherically symmetric and satisfies $u(0) = 1$ with

$$\left(\mathcal{L} - \frac{1}{2a \cdot m(O)} 1_O\right) u = 0$$

for $a := g_O(o')/2$. Now, by the consideration in the beginning of the proof, u must also satisfy

$$u(r+1) = 1 - \frac{1}{a}\sum_{k=0}^{r} \frac{1}{b(\partial B_k(O))}$$

for all $r \in \mathbb{N}_0$. As u is positive we conclude

$$\sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} < \infty.$$

This completes the proof. $\square$

Remark 11.18. Another viewpoint on Theorem 11.17 is that the Green function for weakly spherically symmetric graphs can be calculated explicitly as

$$G(x,o) = m(o)\sum_{n=r}^{\infty} \frac{1}{\frac{1}{2}b(\partial B_n(O))}$$

for $x \in S_r(O)$, $r \in \mathbb{N}_0$ and $o \in O$. In particular,

$$G(x,O) = \sum_{n=r}^{\infty} \frac{1}{\frac{1}{2}b(\partial B_n(O))}$$

for all $x \in S_r(O)$, $r \in \mathbb{N}_0$, from which Theorem 11.17 follows (Exercise).

We now illustrate the theorem above for our two main classes of examples, namely spherically symmetric trees and anti-trees from Examples 11.2 and 11.3. For trees the characterization of recurrence reads as follows.

Example 11.19 (Spherically symmetric trees and recurrence). Let b be a spherically symmetric tree with branching number k. Then b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{\prod_{n=0}^{r} k(n)} = \infty$$

(Exercise).

For anti-trees, rephrasing everything in terms of the sphere growth gives the following characterization.

Example 11.20 (Anti-trees and recurrence). Let b be an anti-tree with sphere size s. Then, b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{s(r)s(r+1)} = \infty$$

(Exercise).

11.5. Stochastic Completeness

We end our investigation of spherically symmetric graphs by characterizing stochastic completeness.

Theorem 11.21 (Volume-area ratio and stochastic completeness). *Let b be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$. Then, b is stochastically complete if and only if*

$$\sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} = \infty$$

holds.

In order to prove the theorem above, we first investigate the boundedness of α-harmonic functions for $\alpha > 0$ by using the recursion formula for solutions found in Lemma 11.12.

Lemma 11.22. *Let (b, c) be a locally finite weakly spherically symmetric graph over (X, m) with respect to a nonempty finite set $O \subseteq X$. Then, the following statements are equivalent:*

(i) *There exists $\alpha > 0$ and a non-trivial spherically symmetric α-harmonic function that is bounded.*
(ii) *For all $\alpha > 0$, all spherically symmetric α-harmonic functions are bounded.*
(iii) *We have*

$$\sum_{r=0}^{\infty} \frac{c(B_r(O)) + m(B_r(O))}{b(\partial B_r(O))} < \infty.$$

Proof. First of all, Lemma 11.12 gives that a spherically symmetric α-harmonic function u is uniquely determined by its value at 0. Thus, for a given $\alpha > 0$, all spherically symmetric α-harmonic functions are bounded if there exists a non-trivial α-harmonic function that is bounded.

Furthermore, for $\alpha > 0$, we let

$$a(\alpha) := \sum_{r=0}^{\infty} \frac{c(B_r(O)) + \alpha m(B_r(O))}{b(\partial B_r(O))}.$$

Obviously, the finiteness of $a(\alpha)$ for some $\alpha > 0$ is equivalent to the finiteness of $a(\alpha)$ for all $\alpha > 0$.

Thus, it remains to show that $a(\alpha) < \infty$ is equivalent to the existence of a non-trivial bounded spherically symmetric α-harmonic function.

By Lemma 11.12, any spherically symmetric u with $(\mathcal{L}+\alpha)u = 0$ satisfies the recursion formula

$$u(r+1) - u(r) = \frac{2}{b(\partial B_r(O))} \sum_{n=0}^{r} \big(c(S_n(O)) + \alpha m(S_n(O))\big) u(n)$$

for all $r \in \mathbb{N}_0$, where we used $q(n)m(S_n(O)) = c(S_n(O))$ for $n \in \mathbb{N}_0$, which follows from the spherical symmetry of q. Now, if $u(0) = 0$, then u is trivial, hence, we may assume that $u(0) \neq 0$ as we are interested in non-trivial solutions.

On the other hand if we assume that $u(0) > 0$, then the recursion formula implies that u is monotonically increasing. In particular, $u(r) \geq u(0) > 0$

for all $r \in \mathbb{N}_0$. Thus, summation in the recursion formula over the spheres gives both the lower bound

$$u(r+1) - u(r) \geq \frac{2\,(c(B_r(O)) + \alpha m(B_r(O)))}{b(\partial B_r(O))} u(0)$$

for $r \in \mathbb{N}_0$ and the upper bound

$$u(r+1) \leq \left(1 + \frac{2(c(B_r(O)) + \alpha m(B_r(O)))}{b(\partial B_r(O))}\right) u(r)$$

for all $r \in \mathbb{N}_0$. The desired equivalence follows easily from these bounds.

Indeed, if $a(\alpha) = \infty$ holds, we find from the lower bound

$$\begin{aligned} u(r) = \sum_{n=0}^{r-1} (u(n+1) - u(n)) &\geq \sum_{n=0}^{r-1} \frac{2\,(c(B_n(O)) + \alpha m(B_n(O)))}{b(\partial B_n(O))} u(0) \\ &\to \infty \end{aligned}$$

as $r \to \infty$. So, u is unbounded in this case. An analogous argument shows that $u(r) \to -\infty$ as $r \to \infty$ if $u(0) < 0$ and $a(\alpha) = \infty$.

On the other hand, if $a(\alpha) < \infty$ holds we find from iteration of the upper bound

$$\begin{aligned} u(r+1) &\leq \left(1 + \frac{2(c(B_r(O)) + \alpha m(B_r(O)))}{b(\partial B_r(O))}\right) u(r) \\ &\leq \prod_{n=0}^{r} \left(1 + \frac{2(c(B_n(O)) + \alpha m(B_n(O)))}{b(\partial B_n(O))}\right). \end{aligned}$$

Now, by $a(\alpha) < \infty$ also

$$\prod_{n=0}^{\infty} \left(1 + \frac{c(B_n(O)) + \alpha m(B_n(O))}{b(\partial B_n(O))}\right) < \infty$$

is valid and the estimate above shows that u is bounded. A similar argument shows that u is strictly negative and bounded below if $u(0) < 0$ and $a(\alpha) < \infty$. This shows the equivalence between (i) and (iii).

Furthermore, since finiteness of $a(\alpha)$ for one $\alpha > 0$ is equivalent to finiteness of $a(\alpha)$ for all $\alpha > 0$, we get that (i) and (ii) are equivalent. This completes the proof. ☐

PROOF OF THEOREM 11.21. If

$$\sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty,$$

then there exists a non-trivial bounded α-harmonic function for $\alpha > 0$ by Lemma 11.22. Thus, the graph is stochastically incomplete by Theorem 10.22.

On the other hand, if the graph is stochastically incomplete, then there exists a positive non-trivial bounded function v which satisfies $(\mathcal{L} + \alpha)v = 0$ for $\alpha > 0$ by Theorem 10.22. We recall that $\mathcal{A}$ denotes the averaging operator given by

$$\mathcal{A}f(x) := \frac{1}{m(S_r(O))} \sum_{y \in S_r(O)} f(y) m(y)$$

for $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Applying this to v gives that $u := \mathcal{A}v$ is a spherically symmetric function with

$$(\mathcal{L} + \alpha)u = \mathcal{L}\mathcal{A}v + \alpha\mathcal{A}v = \mathcal{A}(\mathcal{L} + \alpha)v = 0$$

since $\mathcal{L}$ and $\mathcal{A}$ commute by Lemma 11.8. Therefore, there exists a non-trivial bounded spherically symmetric function u which satisfies $(\mathcal{L} + \alpha)u = 0$ for $\alpha > 0$ and, thus,

$$\sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty$$

by Lemma 11.22. This completes the proof. □

We again illustrate the characterization of stochastic completeness at infinity for our two main classes of examples, namely, spherically symmetric trees and anti-trees from Example 11.2 and 11.3.

Example 11.23 (Spherically symmetric trees and stochastic completeness). Let b be a spherically symmetric tree with branching number k. Then b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{1 + \sum_{n=1}^{r} \prod_{j=0}^{n-1} k(j)}{\prod_{j=0}^{r} k(j)} = \infty$$

(Exercise).

For anti-trees we obtain the following characterization. We note, in particular, that we can use this to construct examples of stochastically incomplete graphs whose balls grow polynomially.

Example 11.24 (Anti-trees and stochastic completeness). Let b be an anti-tree with sphere size s. Then, b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{s(r)s(r+1)} = \infty$$

(Exercise).