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Ergodic Theorems

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LECTURE 1

Ergodic theorems and their background

Around 1880, the Austrian physicist and philosopher L. Boltzmann (1844–1906) formulated his famous *ergodic hypothesis*[1], or *Ergodenhypothese**.

Roughly speaking, he hypothesized that the phase space motion of the ideal gas goes through each physically feasible state. This postulate, called by Maxwell and his followers the *principle of continuity of paths*, was used by Boltzmann to deduce that after restricting to the physically feasible states and independently of the state we start, the phase space motion will visit in the average each fixed region of the phase space proportionally often (time mean) to the volume of the region (phase space mean). The ergodic hypothesis was shortened to the slogan "time mean equals space mean". Even though the original "ergodic hypothesis" was criticized, this form was indispensable for calculations in statistical mechanics, and many were involved in the quest for a mathematical justification.

Among the critics, however, were Lord Kelvin[2] and H. Poincaré[3] who formulated serious doubts concerning the validity of the ergodic hypothesis. Then around 1913 M. Plancherel[4] and A. Rosenthal[5] proved independently that the ergodic hypothesis fails in its original form. Already in 1894 Poincaré indicated a possible rectification of this postulate by requiring that any state comes arbitrarily close, even if not precisely, to any other state. (We will mathematically return to this in one of the lectures.) This became P. and T. Ehrenfest's[6] *quasi ergodic hypothesis*. It is believed, however, that this property was already suggested by Boltzmann.

The ingenious ideas of Boltzmann had not become forgotten and had enormous impact outside of physics as well. The first mathematical confirmation of the hypothesis "time mean equals space mean" is due to J. von Neumann[7], known today

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[1] L. Boltzmann, *Ueber die Eigenschaften monocyklischer und anderer damit verwandter Systeme*, J. Reine Angew. Math. 98 (1885), 68–94.

*Boltzmann used the word "Ergode" without explanation of its meaning. P. and T. Ehrenfest (in the footnote page 30 of [6]) provide the following etymology: ergon $\sim$ "work" and odos $\sim$ "path", "way". This and other interpretations are discussed, e.g., by Mathieu in [15] and Gallavotti in [14].

[2] W. T. Kelvin, *On some test case for the Maxwell–Boltzmann doctrine regarding distribution of energy*, Roy. Soc. Proc. L (1891), 79–88, Nature, Vol. XLIV, 355–358.

[3] H. Poincaré, *Sur une objection à la théorie cinétique des gaz*, C. R. Acad. Sci., Paris 116 (1893), 1017–1021.

[4] M. Plancherel, *Beweis der Unmöglichkeit ergodischer dynamischer systeme*, Ann. Phys. 42 (1913), 1061–1063.

[5] A. Rosenthal, *Beweis der Unmöglichkeit ergodischer Gassysteme*, Ann. Phys. 42 (1913), 796–806.

[6] P. Ehrenfest and T. Ehrenfest, *Begriffliche Grundlagen der Statistischen Auffassung in der Mechanik*, Encykl. d. Math. Wissensch. IV 2 II, Heft 6, 1912. Page 33.

[7] J. von Neumann, *Proof of the quasi-ergodic hypothesis*, Proc. Natl. Acad. Sci. USA 18 (1932), 70–82.

as the *mean ergodic theorem*, and was quickly followed by the "complete" proof of the quasi ergodic hypothesis by G.D. Birkhoff[8] (his result, the *pointwise ergodic theorem*, was actually published before von Neumann's). The paper [13] by C.C. Moore provides some more historical details on these fascinating developments.

Based on these landmark results ergodic theory was established as a mathematical discipline. Today it is a powerful and beautiful area with connections to and interactions with many (partly unexpected) areas of mathematics such as stochastics, functional analysis, number theory, combinatorics, group theory, topology, algebra or graph theory. With this lecture series we give a taste of (classical and modern) ergodic theorems and their connections to other areas.

We shall not pursue the physical interpretation of mathematical truths in these lectures. We leave this to more able hands and do mathematics *l'art pour l'art* (meaning that the applications of mathematics will be in mathematics).

1. Classical Ergodic Theorems

Let $(X, \mathcal{X}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable and **measure-preserving** transformation, the latter meaning that

$$\mu(T^{-1}(B)) = \mu(B) \quad \text{for each } B \in \mathcal{X}.$$

Then, of course, for each positive (meaning ≥ 0) integer n also the transformation $T^n : X \to X$ has both of these properties.

To connect this to the previously mentioned evolution of physical systems one can make the following interpretation. The set X is the **phase space**, $x \in X$ is a possible **state** of the system, the measure $\mu(A)$ of a set A describes the probability that a system is in a state that belongs to A. The time evolution is discrete: the transformation $T : X \to X$ describes the dynamics, i.e., what happens with a given initial state $x \in X$ within one time step. The sequence $x, Tx, T^2x, \ldots$ describes the whole path (phase space motion) that the system goes through as time evolves.

For a measurable function $f : X \to \mathbb{C}$ consider the arithmetic averages

$$A_N f := \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n, \quad N = 1, 2, \ldots,$$

also called **Cesàro averages** or **Cesàro means**.

In our interpretation, $f(x)$ could be a (physical) characteristic of the state x. For example, if $f = \mathbf{1}_A$, the characteristic function of a subset A of the phase space, then $f(x)$ tells whether or not the state x belongs to the region A. For this choice $A_N \mathbf{1}_A(x)$ yields the average number of time steps (until time N) that the phase space motion starting in x spends in the region A. By the ergodic hypothesis this should be equal to the phase space volume $\mu(A)$ of A.

By using the ideas of B.O. Koopman[9], J. von Neumann established his results called today the *mean ergodic theorem*. (We formulate it for the time discrete case.)

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[8] G. D. Birkhoff, *Proof of the ergodic theorem*, Proc. Natl. Acad. Sci. USA 17 (1931), 656–660.

[9] B. O. Koopman, *Hamiltonian systems and transformations in Hilbert space*, Proc. Natl. Acad. Sci. USA 17 (1931), 315–318.

Theorem 1.1 (von Neumann)**.** *For a probability space $(X, \mathcal{X}, \mu)$ consider the Hilbert space $\mathrm{L}^2 := \mathrm{L}^2(X, \mathcal{X}, \mu)$. Let $T : X \to X$ be a measure-preserving transformation, and let P be the orthogonal projection onto the fixed space*

$$F := \{f \in \mathrm{L}^2 : f \circ T = f\}.$$

Then for each $f \in \mathrm{L}^2$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n = Pf \quad \textit{in the } \mathrm{L}^2\textit{-norm.}$$

Suppose that the fixed space F consists of constants only. In this case von Neumann's theorem takes the form

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n = \int_X f \,\mathrm{d}\mu \cdot \mathbf{1} \quad \text{in the } \mathrm{L}^2\text{-norm for each } f \in \mathrm{L}^2$$

(here $\mathbf{1}$ is the constant 1 function). This is an instance of "time mean equals space mean", i.e., a variant of the "ergodic hypothesis". Therefore, we call a transformation T with the above property **ergodic**. Specializing $f = \mathbf{1}_A$ and integrating over $B \in \mathcal{X}$ with respect to μ yield

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

This implies that if $\mu(A) > 0$ and $\mu(B) > 0$, then there is an infinite sequence $(n_k)_{k\in\mathbb{N}}$ such that $\mu(T^{-n_k}(A) \cap B) > 0$ for every $k \in \mathbb{N}$, i.e., a positive portion of B visits the region A after n_k iterations. Since no matter how small the measure of A and B is and where these sets are located, this can be considered as a weak variant of the quasi ergodic hypothesis.

Birkhoff's results provides us more detailed information and assures pointwise convergence.

Theorem 1.2 (Birkhoff)**.** *If $T : X \to X$ is a measure-preserving transformation on a probability space $(X, \mathcal{X}, \mu)$, then for each measurable function $f : X \to \mathbb{C}$ with $\int_X |f| \,\mathrm{d}\mu < \infty$ and for μ-almost all $x \in X$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \quad \textit{exists.}$$

(The original result was formulated for the time-continuous case.) This in combination with von Neumann's result yields for each $A \in \mathcal{X}$ and each ergodic transformation T that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{T^{-n}(A)}(x) = \mu(A) \quad \text{for almost every } x \in X.$$

Since the sum on the left-hand side is precisely the average number of return times to A (or the average time our system spends in A) up to time N, this makes the quasi ergodic hypothesis a theorem for ergodic systems and almost all initial states.

These results of von Neumann and Birkhoff have generalizations that reach well beyond their original scope. We pick one to give you a hint what awaits you in these lectures. Consider the interval $[0, 1]$ equipped with the Lebesgue measure and the transformation $T : [0, 1) \to [0, 1)$, $Tx := x + \alpha - \lfloor x + \alpha \rfloor$ for some fixed irrational number α. This transformation preserves the Lebesgue measure, and can be proved to be ergodic. If $f : [0,1] \to \mathbb{C}$ is continuous with $f(0) = f(1)$ (we call f 1-periodic), then so is $f \circ T$. In analogy to the mean ergodic theorem suppose (and this will be proved) that for each continuous 1-periodic function

$$\frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n$$

converges in the **supremum norm**, in particular, pointwise everywhere. The limit then must be the same as given by von Neumann's theorem, i.e., $\int_0^1 f(t)\,\mathrm{d}t \cdot \mathbf{1}$. It follows that

$$\frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha - \lfloor n\alpha \rfloor) \to \int_0^1 f(t)\,\mathrm{d}t \quad \text{as } N \to \infty.$$

This means that the sequence $(n\alpha - \lfloor n\alpha \rfloor)_{n\in\mathbb{N}}$ is so well-behaved that sampling an arbitrary continuous periodic function f at these points and building the average approximates the integral of f. The sequence $(n\alpha - \lfloor n\alpha \rfloor)_{n\in\mathbb{N}}$ is then called **equidistributed** modulo 1. This property of the sequence $(n\alpha - \lfloor n\alpha \rfloor)_{n\in\mathbb{N}}$ was discovered by P. Bohl[10], W. Sierpiński[11], and H. Weyl[12].

In these lectures we will discuss the previously mentioned ergodic theorems, the necessary tools to prove and generalize them, their applications to topics such as equidistribution of sequences, and much more. Before the real work begins we set the stage and enumerate the minimal requirements needed to follow this course.

2. Preliminaries

The following collection of results and notions should serve as a basis for what is to come, and a guide for you in case you need to familiarize yourself with some of the topics. To do so the book [RCA] by W. Rudin and the book [HFA] by M. Haase are recommended. Another important source will be [RFA].

In these lectures we assume true the basic axioms of set theory and the axiom of choice. We use the notation $\mathbb{N} = \{1, 2, \dots, \}$, and set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. The meaning of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ is as usual.

Banach spaces. Vector spaces, unless otherwise stated, are considered over the complex field $\mathbb{C}$. A mapping $q : V \to \mathbb{R}$ on a vector space V is a **seminorm** if $p(x + y) \le p(x) + p(y)$ and $p(\lambda x) = |\lambda| p(x)$ for each $\lambda \in \mathbb{C}$, $x, y \in V$. A seminorm q is a **norm** if the linear subspace $q^{-1}(\{0\})$ is trivial $\{0\}$. In this case, the pair (V, q), or even V, is called a **normed space**. The usual notation for the norm on the normed space E is $\|\cdot\|_E$, or simply $\|\cdot\|$ if the context allows. A normed space E is **Banach space** if (E, d) is a complete metric space with the metric $d(x, y) := \|x - y\|_E$, $x, y \in E$.

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[10] P. Bohl, *Über ein in der Theorie der säkularen Störungen vorkommendes Problem*, J. reine angew. Math. 135 (1909), 189–283.

[11] W. Sierpiński, *Sur la valeur asymptotique d'une certaine somme*, Bull. Intl. Acad. Polonaise des Sci. et des Lettres (Cracovie) series A (1910), 9–11.

[12] H. Weyl, *Über die Gibbs'sche Erscheinung und verwandte Konvergenzphänomene*, Rendiconti del Circolo Matematico di Palermo 30 (1910), no. 1, 377–407.

For $p \in [1,\infty)$ and for a sequence $x = (x_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ of complex numbers we set

$$\|x\|_p := \Big(\sum_{n\in\mathbb{Z}} |x_n|^p\Big)^{1/p} \quad \text{and} \quad \|x\|_\infty := \sup_{n\in\mathbb{Z}} |x_n|$$

and for $p \in [1,\infty]$

$$\ell^p(\mathbb{Z}) := \big\{x : x \in \mathbb{C}^{\mathbb{Z}},\ \|x\|_p < \infty\big\}.$$

Then $\ell^p(\mathbb{Z})$ is a Banach space with the norm $\|\cdot\|_p$ and the coordinatewise operations. See [RCA, 5.1–5.4], [HFA, Ch. 2,5].

Linear operators and functionals. Given normed spaces E, F and a continuous, linear operator $S : E \to F$, the **operator norm** $\|S\|$ is given by

$$\|S\| := \|S\|_{E\to F} := \sup\{\|Sx\|_F : \|x\|_E \le 1\}.$$

The set of all continuous (bounded), linear operators from E to F is denoted by $\mathscr{L}(E,F)$. If F is a Banach space, then $\mathscr{L}(E,F)$ is a Banach space with the operator norm $\|\cdot\|_{E\to F}$. The operator norm is submultiplicative, in particular $\mathscr{L}(E) := \mathscr{L}(E,E)$ is a normed algebra, with the identity operator I_E as unit.

If $F = \mathbb{C}$, then the Banach space $E' := \mathscr{L}(E,\mathbb{C})$ is called the **dual space** of E, whose elements are called **linear functionals**. Different versions (separation and extension) of the **Hahn–Banach theorem** guarantee existence of linear functionals with various properties. In particular, E' separates the points of E, i.e., for different $x, y \in E$ there is $x' \in E'$ with $x'(x) \ne x'(y)$. We also use the pairing notation for $x \in E$, $x' \in E'$

$$\langle x, x'\rangle := x'(x).$$

The adjoint of $S \in \mathscr{L}(E,F)$ is the operator $S' \in \mathscr{L}(F',E')$ given by $S'x' = x' \circ S$ for $x' \in F'$. See [RFA, Ch. 4], [RCA, 5.16–5.21], [HFA, Ch. 2, 16].

Hilbert spaces. A Banach space H is a **Hilbert space** if there is a scalar product $(\cdot|\cdot) : H \times H \to \mathbb{C}$ such that $\|x\|^2 = (x|x)$ for each $x \in H$. The space $\ell^2(\mathbb{Z})$ is a Hilbert space with the scalar product

$$(x|y) = \sum_{n\in\mathbb{Z}} x_n \overline{y}_n \quad (x, y \in H).$$

Two vectors $x, y \in H$ in a Hilbert space are **orthogonal**, denoted by $x \perp y$, if $(x|y) = 0$. For a subset $A \subset H$ we write $x \perp A$ if $x \perp a$ for every $a \in A$. The orthogonal (complement) of a set $A \subset H$ is $A^\perp := \{x \in H : x \perp A\}$. If F is a closed subspace in H, then $F^\perp$ is indeed a complement of F in the sense that $H = F \oplus F^\perp$, i.e., $F \cap F^\perp = \{0\}$ and $F + F^\perp = H$. The **Riesz–Fréchet theorem** states that for every Hilbert space H and for every $\varphi \in H'$ there is a unique $z \in H$ such that $(x|z) = \langle x, \varphi\rangle$ for all $x \in H$. As a consequence, the mapping $H \to H'$, $z \mapsto (\cdot|z)$ is surjective. It is also isometric and conjugate linear, and we may identify H and H' under this mapping. Given two Hilbert spaces H and K the **adjoint** of an operator $S \in \mathscr{L}(H,K)$ is defined as the unique operator $S^* \in \mathscr{L}(K,H)$ satisfying $(Sx|y)_K = (x|S^*y)_H$ for every $x \in H$ and $y \in K$. An operator $U \in \mathscr{L}(H,K)$ is called **unitary** if $U^*U = I_H$ and $UU^* = I_K$.

For a set I and $n \in I$ let $e_n \in H$ be pairwise orthogonal unit vectors. **Bessels' inequality** states that

$$\|x\|^2 \ge \sum_{n\in I} |(x|e_n)|^2 \quad \text{for every } x \in H.$$

If for every $x \in H$ one has equality here, we call $(e_n)_{n\in I}$ an **orthonormal basis**, and we can write

$$x = \sum_{n\in I} (x|e_n)e_n \quad \text{for each } x \in H,$$

where the sum is an unconditionally convergent series in H. See [RCA, Ch. 4], [HFA, Ch. 1,8, Sec. 12.2].

Weak topologies. Let E be a Banach space. A set $U \subset E'$ is called weak* open if for each $x' \in U$ there are $N \in \mathbb{N}$, $x_1, \ldots, x_N \in E$ and $\varepsilon > 0$ such that

$$U'_{\varepsilon,x_1,\ldots,x_N} := \{y' : |\langle x_j, x' - y'\rangle| < \varepsilon \text{ for } j = 1, \ldots, N\} \subset U.$$

This way a (Hausdorff) topology is defined on E', called the **weak* topology** and denoted by $\sigma(E', E)$. Convergence of sequences in the weak* topology is discussed in Exercise 1.2.

Theorem 1.3 (Banach–Alaoglu)**.** (a) *The unit ball* $\overline{\mathrm{B}}_{E'}(0,1) := \{x' \in E' : \|x'\| \leq 1\}$ *is weak* compact.*

(b) *Suppose E is separable. Then every norm bounded sequence $(x'_n)_{n\in\mathbb{N}}$ possesses a subsequence $(x'_{n_j})_{j\in\mathbb{N}}$ which is convergent in the weak* topology, i.e., there is $x' \in E'$ such that $\langle x, x'_{n_j} - x'\rangle \to$ as $j \to \infty$ for every $x \in E$.*

The **weak topology** $\sigma(E, E')$ is defined by specifying the open sets as follows. A set $U \subset E$ is called weakly open if for each $x \in U$ there are $N \in \mathbb{N}$, $x'_1, \ldots, x'_N \in E'$ and $\varepsilon > 0$ such that

$$U_{\varepsilon,x'_1,\ldots,x'_N} := \{y : |\langle x - y, x'_j\rangle| < \varepsilon \text{ for } j = 1, \ldots, N\} \subset U.$$

If we identify a Hilbert space H with its dual H' as above, then the weak and weak* topologies coincide. In particular the unit ball $\overline{\mathrm{B}}(0, 1)$ of a Hilbert space is weakly compact. Moreover, each bounded sequence $(x_n)_{n\in\mathbb{N}}$ possesses a weakly convergent subsequence (this is true without the separability of H). See [RFA, Ch. 3–4].

Convex sets. Let C be a convex set in a vector space. A point $x \in C$ is called an **extreme point** of C if for each $t \in (0, 1)$, $z, y \in C$ with $x = ty + (1 - t)z$ it follows that $y = z$. The set of extreme points of C is denoted by $\mathrm{Ex}(C)$.

Theorem 1.4 (Krein–Milman)**.** *Let X be a locally convex space (e.g., $X = E$ with the norm topology, $X = E$ with $\sigma(E, E')$, or $X = E'$ with $\sigma(E', E)$), and let $C \subset E$ be a compact, convex subset. Then C is the closed, convex hull of* $\mathrm{Ex}(C)$*, i.e., C is the smallest closed, convex set containing C.*

See [RFA, Thm. 3.23].

Spectrum and related matters. Let E be a Banach space. The **spectrum** of an operator $S \in \mathscr{L}(E)$ is the set

$$\sigma(S) := \{\lambda \in \mathbb{C} : \lambda I - S : E \to E \text{ is not bijective}\},$$

while the **resolvent set** is $\rho(S) := \mathbb{C} \setminus \sigma(S)$. We abbreviate $\lambda I - S$ simply as $\lambda - S$. As a consequence of the **closed graph theorem** (or of the **open mapping theorem**) for $\lambda \in \rho(S)$ the bijective operator $(\lambda - S)^{-1}$ is bounded. The set $\sigma(S) \subset \mathbb{C}$ is compact and non-empty, the mapping $\rho(S) \to \mathscr{L}(E)$, $\lambda \mapsto (\lambda - S)^{-1}$ is continuous (and even has a power series representation around each point in $\rho(S)$).

The closed graph theorem is also responsible for the next result, see Exercise 1.1. Recall that a linear operator $P : E \to E$ on a normed space E is a **projection** if it is idempotent, i.e., $P^2 = P$.

Proposition 1.5. *Let E be a Banach space and $E_0, E_1 \subset E$ closed subspaces such that $E_0 \cap E_1 = \{0\}$. The mapping*

$$P : E_0 + E_1 \to E_0 + E_1, \quad e_0 + e_1 \mapsto e_0$$

is a projection. The subspace $E_0 + E_1$ is closed if and only if P is bounded.

Beside the open mapping theorem also the **principle of uniform boundedness** can be proven with the help of the next result.

Theorem 1.6 (Baire)**.** *Let (X, d) be a complete metric space, and for $n \in \mathbb{N}$ let $G_n \subset X$ be dense open sets. Then $\bigcap_{n\in\mathbb{N}} G_n$ is dense in X.*

See [RCA, 5.5–5.10], [RFA, Thm. 2.2], [HFA, Ch. 15].

Continuous functions. We assume familiarity with basic notions of point set topology at least in metric spaces (open and closed sets, boundary, limits of sequences, Cauchy sequences, continuity, etc). For a set Ω and a function $f : \Omega \to \mathbb{C}$ we set $\|f\|_\infty := \sup_{\omega\in\Omega} |f(\omega)|$. Given a topological space $(\Omega, \mathcal{O})$ (or a metric space (Ω, d)) the space of bounded and continuous functions

$$\mathrm{C_b}(\Omega) := \{f : \Omega \to \mathbb{C} \text{ is continuous and } \|f\|_\infty < \infty\}$$

is a Banach space, and even a unital Banach algebra with the norm $\|\cdot\|_\infty$ and the pointwise operations. If K is a compact space[†], then each continuous function $f : K \to \mathbb{C}$ is bounded (actually, there is $x \in K$ with $|f(x)| = \|f\|_\infty$), and we set $\mathrm{C}(K) := \mathrm{C_b}(K)$.

The next result applied to the closed sets $\{x\}$ and $\{y\}$ $(x \neq y)$ yields that $\mathrm{C}(K)$ **separates the points** of K.

Proposition 1.7 (Urysohn's lemma)**.** *Let K be a compact space, and let $A, B \subset K$ be disjoint, closed subsets. Then there is $f \in \mathrm{C}(K)$ such that $f(K) \subset [0, 1]$ and $f(a) = 1$, $f(b) = 0$ for every $a \in A$, $b \in B$.*

Point separation is also crucial in the next famous result.

Theorem 1.8 (Stone–Weierstraß)**.** *A point separating, conjugation invariant, unital subalgebra of $\mathrm{C}(K)$ is dense in $\mathrm{C}(K)$.*

A function f defined on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is called a **trigonometric polynomial** if it is a finite linear combination of the functions $\mathbf{z}^n : z \mapsto z^n$, $n \in \mathbb{Z}$. For $d \in \mathbb{N}$ and $j = 1, \ldots, d$ consider the functions $\mathbf{z}_j : \mathbb{T}^d \to \mathbb{C}$, $(z_1, \ldots, z_d) \mapsto z_j$. A function of the form $\mathbf{z}^n := \mathbf{z}_1^{n_1} \cdots \mathbf{z}_d^{n_d}$, $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ is called a **trigonometric monomial**, and a finite linear combination of such monomials is a **trigonometric polynomial**; they form a vector space $\mathrm{P}(\mathbb{T}^d)$. The Stone–Weierstraß theorem immediately implies the following.

Proposition 1.9. *The vector space $\mathrm{P}(\mathbb{T}^d)$ of trigonometric polynomials is dense in $\mathrm{C}(\mathbb{T}^d)$.*

[†]The Hausdorff property is included in the definition of a compact space.

For a function $f : \Omega \to \mathbb{R}$ we introduce the notation

$$[f > \alpha] := f^{-1}((\alpha, \infty)), \quad [f < \alpha] := f^{-1}((-\infty, \alpha)), \quad [f = \alpha] = f^{-1}(\{\alpha\}), \quad \text{etc.}$$

The support of a continuous function $f : \Omega \to \mathbb{C}$ is $\operatorname{supp}(f) := \overline{[f \neq 0]}$, where $\overline{A}$ is the closure of A. See [RCA, 2.3-2.13], [HFA, Ch. 3,4].

Measure and integral. We assume familiarity with the elementary theory of positive measures, the Lebesgue integral, and with the most fundamental results: Lebesgue's dominated convergence theorem, Beppo Levi's monotone convergence theorem, Fatou's lemma. The construction of product measures, and the theorems of Fubini and Tonelli are regarded as well-known.

We only fix here notation and terminology. Let $\mathcal{X}$ be a σ-algebra over the set X. We define $\mathscr{L}^0(X, \mathcal{X}) := \{f : X \to \mathbb{C} \text{ measurable}\}$, where $\mathbb{C}$, as everywhere in these lectures, is equipped with the Borel σ-algebra (see also below).

A positive **measure** $\mu : \mathcal{X} \to [0, \infty]$ is a σ-additive function with $\mu(\emptyset) = 0$. In this case (X, μ) is called a **measure space**, if the σ-algebra needs to be stressed we write $(X, \mathcal{X}, \mu)$. The integral of $f \in \mathscr{L}^0(X, \mathcal{X})$ with respect to μ (if it exists) is denoted by

$$\int_X f \, \mathrm{d}\mu \quad \text{or} \quad \int_X f(x) \, \mathrm{d}\mu(x).$$

Given a measure space (X, μ) we define for $p \in [1, \infty)$ and $f \in \mathscr{L}^0(X, \mathcal{X})$

$$\|f\|_p := \Big(\int_X |f|^p \, \mathrm{d}\mu \Big)^{1/p} \quad \text{and} \quad \|f\|_\infty := \inf\{\alpha : |f| \leq \alpha \ \mu\text{-almost everywhere}\}.$$

We also set for $p \in [1, \infty]$

$$\mathscr{L}^p(X, \mu) := \{f \in \mathscr{L}^0(X, \mathcal{X}) : \|f\|_p < \infty\},$$

which are vector spaces and $\|\cdot\|_p$ is a seminorm on $\mathscr{L}^p(X, \mu)$. Define the equivalence relation $\sim_\mu$ by $f \sim_\mu g$ if $[f \neq g]$ is μ-null set. After identifying $\sim_\mu$-equivalent functions we obtain the Banach spaces $\mathrm{L}^p(X, \mu)$ with norm $\|\cdot\|_p$, and the vector space $\mathrm{L}^0(X, \mu)$. Depending on the context we may write $\mathrm{L}^p(X)$, $\mathrm{L}^p(\mu)$, or even L^p. If the σ-algebra needs to be stressed, we write $\mathrm{L}^p(X, \mathcal{X}, \mu)$. If (X, μ) is a finite measure space, then for $1 \leq p \leq r \leq \infty$ we have $\mathrm{L}^r(X, \mu) \subset \mathrm{L}^p(X, \mu)$ with continuous embedding. This can be proven, e.g., with the help of **Hölder's inequality**

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \qquad f, g \in \mathrm{L}^0(X, \mu),\ \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

One of **F. Riesz'** many theorems states that any $\mathscr{L}^p$-Cauchy sequence possesses an almost everywhere convergent subsequence.
See [RCA, Ch. 1–3,7], [HFA, Ch. 7].

3. Measures on compact spaces

To be able to apply methods of functional analysis to measures we extend the notion "measure" to complex valued functions.

Let $\mathcal{X}$ be a σ-algebra over the set X. A (complex) **measure** is a σ-additive function $\mu : \mathcal{X} \to \mathbb{C}$ with $\mu(\emptyset) = 0$. We shall use the terms "positive", "signed" and "complex" to explicitly state where the values of a measure belong to. We also note that by the word "measure" (positive/signed/complex) we (almost) always understand a finite valued one, i.e., we exclude the values $\pm\infty$. The **total variation** $|\mu|$ of a measure μ is defined by

$$|\mu|(A) := \sup\Big\{\sum_{n\in\mathbb{N}} |\mu(A_n)| : (A_n)_{n\in\mathbb{N}} \text{ is a measurable partition of } A\Big\}.$$

We denote by $\mathfrak{M}(X,\mathcal{X})$ the set of all complex measures on $\mathcal{X}$. Here are the most important properties:

Proposition 1.10. (a) *For $\mu \in \mathfrak{M}(X,\mathcal{X})$ and $A \in \mathcal{X}$ we have $|\mu(A)| \le |\mu|(A)$.*
(b) *$|\mu| : \mathcal{X} \to [0,\infty)$ is a finite positive measure.*
(c) *The set $\mathfrak{M}(X,\mathcal{X})$ is a $\mathbb{C}$-vector space and a Banach space if endowed with the norm $\|\cdot\| : \mu \mapsto |\mu|(X)$.*

Let $\mu \in \mathfrak{M}(X,\mathcal{X})$ and let $f : X \to \mathbb{C}$ be a simple function, i.e., of the form $f = \sum_{j=1}^N a_j \mathbf{1}_{A_j}$ with $A_1,\dots,A_N \in \mathcal{X}$ pairwise disjoint and $a_1,\dots,a_N \in \mathbb{C}$. Then we can set

$$\int_X f \,\mathrm{d}\mu := \sum_{j=1}^N a_j \mu(A_j),$$

and quickly prove that this expression is indeed well-defined and that $(f,\mu) \to \int_X f\,\mathrm{d}\mu$ is bilinear. Since

$$\Big|\sum_{j=1}^N a_j\mu(A_j)\Big| \le \sum_{j=1}^N |a_j||\mu(A_j)| \le \sum_{j=1}^N |a_j||\mu|(A_j) = \|f\|_{\mathrm{L}^1(|\mu|)},$$

by density we can extend the integral to $f \in \mathrm{L}^1(|\mu|)$ with linearity in f remaining true. For a bounded measurable function f the mapping $\mu \mapsto \int_X f\,\mathrm{d}\mu$ is linear. See [RCA, Ch. 6].

An important special case is when $X = K$ is a compact space and $\mathcal{X} = \mathcal{B}(K)$ is the Borel σ-algebra generated by the open sets in K, i.e., $\mathcal{B}(K)$ is the smallest σ-algebra that contains every open subset of K. A continuous function $f : K \to \mathbb{C}$ is then measurable. A finite positive measure $\mu : \mathcal{B}(K) \to \mathbb{C}$ is said to be **regular** if for each $B \in \mathcal{B}(K)$

$$\mu(B) = \sup\{\mu(F) : F \subset B,\ F \text{ compact}\} = \inf\{\mu(G) : B \subset G,\ G \text{ open}\}.$$

A complex measure μ is said to be regular if $|\mu|$ is regular. The set of regular, complex Borel measures is denoted by $\mathrm{M}(K)$. It can be shown that $\mathrm{M}(K)$ is a Banach space with the total variation norm, and if K is metrizable, then any finite positive measure is regular, i.e., $\mathrm{M}(K) = \mathfrak{M}(K,\mathcal{B}(K))$. As a consequence of regularity and Urysohn's lemma we obtain for signed measures $\mu,\nu \in \mathrm{M}(K)$ that

$$\mu \le \nu \quad \Leftrightarrow \quad \text{for every positive } f \in \mathrm{C}(K)\text{:} \quad \int_K f\,\mathrm{d}\mu \le \int_K f\,\mathrm{d}\nu,$$

and for general complex measures $\mu,\nu \in \mathrm{M}(K)$ that

$$\mu = \nu \quad \Leftrightarrow \quad \text{for every } f \in \mathrm{C}(K)\text{:} \quad \int_K f\,\mathrm{d}\mu = \int_K f\,\mathrm{d}\nu.$$

The next is a fundamental theorem and allows the application of functional analytic techniques when studying measures on compact spaces.

Theorem 1.11. *The mapping $J : \mathrm{M}(K) \to \mathrm{C}(K)'$ defined by*

$$(J\mu)f := \int_K f \,\mathrm{d}\mu$$

is an isometric isomorphism of Banach spaces.

By virtue of this result we identify the Banach spaces $\mathrm{M}(K)$ and $\mathrm{C}(K)'$, and write

$$\langle f, \mu \rangle = \int_K f \,\mathrm{d}\mu.$$

See [RCA, 2.14–2.18 and Thm. 6.19].

The **support** $\mathrm{supp}(\mu)$ of a positive measure $\mu \in \mathrm{M}(K)$ is the closed set

$$\mathrm{supp}(\mu) := \big\{x \in K : \mu(U) > 0 \text{ for each open set } U \text{ with } x \in U\big\}$$

The support of a measure is easily seen to be a closed set. If $\mathrm{supp}(\mu)$ is a singleton, then μ is a scalar multiple of a **Dirac measure**, i.e., $\mu = c\delta_a$ for some $c \in \mathbb{C}$ and $a \in K$, where $\delta_a(A) = 1$ if $a \in A$ and $\delta_a(A) = 0$ if $a \notin A$.

Proposition 1.12. *Let K be a compact space and let $\mu \in \mathrm{M}(K)$ be positive.*

(a) *We have $\mu(\mathrm{supp}(\mu)^c) = 0$, and if a measurable function $f : K \to \mathbb{C}$ vanishes on $\mathrm{supp}(\mu)$, then*

$$\langle f, \mu \rangle = \int_K f \,\mathrm{d}\mu = 0.$$

(b) *If $L \subset K$ is a closed subset and for every $f \in \mathrm{C}(K)$ with $f|_L = 0$ also $\langle f, \mu \rangle = 0$ holds, then $\mathrm{supp}(\mu) \subset L$.*

We leave the proof as Exercise 1.6. The next result is a functional analytic description of the support of a measure.

Proposition 1.13. *Let $K \subset \mathbb{C}$ be a non-empty compact set, and let $\mu \in \mathrm{M}(K)$ be positive. Consider the multiplication operator $M_{\mathbf{z}} : \mathrm{L}^2(K, \mu) \to \mathrm{L}^2(K, \mu)$ (recall: $\mathbf{z} : z \mapsto z$). Then $M_{\mathbf{z}}$ is bounded, linear and $\sigma(M_{\mathbf{z}}) = \mathrm{supp}(\mu)$.*

The proof is left as Exercise 1.7. We close this lecture with the following important information about the set $\mathrm{M}_1(K)$ of regular, Borel probability measures on K.

Proposition 1.14. *Let K be a compact space.*

(a) *The set $\mathrm{M}_1(K)$ is a weak* compact, convex subset of $\mathrm{M}(K)$.*
(b) *The extreme points of $\mathrm{M}_1(K)$ are precisely the Dirac measures.*
(c) *The convex set $\mathrm{M}_1(K)$ is the closed convex hull of the Dirac measures.*

Proof. (a) We have

$$\mathrm{M}_1(K) = \{\mu \in \mathrm{M}(K) : \forall f \in \mathrm{C}(K),\ f \geq 0\ : \langle f, \mu \rangle \geq 0 \text{ and } \langle \mathbf{1}, \mu \rangle = 1\},$$

implying that $\mathrm{M}_1(K)$ is weak* closed in $\mathrm{M}(K)$ and convex. Since the closed unit ball $\overline{\mathrm{B}}_{\mathrm{M}(K)}(0,1)$ is weak* compact (Banach–Alaoglu, Theorem 1.3) and contains $\mathrm{M}_1(K)$, the compactness of the latter set follows.

(b) Let $a \in K$ and consider the Dirac measure δ_a. Suppose $\delta_a = (1-t)\mu_0 + t\mu_1$ for some $t \in (0,1)$ and $\mu_0, \mu_1 \in \mathrm{M}_1(K)$. It follows $\mu_j(K \setminus \{a\}) = 0$, and hence $\mu_j = \delta_a$ for $j = 1, 2$. This shows $\delta_a \in \mathrm{Ex}(\mathrm{M}_1(K))$. Let $\mu \in \mathrm{Ex}(\mathrm{M}_1(K))$. We claim that if $A \in \mathcal{B}(K)$, then $\mu(A) \in \{0, 1\}$. Indeed, if $\mu(A) \in (0,1)$ held for some $A \in \mathcal{B}(K)$, then with the measures $\mu_0(B) := \mu(A \cap B)/\mu(A)$, $\mu_1(B) := \mu(A^c \cap B)/\mu(A^c)$ we would have $\mu_0, \mu_1 \in \mathrm{M}_1(K)$ and $\mu = (1-\mu(A))\mu_1 + \mu(A)\mu_0$, a contradiction. Thus the claim is proven. If $a, b \in \operatorname{supp}(\mu)$ and $a \neq b$, then there are disjoint open sets U, V with $a \in U$, $b \in V$ and $\mu(U) > 0$, $\mu(V) > 0$, i.e., $\mu(U) = \mu(V) = 1$. This is impossible, so that $\operatorname{supp}(\mu)$ is a singleton and μ is a Dirac measure.

(c) Follows from (a), (b) and the Krein–Milman Theorem 1.4. ∎

Exercises

Exercise 1.1 (Bounded projections)**.** Prove Proposition 1.5.

Exercise 1.2 (Weak and weak* limits of sequences)**.** Let E be a Banach space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in E is **weakly convergent** to $x \in E$ if $\langle x_n, x'\rangle \to \langle x, x'\rangle$ as $n \to \infty$ for each $x' \in E'$. We denote this by writing $x_n \xrightarrow{\sigma} x$. Prove the following assertions:

(a) If $x_n \xrightarrow{\sigma} x$ and $x_n \xrightarrow{\sigma} y$, then $x = y$.

(b) If $x_n \xrightarrow{\sigma} x$, $y_n \xrightarrow{\sigma} y$ and for the scalar sequence $(\lambda_n)_{n\in\mathbb{N}}$ we have $\lambda_n \to \lambda$, then $x_n + \lambda_n y_n \xrightarrow{\sigma} x + \lambda y$.

Recall that a sequence $(x'_n)_{n\in\mathbb{N}}$ in E' is weak* convergent to $x' \in E'$ if $\langle x, x'_n\rangle \to \langle x, x'\rangle$ as $n \to \infty$ for each $x \in E$. We denote this by writing $x'_n \xrightarrow{\sigma^*} x'$. Formulate and prove the statements analogous to (a) and (b) for the weak* convergence.

Exercise 1.3 (Pointwise convergence)**.** Let E, F be Banach spaces and let $D \subset E$ be a dense subset. For $n \in \mathbb{N}$ let $S_n \in \mathscr{L}(E, F)$. Prove the equivalence of the following assertions:

(i) $\sup_{n\in\mathbb{N}} \|S_n\| < \infty$ and for each $x \in D$ the sequence $(S_n x)_{n\in\mathbb{N}}$ converges in F.

(ii) For each $x \in E$ the sequence $(S_n x)_{n\in\mathbb{N}}$ is convergent in F.

Prove that under these equivalent conditions the limit operator S given by $Sx := \lim_{n\to\infty} S_n x$ belongs to $\mathscr{L}(E, F)$.

Exercise 1.4 (Extreme points)**.** Consider the $\mathbb{R}$-vector space $V = \mathbb{R}^2$. Determine the extreme points of the following convex sets.

(a) $B_1 := \{(x, y) \in V : |x| + |y| \leq 1\}$,

(b) $B_2 := \{(x, y) \in V : |x|^2 + |y|^2 \leq 1\}$,

(c) $B_\infty := \{(x, y) \in V : |x|, |y| \leq 1\}$.

Exercise 1.5. Prove that $\mathfrak{M}(X, \mathcal{X})$ is a Banach space with the total variation norm (accepting that $|\mu|(A) < \infty$ for each $A \in \mathcal{X}$).

Exercise 1.6 (Support of measures)**.** Prove Proposition 1.12.

Exercise 1.7 (Spectrum and support)**.** Prove Proposition 1.13.

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LECTURE 2

Measure-preserving systems

In this lecture we introduce the basic objects of ergodic theory, namely invariant measures, measure-preserving transformations/systems and the corresponding Koopman operator. As a first surprising result, we present Poincaré's recurrence theorem. We also discuss various examples showing connections to other areas such as group theory, stochastics, number theory and topology.

1. Measure-preserving systems and recurrence

The following definition is fundamental for these lectures.

Definition 2.1. (a) Let (X, μ) be a probability space, and let $T : X \to X$ be a measurable transformation. Then T is called **μ-preserving** (and μ is called **T-invariant**) if

$$\mu(T^{-1}B) = \mu(B)$$

holds for every measurable $B \subset X$. (Here $T^{-1}B$ is the preimage of B under T.) In this case, the triple (X, μ, T) is called a **measure-preserving system**. If the underlying σ-algebra $\mathcal{X}$ needs to be stressed, we write $(X, \mathcal{X}, \mu, T)$.

(b) More generally, a measurable map $T : X \to Y$ between two measure spaces (X, μ) and (Y, ν) is called **measure-preserving** if $\mu(T^{-1}B) = \nu(B)$ for every measurable $B \subset Y$.

(c) Let μ be a positive measure on the σ-algebra $\mathcal{X}$ over X, let $\mathcal{Y}$ be a σ-algebra over Y, and let $T : X \to Y$ be measurable. The **push-forward** $T_*\mu$ of μ is defined by $T_*\mu(B) := \mu(T^{-1}B)$ for $B \in \mathcal{Y}$.

Remark 2.2. (a) A measurable map $T : X \to Y$ between two measure spaces (X, μ) and (Y, ν) is measure-preserving if and only if $T_*\mu = \nu$.

(b) If T is μ-preserving, then the images of sets may have larger measure than the sets themselves:

$$\inf\{\mu(C) : T(B) \subset C,\ C \text{ measurable}\} \geq \mu(B) \quad \text{for each measurable } B \subset X.$$

This inequality can be strict, see, for instance, Example 2.16.

The following famous recurrence result is remarkably easy to prove.

Theorem 2.3 (Poincaré's recurrence theorem)**.** *Let (X, μ, T) be a measure-preserving system and let $A \subset X$ satisfy $\mu(A) > 0$. There is $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A) > 0$.*

Proof. Let $A \subset X$ be measurable such that $\mu(A \cap T^{-n}A) = 0$ for every $n \in \mathbb{N}$. Since T is μ-preserving, we thus have

$$\mu(T^{-k}A \cap T^{-(n+k)}A) = 0 \quad \text{for every } n \in \mathbb{N},\ k \in \mathbb{N}_0.$$

This means that the sets $A, T^{-1}A, T^{-2}A, \ldots$ are pairwise disjoint up to null sets. But they all have the same measure $\mu(A)$. Since μ is a finite measure, this can only happen if $\mu(A) = 0$. ∎

Poincaré's result shows in particular that for every $A \subset X$ with positive measure there is at least one $x \in A$ which is **recurrent** to A, i.e., satisfies $T^n x \in A$ for at least one $n \in \mathbb{N}$. In fact, *almost every* $x \in A$ is **infinitely recurrent** to A, i.e., comes back to A infinitely often.

Corollary 2.4 (Infinite recurrence)**.** *Let (X, μ, T) be a measure-preserving system and let $A \subset X$ be with $\mu(A) > 0$. Then almost every $x \in A$ satisfies $T^n x \in A$ for infinitely many $n \in \mathbb{N}$.*

Proof. We first show that almost every $x \in A$ is recurrent to A. Consider the measurable set

$$B := A \setminus \bigcup_{k=1}^{\infty} T^{-k}A = A \cap \Big(\bigcap_{k=1}^{\infty} T^{-k}A^c \Big)$$

of all points which never return to A. For every $n \in \mathbb{N}$ we have

$$B \cap T^{-n}B \subset A \cap T^{-n}A \cap \Big(\bigcap_{k=1}^{\infty} T^{-k}A^c \Big) = \emptyset.$$

Thus Poincaré's recurrence theorem implies $\mu(B) = 0$.

We now show that almost every $x \in A$ is infinitely recurrent to A. By the above we have

$$A \subset \bigcup_{n=1}^{\infty} T^{-n}A \cup B \tag{2.1}$$

with the above null set B. For each $k \in \mathbb{N}$ it follows that

$$T^{-1}A \subset \bigcup_{n=2}^{\infty} T^{-n}A \cup T^{-1}B, \ \ldots \ , T^{-(k-1)}A \subset \bigcup_{n=k}^{\infty} T^{-n}A \cup T^{-(k-1)}B.$$

Inserting this into (2.1) leads to

$$A \subset \bigcup_{n=k}^{\infty} T^{-n}A \cup N_k$$

for every k, where N_k is a null set. Since $\cup_{k=1}^{\infty} N_k$ is a null set, almost every $x \in A$ is infinitely recurrent to A. ∎

The following lemma helps proving the measure-preserving property of a transformation.

Lemma 2.5. *Let $\mathcal{X}$ be a σ-algebra over the set X, let $\mathcal{E}$ be a generator of $\mathcal{X}$, i.e., $\mathcal{X}$ is the smallest σ-algebra over X containing $\mathcal{E}$, and suppose that $\mathcal{E}$ is closed under $\cap$. Let $T : X \to X$ be a measurable transformation. Then a finite, positive measure μ on $\mathcal{X}$ is T-invariant if and only if $\mu(T^{-1}B) = \mu(B)$ for every $B \in \mathcal{E}$.*

Sketch of proof. Let $\mathcal{D} := \{B \subset \mathcal{X} : \ \mu(T^{-1}B) = \mu(B)\}$. It is easy to see that $\mathcal{D}$ has the following three properties.

(1) $X \in \mathcal{D}$.
(2) If $A \in \mathcal{D}$, then $A^c \in \mathcal{D}$.

(3) If $A_n \in \mathcal{D}$ for each $n \in \mathbb{N}$ and these sets are pairwise disjoint, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

Since by assumption $\mathcal{E} \subset \mathcal{D}$, Dynkin's theorem, see [BPM, Thm. 3.2], implies that the σ-algebra $\mathcal{X}$ generated by $\mathcal{E}$ is contained in $\mathcal{D}$. ∎

Here is another useful characterization of the measure-preserving property.

Proposition 2.6. *Let (X, μ), (Y, ν) be probability spaces and let $T : X \to Y$ be measurable.*

(a) *For every bounded (or positive) measurable function $f : Y \to \mathbb{C}$*

$$\int_X f \circ T \,\mathrm{d}\mu = \int_Y f \,\mathrm{d}T_*\mu.$$

(b) *The transformation T is measure-preserving if and only if*

$$\int_X f \circ T \,\mathrm{d}\mu = \int_Y f \,\mathrm{d}\nu \tag{2.2}$$

for every bounded, measurable function $f : Y \to \mathbb{C}$. In this case, (2.2) *holds for every $f \in \mathscr{L}^1(Y, \nu)$.*

Proof. (a) For a measurable set A and $f = \mathbf{1}_A$ we have the asserted equality by the definition of the push-forward measure, since $\mathbf{1}_A \circ T = \mathbf{1}_{T^{-1}A}$. By linearity we obtain the statement for simple functions. If $f : X \to \mathbb{C}$ is a bounded (or positive) measurable function, then there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging pointwise to f with $|f_n| \le |f_{n+1}| \le |f|$ for every $n \in \mathbb{N}$. We obtain, by Lebesgue's or Beppo Levi's theorem, that

$$\int_X f \circ T \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \circ T \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}T_*\mu = \int_X f \,\mathrm{d}T_*\mu.$$

(b) Suppose (2.2) is satisfied for every bounded, measurable function. Let $A \subset Y$ be measurable and consider $f := \mathbf{1}_A \in \mathrm{L}^\infty(Y, \nu)$. Then by the hypothesis

$$\nu(A) = \int_Y \mathbf{1}_A \,\mathrm{d}\nu = \int_X \mathbf{1}_A \circ T \,\mathrm{d}\mu = \int_X \mathbf{1}_{T^{-1}A} \,\mathrm{d}\mu = \mu(T^{-1}A),$$

thus T is measure-preserving.

Next, suppose that T is measure-preserving, i.e., $T_*\mu = \mu$. Each $f \in \mathrm{L}^1(X, \mu)$ can be decomposed into real and imaginary, and then into positive and negative parts, all of which are positive and belong to $\mathscr{L}^1(Y, \nu)$. By part (a) and by linearity we obtain (2.2). ∎

The following definition of invertibility is natural, although it might appear somewhat technical at first glance.

Definition 2.7. (a) Let (X, μ) and (Y, ν) be two probability spaces. A measurable map $\theta : X \to Y$ is called **essentially invertible** if there are measurable sets $X' \subset X$ and $Y' \subset Y$ of full measure such that $\theta : X' \to Y'$ is bijective and $\theta^{-1} : Y' \to X'$ is measurable (with respect to the trace σ-algebras).

(b) A measure-preserving system (X, μ, T) is called **invertible** if T is essentially invertible.

(c) Two measure-preserving systems (X, μ, T) and (Y, ν, S) are called **isomorphic** if there is an essentially invertible map $\theta : X \to Y$ such that $S \circ \theta = \theta \circ T$ μ-a.e..

Remark 2.8. (a) A measurable map $\theta : X \to Y$ is essentially invertible if and only if there exists a measurable map $\eta : Y \to X$ with $\eta \circ \theta = \mathrm{id}_X$ a.e. and $\theta \circ \eta = \mathrm{id}_Y$ a.e., see Exercise 2.1. In this case η is called the **essential inverse** of θ. Two essential inverses of θ coincide except on a null set, and we denote them by the same symbol θ^{-1}.

(b) If a measure-preserving system (X, μ, T) is invertible, then the transformation T^{-1} is automatically μ-preserving as well (why?) and thus gives rise to a new measure-preserving system (X, μ, T^{-1}) called the **inverse system**. Intuitively, passing from T to T^{-1} corresponds to reversing the time.

2. Examples

We present here some fundamental examples of measure-preserving systems. This list of examples will be continually augmented during these lectures.

Example 2.9 (Finite state space)**.** Let X be a finite set and let μ be the normalized counting measure on X. Then a transformation $T : X \to X$ is μ-preserving if and only if T is a bijection, in which case T is invertible.

Example 2.10 (Torus rotation)**.** Consider the unit circle

$$\mathbb{T} := \{z : z \in \mathbb{C},\ |z| = 1\},$$

which is a closed and bounded, hence compact subset of $\mathbb{C}$. On the Borel σ-algebra $\mathcal{B}(\mathbb{T})$ of $\mathbb{T}$ we consider the normalized arclength-measure m, for which $\mathrm{m}(\mathbb{T}) = 1$. Note that $\mathbb{T}$ with the multiplication of complex numbers is a compact, commutative group, and the measure m is invariant under each left rotation

$$\tau_a : \mathbb{T} \to \mathbb{T}, \quad x \mapsto ax.$$

For any $a \in \mathbb{T}$ we therefore obtain the invertible measure-preserving system $(\mathbb{T}, \mathrm{m}, \tau_a)$, abbreviated as $(\mathbb{T}, \mathrm{m}, a)$. Note that for some fixed $a \in \mathbb{T}$ there might be many other τ_a-invariant measures than m, see Exercise 2.2.

The additive analogue of the previous example is the following.

Example 2.11 (Translation mod 1)**.** Let $\alpha \in [0, 1)$. Consider $[0, 1)$ equipped with the Lebesgue measure λ and $T_\alpha : [0, 1) \to [0, 1)$ given by

$$T_\alpha x = x + \alpha - \lfloor x + \alpha \rfloor.$$

Instead of $y - \lfloor y \rfloor$ we often write $y \bmod 1$. Then $([0, 1), \lambda, T_\alpha)$ is an invertible measure-preserving system (why?), abbreviated as $([0, 1), \lambda, \alpha)$.

This translation system is isomorphic to the rotation system $(\mathbb{T}, \mathrm{m}, a)$ from Example 2.10 via the map $\theta : [0, 1) \to \mathbb{T}$ given by $\theta(x) := \mathrm{e}^{2\pi \mathrm{i} x}$ for $x \in [0, 1)$.

These two examples are special instances from the following important class.

Example 2.12 (Group rotation)**.** Consider a group G which is also equipped with a topology (more restrictively with a metric) such that the multiplication

$$G \times G \to G, \quad (x, y) \mapsto xy$$

and the inversion

$$G \to G, \quad x \mapsto x^{-1}$$

are continuous. Then G is called a **topological group**. For each $a \in G$, as in Example 2.10, we can consider the left rotation

$$\tau_a : G \to G, \quad x \mapsto ax.$$

Since the group is not assumed to be commutative, it makes sense to define the right rotation as well

$$\rho_a : G \to G, \quad x \mapsto xa.$$

These are all continuous, hence measurable mappings with respect to the Borel σ-algebra $\mathcal{B}(G)$. It is known that if G is a compact topological group (such as $\mathbb{T}$, $\mathbb{T}^d$ or their additive analogues), then there is unique regular probability measure on $\mathcal{B}(G)$ invariant under each left and right rotation, see, e.g, [EFHN, Thm. G.10] or [HMT, Ch. XI]. This measure is called the **Haar measure** of G and denoted by m_G. For each $a \in G$ we obtain two invertible measure-preserving systems $(G, \mathrm{m}_G, \tau_a)$ and $(G, \mathrm{m}_G, \rho_a)$. The measure-preserving systems corresponding to the left rotations τ_a are abbreviated as (G, m_G, a).

Important special cases of group rotations are the higher dimensional torus rotations $(\mathbb{T}^d, \mathrm{m}^d, a)$, where $\mathrm{m}^d = \mathrm{m} \otimes \cdots \otimes \mathrm{m}$ is the product measure on $\mathcal{B}(\mathbb{T}^d)$ and $a \in \mathbb{T}^d$.

Example 2.13 (Skew rotation)**.** Consider a torus rotation system $(\mathbb{T}, \mathrm{m}, a)$. Since m is invariant under each (left=right) rotation τ_x, the transformation

$$T_a : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (ax, xy)$$

is m^2-preserving. Indeed, by Fubini's theorem, for every bounded, measurable function $f : \mathbb{T}^2 \to \mathbb{C}$

$$\begin{aligned} \int\limits_{\mathbb{T}^2} f(ax, xy)\, \mathrm{d}x\, \mathrm{d}y &= \int\limits_{\mathbb{T}} \Big(\int\limits_{\mathbb{T}} f(ax, xy)\, \mathrm{d}y \Big)\, \mathrm{d}x = \int\limits_{\mathbb{T}} \Big(\int\limits_{\mathbb{T}} f(ax, y)\, \mathrm{d}y \Big)\, \mathrm{d}x \\ &= \int\limits_{\mathbb{T}} \Big(\int\limits_{\mathbb{T}} f(ax, y)\, \mathrm{d}x \Big)\, \mathrm{d}y = \int\limits_{\mathbb{T}^2} f(x, y)\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

holds, so by Proposition 2.6 T_a is measure-preserving. (Integration with respect to the Haar measure is often denoted as above. We thus obtain the measure-preserving system $(\mathbb{T}^2, \mathrm{m}^2, T_a)$, called a **skew rotation**

The additive version of it is the **skew shift** on $[0, 1)^2$ with the 2-dimensional Lebesgue (= Haar) measure given by

$$(x, y) \mapsto (x + \alpha \bmod 1, x + y \bmod 1)$$

for a fixed $\alpha \in [0, 1)$.

A fundamentally different class of examples is described next.

Example 2.14 (Shifts)**.** Let (Y, ν) be a probability space and consider the infinite product $X = Y^{\mathbb{Z}}$, $\mathcal{X}$ the product σ-algebra generated by **cylinder sets**, i.e., sets of the form

$$A = \cdots \times Y \times Y \times A_{-N} \times \cdots \times A_0 \times A_1 \times \cdots A_N \times Y \times \cdots \times Y \cdots$$

with measurable $A_{-N}, \ldots A_N \subset Y$, and product measure μ given by

$$\mu(A) = \prod_{j=-N}^{N} \nu(A_j)$$

on such cylinder sets, and μ is uniquely determined by this property, see [HMT, §38]. Define now the left shift

$$T : X \to X, \quad (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}.$$

Then (X, μ, T) is an invertible measure-preserving system, which we call the **two-sided shift with state space** (Y, ν) (use Lemma 2.5). If one replaces in the above the set of integers by the natural numbers, and modifies the definition of cylinder sets accordingly, one obtains the **one-sided shift with state space** (Y, ν).

Note that if Y is a metric space with a bounded metric d and $\mathcal{Y} = \mathcal{B}(Y)$, then X is also a metric space with $d_X(x, y) := \sum_{n=-\infty}^{\infty} 2^{-|n|} d(x_n, y_n)$, and the product σ-algebra $\mathcal{X}$ is the Borel σ-algebra. We leave the proof to the reader.

The particular case when Y is finite deserves additional attention.

Example 2.15 (Bernoulli shift)**.** If $Y = \{0, 1 \ldots, k-1\}$ for some fixed $k \in \mathbb{N}$ and $\mu(\{j\}) = p_j$ for some **probability vector** $p \in \mathbb{R}^k$, then the corresponding shift from the previous Example 2.14 is called a **Bernoulli shift** and is denoted by $B(p_0, \ldots, p_{k-1})$. Here, Y is called the **alphabet** with k **letters**, and elements of $X = Y^{\mathbb{Z}}$ (or $X = Y^{\mathbb{N}}$, respectively) are called **infinite words**.

Example 2.16 (Bernoulli shift and the doubling map)**.** Consider the one-sided Bernoulli shift $B(1/2, 1/2)$ from Example 2.15 and the interval $[0, 1]$ with the Lebesgue measure λ. The **doubling map** T on $[0, 1]$ given by $Tx := 2x \bmod 1$ preserves λ. The map $\theta : \{0, 1\}^{\mathbb{N}_0} \to [0, 1]$ given by

$$\theta(t_0, t_1, \ldots) := \sum_{n=0}^{\infty} \frac{t_n}{2^{n+1}}$$

corresponds to the binary representation of numbers and provides an isomorphism between the Bernoulli shift and the measure-preserving system $([0, 1], \lambda, T)$, see Exercise 2.4. (Intuitively it is clear that shifting the binary representation of a number to the left corresponds to the doubling the number modulo 1). Note that the isomorphism θ is not bijective due to the fact that the binary representation is unique for almost all but not all numbers.

It was a long standing open question in ergodic theory whether the (two-sided) Bernoulli shifts $B(1/n, \ldots, 1/n)$ and $B(1/m, \ldots, 1/m)$ are non-isomorphic whenever $n \neq m$. This was confirmed by A.N. Kolmogorov only in 1958 using the notion of the measure-theoretic entropy (see [5]), now known as the Kolmogorov–Sinai entropy. However, contrary to what the first impression would tell, the Bernoulli shifts $B(1/4, 1/4, 1/4, 1/4)$ and $B(1/2, 1/8, 1/8, 1/8, 1/8)$ are isomorphic. This was proven by L.D. Mešalkin in 1959, see [6]. This isomorphism problem for the Bernoulli shifts was completely settled by D.S. Ornstein in 1970, see [7]. The interested reader can consult, for example, the survey [8] by B. Weiss.

The following shows the connection between measure-preserving shift systems and stationary stochastic processes.

Example 2.17 (Stationary stochastic processes)**.** Let (Ω, P) be a probability space and for $j \in \mathbb{Z}$ let $f_j : \Omega \to \mathbb{R}$ be measurable functions, i.e., random variables. The sequence $(f_j)_{j\in\mathbb{Z}}$ is called a stochastic process and we suppose that it is *stationary*.

This means that for every $m \in \mathbb{N}$, every $n_1, \ldots, n_m \in \mathbb{Z}$, all Borel sets $B_1, \ldots, B_m \subset \mathbb{R}$ and every $k \in \mathbb{Z}$ one has

$$\mathrm{P}\Big(\bigcap_{j=1}^{m} f_{n_j}^{-1}(B_j)\Big) = \mathrm{P}\Big(\bigcap_{j=1}^{m} f_{n_j+k}^{-1}(B_j)\Big). \tag{2.3}$$

We now construct an associated measure-preserving system. Consider $X := \mathbb{R}^{\mathbb{Z}}$ with the Borel algebra (i.e., the product σ-algebra with $\mathcal{B}(\mathbb{R})$ in each of the components) and define $f : \Omega \to X$ by

$$f(\omega) := (\ldots, f_{-1}(\omega), f_0(\omega), f_1(\omega)\ldots), \quad \text{i.e.,} \quad (f(\omega))_j = f_j(\omega).$$

For a Borel set $A \subset X$ set $\mu(A) := \mathrm{P}(f^{-1}(A))$, i.e., μ is the push-forward probability measure corresponding to f. Consider finally the left shift on $\mathbb{R}^{\mathbb{Z}}$ given by $T(x_j)_{j\in\mathbb{Z}} := (x_{j+1})_{j\in\mathbb{Z}}$. Then (X, μ, T) is a measure-preserving system. To see this take a cylinder set $B \subset \mathbb{R}^{\mathbb{Z}}$ and observe that $f^{-1}(B)$ is of the form $\cap_{j=1}^{m} f_{n_j}^{-1}(B_j)$. Thus the stationarity property (2.3) together with the definition of μ implies

$$\mu(B) = \mathrm{P}(f^{-1}(B)) = \mathrm{P}(f^{-1}(T^{-1}B)) = \mu(T^{-1}B),$$

i.e., T is indeed μ-preserving.

Thus a stationary stochastic process defines a measure-preserving shift system in fairly straightforward manner. We now show that one can go in the converse direction.

Example 2.18. Let $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \mu, T)$ be a measure-preserving system, where T denotes the left shift on $\mathbb{R}^{\mathbb{Z}}$. We invert the procedure described in the previous example and construct a corresponding stationary stochastic process (to which Example 2.17 associates precisely $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \mu, T)$). Define $\Omega := \mathbb{R}^{\mathbb{Z}}$, $\mathrm{P} := \mu$ and $f_j := \pi_j$, $j \in \mathbb{Z}$, where $\pi_j : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is the projection onto the jth coordinate. Then every π_j is measurable since $\pi_j^{-1}(B)$ is a cylinder set for every Borel subset $B \subset \mathbb{R}$. We thus obtain the stochastic process $(f_j)_{j\in\mathbb{Z}}$. Moreover, this process is stationary by the measure-preserving property of T

$$\mu\Big(\bigcap_{j=1}^{m} \pi_{n_j}^{-1}(B_j)\Big) = \mu\Big(T^{-k}\bigcap_{j=1}^{m} \pi_{n_j}^{-1}(B_j)\Big) = \mu\Big(\bigcap_{j=1}^{m} \pi_{n_j+k}^{-1}(B_j)\Big).$$

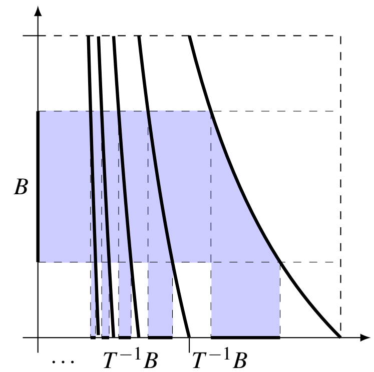
Finally, $(\mathbb{R}^{\mathbb{Z}}, \mu, T)$ arises from $(f_j)_{j\in\mathbb{Z}}$ as in Example 2.17. Indeed, since $f : \Omega \to \mathbb{R}^{\mathbb{Z}}$ is the identity by construction, we have $\mu = f_*\mathrm{P}$.

To sum up the previous two examples we can say that to study stationary stochastic processes is the same as to study shift-invariant probability measures on $\mathbb{R}^{\mathbb{Z}}$. Note that the same holds if we replace $\mathbb{Z}$ by $\mathbb{N}$ or $\mathbb{N}_0$ (check it!).

Next we turn to one of the many connections to number theory.

Example 2.19 (Gauss transformation)**.** Let $X := [0, 1)$ and let $T : X \to X$ denote the **Gauss transformation** defined by $T(0) := 0$ and

$$Tx := \tfrac{1}{x} - \left\lfloor \tfrac{1}{x} \right\rfloor \quad \text{for } x \neq 0.$$



We see that $Tx = \frac{1}{x} - n$ whenever $x \in (\frac{1}{n+1}, \frac{1}{n}]$ and therefore

$$T^{-1}\{y\} = \left\{ \tfrac{1}{y+n} : n \in \mathbb{N} \right\} \quad \text{for } y \in (0,1). \tag{2.4}$$

We finally define the measure μ by

$$\mu(A) := \frac{1}{\ln 2} \int_A \frac{\mathrm{d}x}{x+1}, \quad A \in \mathcal{B}([0,1]).$$

It is Exercise 2.5 to show that (X, μ, T) is a measure-preserving system, it is called the **Gauss system**.

This system is directly connected to simple continued fractions which we briefly discuss here. A **(simple) continued fraction** is a formal expression of the form

$$[a_0; a_1, a_2, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

with $a_0 \in \mathbb{Z}$ and $(a_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{N}$. Continued fractions lead to a way of representing a real number as a sequence of integers different from the standard expansion using base 10 (or any other base). For a given $x \in \mathbb{R}$ the corresponding continued fraction is obtained by the following inductive procedure.

We set $a_0 := \lfloor x \rfloor$, $r_0 := x - a_0$. Suppose $a_0 \in \mathbb{Z}$, $a_1, \ldots, a_n \in \mathbb{N}$ and $r_0, \ldots, r_n \in [0,1)$ have been defined. If $r_n = 0$, we stop and the construction of the continued fraction $[a_0; a_1, \ldots, a_n]$ is finished. Otherwise, define $a_{n+1} := \lfloor \frac{1}{r_n} \rfloor$ and $r_{n+1} := \frac{1}{r_n} - a_{n+1}$, and continue repeating this procedure indefinitely. In this way we obtain a finite or infinite sequence of numbers a_n with $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{N}$ and write $x = [a_0; a_1, a_2, \ldots]$. For example, the golden ratio ϕ satisfies $\phi = [1; 1, 1, 1, \ldots]$ (Exercise 2.5).

Already the ancient greeks (notably Euclid, Archimedes of Syracuse and Diophantos of Alexandria) knew about these miraculous objects, but the first treatment that went beyond sheer numerical tricks is due to R. Bombelli[1]. The systematic study was started by P. Cataldi[2], and he is considered by many as *the real discoverer*. A number of giants of mathematics were involved in the development of the theory of continued fractions, such as Euler, Lagrange, Galois, and, of course,

[1] R. Bombelli, *L'algebra parte maggiore dell'arithmetica divisa in tre libri*, 1572, Bologna.

[2] P. A. Cataldi, *Trattato del modo brevissimodi trovare la radice quadra delli numeri*, 1613, Bologna.

Gauss. Here we list some properties of continued fractions to give a flavour of their theory. The interested reader is referred, e.g., to [3] and [4].

(1) A real number x is rational if and only if its continued fraction representation is finite, see Exercise 2.5. Moreover, the continued fraction of x is eventually periodic if and only if x is a quadratic irrational.

(2) Conversely, every infinite sequence $(a_j)_{j\in\mathbb{N}_0}$ with $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{N}$ leads to a continuous fraction of an irrational number. Thus we have a bijection between irrational numbers and infinite continuous fractions. Note that the representation of rational numbers is not unique since

$$[a_0; a_1, a_2, \ldots, a_n + 1] = [a_0; a_1, a_2, \ldots, a_n, 1].$$

(3) Let $x = [a_0; a_1, a_2, \ldots]$. Define the $n^{\mathbf{th}}$ **convergent** of the continued fraction as the reduced fraction

$$\frac{p_n}{q_n} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_n}}},$$

p_n, q_n being coprime. Then one has

$$\frac{p_2}{q_2} \le \frac{p_4}{q_4} \le \ldots \le x \le \ldots \le \frac{p_3}{q_3} \le \frac{p_1}{q_1}$$

with $\frac{p_{2n}}{q_{2n}} \nearrow x$ and $\frac{p_{2n+1}}{q_{2n+1}} \searrow x$. For every $n \in \mathbb{N}$, the rational number $\frac{p_n}{q_n}$ satisfies

$$\left| x - \frac{p_n}{q_n} \right| \le \frac{1}{q_n^2}$$

and is the best approximation of x by rationals with denominator less than or equal to q_n. Generally, bounded continued fractions (i.e., bounded sequences $(a_n)_{n\in\mathbb{N}_0}$) correspond to so-called *badly approximable* numbers, the golden ratio representing the worst case.

The next proposition explains the relation of continued fractions to the Gauss transformation, see Exercise 2.5.

Proposition 2.20. *Consider the Gauss system* $([0, 1), \mu, T)$*, and let* $x \in [0, 1) \setminus \mathbb{Q}$ *have the continued fraction representation* $x = [a_0; a_1 \ldots]$*. Then for each* $n \in \mathbb{N}$

$$a_n = \left\lfloor \frac{1}{T^{n-1}x} \right\rfloor.$$

An analogous statement holds for rational numbers as well.

3. The Koopman operator

We associate to each measure-preserving system a *linear* operator.

Definition 2.21. Let (X, μ, T) be a measure-preserving system. For a measurable function $f : X \to \mathbb{C}$ define

$$S_T f := f \circ T.$$

Then $S_T f$ is again a measurable function and S_T defines a linear operator on the vector space $\mathscr{L}^0(X)$ of measurable functions. The operator S_T acts actually on $\sim_\mu$-equivalence classes of functions, and hence defines a linear operator $S_T : \mathrm{L}^0(X, \mu) \to \mathrm{L}^0(X, \mu)$, called the **Koopman operator** (induced by T).

In the same manner, if $T : X \to Y$ is measure-preserving for the probability spaces (X, μ) and (Y, ν), a linear operator $S_T : \mathrm{L}^0(Y, \nu) \to \mathrm{L}^0(X, \mu)$ is defined by $S_T f := f \circ T$. Again S_T can be called the Koopman operator induced by T. By Proposition 2.6 the operator $S_T : \mathrm{L}^p(Y, \nu) \to \mathrm{L}^p(X, \mu)$ is a linear isometry for every $p \in [1, \infty)$. The same holds also for $p = \infty$, and the Koopman operator has important algebraic properties, see Exercise 2.6.

Proposition 2.22. *Let (X, μ), (Y, ν) be probability spaces, and let $T : X \to Y$ be measure-preserving. Let $p \in [1, \infty]$ and let $f, g \in \mathrm{L}^0(Y, \nu)$.*
(a) *$f \geq g$ implies $S_T f \geq S_T g$. In particular, $f \geq 0$ implies $S_T f \geq 0$ (positivity).*
(b) *$S_T(f \cdot g) = S_T f \cdot S_T g$ (multiplicativity).*
(c) *$S_T |f| = |S_T f|$.*
(d) *$S_T \mathbf{1} = \mathbf{1}$. More generally, $S_T \mathbf{1}_A = \mathbf{1}_{T^{-1}A}$ for every measurable $A \subset Y$.*

4. Topological dynamical systems and invariant measures

An important class of measure-preserving systems arises from continuous maps on compact spaces.

Definition 2.23. Let K be a compact space (metric if you like), and let $T : K \to K$ be a continuous mapping. The pair (K, T) is called a **topological (dynamical) system**. It is called **invertible** if T is bijective (in which case the inverse of T is automatically continuous).

Example 2.24. (1) If $L = \{0, \ldots, k-1\}$ and $K = L^{\mathbb{Z}}$ or $K = L^{\mathbb{N}}$, then K is a compact space, and the shift T, $(x_n) \mapsto (x_{n+1})$ from Example 2.15 is a continuous mapping. So (K, T) is a topological system.
(2) For $K = \mathbb{T}$ and $T = \tau_a$, $x \mapsto ax$, from Example 2.10, $(\mathbb{T}, \tau_a)$ is topological system, abbreviated as $(\mathbb{T}, a)$.
(3) More generally, given a compact (metric) topological group and $a \in G$, we obtain the topological system (G, τ_a), abbreviated as (G, a).

Similarly to the case of measure-preserving transformations, a continuous mapping $T : K \to L$ between compact spaces induces a linear contraction

$$S_T : \mathrm{C}(L) \to \mathrm{C}(K), \quad S_T f := f \circ T.$$

If (K, T) is a topological system S_T is called **the Koopman operator**. It has the same algebraic properties as the Koopman operator on L^∞ (cf. Proposition 2.22).

Recall that the dual space of $\mathrm{C}(K)$ is identified with the space $\mathrm{M}(K)$ of regular Borel measures on K via the mapping $\mu \mapsto \int_K \cdot \, \mathrm{d}\mu$ (see Theorem 1.11).

Let $T : K \to K$ be a continuous transformation. For $\mu \in \mathrm{M}(K)$ we define the **push-forward measure** by $T_* \mu(B) := \mu(T^{-1}B)$ for $B \in \mathcal{B}(K)$. The measure μ is called **T-invariant** if $T_* \mu = \mu$. As in Proposition 2.6 one can show (exercise) that for each $f \in \mathrm{C}(K)$

$$\int_K f \, \mathrm{d}T_* \mu = \int_K f \circ T \, \mathrm{d}\mu.$$

From this we conclude

$$\langle f, S_T' \mu \rangle = \langle S_T f, \mu \rangle = \int_K S_T f \, \mathrm{d}\mu = \int_K f \circ T \, \mathrm{d}\mu = \int_K f \, \mathrm{d}T_* \mu = \langle f, T_* \mu \rangle$$

for every $f \in \mathrm{C}(K)$, i.e., $T_* \mu = S_T' \mu$. We thus obtain the following important fact.

Proposition 2.25. *Let (K,T) be a topological system. A measure $\mu \in \mathrm{M}(K)$ is T-invariant if and only if $\mu \in \ker(I - S_T')$.*

The following theorem shows in particular that every topological system gives rise to at least one measure-preserving system.

Theorem 2.26 (Krylov–Bogoljubov)**.** *Let (K,T) be a topological system and let $f \in \mathrm{C}(K)$ satisfy $f \neq 0$ and $S_T f = f$. Then there exists a T-invariant probability measure μ on K with $\langle \mu, f\rangle \neq 0$.*

Proof. Let $x \in K$ such that $f(x) \neq 0$ and consider the probability measures

$$\nu_n := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j x}.$$

Since $\mathrm{M}_1(K)$ is weak* compact by Proposition 1.14, the sequence $(\nu_n)_{n\in\mathbb{N}}$ possesses a cluster point $\mu \in \mathrm{M}_1(K)$. For $g \in \mathrm{C}(K)$ we have

$$\begin{aligned}\Big|\langle g\circ T, \nu_n\rangle - \langle g, \nu_n\rangle\Big| &= \Big|\frac{1}{n}\sum_{j=0}^{n-1}\Big(\langle g, \delta_{T^{j+1}x}\rangle - \langle g, \delta_{T^j x}\rangle\Big)\Big| \\ &= \frac{1}{n}\Big|\langle g, \delta_{T^n x}\rangle - \langle g, \delta_x\rangle\Big| \leq \frac{2\|g\|_\infty}{n}.\end{aligned}$$

Since μ is a cluster point, we obtain $|\langle g\circ T, \mu\rangle - \langle g, \mu\rangle| = 0$ for each $g \in \mathrm{C}(K)$, implying that μ is T-invariant. Since $S_T f = f$ and $\langle f, \nu_n\rangle = \langle f, \delta_x\rangle \neq 0$, we obtain $\langle f, \mu\rangle = \langle f, \delta_x\rangle \neq 0$, as required. ∎

Notation 2.27. From now on, following many authors, most of the time we will write T for both a measure-preserving transformation and the induced Koopman operator. Thus we use the convention $(Tf)(x) = f(Tx)$.

An exception will be the Koopman operator of the left (or right) rotation by a on a compact topological group from Example 2.12 which we denote by L_a or R_a, respectively.

Exercises

Exercise 2.1 (Essential invertibility)**.** Prove the assertion of Remark 2.8(a).

Exercise 2.2 (Invariant measures for rotations)**.** (a) Let $a \in \mathbb{T}$ be irrational, i.e., not a root of unity. Show that the set $\{1, a, a^2, a^3, \ldots\}$ is dense in $\mathbb{T}$.

(b) Let $a \in \mathbb{T}$ be fixed. Describe all τ_a-invariant measures on $\mathbb{T}$, where τ_a is the rotation by a discussed in Example 2.10. (Hint: Distinguish the cases of rational and irrational a. For irrational a use (a).)

What changes if one replaces the rotation from Example 2.10 by the translation from Example 2.11?

Exercise 2.3 (Bernoulli shifts)**.** Prove that for $k \geq 2$ and a probability vector $p \in \mathbb{R}^k$ with $p_j > 0$ for each $j \in \{0, \ldots, k-1\}$ the one-sided Bernoulli shift is not invertible.

Exercise 2.4. Prove that the doubling map preserves the Lebesgue measure. Show that the Bernoulli shift and the doubling map from Example 2.16 are isomorphic. What if one replaces $B(1/2, 1/2)$ by $B(1/n, \ldots, 1/n)$ for an arbitrary $n \in \{2, 3, \ldots\}$?

Exercise 2.5 (Gauss system and continued fractions)**.** (a) Show that the Gauss system defined in Example 2.19 is measure-preserving. (Hint: Use (2.4).)

(b) Show that a real number is rational if and only if its continued fraction representation is finite.

(c) Show that the golden ratio satisfies $\phi = [1; 1, 1, 1, \ldots]$. Which number has the representation $[1; 2, 2, 2, \ldots]$?

(d) Prove Proposition 2.20 by induction.

Exercise 2.6. Prove that the Koopman operator of a measure-preserving system is isometric on the space $\mathrm{L}^\infty(X, \mu)$. Prove also the algebraic properties from Proposition 2.22 of Koopman operators on $\mathrm{L}^0(X, \mu)$ and also on $\mathrm{C}(K)$.

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LECTURE 3

Minimality and ergodicity

Given a dynamical system, topological or measure-preserving, in this lecture we study proper regions of the state space which stay invariant under the dynamics. More precisely, we are interested in the absence of such regions, since their existence would contradict the "ergodic hypothesis" as formulated by Boltzmann. Indeed, in this case the phase space motion is confined to that region hence cannot reach every other state. Also "time mean equals space mean" would fail since the phase space motion does not spend any time outside of this invariant region.

1. Invariant sets for topological systems

We begin with some definitions for a topological system (K, T). For a point $x \in K$ and a subset $A \subset \mathbb{N}_0$ (or $A \subset \mathbb{Z}$ if T is invertible) we define

$$T^A x := \{T^n x : n \in A\},$$

and call the set $\mathrm{orb}_+(x) := T^{\mathbb{N}_0} x$ the **orbit** of x. A subset $F \subset K$ is called (forward) **invariant** if $\mathrm{orb}_+(x) \subset F$ for each $x \in F$. If T is invertible, then the **two-sided orbit** of x is $\mathrm{orb}(x) := T^{\mathbb{Z}} x$, and a subset $F \subset K$ is **two-sided invariant** if $\mathrm{orb}(x) \subset F$ for each $x \in F$. A closed, invariant set F yields, by restriction to F, a **subsystem** $(F, T|_F)$, denoted by (F, T). In the same manner two-sided invariant sets of an invertible system yield invertible subsystems. If the system (K, T) is invertible, then any non-empty, closed, invariant set contains a closed, non-empty, two-sided invariant set (see Exercise 3.1). A point $x \in K$ is called **topologically transitive** if $\overline{\mathrm{orb}_+(x)} = K$, and **topologically two-sided transitive** if $\overline{\mathrm{orb}(x)} = K$. Finally, the system (K, T) is called **minimal** if the only closed, invariant sets are the trivial ones, $\emptyset$ and K.

Remark 3.1. The closure of an invariant set is easily seen to be invariant, so in particular $\overline{\mathrm{orb}_+(x)}$ is an invariant set, and the system (K, T) is minimal if and only if each of the points $x \in K$ is topologically transitive. Analogously, an invertible system (K, T) is minimal if and only if each of the points $x \in K$ is topologically two-sided transitive (see Exercise 3.2). This gives an informal relation to the *quasi ergodic hypothesis* of the Ehrenfests since in such systems the orbit of every point comes arbitrarily close to any other point.

We introduce the following terminology. Let (K, T) and (L, S) be topological systems. A **homomorphism** between the (K, T) and (L, S) is a continuous map $\pi : K \to L$ such that $\pi \circ T = S \circ \pi$. We denote this by writing $\pi : (K, T) \to (L, S)$. Such a homomorphism is called an **automorphism** if it is also a homeomorphism between K and L. Automorphisms take orbits to orbits and closed, invariant sets to closed, invariant sets. This can be exploited in the following characterization of minimality for group rotations.

Proposition 3.2. *Let G be a compact group and let $a \in G$. The following assertions are equivalent.*

(i) *The group rotation system (G, a) is minimal.*
(ii) *The subsemigroup $\{a^n : n \in \mathbb{N}_0\}$ is dense in G.*
(iii) *The cyclic subgroup $\langle a\rangle = \{a^n : n \in \mathbb{Z}\}$ is dense in G.*

Minimality, therefore, is a strong condition for such systems. For example, it implies that the group G is abelian, see Exercise 3.3.

Proof. The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are clear. To prove that (iii) implies (i) let $x \in G$. Then $\mathrm{orb}(x) = \rho_x \mathrm{orb}(1) = \rho_x\langle a\rangle$, where ρ_x is the multiplication by x from the right and hence an automorphism of (G, a). This implies $\overline{\mathrm{orb}}(x) = G$, i.e., every point in G is topologically two-sided transitive. By Remark 3.1, the proof is complete. ∎

Important examples of minimal systems are provided by certain torus rotations.

Proposition 3.3 (Kronecker)**.** *The system $(\mathbb{T}, a)$ is minimal if and only if $a \in \mathbb{T}$ is not a root of unity, i.e., if the cyclic subgroup $\langle a\rangle$ is infinite.*

Proof. If a is a root of unity, then $\mathrm{orb}_+(1)$ is finite and invariant, so that $(\mathbb{T}, a)$ cannot be minimal. If a is not a root of unity, then by Exercise 2.2(a) $\mathrm{orb}_+(1)$ is dense in $\mathbb{T}$, and by Proposition 3.2 we obtain the minimality of $(\mathbb{T}, a)$. ∎

The following result is a useful characterization of minimality using open sets instead of points.

Proposition 3.4 (Characterization of minimality)**.** *A topological system (K, T) is minimal if and only if for each non-empty, open set $U \subset K$ there is $N \in \mathbb{N}_0$ such that*

$$K = \bigcup_{n=0}^{N} T^{-n}U. \tag{3.1}$$

Proof. Suppose (K, T) is minimal, and let $U \subset K$ be non-empty and open. Consider the set

$$F := \bigcap_{n\in\mathbb{N}_0} T^{-n}(K \setminus U)$$

of points which never visit U. This is a closed, invariant set and $F \neq K$ since $U \cap F = \emptyset$. It follows that $F = \emptyset$, i.e., $K = \bigcup_{n\in\mathbb{N}_0} T^{-n}U$, and by compactness we can take a finite subcover. This proves (3.1).

Conversely, let F be a non-trivial, closed, invariant subset of K, then $U := K\setminus F$ is non-empty, open and violates condition (3.1). ∎

It turns out that in a minimal system every point visits each non-empty, open set even infinitely often. Moreover, the sequence (or set) of visiting times has *some structure*. This is the content of the next proposition.

Proposition 3.5. *Let (K, T) be minimal, and let $U \subset K$ be a non-empty, open set. Then for each $x \in K$ the set*

$$R_U(x) := \{n \in \mathbb{N} : T^n x \in U\}$$

of return times to U is* **syndetic** *(or* **relatively dense***), i.e., there is $N \in \mathbb{N}$ such that $[k, k+N] \cap R_U(x) \neq \emptyset$ for every $k \in \mathbb{N}$.*

Proof. By Proposition 3.4 there is $N \in \mathbb{N}$ such that $K = \bigcup_{j=0}^{N} T^{-j}U$. This implies that for each $x \in K$ and $k \in \mathbb{N}$ there is $j \in \{0, \dots, N\}$ such that $T^k x \in T^{-j}U$, meaning $x \in T^{-(k+j)}U$. It follows that $R_U(x) \cap [k, k+N] \neq \emptyset$. ∎

A point $x \in K$ is called **almost periodic** if for each non-empty, open set $U \subset K$ the set of return times $R_U(x)$ is syndetic. By the above, in a minimal system every point is almost periodic.

Proposition 3.6 (Characterization of almost periodic points)**.** *For a topological system (K, T) and a point $x \in K$ the following assertions are equivalent.*

(i) *The point x is almost periodic.*
(ii) *The set $\overline{\mathrm{orb}}_+(x)$ is minimal.*
(iii) *The point x is contained in a minimal subsystem.*

Proof. The implication (ii)⇒(iii) is trivial and (iii)⇒(i) follows from Proposition 3.5, so we only need to prove (i)⇒(ii). Let U be an open set containing x, and let $F := \overline{\mathrm{orb}}_+(x)$. Since x is almost periodic, there is $N \in \mathbb{N}$ such that $T^k x \in \bigcup_{j=0}^{N} T^{-j}U \subset \bigcup_{j=0}^{N} T^{-j}\overline{U}$ for each $k \in \mathbb{N}$. It follows that $F \subset \bigcup_{j=0}^{N} T^{-j}\overline{U}$. As a consequence, for any $y \in F$ there is $j \in \{0, \dots, N\}$ with $T^j y \in \overline{U}$. This, being true for every open neighbourhood U of x, implies $x \in \overline{\mathrm{orb}}_+(y)$ and $\overline{\mathrm{orb}}_+(x) = \overline{\mathrm{orb}}_+(y)$. Minimality of $\overline{\mathrm{orb}}_+(x)$ follows. ∎

Here is a strong form of recurrence for points in minimal systems.

Proposition 3.7 (Almost periodic points in metrizable systems are recurrent)**.** *Let (K, T) be a topological system with metrizable K, and let $x \in K$ be almost periodic. Then x is* **recurrent***, i.e., there is a subsequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ such that $T^{n_k} x \to x$ as $k \to \infty$.*

We leave the proof as Exercise 3.4.

Proposition 3.8. *Let (K, T) be a topological system.*

(a) *Let (L, S) be another topological system and let $\pi : (K, T) \to (L, S)$ be a homomorphism. If $x \in K$ is almost periodic in (K, T), then so is $\pi(x)$ in (L, S).*
(b) *Suppose (L, T) is a subsystem of (K, T). If $x \in K$ is almost periodic in (K, T), then it is almost periodic in (L, T).*

We leave the proof of this result again as Exercise 3.4.

Corollary 3.9. *In a compact group rotation system (G, a) every point is almost periodic.*

Proof. The subgroup $H := \overline{\langle a \rangle}$ provides the group rotation subsystem (H, a), which is minimal by Proposition 3.2. This implies that 1 is almost periodic, thus for each $x \in K$ the homomorphic image $x = \rho_x(1)$ is almost periodic. ∎

The following result is known as Birkhoff's Recurrence Theorem[1].

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*Of course, if $x \notin U$ this should be called the set of *visiting* times, but the terminology *return times* is more widespread.

[1] G. D. Birkhoff, *Quelques théorèmes sur le mouvement des systèmes dynamiques*, S. M. F. Bull. 40 (1912), 305–323.

Theorem 3.10 (Birkhoff)**.** *Each topological system (K, T) contains a minimal subsystem and therefore an almost periodic point. In particular, if K is metrizable, then it contains at least one recurrent point.*

Proof. Consider the set $\mathcal{F} := \{F : F \subset K \text{ closed, invariant, } F \neq \emptyset\}$, which is partially ordered by inclusion $A \subset B$. This partially ordered sets satisfies the conditions of Zorn's lemma, and therefore has a minimal element F, which provides a minimal subsystem (F, T). By Proposition 3.6, any point $x \in F$ is almost periodic. The rest follows from Proposition 3.7. ∎

2. Invariant sets for measure-preserving systems

Because of the presence of null sets (which are anyway negligible), the whole business of invariant sets for measure-preserving systems becomes somewhat more subtle. We begin with some terminology.

Let (X, μ) be a probability space and $A, B \subset X$ measurable. We say that $A \subset B$ **up to a null set** if $\mu(B \setminus A) = 0$. Analogously, we say that $A = B$ up to a null set if $\mu(A \triangle B) = 0$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B. The simple inequality

$$|\mu(A) - \mu(B)| \leq \mu(A \triangle B) \tag{3.2}$$

will be very useful. Note that being equal up to a null set defines an equivalence relation on measurable sets.

Definition 3.11 (Invariant sets)**.** Let (X, μ, T) be a measure-preserving system. A measurable set $A \subset X$ is called (T-)**invariant** if $A = T^{-1}A$ up to a null set.

Remark 3.12. By the measure-preserving property, a measurable set A is invariant if and only if $T^{-1}(A) \subset A$ holds up to a null set, and this holds if and only if $A \subset T^{-1}(A)$ holds up to a null set, see Exercise 3.7

The set of invariant sets is denoted by $\mathcal{X}_T$, and it is a sub-σ-algebra of the underlying σ-algebra $\mathcal{X}$ (see Exercise 3.7), called the **invariant sub-σ-algebra**. In particular, we have $A^c \in \mathcal{X}_T$ if $A \in \mathcal{X}_T$. A non-trivial invariant set A thus gives rise to two systems $(X, \mu_A/\mu(A), T)$ and $(X, \mu_{A^c}/\mu(A^c), T)$, where for $B \in \mathcal{X}$ the measure μ_B is defined by

$$\mu_B(C) := \mu(B \cap C) \quad \text{for each } C \in \mathcal{X}.$$

Definition 3.13 (Ergodicity)**.** The measure-preserving system (X, μ, T), and also the transformation T, are called **ergodic** if $\mu(A) \in \{0, 1\}$ holds for each invariant set $A \subset X$.

In Lecture 1 we defined ergodicity via a fixed space condition. In Proposition 3.21 below we will show that this is equivalent to the previous definition.

We now present the first examples of ergodic (and non-ergodic) systems.

Example 3.14 (Finite systems)**.** Let X be a finite set with the normalized counting measure μ and let $T : X \to X$ be bijective. Then T is ergodic if and only if it is a cyclic permutation.

Example 3.15 (Identity is (almost) never ergodic)**.** Let (X, μ) be a probability space. Then $\mathrm{id} : X \to X$ is ergodic if and only if there is no measurable set $A \subset X$ with $0 < \mu(A) < 1$. This shows that, in contrary to the topological case where every system contains a minimal subsystem, not every measure-theoretic system has an ergodic subsystem. For example, $([0,1], \lambda, \mathrm{id})$ has no ergodic subsystem.

An important class of ergodic transformations are (Bernoulli) shifts (cf. Example 2.14). We need the following auxiliary result, for a proof see [BPM, Thm. 11.4 and Cor.].

Lemma 3.16. *Let $(X, \mathcal{X}, \mu)$ be a probability space and let $\mathcal{E}$ be an algebra of sets and a generator of the σ-algebra $\mathcal{X}$. Then for every set $A \in \mathcal{X}$ and for every $\varepsilon > 0$ there is $B \in \mathcal{E}$ such that $\mu(A \triangle B) < \varepsilon$.*

Proposition 3.17 (Shifts are ergodic)**.** *Let (Y, ν) be a probability space. Both the one-sided and the two-sided shift with state space (Y, ν) are ergodic.*

Proof. We consider only the case of one-sided shifts (X, μ, T), the invertible case being analogous. Let $\mathcal{E}$ be the algebra of finite unions of cylinder sets (which generates the product σ-algebra $\mathcal{X}$). Let $A \in \mathcal{X}$ be an invariant set, let $\varepsilon > 0$ be arbitrary, and let $B \in \mathcal{E}$ be such that $\mu(A \triangle B) < \varepsilon$ (use Lemma 3.16). Then B is of the form

$$B = B_N \times Y \times Y \cdots, \quad \text{where } B_N \subset Y^N.$$

Note that $T^{-n}B = Y^n \times B_N \times Y \times Y \cdots$, so for every $n > N$ we have

$$\mu(B \cap T^{-n}B) = \mu(B) \cdot \mu(T^{-n}B) = \mu(B)^2$$

by the definition of the product measure. Thus we obtain by (3.2) for every $n > N$

$$\begin{aligned} |\mu(A) - \mu(A)^2| &\le |\mu(A) - \mu(B \cap T^{-n}B)| + |\mu(B \cap T^{-n}B) - \mu(A)^2| \\ &\le \mu(A \triangle (B \cap T^{-n}B)) + |\mu(B)^2 - \mu(A)^2| \\ &\le \mu(A \triangle B) + \mu(A \triangle T^{-n}B) + |\mu(B)^2 - \mu(A)^2| \\ &= \mu(A \triangle B) + \mu(T^{-n}A \triangle T^{-n}B) + |\mu(B)^2 - \mu(A)^2| \\ &\le 2\varepsilon + |\mu(B)^2 - \mu(A)^2| \le 2\varepsilon + 2|\mu(A) - \mu(B)| \le 4\varepsilon. \end{aligned}$$

This, being true for every $\varepsilon > 0$, implies $\mu(A) = \mu(A)^2$, hence $\mu(A) \in \{0, 1\}$. ∎

Proposition 3.18 (Characterization of ergodicity)**.** *For a measure-preserving system (X, μ, T) the following assertions are equivalent.*

(i) *The system is ergodic.*

(ii) *Every $A \subset X$ with $\mu(A) > 0$ satisfies*

$$\bigcap_{k \in \mathbb{N}_0} \bigcup_{n \ge k} T^{-n}A = X \quad \textit{up to a null set.}$$

(iii) *Every $A \subset X$ with $\mu(A) > 0$ satisfies*

$$\bigcup_{n \in \mathbb{N}_0} T^{-n}A = X \quad \textit{up to a null set.}$$

(iv) *For each pair of sets $A, B \subset X$ with $\mu(A), \mu(B) > 0$ there is $n \in \mathbb{N}$ with*

$$\mu(T^{-n}A \cap B) > 0.$$

Proof. Suppose (i) and $\mu(A) > 0$. For $k \in \mathbb{N}_0$ define $A_k := \bigcup_{n \ge k} T^{-n}A$. Then $T^{-1}(A_k) = A_{k+1} \subset A_k$ and therefore A_k is an invariant set with $\mu(A_k) \ge \mu(T^{-k}A) = \mu(A) > 0$. The assumption implies $\mu(A_k) = 1$, and by intersecting $\mu(\bigcap_{k \in \mathbb{N}} A_k) = 1$, i.e., (ii) follows. The implication (ii)⇒(iii) is trivial.

Assume (iii) and let $B \subset X$ be measurable satisfying $\mu(T^{-n}A \cap B) = 0$ for each $n \in \mathbb{N}$. For $k = 0, 1$ let again $A_k := \bigcup_{n \geq k} T^{-n}A$. Since $T^{-1}A_0 = A_1 \subset A_0$, it follows that $A_0 = A_1$ up to a null set. We obtain

$$0 = \mu\left(\bigcup_{n\in\mathbb{N}} (B \cap T^{-n}A)\right) = \mu(B \cap A_1) = \mu(B \cap A_0) = \mu(B \cap X) = \mu(B),$$

proving (iv).

To see the implication (iv)⇒(i) take $B := A^c$ in (iv) for an invariant set $A \subset X$, to conclude that $\mu(A) \in \{0, 1\}$. ■

Remark 3.19. Condition (iii) in the above characterization means that almost every $x \in X$ visits every set A with positive measure at least once, and condition (ii) means that almost every $x \in X$ visits such A infinitely often.

Example 3.20 (Recurrence in random literature)**.** We now discuss a concrete consequence of Proposition 3.18 which might appear somewhat counterintuitive at the first glance. Imagine someone typing randomly on a typewriter which has 90 different typesetting symbols. The work starts now. This can be modeled by the one-sided Bernoulli shift $B(1/90, \ldots, 1/90)$ which is ergodic by Proposition 3.17. An infinite piece of literature is then described by each of the sequences $x \in \{0, \ldots, 89\}^{\mathbb{N}}$. Some of these sequences correspond to total nonsense, others contain poems of Goethe, romans of Joyce (or your favorite book), yet another contain some digits of π, or a mixture of these, etc. Let $a_1 a_2 \cdots a_N$ be the random text that can be seen in the background of the poster of this Internet Seminar. Since the cylinder set

$$A := \left\{x \in \{0, \ldots, 89\}^{\mathbb{N}} : x_1 = a_1, \ldots, x_N = a_N\right\}$$

has positive measure (precisely $1/90^N$), we see, by Proposition 3.18, that almost every $x \in \{0, \ldots, 89\}^{\mathbb{N}}$ visits A infinitely many times under the shift-dynamics, meaning that the announcement of this Internet Seminar occurs in almost every infinite word infinitely often. This means that the person will almost surely type the poster infinitely often. (The poster was, however, produced by different methods.) What changes (and what does it mean for the typist) if we replace here the one-sided Bernoulli shift by a two-sided one?

3. Koopman operator and ergodicity

For a linear operator $S : V \to V$ on a vector space V we defined the **fixed space** by

$$\mathrm{Fix}(S) := \{f \in V : Sf = f\},$$

and call each $f \in \mathrm{Fix}(S)$ **invariant under** S. Note that for a measure-preserving system, constant functions are always invariant under the Koopman operator.

Proposition 3.21 (Ergodicity via invariant functions)**.** *Let (X, μ, T) be a measure-preserving system and let $p \in [1, \infty)$. The Koopman operator on $\mathrm{L}^0(X, \mu)$ is also denoted by T. The following assertions are equivalent.*

(i) *The system (X, μ, T) is ergodic.*
(ii) *If $f : X \to \mathbb{C}$ is measurable with $f = f \circ T$ μ-a.e., then there is $c \in \mathbb{C}$ such that $f = c\mathbf{1}$ μ-a.e.*
(iii) *Every invariant function $f \in \mathrm{L}^p(X, \mu)$ is constant.*

(iv) *Every invariant function* $f \in \mathrm{L}^\infty(X, \mu)$ *is constant.*

Proof. (i)⇒(ii): Let $f : X \to \mathbb{C}$ be measurable with $Tf = f$ μ-a.e., and assume without loss of generality that f is real-valued (otherwise we pass to the real and imaginary parts). For $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$ define the measurable set

$$A_{k,n} := [\tfrac{k}{2^n} \leq f < \tfrac{k+1}{2^n}].$$

This set is T-invariant, and thus either $\mu(A_{k,n}) = 0$, or $\mu(A_{k,n}) = 1$ by (i). Since for each fixed $n \in \mathbb{N}_0$ the sets $A_{k,n}$ are pairwise disjoint for $k \in \mathbb{Z}$ with $\bigcup_{k\in\mathbb{Z}} A_{k,n} = X$, there is precisely one $k(n) \in \mathbb{Z}$ with $\mu(A_{k(n),n}) = 1$. Then the set

$$Z := \bigcap_{n\in\mathbb{N}_0} A_{k(n),n}$$

has full measure, and for $x, y \in Z$ we have $|f(x) - f(y)| \leq 1/2^n$ for every $n \in \mathbb{N}$, i.e., $f(x) = f(y)$.

The implications (ii)⇒(iii)⇒(iv) are trivial. So it remains to prove (i) assuming (iv). Let $A \subset X$ be invariant. We have $T\mathbf{1}_A = \mathbf{1}_{T^{-1}A} = \mathbf{1}_A$, i.e., the function $\mathbf{1}_A$ is invariant under T. By (iv), $\mathbf{1}_A$ must be constant, so that $\mu(A) \in \{0, 1\}$. ∎

The study of the ergodicity of torus rotations becomes now very easy.

Proposition 3.22. *The torus rotation* $(\mathbb{T}^d, \mathrm{m}^d, a)$ *with* $a = (a_1, \ldots, a_d) \in \mathbb{T}^d$ *is ergodic if and only if* $a_1, \ldots, a_d$ *are* ***rationally independent****, i.e., satisfy*

$$\Big((n_1, \ldots, n_d) \in \mathbb{Z}^d \quad \text{and} \quad a_1^{n_1} a_2^{n_2} \cdots a_d^{n_d} = 1\Big) \quad \Longrightarrow \quad n_1 = n_2 = \cdots = n_d = 0.$$

Proof. The monomials $\mathbf{z}^n$ with $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ given by $\mathbf{z}^n(z_1, \ldots, z_d) := z_1^{n_1} \cdots z_d^{n_d}$ form an orthonormal system in $\mathrm{L}^2(\mathbb{T}^d, \mathrm{m}^d)$ and, by Proposition 1.9, even an orthonormal basis. Let f be an invariant function of the Koopman operator T on $\mathrm{L}^2(\mathbb{T}^d, \mathrm{m}^d)$. Then expanding f with respect to the orthonormal basis leads to

$$\sum_{n\in\mathbb{Z}^d} b_n \mathbf{z}^n = f = Tf = \sum_{n\in\mathbb{Z}^d} a^n b_n \mathbf{z}^n,$$

as an $\mathrm{L}^2(\mathbb{T}^d, \mathrm{m}^d)$-convergent series, where $a^n := a_1^{n_1} \cdots a_d^{n_d}$. By comparing the coefficients we obtain $b_n = a^n b_n$ for every $n \in \mathbb{Z}^d$.

If $f \notin \mathbb{C}\mathbf{1} = \mathbb{C}\mathbf{z}^0$, then there is $n \in \mathbb{Z}^d \setminus \{0\}$ with $b_n \neq 0$, implying $1 = a^n = a_1^{n_1} \cdots a_d^{n_d}$. This proves that in case of rational independence, $\mathrm{Fix}(T) = \mathbb{C}\mathbf{1}$, and ergodicity follows by Proposition 3.21.

For the converse implication, assume that $a_1, \ldots, a_d$ satisfy $a_1^{n_1} \cdots a_d^{n_d} = 1$ with $n = (n_1, \ldots, n_d) \neq (0, \ldots, 0)$. Then the corresponding monomial $\mathbf{z}^n$ is not constant and satisfies $T\mathbf{z}^n = \mathbf{z}^n$, implying that the system (X, μ, T) is not ergodic by Proposition 3.21. ∎

The following gives information on the spectrum of the Koopman operator of an ergodic system. Recall for a vector space V and a linear operator $S : V \to V$ the notation

$$P\sigma(S) := \big\{\lambda \in \mathbb{C} : \ Sf = \lambda f \text{ for some } f \in V \setminus \{0\}\big\}$$

for the set of eigenvalues of S, called the **point spectrum** of S.

Proposition 3.23 (Point spectrum of the Koompan operator for ergodic systems)**.** *Let (X, μ, T) be an ergodic measure-preserving system, and let T denote the Koopman operator on $\mathrm{L}^0(X, \mu)$. Then the set $P\sigma(T) \cap \mathbb{T}$ of unimodular eigenvalues is a subgroup of $\mathbb{T}$, each such eigenvalue has one-dimensional eigenspace, and a corresponding eigenfunction can be taken unimodular.*

Since constant functions are always T-invariant, $1 \in P\sigma(T)$ and, in particular, $P\sigma(T) \neq \emptyset$.

Proof. Let $\lambda \in P\sigma(T) \cap \mathbb{T}$ with corresponding eigenfunction f. Since $|f \circ T| = |f| \circ T$, we have that $|f| \in \operatorname{Fix}(T)$, so that, by ergodicity and by Proposition 3.21, $|f|$ is a non-zero constant. Hence after scaling f can be taken to be unimodular.

Let $\mu \in P\sigma(T) \cap \mathbb{T}$ have an eigenfunction g which, together with f, is assumed to be unimodular. By the algebraic properties of the Koopman operator (see Proposition 2.22) we obtain that $T(f\overline{g}) = T(f)\overline{T(g)} = (\lambda\overline{\mu})f\overline{g}$. Since $f\overline{g}$ is unimodular and hence non-zero, it is an eigenfunction to the eigenvalue $\lambda\overline{\mu}$. If $\mu = \lambda$, ergodicity yields $f\overline{g} = \mathbf{1}$, i.e., $f = g$, showing the one-dimensionality of the eigenspace. Moreover, we see that for $\lambda, \mu \in P\sigma(T) \cap \mathbb{T}$ also $\overline{\mu}$ and $\lambda\mu$ belong to $P\sigma(T) \cap \mathbb{T}$, i.e., this set is subgroup of $\mathbb{T}$. ∎

4. More on invariant measures for topological systems

We now study the relation between topological and measure-preserving systems from the point of view of ergodicity properties. For this purpose we introduce the notion of absolute continuity with two central results, namely Lebesgue's decomposition and the Radon–Nikodym theorem. For details we refer to Rudin's book [RCA, 6.7–6.14].

Let $(X, \mathcal{X})$ be a measurable space, let $\mu \in \mathfrak{M}(X, \mathcal{X})$ be a positive measure, and $f \in \mathrm{L}^1(X, \mathcal{X}, \mu) =: \mathrm{L}^1(\mu)$. Then the measure ν defined for $A \in \mathcal{X}$ by

$$\nu(A) := \int_A f \, \mathrm{d}\mu$$

satisfies $\nu(A) = 0$ whenever $\mu(A) = 0$. We use the notation $f \cdot \mu$ fo this measure ν. The proof of the following lemma is a routine exercise, see, e.g., [RCA, Thm. 1.29]

Lemma 3.24. *If $\mu \in \mathfrak{M}(X, \mathcal{X})$ is a positive measure and $\nu = f \cdot \mu$ for some $f \in \mathrm{L}^1(\mu)$, then for every bounded (or positive) measurable function $g : X \to \mathbb{C}$*

$$\int_X g \, \mathrm{d}\nu = \int_X gf \, \mathrm{d}\mu.$$

Definition 3.25. (a) Let $\mu, \nu \in \mathfrak{M}(X, \mathcal{X})$ be two positive measures. If each μ-null set is at the same time a ν-null set, then ν is said to be **absolutely continuous** with respect to μ. We denote this by writing $\nu \ll \mu$. If additionally $\mu \ll \nu$, then we call the measures μ and ν **equivalent**, denoted by $\mu \simeq \nu$.

(b) Let $\mu, \nu \in \mathfrak{M}(X, \mathcal{X})$ be two complex measures. We say that ν is absolutely continuous with respect to μ if $|\nu| \ll |\mu|$.

(c) A pair of complex measures $\mu, \nu \in \mathfrak{M}(X, \mathcal{X})$ is called **(mutually) singular** if there is a set $A \in \mathcal{X}$ with $|\mu|(A) = |\nu|(A^c) = 0$. We denote this relation by writing $\mu \perp \nu$.

We see that the measure $f \cdot \mu$ from the above is absolutely continuous with respect to μ.

Theorem 3.26 (Lebesgue decomposition & Radon–Nikodym theorem)**.** *Let $\nu, \mu \in \mathfrak{M}(X, \mathcal{X})$ and let μ be positive.*

(a) *There are unique measures ν_a and ν_s with $\nu_s \perp \mu$, $\nu_a \ll \mu$ and $\nu = \nu_a + \nu_s$.*
(b) *There is $f \in \mathrm{L}^1(\mu)$ such that $\nu_a = f \cdot \mu$, i.e.*

$$\nu_a(A) = \int_A f \,\mathrm{d}\mu \quad \text{for each } A \in \mathcal{X}.$$

Moreover, if ν is positive, then so are ν_a, ν_s and f.

If $\nu \ll \mu$, then $\nu_s = 0$ in part (a), so that $\nu = f \cdot \mu$. The function f is called the **Radon–Nikodym derivative** of ν with respect to μ.

We now come return to topological systems and take closer look invariant measure. Let (K, T) be a topological system. Let $\mathrm{M}_1(K, T)$ be the set of T-invariant regular, Borel, probability measures. By the Krylov–Bogoljubov theorem (see Lecture 2) $\mathrm{M}_1(K, T)$ is non-empty. An invariant, probability measure $\mu \in \mathrm{M}_1(K, T)$ is called **ergodic** if $(K, \mathcal{B}(K), \mu, T)$ is an ergodic measure-preserving system.

Proposition 3.27 (Ergodicity of invariant measures)**.** *Let (K, T) be a topological system and let $\mu, \nu \in \mathrm{M}_1(K, T)$.*

(a) *If μ is ergodic and $\nu \ll \mu$, then $\mu = \nu$.*
(b) *If $\mu \neq \nu$ and both are ergodic, then they are mutually singular.*
(c) *The measure $\mu \in \mathrm{M}_1(K, T)$ is ergodic if and only if μ is an extreme point of the (weak* compact, convex) set $\mathrm{M}_1(K, T)$.*

Proof. (a) Since $\nu \ll \mu$ there is $f \in \mathrm{L}^1(\mu)$ with $f \geq 0$ and $\nu = f \cdot \mu$ (Theorem 3.26). We prove that $f = \mathbf{1}$ in $\mathrm{L}^1(\mu)$. We show first that $A := [f < 1]$ is an invariant set for (K, μ, T). To see this we compute

$$\int_{A \cap T^{-1}A} f \,\mathrm{d}\mu + \int_{A \setminus T^{-1}A} f \,\mathrm{d}\mu = \nu(A) = \nu(T^{-1}A) = \int_{A \cap T^{-1}A} f \,\mathrm{d}\mu + \int_{T^{-1}A \setminus A} f \,\mathrm{d}\mu.$$

As a consequence we obtain

$$\int_{A \setminus T^{-1}A} f \,\mathrm{d}\mu = \int_{T^{-1}A \setminus A} f \,\mathrm{d}\mu. \tag{3.3}$$

Since $A \setminus T^{-1}A \subset [f < 1]$, $T^{-1}A \setminus A \subset [f \geq 1]$ and

$\mu(A \setminus T^{-1}A) = \mu(A) - \mu(A \cap T^{-1}A) = \mu(T^{-1}A) - \mu(A \cap T^{-1}A) = \mu(T^{-1}A \setminus A)$,

we conclude from (3.3) that $\mu(A \setminus T^{-1}A) = 0 = \mu(T^{-1}A \setminus A)$, implying $A = T^{-1}A$ up to a μ-null set, and the invariance of A is proven.

By ergodicity of μ we have $\mu(A) \in \{0, 1\}$. If $\mu(A) = 1$ were true, then the inequality $\nu(X) = \int_X f \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu < \mu(A) = 1$ would provide a contradiction. Thus $\mu(A) = 0$ must hold, meaning that $f \geq \mathbf{1}$ μ-almost everywhere. Since $\nu = f \cdot \mu$ is a probability measure, we conclude $f = \mathbf{1}$ μ-almost everywhere.

(b) By the Lebesgue decomposition $\nu = \nu_a + \nu_s$ holds with uniquely determined positive measures ν_a, ν_s such that $\nu_a \ll \mu$ and $\nu_s \perp \mu$. Since $\nu = T_*\nu = T_*\nu_a +$

$T_*\nu_s$ and $T_*\nu_a \ll T_*\mu = \mu$, $T_*\nu_s \perp T_*\mu = \mu$, the uniqueness of the Lebesgue decomposition implies $T_*\nu_a = \nu_a$. Therefore ν_a is an invariant measure with $\nu_a \ll \mu$. If $\nu_a = 0$, then $\nu = \nu_s \perp \mu$, and we are done. If $\nu_a \neq 0$, the probability measure $\nu_a/\|\nu_a\|$ is absolutely continuous with respect to μ, so by part (a) $\nu_a/\|\nu_a\| = \mu$. This implies $\mu \ll \nu_a \leq \nu$. Since also ν is ergodic, this implies again by part (a) $\mu = \nu$.

(c) That $\mathrm{M}_1(K,T)$ is convex and weak* compact is left as Exercise 3.9. Let $\mu \in \mathrm{M}_1(K,T)$ and let A be an invariant set of (K,μ,T). Then also A^c is T-invariant. If $\mu(A) \in (0,1)$, then both $\mu_A/\mu(A)$ and $\mu_{A^c}/\mu(A^c)$ are T-invariant, probability measures, and μ is a non-trivial convex combination of these, showing $\mu \notin \mathrm{Ex}(\mathrm{M}_1(K,T))$. Conversely, suppose that (K,μ,T) is ergodic and $\mu = (1-t)\mu_1 + t\mu_2$ for some $t \in (0,1)$ and $\mu_1,\mu_2 \in \mathrm{M}_1(K,T)$. Then $\mu_1 \ll \mu$, so from part (a) we deduce $\mu_1 = \mu$, and hence $\mu_2 = \mu$. ∎

Remark 3.28. By the Krein–Milman theorem, see Lecture 1, $\mathrm{Ex}(\mathrm{M}_1(K,T))$ is non-empty, i.e., there is always an ergodic measure $\mu \in \mathrm{M}_1(K,T)$. Furthermore, we see that $\mathrm{M}_1(K,T)$ is a singleton if and only if there is a unique *ergodic* measure $\mu \in \mathrm{M}_1(K,T)$. In the latter case we call the topological system (K,T) **uniquely ergodic**.

Next we connect invariant measures and invariant sets for topological systems.

Proposition 3.29 (Invariant measures versus invariant sets)**.** *Let (K,T) be a topological system.*

(a) *For $\mu \in \mathrm{M}_1(K,T)$ we have $T\mathrm{supp}(\mu) = \mathrm{supp}(\mu)$. In particular, $\mathrm{supp}(\mu)$ is a closed, invariant set in the topological system (K,T).*
(b) *Let $L \subset K$ be a closed, invariant set. Then there is an ergodic measure $\mu \in \mathrm{M}_1(K,T)$ with $\mathrm{supp}(\mu) \subset L$.*
(c) *The system (K,T) is minimal if and only if $\mathrm{supp}(\mu) = K$ for every ergodic $\mu \in \mathrm{M}_1(K,T)$.*

Proof. (a) We first prove that $T\mathrm{supp}(\mu) \subset \mathrm{supp}(\mu)$. Let $y \in T\mathrm{supp}(\mu)$ and let U be an open set with $y \in U$. Then there is $x \in \mathrm{supp}(\mu)$ with $Tx = y$, and by continuity there is an open set V with $x \in V$ and $TV \subset U$. Since $V \subset T^{-1}(TV) \subset T^{-1}U$ and by the definition of the support, we obtain $0 < \mu(V) \leq \mu(T^{-1}U) = \mu(U)$. It follows that $y \in \mathrm{supp}(\mu)$.

To see the converse inclusion $\mathrm{supp}(\mu) \subset T\mathrm{supp}(\mu)$ let $f \in \mathrm{C}(K)$ vanish on the compact set $T\mathrm{supp}(\mu)$ but otherwise arbitrary. Then

$$\int_K f \,\mathrm{d}\mu = \int_K f \,\mathrm{d}T_*\mu = \int_K f \circ T \,\mathrm{d}\mu = \int_{\mathrm{supp}(\mu)} f \circ T \,\mathrm{d}\mu = 0.$$

By Proposition 1.12 we obtain that $\mathrm{supp}(\mu) \subset T\mathrm{supp}(\mu)$.

(b) By Remark 3.28 there is an ergodic $\nu \in \mathrm{M}_1(L,T)$. For $B \in \mathcal{B}(K)$ define $\mu(B) := \nu(B \cap L)$. Then $\mu \in \mathrm{M}_1(K,T)$ (why?) and $\mathrm{supp}(\mu) \subset L$. To show that μ is ergodic, let $A \subset K$ be an invariant set for (K,μ,T). Then $0 = \mu(A \setminus T^{-1}A) = \nu((A \cap L) \setminus (T^{-1}(A \cap L)))$. The ergodicity of ν implies $\mu(A) = \nu(A \cap L) \in \{0,1\}$.

(c) To show that under the asserted condition (K,T) is minimal we can apply part (b). Conversely, suppose that (K,T) is minimal. Then by part (a) $\mathrm{supp}(\mu) = K$ must hold for every invariant, in particular, for every ergodic measure μ. ∎

We finish this lecture with a higher-dimensional analogue of Kronecker's theorem. We present a proof motivated by what has been said in Lecture 1.

Proposition 3.30. *For $a \in \mathbb{T}^d$ consider the torus rotation $(\mathbb{T}^d, a)$, and denote its Koopman operator on $\mathrm{C}(\mathbb{T}^d)$ by L_a. Then for each $f \in \mathrm{C}(\mathbb{T}^d)$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} L_a^n f$$

exists in $\mathrm{C}(\mathbb{T}^d)$. If $a_1, \ldots, a_d$ are rationally independent, then the limit equals $\int_{\mathbb{T}^d} f \,\mathrm{dm}^d \cdot \mathbf{1}$.

Proof. For $N \in \mathbb{N}$ let $A_N := \frac{1}{N}\sum_{n=0}^{N-1} L_a^n$. Then $A_N \in \mathscr{L}(\mathrm{C}(\mathbb{T}^d))$ with $\|A_N\| \le 1$ for each $N \in \mathbb{N}$. Therefore, it suffices to prove the convergence of $A_N f$ for f in the dense subspace $\mathrm{P}(\mathbb{T}^d)$ of trigonometric polynomials (cf. Exercise 1.3). We may also restrict our attention to trigonometric monomials $\mathbf{z}^m$, where $m \in \mathbb{Z}^d$. We then have $L_a \mathbf{z}^m = a_1^{m_1} \cdots a_d^{m_d} \cdot \mathbf{z}^m$. If $b := a_1^{m_1} \cdots a_d^{m_d} = 1$, we readily see that $A_N \mathbf{z}^m = \mathbf{z}^m$ for each $N \in \mathbb{N}$. Otherwise we obtain

$$A_N \mathbf{z}^m = \frac{1}{N} \sum_{n=0}^{N-1} b^n \mathbf{z}^m = \frac{1}{N} \frac{b^N - 1}{b-1} \mathbf{z}^m \to 0 \quad \text{as } N \to \infty.$$

The last assertion follows by linearity and by the above calculation, since in the case of rational independence for every $m \in \mathbb{Z}^d \setminus \{0\}$ the limit of $A_N \mathbf{z}^m$ equals $0 = \int_{\mathbb{T}^d} \mathbf{z}^m \,\mathrm{dm}^d \cdot \mathbf{1}$, and the limit of $A_N \mathbf{z}^0 = \mathbf{1}$ the limit equals $\mathbf{1}$. ∎

As a corollary we obtain the promised result.

Theorem 3.31 (Kronecker)**.** *The torus rotation $(\mathbb{T}^d, a)$ with $a = (a_1, \ldots, a_d) \in \mathbb{T}^d$ is minimal if and only if $a_1, \ldots, a_d$ are rationally independent. In this case $(\mathbb{T}^d, a)$ is uniquely ergodic, and the Haar measure is the unique (ergodic) invariant measure.*

Proof. If $a_1^{m_1} \cdots a_d^{m_d} = 1$ for some $m \in \mathbb{Z}^d \setminus \{0\}$, then

$$F := \{z \in \mathbb{T}^d : z_1^{m_1} \cdots z_d^{m_d} = 1\}$$

is a non-trivial, closed, invariant set. Therefore $(\mathbb{T}^d, a)$ is not minimal.

Conversely, suppose $a_1^{m_1} \cdots a_d^{m_d} \neq 1$ for each $m \neq 0$. Proposition 3.22 and the hypothesis yield that m^d is an ergodic measure for $(\mathbb{T}^d, a)$. Let $\mu \in \mathrm{M}_1(\mathbb{T}^d, a)$ be any ergodic measure. By the previous proposition and by the dominated convergence theorem $\langle f, \mu\rangle = \langle A_N f, \mu\rangle \to \langle f, \mathrm{m}^d\rangle\langle \mathbf{1}, \mu\rangle = \langle f, \mathrm{m}^d\rangle$ as $N \to \infty$ for every $f \in \mathrm{C}(\mathbb{T}^d)$. This implies $\mu = \mathrm{m}^d$. Since $\mathrm{supp}(\mathrm{m}^d) = \mathbb{T}^d$, Proposition 3.29 yields that $(\mathbb{T}^d, a)$ is minimal. ∎

The following characterization can be proven analogously to the handled special case of $\mathbb{T}^d$ using more theory of compact, abelian groups, see, e.g., [EFHN, Thm. 10.13 and Prop. 14.21].

Theorem 3.32 (Ergodicity and minimality for rotations)**.** *Let (G, a) be a rotation system. Then the following assertions are equivalent.*

(i) *(G, a) is minimal.*
(ii) *(G, m_G, a) is ergodic.*
(iii) *$\{a^n : n \in \mathbb{N}_0\}$ is dense in G.*

In this case, G is abelian, (G, a) is uniquely ergodic, and m_G is the unique invariant probability measure.

Exercises

Exercise 3.1 (Invariant sets)**.** Let (K, T) be an invertible topological system and let $A \subset K$ be a non-empty, closed, invariant set. Show that there exists a non-empty, closed, two-sided invariant set $B \subset A$.

Exercise 3.2 (Minimality)**.** Prove the characterization of minimality via the orbits as explained in Remark 3.1.

Exercise 3.3 (Group rotations)**.** Prove that if a group rotation (G, a) is minimal, then G is abelian. Is the converse true?

Exercise 3.4. Prove Propositions 3.7 and 3.8.

Exercise 3.5 (Periodic points)**.** Let (K, T) be a topological system. A point $x \in K$ is called **periodic** in (K, T) if there is $p \in \mathbb{N}$ such that $T^p x = x$. Show that a periodic point is almost periodic. Is the converse true?

Exercise 3.6. Consider the compact space $K = \{0, 1\}^{\mathbb{N}}$ and the one-sided, shift system (K, T) from Example 2.24.

(a) Prove that a point $x \in K$ is almost periodic if and only if for each finite subword y of x there is $\ell \in \mathbb{N}$ such that the gap between each two subsequent occurrences of y is of length at most ℓ.
(b) Give an example of an almost periodic but not periodic point $x \in K$ (cf. Exercise 3.5).
(c) Give an example of a point $x \in K$ which is recurrent but not almost periodic (cf. Proposition 3.7).

Exercise 3.7 (Invariant sets)**.** Let (X, μ, T) be a measure-preserving system. Prove that a set $A \in \mathcal{X}$ is invariant if and only if $A \subset T^{-1}A$ up to a null set, and if and only if $T^{-1}A \subset A$ up to a null set. Prove that the T-invariant sets form a σ-algebra.

Exercise 3.8 (Ergodicity)**.** Prove that a measure-preserving system (X, μ, T) is ergodic if and only if each measurable set $A \subset X$ with $T^{-1}A = A$ satisfies $\mu(A) \in \{0, 1\}$.

Exercise 3.9 (Invariant measures)**.** Let (K, T) be a topological system. Show that $\mathrm{M}_1(K, T)$ of all T-invariant regular, Borel, probability measures on K is a convex and weak* compact subset of $\mathrm{M}(K)$.

Exercise 3.10 (Rotations on the unit disc)**.** Consider the closed unit disc $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and let $a \in \mathbb{T}$ be not a root of unity. The rotation $\tau_a : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, $z \mapsto az$ is continuous, and gives rise to the invertible topological system $(\overline{\mathbb{D}}, \tau_a)$. Describe all closed, τ_a-invariant sets, all invariant probability measures, and determine the ergodic ones. Can you give a relation between Proposition 1.14(c) and Proposition 3.27(c) in this situation?

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LECTURE 4

Fourier transform of measures

In this lecture we study the Banach space $\mathrm{M}(\mathbb{T})$ of complex, Borel measures on the torus in more detail, and for this purpose we introduce the Fourier transform as a fundamental and extremely powerful tool.

1. Fourier transform

Recall that for $n \in \mathbb{Z}$ the function $\mathbf{z}^n : \mathbb{T} \to \mathbb{T}$ is defined by $z \mapsto z^n$ and the linear combinations of such functions, the trigonometric polynomials, form a dense subspace of $\mathrm{C}(\mathbb{T})$ (see Corollary 1.9), and hence of $\mathrm{L}^2(\mathbb{T}, \mu)$ for each positive measure $\mu \in \mathrm{M}(\mathbb{T})$. Recall also the following estimate for a measure $\mu \in \mathrm{M}(\mathbb{T})$ and $f \in \mathrm{L}^1(\mathbb{T}, |\mu|)$

$$\Big|\int_{\mathbb{T}} f \,\mathrm{d}\mu\Big| \le \int_{\mathbb{T}} |f| \,\mathrm{d}|\mu| \le \|f\|_{\mathrm{L}^1(\mathbb{T},|\mu|)}.$$

Definition 4.1. Let $\mu \in \mathrm{M}(\mathbb{T})$, and let $n \in \mathbb{Z}$. We call

$$\hat{\mu}(n) := \int_{\mathbb{T}} \mathbf{z}^{-n} \,\mathrm{d}\mu$$

the n**th Fourier coefficient** of μ. The sequence $\hat{\mu} = (\hat{\mu}(n))_{n\in\mathbb{Z}}$ is called the **Fourier transform of** μ, and the mapping $\mathscr{F} : \mu \mapsto \hat{\mu}$ the **Fourier transform**.

Proposition 4.2. *The Fourier transform $\mathscr{F} : \mathrm{M}(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$ is an injective, linear contraction.*

Proof. It is clear that $\mathscr{F} : \mathrm{M}(\mathbb{T}) \to \mathbb{C}^{\mathbb{Z}}$ is linear. For each $n \in \mathbb{Z}$ we have $|\hat{\mu}(n)| \le \langle |\mathbf{z}^{-n}|, |\mu|\rangle \le |\mu|(\mathbb{T}) = \|\mu\|$, so $\mathscr{F}$ maps into $\ell^\infty(\mathbb{Z})$ and it is a contraction. To show injectivity, suppose $\hat{\mu}(n) = 0$ for each $n \in \mathbb{Z}$. Then $\langle p, \mu\rangle = 0$ holds for every trigonometric polynomial $p \in \mathrm{P}(\mathbb{T})$. Since $\mathrm{P}(\mathbb{T})$ is dense in $\mathrm{C}(\mathbb{T})$, we obtain $\mu = 0$. ∎

Before we relate the Fourier transform of measures to the Fourier transform of functions, we record here the following general fact that will be needed also in the later lectures (see [RCA, Thms. 6.9–6.13]).

Proposition 4.3. *Let X be a set, let $\mathcal{X}$ be a σ-algebra over X, let $\mu \in \mathrm{M}(X, \mathcal{X})$ be a complex measure, and let $f \in \mathrm{L}^1(X, |\mu|)$. For $A \in \mathcal{X}$ define*

$$\nu(A) := \int_A f \,\mathrm{d}\mu.$$

Then the following assertions hold.

(a) *ν is a complex measure which is absolutely continuous with respect to μ. We use the notation $f \cdot \mu$ for this measure ν.*

(b) *For every bounded measurable function $g : X \to \mathbb{C}$*

$$\int_X g \,\mathrm{d}\nu = \int_X gf \,\mathrm{d}\mu.$$

The same assertion holds for each $g : X \to \mathbb{C}$ with $fg \in \mathrm{L}^1(X, |\mu|)$.

(c) *There is a Borel measurable function $h : X \to \mathbb{T}$ such that $\mu = h \cdot |\mu|$. This is called the* ***polar decomposition*** *of μ. Moreover, we have $|\mu| = \overline{h} \cdot \mu$.*

(d) *The total variations satisfy $|\nu| = |f| \cdot |\mu|$. In particular, $\|\nu\| = \|f\|_{\mathrm{L}^1(X,|\mu|)}$.*

(e) *Suppose $\mu \in \mathrm{M}(X, \mathcal{X})$ is a positive measure. The mapping $J_\mu : \mathrm{L}^1(X, \mu) \to \mathrm{M}(X, \mathcal{X})$, $J_\mu f := f \cdot \mu$ is a linear isometry preserving the modulus, i.e., $J_\mu|f| = |f \cdot \mu|$. Moreover, f is real-valued if and only if $J_\mu f$ is a signed measure, and $f \geq 0$ if and only if $f \cdot \mu$ is a positive measure.*

If we identify $\mathrm{L}^1(\mathbb{T}, \mathrm{m})$ with a closed subspace of $\mathrm{M}(\mathbb{T})$ (under the isometry J_m from the previous proposition, where m is the Haar measure on $\mathbb{T}$) we get back the classical definition of the Fourier transform of L^1-functions given by

$$\hat{f}(n) = \int_\mathbb{T} f\mathbf{z}^{-n} \,\mathrm{dm} \quad \text{for } f \in \mathrm{L}^1(\mathbb{T}, \mathrm{m}) \text{ and } n \in \mathbb{Z},$$

or, in a more familiar way (after identifying $[0, 1)$ with addition mod 1 with $\mathbb{T}$, cf. Example 2.11),

$$\hat{f}(n) = \int_{[0,1]} f(s)\mathrm{e}^{-2\pi ins} \,\mathrm{d}s \quad \text{for } f \in \mathrm{L}^1([0, 1], \lambda) \text{ and } n \in \mathbb{Z}.$$

Thus Definition 4.1 consistently extends the notion of Fourier transform of functions. Since $(\mathbf{z}^n)_{n\in\mathbb{Z}}$ is an orthonormal basis in $\mathrm{L}^2(\mathbb{T}, \mathrm{m})$, we immediately obtain the following important fact, which we record here for completeness.

Proposition 4.4 (Parseval's identity)**.** *The Fourier transform $\mathscr{F} : \mathrm{L}^2(\mathbb{T}, \mathrm{m}) \to \ell^2(\mathbb{Z})$ is a unitary operator.*

Next come some basic computational rules for the Fourier transform, see Exercise 4.1.

Proposition 4.5 (Properties of the Fourier transform)**.** *Let $\mu, \nu \in \mathrm{M}(\mathbb{T})$.*

(a) *For the rotation $\tau_a : \mathbb{T} \to \mathbb{T}$ by $a \in \mathbb{T}$ we have $\widehat{(\tau_a)_*\mu}(n) = a^{-n}\hat{\mu}(n)$ for every $n \in \mathbb{Z}$.*

(b) *For $m, n \in \mathbb{Z}$ we have $\widehat{\mathbf{z}^m \cdot \mu}(n) = \hat{\mu}(n - m)$.*

(c) *For $n \in \mathbb{Z}$ we have $\hat{\overline{\mu}}(n) = \overline{\hat{\mu}}(-n)$, where $\overline{\mu}$ is the complex conjugate of the measure μ.*

Our next aim is to study the asymptotic behavior of the sequence $(\hat{\mu}(n))_{n\in\mathbb{Z}}$. A measure $\mu \in \mathrm{M}(\mathbb{T})$ is called **Rajchman** if

$$\lim_{|n|\to\infty} \hat{\mu}(n) = 0.$$

For example, for $f \in \mathrm{L}^2(\mathbb{T}, \mathrm{m})$ the measure $f \cdot \mathrm{m}$ is Rajchman by, e.g., Parseval's identity. Rajchman measures form a closed subspace of $\mathrm{M}(\mathbb{T})$, see Exercise 4.2.

Proposition 4.6. *Let $\mu \in \mathrm{M}(\mathbb{T})$ be a Rajchman measure. Then every $\nu \in \mathrm{M}(\mathbb{T})$ with $\nu \ll \mu$ is a Rajchman measure.*

Proof. Since μ is a Rajchman measure, so is the measure $\mathbf{z}^k \cdot \mu$ for each fixed $k \in \mathbb{Z}$ (see Proposition 4.5). As a consequence, $p \cdot \mu$ is a Rajchman measure for each trigonometric polynomial $p \in \mathrm{P}(\mathbb{T})$. Since $\nu \ll |\mu|$, by the Radon–Nikodym theorem (Theorem 3.26), we can write $\nu = g \cdot |\mu|$ for some $g \in \mathrm{L}^1(\mathbb{T}, |\mu|)$. We have the polar decomposition $\mu = h \cdot |\mu|$ for some measurable $h : \mathbb{T} \to \mathbb{T}$, so that $\nu = g\overline{h} \cdot \mu$. For $f := g\overline{h}$ and for every trigonometric polynomial p we have

$$\begin{aligned}\limsup_{|n|\to\infty} \big|\mathscr{F}(f \cdot \mu)(n)\big| &\leq \limsup_{|n|\to\infty} \big|\mathscr{F}((f-p) \cdot \mu)(n)\big| \leq \|\mathscr{F}((f-p)\cdot\mu)\|_\infty \\ &\leq \|f-p\|_{\mathrm{L}^1(\mathbb{T},|\mu|)}.\end{aligned}$$

With an appropriate choice of $p \in \mathrm{P}(\mathbb{T})$ the term $\|f-p\|_{\mathrm{L}^1(\mathbb{T},|\mu|)}$ can be made arbitrarily small. This proves the assertion. ∎

Since m is a Rajchman measure (why?), we obtain the classical Riemann–Lebesgue Lemma stating that all measures $\mu \in \mathrm{M}(\mathbb{T})$ which are absolutely continuous with respect to m are Rajchman.

Theorem 4.7 (Riemann–Lebesgue Lemma)**.** *For $f \in \mathrm{L}^1(\mathbb{T}, \mathrm{m})$ we have*

$$\lim_{|n|\to\infty} \hat{f}(n) = 0.$$

Remark 4.8. Since μ, $\overline{\mu}$ are equivalent measures, and since $\hat{\overline{\mu}}(-n) = \overline{\hat{\mu}}(n)$ for each $n \in \mathbb{Z}$, the proof of Proposition 4.6 yields (considering $\limsup_{n\to\infty}$ instead of $\limsup_{|n|\to\infty}$) that

$$\mu \text{ is Rajchman} \iff \lim_{n\to\infty} \hat{\mu}(n) = 0 \iff \lim_{n\to-\infty} \hat{\mu}(n) = 0.$$

2. Discrete and continuous measures

The Fourier transform of a measure $\mu \in \mathrm{M}(\mathbb{T})$ is very useful when studying qualitative properties of μ. The following is an elementary example for this.

Proposition 4.9. *For $\mu \in \mathrm{M}(\mathbb{T})$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \hat{\mu}(n) = \mu(\{1\}).$$

Proof. The assertion follows from the formula $\sum_{n=0}^{N-1} z^n = \frac{z^N - 1}{z-1}$ for $z \neq 1$, and from Lebesgue's theorem. ∎

Therefore, the limit of arithmetic averages of the Fourier coefficients of μ tell us whether μ contains δ_1 as a component. As we shall see shortly, with similar arguments we can see from the Fourier coefficients whether μ vanishes on each at most countable set. We give a name to this latter property.

Definition 4.10. A measure $\mu \in \mathrm{M}(\mathbb{T})$ is called **continuous** if $\mu\{a\} = 0$ for each $a \in \mathbb{T}$, i.e., if μ vanishes on each at most countable set (from now on we use the abbreviation $\mu\{a\}$ for $\mu(\{a\})$). A measure $\mu \in \mathrm{M}(\mathbb{T})$ is called **discrete** if there is an at most countable set $A \subset \mathbb{T}$ and a function $c : A \to \mathbb{C} \setminus \{0\}$ such that

$$\mu = \sum_{a\in A} c_a \delta_a.$$

In this case, the elements of A are called **atoms** of μ. The series here is unconditionally convergent in $\mathrm{M}(\mathbb{T})$.

A measure which is continuous and discrete at the same time is identically 0. The following decomposition is classical and will be of extreme importance in the next lecture.

Proposition 4.11 (Continuous–discrete decomposition)**.** (a) *The subsets*

$$\mathrm{M}_c(\mathbb{T}) := \{\mu : \mu \in \mathrm{M}(\mathbb{T}) \textit{ is continuous}\}$$

and

$$\mathrm{M}_d(\mathbb{T}) := \{\mu : \mu \in \mathrm{M}(\mathbb{T}) \textit{ is discrete}\}$$

are closed subspaces of $\mathrm{M}(\mathbb{T})$*.*

(b) *For each* $\mu \in \mathrm{M}(\mathbb{T})$ *there is a unique continuous measure* μ_c *and a unique discrete measure* μ_d *such that* $\mu = \mu_c + \mu_d$*. Moreover, we have*

$$\|\mu_d\| = |\mu_d|(\mathbb{T}) = \sum_{a\in\mathbb{T}} |\mu\{a\}|.$$

Proof. (a) The proof is left as Exercise 4.5.

(b) For $\mu \in \mathrm{M}(\mathbb{T})$ the set $A := \{a : a \in \mathbb{T},\ \mu\{a\} \neq 0\}$ is at most countable. The measure $\mu_d := \sum_{a\in A} \mu\{a\}\delta_a$ is discrete, and $\mu_c = \mu - \mu_d$ is continuous. Uniqueness follows from part (a). ∎

The following result of Wiener connecting continuity of a measure μ to the asymptotic behavior of $\hat{\mu}$, has an elementary proof and, as we shall see, a number of important applications.

Proposition 4.12 (Wiener's Lemma)**.** *For* $\mu \in \mathrm{M}(\mathbb{T})$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\mu}(n)|^2 = \sum_{a\in\mathbb{T}} |\mu\{a\}|^2. \tag{4.1}$$

In particular, μ *is continuous if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\mu}(n)|^2 = 0.$$

Proof. Let $\Delta = \{(z,z) : z \in \mathbb{T}\} \subset \mathbb{T}\times\mathbb{T}$ be the diagonal. Fubini's theorem and Lebesgue's dominated convergence theorem imply

$$\begin{aligned}
\frac{1}{N} \sum_{n=0}^{N-1} |\hat{\mu}(n)|^2 &= \frac{1}{N} \sum_{n=0}^{N-1} \int_{\mathbb{T}} z^{-n}\,\mathrm{d}\mu(z) \int_{\mathbb{T}} \overline{w}^{-n}\,\mathrm{d}\overline{\mu}(w) \\
&= \int_{\mathbb{T}\times\mathbb{T}} \frac{1}{N} \sum_{n=0}^{N-1} (z\overline{w})^{-n}\,\mathrm{d}(\mu\otimes\overline{\mu})(z,w) \\
&= \int_{\Delta} \mathbf{1}\,\mathrm{d}(\mu\otimes\overline{\mu})(z,w) + \int_{\Delta^c} \frac{1}{N}\frac{(z\overline{w})^N - 1}{z\overline{w} - 1}\,\mathrm{d}(\mu\otimes\overline{\mu})(z,w) \\
&\to \int_{\mathbb{T}\times\mathbb{T}} \mathbf{1}_\Delta\,\mathrm{d}(\mu\otimes\overline{\mu})(z,w) \quad \text{as } N\to\infty \\
&= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \mathbf{1}_{\{w\}}(z)\,\mathrm{d}\mu(z) \right) \mathrm{d}\overline{\mu}(w) = \int_{\mathbb{T}} \mu\{w\}\,\mathrm{d}\overline{\mu}(w) = \sum_{a\in\mathbb{T}} |\mu\{a\}|^2.
\end{aligned}$$

The final assertion follows directly from (4.1). ∎

A sequence $(x_n)_{n\in\mathbb{N}_0}$ in a Banach space E is called **Cesàro convergent** to x if the arithmetic averages, or **Cesàro averages**,

$$\frac{1}{N}\sum_{n=0}^{N-1} x_n \quad \text{converge to } x \text{ as } N \to \infty,$$

in this case x is called the **Cesàro limit** of the sequence. Thus for a continuous measure μ the sequence $(|\hat{\mu}(n)|^2)_{n\in\mathbb{N}_0}$ is Cesàro convergent to 0. Here are some elementary properties of Cesàro convergence, see Exercise 4.4.

Proposition 4.13 (Properties of the Cesàro limit)**.** *Let E be a Banach space and let $(x_n)_{n\in\mathbb{N}_0}$ and $(y_n)_{n\in\mathbb{N}_0}$ be two sequences in E.*

(a) *The sequence $(x_n)_{n\in\mathbb{N}_0}$ has at most one Cesàro limit.*

(b) *The Cesàro limit is shift invariant, i.e.,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} x_n = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n$$

holds whenever one of these limits exists.

(c) *If $(x_n)_{n\in\mathbb{N}_0}$ is convergent to x, then it is Cesàro convergent to x.*

(d) *Suppose $(x_n)_{n\in\mathbb{N}_0}$ is a periodic sequence, i.e., one that satisfies $x_n = x_{n+m}$ for some $m \in \mathbb{N}$ and each $n \in \mathbb{N}_0$. Then $(x_n)_{n\in\mathbb{N}_0}$ is Cesàro convergent to*

$$\frac{1}{m}\sum_{n=0}^{m-1} x_n.$$

(e) *Suppose that the sequences $(\lambda_n)_{n\in\mathbb{N}_0}$ in $\mathbb{C}$ and $(x_n)_{n\in\mathbb{N}_0}$ are bounded, one of them is convergent, the other one Cesàro convergent with respective limits λ and x. Then $(\lambda_n x_n)_{n\in\mathbb{N}_0}$ is Cesàro convergent to λx.*

For a Rajchman measure we have $|\hat{\mu}(n)|^2 \to 0$, so as a consequence we see that every Rajchman measure $\mu \in \mathrm{M}(\mathbb{T})$ is continuous. However, not every continuous measure is Rajchman as the following example shows.

Example 4.14 (Cantor–Lebesgue measure)**.** Consider the ternary Cantor set $C \subset [0,1]$ and the compact space $K := \{0,1\}^{\mathbb{N}}$. Then

$$\theta : K \to C, \quad (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} \frac{2x_n}{3^n}$$

is a homeomorphism. Consider the Bernoulli shift $B(1/2,1/2)$ on K with μ the corresponding measure. Let $S : C \to C$ be given by $Sx = 3x - \lfloor 3x \rfloor$, then S is measurable, the push-forward measure $\theta_*\mu$ is S-invariant and θ is an isomorphism between the measure-preserving systems $B(1/2,1/2)$ and $(C, \theta_*\mu, S)$. Now $\theta_*\mu$ can be extended to the Borel σ-algebra $\mathcal{B}([0,1])$ by 0 on subsets of $[0,1] \setminus C$. This extension is called the **Cantor–Lebesgue measure** and is denoted by μ_{CL}. By construction $\operatorname{supp}(\mu_{\mathrm{CL}}) = C$ and it is easy to see that μ_{CL} is a continuous measure. We can consider the push-forward of μ_{CL} under the map $s \mapsto \mathrm{e}^{2\pi \mathrm{i} s}$, which is also denoted by μ_{CL}. By Exercise 4.3 for each $n \in \mathbb{Z}$

$$\hat{\mu}_{\mathrm{CL}}(n) = \hat{\mu}_{\mathrm{CL}}(3n).$$

Since $\mu_{\mathrm{CL}} \neq \mathrm{m}$, there is $n \in \mathbb{Z} \setminus \{0\}$ such that $\hat{\mu}_{\mathrm{CL}}(n) \neq 0$. It follows that μ_{CL} is not a Rajchman measure.

Every measure $\mu \in \mathrm{M}(\mathbb{T})$ satisfies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\mu}(n)|^2 \le \|\hat{\mu}\|_\infty^2 \le \|\mu\|^2.$$

Thus the case when we have equality here can be regarded as extremal.

Proposition 4.15. *For a measure $\mu \in \mathrm{M}(\mathbb{T})$ the following assertions are equivalent.*

(i) $\lim\limits_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\mu}(n)|^2 = \|\mu\|^2$.

(ii) *The support* $\mathrm{supp}(\mu)$ *has at most one element, i.e., μ is a scalar multiple of a Dirac measure.*

Proof. (i)⇒(ii): The hypothesis and Proposition 4.12 yield

$$\|\mu\|^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\mu}(n)|^2 = \sum_{a \text{ atom}} |\mu\{a\}|^2 \le \|\mu\| \sum_{a \text{ atom}} |\mu\{a\}| \le \|\mu\|^2,$$

implying $|\mu\{a\}|^2 = \|\mu\| \cdot |\mu\{a\}|$ or, equivalently, $|\mu\{a\}| \in \{0, \|\mu\|\}$ for every atom a. We conclude that μ is a scalar multiple of a Dirac measure.

(ii)⇒(i) follows from a simple computation for $\mu = c\delta_a$ ∎

3. Upper and lower density

Let $\mu \in \mathrm{M}(\mathbb{T})$ be a continuous measure. By Wiener's Lemma, it is intuitively clear that for a given number $r > 0$ the inequality $|\hat{\mu}(n)| > r$ cannot hold for n from a too large subset of $\mathbb{Z}$. To understand what "too large" means here, and to describe better in which sense the Fourier coefficients $\hat{\mu}(n)$ converge to 0, we introduce the following notions.

Definition 4.16. (a) The **upper density** $\overline{\mathrm{d}}(A)$ of a subset $A \subset \mathbb{N}_0$ is

$$\overline{\mathrm{d}}(A) := \limsup_{N\to\infty} \frac{|A \cap \{1, \dots, N\}|}{N} = \limsup_{N\to\infty} \frac{|A \cap \{0, \dots, N-1\}|}{N},$$

where $|B|$ denotes the number of elements in B (why does the second equality hold?). Analogously, the **lower density** $\underline{\mathrm{d}}(A)$ is defined by $\liminf$ replacing $\limsup$. Obviously, $0 \le \underline{\mathrm{d}}(A) \le \overline{\mathrm{d}}(A) \le 1$. If $\overline{\mathrm{d}}(A) = \underline{\mathrm{d}}(A)$, this common value (given by the limit) is called the **density** of A.

(b) We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space Ω **converges in density** to $x \in \Omega$ if there is a subset $J = \{n_1 < n_2 < \cdots\}$ with density 1 and such that $x_{n_j} \to x$ as $j \to \infty$. We denote this by writing

$$x = \operatorname*{D-lim}_{n\to\infty} x_n \quad \text{or} \quad x_n \xrightarrow{\mathrm{D}} x.$$

Here are some essential properties.

Proposition 4.17. (a) *For each subset $A \subset \mathbb{N}_0$ we have $\overline{\mathrm{d}}(A^c) = 1 - \underline{\mathrm{d}}(A)$.*

(b) *For each pair of subsets $A, B \subset \mathbb{N}_0$*

$$\overline{\mathrm{d}}(A \cup B) \le \overline{\mathrm{d}}(A) + \overline{\mathrm{d}}(B).$$

(c) *For each pair of disjoint subsets $A, B \subset \mathbb{N}_0$*

$$\underline{\mathrm{d}}(A \cup B) \ge \underline{\mathrm{d}}(A) + \underline{\mathrm{d}}(B).$$

(d) *If* $\mathrm{d}(A) = 1$, *then for each* $B \subset \mathbb{N}_0$

$$\overline{\mathrm{d}}(A \cap B) = \overline{\mathrm{d}}(B) \quad \textit{and} \quad \underline{\mathrm{d}}(A \cap B) = \underline{\mathrm{d}}(B).$$

(e) *If* $A_1, \dots, A_k \subset \mathbb{N}_0$ *satisfy* $\mathrm{d}(A_1) = \dots = \mathrm{d}(A_k) = 1$, *then* $\mathrm{d}(A_1 \cap \dots \cap A_k) = 1$.

(f) *If* $A_k \subset \mathbb{N}_0$ *and* $\mathrm{d}(A_k) = 1$ *for every* $k \in \mathbb{N}$, *then there is a subset* $A \subset \mathbb{N}$ *such that* $\mathrm{d}(A) = 1$ *and for each* $k \in \mathbb{N}$ *the set* $A \setminus A_k$ *is finite.*

Proof. The proof of (a)–(e) is left as Exercise 4.6.

(f) We may suppose by (e) that $A_{k+1} \subset A_k$ for every $k \in \mathbb{N}$. By the hypothesis for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\frac{|A_k \cap \{1, \dots, m\}|}{m} > 1 - \frac{1}{k} \quad \text{for every } m \geq n_k. \tag{4.2}$$

Moreover, we can assume that $(n_k)_{k \in \mathbb{N}}$ is strictly increasing. Define

$$A := \bigcap_{k \in \mathbb{N}} \Big(A_k \cup \{1, \dots, n_k\} \Big).$$

Then, of course, $A \setminus A_k$ is finite for every $k \in \mathbb{N}$. We claim that $\underline{\mathrm{d}}(A) = 1$. Set $n_0 := 0$ and let $m \in \mathbb{N}$ be arbitrary. Take $k \in \mathbb{N}_0$ such that $n_k < m \leq n_{k+1}$. Let $x \in A_k \cap \{1, \dots, m\}$, then, by monotonicity, $x \in A_j \cap \{1, \dots, m\}$ for each $j \in \{1, \dots, k\}$. On the other hand, for $j > k$ we have $x \in \{1, \dots, n_k\} \subset \{1, \dots, n_j\}$. Altogether we obtain $x \in A \cap \{1, \dots, m\}$ and thus by (4.2)

$$\frac{|A \cap \{1, \dots, m\}|}{m} \geq \frac{|A_k \cap \{1, \dots, m\}|}{m} \geq 1 - \frac{1}{k}.$$

It follows that $\underline{\mathrm{d}}(A) = 1$ implying $\mathrm{d}(A) = 1$. ∎

The next very useful result relates convergence of Cesàro averages and convergence in density.

Proposition 4.18 (Koopman–von Neumann Lemma)**.** (a) *For a bounded sequence* $(a_n)_{n \in \mathbb{N}}$ *in* $[0, \infty)$ *the following are equivalent.*

(i) $\operatorname*{D\text{-}lim}_{n \to \infty} a_n = 0$.

(ii) $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n = 0$.

(iii) $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n^2 = 0$.

(b) *For a bounded sequence* $(b_n)_{n \in \mathbb{N}}$ *in* $(-\infty, 1]$ *the following are equivalent.*

(i) $\operatorname*{D\text{-}lim}_{n \to \infty} b_n = 1$.

(ii) $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_n = 1$.

Moreover, if $b_n \geq 0$ *for each* $n \in \mathbb{N}$, *then these assertions are equivalent to:*

(iii) $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_n^2 = 1$.

Proof. (a) We first prove (i)⇒(ii). Let $J \subset \mathbb{N}$ have density 1 such that $\lim_{n \to \infty, n \in J} a_n = 0$. Let $\varepsilon > 0$ be arbitrary, and let $N \in \mathbb{N}$ be such that for each $n \in J$ with $n \geq N$ we have $a_n \leq \varepsilon$. The following estimates hold for each such n:

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^{n} a_j &= \frac{1}{n} \sum_{j=1}^{N} a_j + \frac{1}{n} \sum_{j=N+1}^{n} a_j \leq \sup_{i \in \mathbb{N}} a_i \frac{N}{n} + \frac{1}{n} \sum_{\substack{j=N+1 \\ j \in J}}^{n} a_j + \frac{1}{n} \sum_{\substack{j=N+1 \\ j \notin J}}^{n} a_j \\ &\leq \sup_{i \in \mathbb{N}} a_i \frac{N}{n} + \varepsilon + \sup_{i \in \mathbb{N}} a_i \frac{|\{1, \dots, n\} \setminus J|}{n}. \end{aligned}$$

Taking $\limsup_{n\to\infty}$ of both sides, we obtain by using Proposition 4.17(a)

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} a_j \leq \varepsilon + \overline{\mathrm{d}}(\mathbb{N}\setminus J) = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the assertion follows.
(ii)⇒(i): For $k \in \mathbb{N}$ let $B_k := \{n \in \mathbb{N} : a_n \geq 1/k\}$. Then

$$\frac{|B_k \cap \{1,\dots,n\}|}{n} \leq \frac{1}{n}\sum_{j=1}^{n} k a_j \to 0 \quad \text{as } n\to\infty$$

by hypothesis. This means $\mathrm{d}(B_k) = 0$ for each $k \in \mathbb{N}$. By Proposition 4.17 (applied to $A_k = B_k^c$) there is $A \subset \mathbb{N}$ with density 1 such that $A \cap A_k$ is finite for each $k \in \mathbb{N}$, implying $\lim\limits_{\substack{n\to\infty\\ n\in A}} a_n = 0$.
(i)⇔(iii) follows from (i)⇔(ii) since

$$\text{D-}\lim_{n\to\infty} a_n = 0 \quad\iff\quad \text{D-}\lim_{n\to\infty} a_n^2 = 0.$$

(b) The equivalence (i)⇔(ii) follows from (a) by considering $a_n := 1 - b_n$, while (i)⇔(iii) can be proven by the same arguments as in part (a). ∎

Returning to Fourier coefficients of measures we can state the following as an immediate consequence of the previous discussion.

Proposition 4.19. *Let $\mu \in \mathrm{M}(\mathbb{T})$ be a probability measure. Then the following equivalences hold.*
(a) $\text{D-}\lim\limits_{n\to\infty} |\hat{\mu}(n)| = 1 \iff$ *μ is a Dirac measure.*
(b) $\text{D-}\lim\limits_{n\to\infty} |\hat{\mu}(n)| = 0 \iff$ *μ is continuous.*

4. Positive definite sequences

We are now interested in whether a given complex sequence is a Fourier transform of a positive measure. Of course, if so, the sequence needs to be bounded. The following definition gives an additional property that such sequences must have.

Definition 4.20. A function $c : \mathbb{Z} \to \mathbb{C}$ (i.e., a sequence $(c(n))_{n\in\mathbb{Z}}$) is called **positive definite**, if for all complex scalar sequences $(x_n)_{n\in\mathbb{N}}$ with finite support

$$\sum_{i,j\in\mathbb{N}} c(i-j)x_i\overline{x_j} \geq 0.$$

A sequence $(c(n))_{n\in\mathbb{Z}}$ is therefore positive definite if and only if for each $N \in \mathbb{N}$ the matrix $(c(i-j))_{i,j=1}^{N}$ is positive semidefinite. Some further elementary properties are listed in Exercise 4.9.

Example 4.21. The Fourier transform $\hat{\mu}$ of a positive measure $\mu \in \mathrm{M}(\mathbb{T})$ is positive definite. Indeed, for $(x_n)_{n\in\mathbb{N}}$ with finite support

$$\begin{aligned}\sum_{i,j}\hat{\mu}(i-j)x_i\overline{x_j} &= \sum_{i,j}\int_{\mathbb{T}} \mathbf{z}^{j-i}x_i\overline{x_j}\,\mathrm{d}\mu = \int_{\mathbb{T}}\sum_i \mathbf{z}^{-i}x_i \cdot \sum_j \overline{\mathbf{z}^{-j}x_j}\,\mathrm{d}\mu\\ &= \int_{\mathbb{T}}\Bigl|\sum_k \mathbf{z}^k x_k\Bigr|^2\,\mathrm{d}\mu \geq 0.\end{aligned}$$

Our ultimate aim is to show that each positive definite sequence is the Fourier transform of a positive measure. For this we need some preparation, which takes up the rest of this section.

Another class of positive definite sequences can be given using the convolution. The **convolution** of two sequences $a, b \in \ell^1(\mathbb{Z})$ is the sequence $a * b$ defined by

$$(a * b)(n) = \sum_{k \in \mathbb{Z}} a(n-k)b(k).$$

Note that for each $n \in \mathbb{Z}$ the previous series is absolutely convergent.

Example 4.22. For $a \in \ell^1(\mathbb{Z})$ let the sequence $\tilde{a}$ be defined by $\tilde{a}(j) := \overline{a}(-j)$ for $j \in \mathbb{Z}$. Then $a * \tilde{a}$ is positive definite. Indeed, by substituting $k = l - j$ and then by interchanging summation(!) we obtain for $(x_n)_{n \in \mathbb{N}}$ with finite support

$$\begin{aligned}\sum_{i,j} (a * \tilde{a})(i-j) x_i \overline{x_j} &= \sum_{i,j} \sum_k a(i-j-k)\overline{a(-k)} x_i \overline{x_j} \\ = \sum_{i,j} \sum_l a(i-l) x_i \overline{a(j-l)}\, \overline{x_j} &= \sum_l \sum_i a(i-l) x_i \cdot \overline{\sum_j a(j-l) x_j} \geq 0.\end{aligned}$$

As a next step we prove that the product of positive definite sequences is positive definite. For this some facts from linear algebra need to be recalled. The **Hadamard product** $A \circ B$ of two complex matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same dimension is given by

$$(A \circ B)_{ij} = (a_{ij} b_{ij}).$$

Note that

$$(A, B) \mapsto A \circ B \quad \text{is bilinear.}$$

The proof of the following elementary result is left as Exercise 4.8.

Lemma 4.23. *Let A, B be complex $n \times m$-matrices, and let C, D be complex $m \times k$-matrices. Then*

$$(A \circ B)(C \circ D) = (AC) \circ (BD).$$

Proposition 4.24 (Schur)**.** *The Hadamard product of complex positive semidefinite quadratic matrices of the same dimension is positive semidefinite.*

Proof. Let A, B be positive semidefinite $N \times N$ matrices, and let $a_1, \dots, a_N \in \mathbb{C}^N$ and $b_1, \dots, b_N \in \mathbb{C}^N$ be the pairwise orthogonal eigenvectors of A and B, respectively, with corresponding eigenvalues $\alpha_1, \dots, \alpha_N \geq 0$ and $\beta_1, \dots, \beta_N \geq 0$. We can write

$$A = \sum_{j=1}^N \alpha_j a_j a_j^\top \quad \text{and} \quad B = \sum_{k=1}^N \beta_k b_k b_k^\top.$$

Therefore, by Lemma 4.23 and by bilinearity it follows that

$$A \circ B = \sum_{j,k=1}^N \alpha_j \beta_k (a_j a_j^\top) \circ (b_k b_k^\top) = \sum_{j,k=1}^N \alpha_j \beta_k (a_j \circ b_k)(a_j \circ b_k)^\top,$$

proving the statement. ∎

An immediate consequence of Schur's theorem is the following fundamental result.

Proposition 4.25. *The coordinatewise product of positive definite sequences is again positive definite.*

We can now prove the promised result.

Theorem 4.26 (Bochner–Herglotz)**.** *A sequence $c = (c(k))_{k\in\mathbb{Z}}$ in $\mathbb{C}$ is positive definite if and only if it is the Fourier transform of a positive measure μ on $\mathbb{T}$, i.e.,*

$$\hat{\mu}(k) = c(k) \quad \textit{for each } k \in \mathbb{Z}.$$

Proof. One implication is already known. For the other suppose that c is a positive definite sequence. For $n \in \mathbb{N}$ consider the characteristic sequences $\mathbf{1}_{[0,n)}$ and $\mathbf{1}_{(-n,0]}$ of the intervals $[0,n) := \{0,\dots,n-1\} \subset \mathbb{Z}$ and $(-n,0] = \{-n+1,\dots,0\}$, respectively. Then by Example 4.22 and Proposition 4.25 the sequence $a_n = \frac{1}{n}\mathbf{1}_{[0,n)} * \mathbf{1}_{-(n,0]} \cdot c$ is positive definite. For fixed $k \in \mathbb{Z}$ we obviously have

$$a_n(k) = c(k)\max\bigl\{1 - \tfrac{|k|}{n}, 0\bigr\} \to c(k) \quad \text{as } n \to \infty.$$

Clearly, $a_n = (a_n(k))_{k\in\mathbb{Z}}$ has support in $\{-n,\dots,0,\dots,n\}$, so that

$$h_n := \sum_k a_n(k)\mathbf{z}^k$$

is a trigonometric polynomial. We first claim that it is positive. Let $n \in \mathbb{N}$ be fixed. For $z \in \mathbb{T}$ and $N \in \mathbb{N}$ with $N \geq n$ we have by positive definiteness

$$\begin{aligned} 0 &\leq \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} a_n(i-j)z^i\overline{z}^j = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} a_n(i-j)z^{i-j} \\ &= \frac{1}{N}\sum_{k=-N}^{N} a_n(k)z^k\Bigl(N - |k|\Bigr) = \sum_{k=-n}^{n} a_n(k)z^k\Bigl(1 - \frac{|k|}{N}\Bigr), \end{aligned}$$

and here the right-hand side converges to $h_n(z)$ as $N \to \infty$.

We now consider the measures $\mu_n := h_n \cdot \mathrm{m}$. By Proposition 4.3

$$\|\mu_n\| = \int_{\mathbb{T}} |h_n|\mathrm{dm} = \int_{\mathbb{T}} h_n\mathrm{dm} = a_n(0) = c(0),$$

so that, by Proposition 1.14 and Theorem 1.3(b), the sequence $(\mu_n)_{n\in\mathbb{N}}$ has a subsequence $(\mu_{n_j})_{j\in\mathbb{N}}$ with $\mu_{n_j} \to \mu$ in the weak* topology as $j \to \infty$ for some positive $\mu \in \mathrm{M}(\mathbb{T})$. For each fixed $k \in \mathbb{Z}$ we obtain

$$\hat{\mu}(k) = \langle \mathbf{z}^{-k}, \mu\rangle = \lim_{j\to\infty}\langle \mathbf{z}^{-k}, \mu_{n_j}\rangle = \lim_{j\to\infty}\int_{\mathbb{T}} \mathbf{z}^{-k}h_{n_j}\,\mathrm{dm} = \lim_{j\to\infty} a_{n_j}(k) = c(k). \qquad \blacksquare$$

Exercises

Exercise 4.1 (Properties of the Fourier transform). Prove Proposition 4.5

Exercise 4.2 (Rajchman measures). Prove that Rajchman measures form a closed subspace of $\mathrm{M}(\mathbb{T})$.

Exercise 4.3 (Cantor–Lebesgue measure). Work out the details of Example 4.14.

Exercise 4.4 (Cesàro limit). Prove Proposition 4.13.

Exercise 4.5 (Discrete and continuous measures). Prove that for a discrete measure with set of atoms A

$$\|\mu\| = \sum_{a\in A} |\mu\{a\}|.$$

Prove, furthermore, that $\mathrm{M}_c(\mathbb{T})$ and $\mathrm{M}_d(\mathbb{T})$ are closed subspaces of $\mathrm{M}(\mathbb{T})$.

Exercise 4.6 (Upper and lower density). Prove the assertions (a)–(e) from Proposition 4.17. Prove also that for each pair of real numbers $\alpha, \beta \in [0,1]$ with $\alpha \le \beta$ there is a set $A \subset \mathbb{N}_0$ such that $\underline{\mathrm{d}}(A) = \alpha$ and $\overline{\mathrm{d}}(A) = \beta$.

Exercise 4.7 (Density and gaps). Prove that a syndetic set $A \subset \mathbb{N}$ has positive lower density. On the other hand, show that if A has positive upper density and $A = \{k_n : n \in \mathbb{N}\}$ for a subsequence $(k_n)_{n\in\mathbb{N}}$, then

$$\liminf_{n\to\infty} (k_{n+1} - k_n) < \infty.$$

Give an example of a syndetic set A with $\underline{\mathrm{d}}(A) < \overline{\mathrm{d}}(A)$.

Exercise 4.8 (Hadamard product). Prove Lemma 4.23.

Exercise 4.9 (Positive definite sequences). Show, using the definition only, that a positive definite sequence $c \in \mathbb{C}^{\mathbb{Z}}$ is bounded and satisfies $c(0) \ge 0$ and $|c(n)| \le c(0)$ for each $n \in \mathbb{Z}$. Prove also that $c = (c(n))_{n\in\mathbb{Z}}$ is positive definite if and only if for some, and then for each $m \in \mathbb{Z}$, the sequence $(c(n+m))_{n\in\mathbb{Z}}$ is positive definite.

References

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LECTURE 5

The spectral theorem

In this lecture we introduce a powerful technique to study unitary operators on Hilbert spaces. Recall that a linear operator $U : H \to K$ between two Hilbert spaces is called **unitary** if $UU^* = I_K$ and $U^*U = I_H$. A unitary operator U is surjective and preserves scalar products, in particular, it is an isometry. Conversely, the following, well-known fact implies that every surjective isometry between two Hilbert spaces preserves scalar products and hence is unitary (why?).

Proposition 5.1 (Polarization identity)**.** *Let H be a Hilbert space and let $b : H \times H \to H$ be a sesquilinear form. Then for every $x, y \in H$*

$$b(x,y) = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k b(x + \mathrm{i}^k y, x + \mathrm{i}^k y).$$

Relevant examples of unitary operators are provided by invertible measure-preserving systems. Their Koopman operator is unitary on the corresponding L^2-space (why?). Another important class of unitary operators is given in the next example, see Exercise 5.1.

Example 5.2. (1) Let (X, μ) be a measure space, and let $m : X \to \mathbb{T}$ be a measurable function. Then the multiplication operator $M_m : \mathrm{L}^2(X,\mu) \to \mathrm{L}^2(X,\mu)$, $M_m : f \mapsto mf$, is unitary.

(2) In particular, for every positive measure $\mu \in \mathrm{M}(\mathbb{T})$ the multiplication operator $M_{\mathbf{z}}$ is unitary on $\mathrm{L}^2(\mathbb{T},\mu)$.

Our aim in this lecture is to prove that the previous example represents the general case of unitary operators. For this purpose, to each unitary operator U we shall assign a family of measures on the spectrum of U such that the corresponding Fourier coefficients completely describe the sesquilinear forms associated to the powers of U. The following fact is well known, and tells that these measures should be sought in $\mathrm{M}(\mathbb{T})$.

Proposition 5.3. *The spectrum of a unitary operator $U \in \mathscr{L}(H)$ is a subset of $\mathbb{T}$.*

1. Scalar spectral measures

Recall from Lecture 4 that for $n \in \mathbb{Z}$ the nth Fourier coefficient $\hat{\mu}(n)$ of a measure $\mu \in \mathrm{M}(\mathbb{T})$ is

$$\hat{\mu}(n) = \int_{\mathbb{T}} \mathbf{z}^{-n}\,\mathrm{d}\mu.$$

Theorem 5.4 (Scalar spectral measures)**.** *Let U be a unitary operator on a Hilbert space H. For each pair of vectors $x, y \in H$ there is a unique measure $\sigma_{x,y} \in \mathrm{M}(\mathbb{T})$*

with

$$\hat{\sigma}_{x,y}(n) = \int_{\mathbb{T}} \mathbf{z}^{-n}\,\mathrm{d}\sigma_{x,y} = (x|U^n y).$$

Moreover, for every $x \in H$ the measure $\sigma_x := \sigma_{x,x}$ is positive.

Proof. For given $x \in H$ define $c(n) := (x|U^n x)$ for $n \in \mathbb{Z}$. We show that $(c(n))_{n\in\mathbb{Z}}$ is a positive definite sequence. For each finitely supported sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$

$$\sum_{i,j\in\mathbb{N}} c(i-j) y_i \overline{y_j} = \sum_{i,j\in\mathbb{N}} (U^j x|U^i x) y_i \overline{y_j} = \Big(\sum_{j\in\mathbb{N}} \overline{y_j} U^j x \Big| \sum_{i\in\mathbb{N}} \overline{y_i} U^i x\Big) \geq 0.$$

Therefore, by the Bochner–Herglotz Theorem, see Theorem 4.26, there is a positive measure $\sigma_{x,x} \in \mathrm{M}(\mathbb{T})$ with $c(n) = \hat{\sigma}_{x,x}(n)$ for each $n \in \mathbb{Z}$. For $y \in H$ we now set

$$\sigma_{x,y} := \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \sigma_{x+\mathrm{i}^k y}.$$

The linearity of $\mu \mapsto \langle \mathbf{z}^{-n}, \mu\rangle$ and the polarization identity (Proposition 5.1) applied to the sesquilinear form $(x,y) \mapsto (x|U^n y)$ yield $\hat{\sigma}_{x,y}(n) = (x|U^n y)$. Uniqueness follows from the injectivity of the Fourier transform, see Proposition 4.2. ∎

The measures $\sigma_{x,y} \in \mathrm{M}(\mathbb{T})$ from the previous proposition are called **scalar spectral measures** associated to U, and we study them in the following.

For a trigonometric polynomial $p \in \mathrm{P}(\mathbb{T})$ with

$$p = \sum_{j=-N}^{N} a_j \mathbf{z}^j$$

we set

$$p(U) := \sum_{j=-N}^{N} a_j U^j = \sum_{j=1}^{N} a_{-j} (U^*)^j + \sum_{j=0}^{N} a_j U^j.$$

Note that the mapping $\mathrm{P}(\mathbb{T}) \to \mathscr{L}(H)$, $p \mapsto p(U)$ is linear and multiplicative. In particular we have $(\mathbf{z}p)(U) = Up(U)$. For each pair $x, y \in H$ of vectors and each pair of trigonometric polynomials $p, q \in \mathrm{P}(\mathbb{T})$ a short calculation yields

$$(p(U)x|q(U)y) = \int_{\mathbb{T}} p\overline{q}\,\mathrm{d}\sigma_{x,y}. \tag{5.1}$$

Proposition 5.5 (Properties of scalar spectral measures)**.** *Let $U \in \mathscr{L}(H)$ be a unitary operator with scalar spectral measures $(\sigma_{x,y})_{x,y\in H}$.*

(a) *The mapping*

$$H \times H \to \mathrm{M}(\mathbb{T}), \quad (x,y) \mapsto \sigma_{x,y}$$

is sesquilinear, and we have $\sigma_{x,y} = \overline{\sigma_{y,x}}$ for every $x, y \in H$. Moreover,

$$\|\sigma_x\| = \|x\|^2 \quad \textit{for each } x \in H.$$

(b) *For a trigonometric polynomial $p \in \mathrm{P}(\mathbb{T})$*

$$\|p(U)x\| = \|p\|_{\mathrm{L}^2(\sigma_x)} \quad \textit{for each } x \in H.$$

(c) *For all $x, y \in H$ and all $p, q \in \mathrm{P}(\mathbb{T})$*

$$\sigma_{p(U)x,q(U)y} = p\overline{q}\cdot\sigma_{x,y} \quad \textit{and} \quad \sigma_{p(U)x} = |p|^2\cdot\sigma_x.$$

(See Proposition 4.3(a) for the definition of measures of the form $f \cdot \mu$.)

(d) *For all $x, y \in H$ and all $p, q \in \mathrm{P}(\mathbb{T})$*

$$\Big|\int_{\mathbb{T}} p\overline{q}\,\mathrm{d}\sigma_{x,y}\Big| \leq \|p\|_{\mathrm{L}^2(\sigma_x)}\|q\|_{\mathrm{L}^2(\sigma_y)}.$$

(e) *For each pair $x, y \in H$*

$$\|\sigma_{x,y}\| \leq \|x\|\|y\|.$$

In particular, the mappings $H \times H \to \mathrm{M}(\mathbb{T})$, $(x, y) \mapsto \sigma_{x,y}$ and $H \to \mathrm{M}(\mathbb{T})$, $x \mapsto \sigma_x$ are continuous.

(f) *For each pair $x, y \in H$ there is $h_{x,y} \in \mathrm{L}^2(\sigma_x)$ such that $\sigma_{x,y} = h_{x,y} \cdot \sigma_x$.*

Proof. (a) is clear from the definition of σ_x and from the fact that $\|\sigma_x\| = \langle \mathbf{z}^0, \sigma_x\rangle$, since σ_x is positive.

(b) For $x \in H$ and a trigonometric polynomial $p \in \mathrm{P}(\mathbb{T})$ we have by (5.1) that

$$\|p(U)x\|^2 = (p(U)x|p(U)x) = \int_{\mathbb{T}} |p|^2\,\mathrm{d}\sigma_x.$$

(c) By (5.1) we obtain for $p, q \in \mathrm{P}(\mathbb{T})$ and for $n \in \mathbb{Z}$ that

$$\int_{\mathbb{T}} \mathbf{z}^n\,\mathrm{d}\sigma_{p(U)x,q(U)y} = (U^n p(U)x|q(U)y) = \int_{\mathbb{T}} \mathbf{z}^n p\overline{q}\,\mathrm{d}\sigma_{x,y}.$$

By the uniqueness of the scalar spectral measures we obtain $\sigma_{p(U)x,q(U)y} = p\overline{q}\cdot\sigma_{x,y}$, whence also the last assertion follows immediately.

(d) By (b)

$$\Big|\int_{\mathbb{T}} p\overline{q}\,\mathrm{d}\sigma_{x,y}\Big| = |(p(U)x|q(U)y)| \leq \|p(U)x\| \cdot \|q(U)y\| = \|p\|_{\mathrm{L}^2(\sigma_x)}\|q\|_{\mathrm{L}^2(\sigma_y)}.$$

(e) By the assertions in (a) and (d) we have

$$|\langle p, \sigma_{x,y}\rangle| \leq \|p\|_{\mathrm{L}^2(\sigma_x)}\|\mathbf{1}\|_{\mathrm{L}^2(\sigma_y)} \leq \|p\|_\infty\|x\|\|y\|.$$

Taking the supremum with respect to $p \in \mathrm{P}(\mathbb{T})$ with $\|p\|_\infty \leq 1$, we obtain the asserted inequality (use that $\mathrm{P}(\mathbb{T})$ is dense in $\mathrm{C}(\mathbb{T})$, Proposition 1.9).

(f) By part (d) the mapping $\mathrm{P}(\mathbb{T}) \to \mathbb{C}$, $p \mapsto \int_{\mathbb{T}} p\,\mathrm{d}\sigma_{x,y}$ is a continuous linear functional, hence can be uniquely extended to a continuous linear functional on $\mathrm{L}^2(\sigma_x)$ (recall that $\mathrm{P}(\mathbb{T})$ is dense in $\mathrm{L}^2(\sigma_x)$). The Riesz–Fréchet theorem yields the function $h_{x,y} \in \mathrm{L}^2(\sigma_x)$ with the required properties. ■

As a consequence, for each fixed, bounded and measurable $f : \mathbb{T} \to \mathbb{C}$ the mapping $b_f : H \times H \to \mathbb{C}$ defined by

$$b_f(x, y) := \int_{\mathbb{T}} f\,\mathrm{d}\sigma_{x,y}$$

is sesquilinear, conjugate symmetric and continuous. Indeed, continuity follows from

$$|b_f(x, y)| \leq \|f\|_\infty|\sigma_{x,y}| \leq \|f\|_\infty\|x\|\|y\|.$$

Continuous sesquilinear forms have the following property, which can be proven easily with the help of the Riesz–Fréchet theorem.

Proposition 5.6. *Let H and K be Hilbert spaces and let $b : H \times K \to \mathbb{C}$ be a continuous sesquilinear form. Then there is a unique operator $S \in \mathscr{L}(H, K)$ such that for each $x \in H$ and $y \in K$*

$$b(x, y) = (Sx|y)_K.$$

This proposition, applied to the form b_f from the above, yields a unique bounded linear operator, denoted by $f(U)$, such that

$$\int_{\mathbb{T}} f \,\mathrm{d}\sigma_{x,y} = b_f(x, y) = (f(U)x|y) \quad \text{for each } x, y \in H.$$

We now collect some properties of the operators $f(U)$ and the corresponding measures $\sigma_{f(U)x,g(U)y}$.

Proposition 5.7. *Let U be a unitary operator on a Hilbert space H with scalar spectral measures $(\sigma_{x,y})_{x,y\in H}$. For each pair $f, g : \mathbb{T} \to \mathbb{C}$ of bounded and measurable functions and each pair $x, y \in H$ of vectors we have*

(a) $\sigma_{f(U)x,g(U)y} = f\overline{g} \cdot \sigma_{x,y}$,
(b) $\left|\int_{\mathbb{T}} f\overline{g} \,\mathrm{d}\sigma_{x,y}\right| \leq \|f\|_{\mathrm{L}^2(\sigma_x)}\|g\|_{\mathrm{L}^2(\sigma_y)}$,
(c) $\|f(U)x\| = \|f\|_{\mathrm{L}^2(\sigma_x)}$.

Proof. (a) For each $n \in \mathbb{Z}$ we have that

$$\hat{\sigma}_{f(U)x,y}(n) = (f(U)x|U^n y) = \int_{\mathbb{T}} f \,\mathrm{d}\sigma_{x,U^n y} = \int_{\mathbb{T}} f\mathbf{z}^{-n} \,\mathrm{d}\sigma_{x,y} = \widehat{f \cdot \sigma_{x,y}}(n),$$

hence, by the injectivity of the Fourier transform, it follows that $\sigma_{f(U)x,y} = f \cdot \sigma_{x,y}$. By similar reasonings we obtain $\sigma_{f(U)x,g(U)y} = f\overline{g} \cdot \sigma_{x,y}$.

(b) Take trigonometric polynomials $p_n \in \mathrm{P}(\mathbb{T})$ with $p_n \to f$ in $\mathrm{L}^2(\sigma_x)$ and let $q \in \mathrm{P}(\mathbb{T})$. Then we have

$$\Big|\int_{\mathbb{T}} p_n q \,\mathrm{d}\sigma_{x,y}\Big| \leq \|p_n\|_{\mathrm{L}^2(\sigma_x)}\|q\|_{\mathrm{L}^2(\sigma_y)}.$$

The right-hand side converges to $\|f\|_{\mathrm{L}^2(\sigma_x)}\|q\|_{\mathrm{L}^2(\sigma_y)}$. For the left-hand side we have by Proposition 5.5 and by the continuity of the scalar product on $\mathrm{L}^2(\sigma_x)$ that

$$\Big|\int_{\mathbb{T}} p_n q \,\mathrm{d}\sigma_{x,y}\Big| = \Big|\int_{\mathbb{T}} p_n q h_{x,y} \,\mathrm{d}\sigma_x\Big| \to \Big|\int_{\mathbb{T}} f q h_{x,y} \,\mathrm{d}\sigma_x\Big|$$

as $n \to \infty$. Altogether, we obtain

$$\Big|\int_{\mathbb{T}} f q \,\mathrm{d}\sigma_{x,y}\Big| \leq \|f\|_{\mathrm{L}^2(\sigma_x)}\|q\|_{\mathrm{L}^2(\sigma_y)}.$$

We repeat this argument for a sequence of trigonometric polynomials $(q_n)_{n\in\mathbb{N}}$ with $q_n \to \overline{g}$, and obtain the statement.

(c) By part (a) we have $\|f(U)x\|^2 = \int_{\mathbb{T}} |f|^2 \,\mathrm{d}\sigma_x = \|f\|^2_{\mathrm{L}^2(\sigma_x)}$. ∎

2. Cyclic subspaces

Given a unitary operator U on a Hilbert space H and a vector $x \in H$ we define

$$Z(x) := \overline{\mathrm{lin}}\{U^n x : n \in \mathbb{Z}\} = \overline{\{p(U)x : p \in \mathrm{P}(\mathbb{T})\}},$$

and call $Z(x)$ the **cyclic subspace** generated by x. Evidently, the cyclic subspace $Z(x)$ and its orthogonal complement $Z(x)^\perp$ both **reduce** U, i.e., they are invariant under U and U^*. The restriction of U to reducing subspaces is again unitary (why?). A vector $x \in H$ is called **cyclic** if $Z(x) = H$.

By Proposition 5.5(b), for fixed $x \in H$ the mapping $\mathrm{P}(\mathbb{T}) \to Z(x)$, $p \mapsto p(U)x$ is isometric, so that it has a unique extension W to $\mathrm{L}^2(\sigma_x)$, which is isometric and surjective onto $Z(x)$. We introduce the notation $f(U)x := Wf \in Z(x)$, which is then consistent with the definition of $f(U)$ if f is bounded and measurable. Note that for $f \in \mathrm{L}^2(\sigma_x)$ there is, in general, no such bounded operator as $f(U)$.

Proposition 5.8. *Let $U \in \mathscr{L}(H)$ be a unitary operator, and let $x \in H$. Then*

$$Z(x) = \{f(U)x : f \in \mathrm{L}^2(\sigma_x)\},$$

and the mapping $W : \mathrm{L}^2(\sigma_x) \to Z(x)$, $f \mapsto f(U)x$ is unitary and ***intertwines*** *the restricted operator $U|_{Z(x)}$ and the multiplication $M_{\mathbf{z}}$ by $\mathbf{z}$ on $\mathrm{L}^2(\sigma_x)$, i.e., $U|_{Z(x)}W = WM_{\mathbf{z}}$. In particular, $U|_{Z(x)}$ and $M_{\mathbf{z}}$ are* ***unitarily equivalent****.*

Proof. That the mapping $\mathrm{P}(\mathbb{T}) \to \mathrm{lin}\{U^n x : n \in \mathbb{Z}\}$, $p \mapsto p(U)x$ intertwines the two mentioned operators is clear. By approximation, it follows that W is intertwining as well. Everything else has been already discussed in the preceding paragraph. ∎

The next result tells that the scalar spectral measures are supported in the spectrum of the operator. (We use here the obvious fact that unitarily equivalent operators have the same spectrum).

Corollary 5.9. *For each $x \in H$ we have $\mathrm{supp}(\sigma_x) \subset \sigma(U)$. If x is cyclic vector, then $\mathrm{supp}(\sigma_x) = \sigma(U)$.*

Proof. We have $\sigma(U|_{Z(x)}) \subset \sigma(U)$. On the other hand, Propositions 1.13 and 5.8 yield the equality $\mathrm{supp}(\sigma_x) = \sigma(U|_{Z(x)})$. ∎

Also, as a direct consequence of the above, if U has a cyclic vector, then U is unitarily equivalent to a multiplication operator. More precisely, we have the following important result, which will be used frequently throughout this course.

Theorem 5.10 (Spectral theorem for operators with cyclic vectors, multiplication form)**.** *Let $U \in \mathscr{L}(H)$ be a unitary operator with a cyclic vector $x \in H$. Then U is unitarily equivalent to a multiplication operator $M = M_{\mathbf{z}}$ on $\mathrm{L}^2(\mathbb{T}, \mu)$ with a positive Borel measure $\mu \in \mathrm{M}(\mathbb{T})$. Moreover, $\sigma(M) = \mathrm{supp}(\mu)$.*

A more general form of the spectral theorem is treated in Exercise 5.7.

Before describing a first connection between the Hilbert space structure of the cyclic subspaces and the *lattice structure* of $\mathrm{M}(\mathbb{T})$, we present some general facts from measure theory (the proof is left as Exercise 5.4).

Lemma 5.11. *Let $\mathcal{X}$ be a σ-algebra over the set X, and let $\lambda, \mu, \nu \in \mathfrak{M}(X, \mathcal{X})$.*

(a) *$\mu \ll \nu$ and $\nu \ll \lambda$ implies $\mu \ll \lambda$.*
(b) *$\simeq$ is an equivalence relation on $\mathfrak{M}(X, \mathcal{X})$.*
(c) *If $\mu \perp \nu$ and $\mu \ll \nu$, then $\mu = 0$.*

(d) *If* $\mu \ll \nu$ *and* $\nu \perp \lambda$*, then* $\mu \perp \lambda$*.*
(e) *If* $\mu \perp \nu$*, then* $|\mu + \nu| = |\mu| + |\nu|$*.*
(f) *If* $\lambda \ll \mu + \nu$*, then there are* $\mu', \nu' \in \mathfrak{M}(X, \mathcal{X})$ *with* $\lambda = \mu' + \nu'$ *and* $\mu' \ll \mu$*,* $\nu' \ll \nu$*.*
(g) *The sets* $\{\sigma \in \mathfrak{M}(X, \mathcal{X}) : \sigma \ll \mu\}$ *and* $\{\sigma \in \mathfrak{M}(X, \mathcal{X}) : \sigma \perp \mu\}$ *are norm closed subspaces in* $\mathfrak{M}(X, \mathcal{X})$*.*

We now return to unitary operators, and relate the cyclic subspaces and the properties of the scalar spectral measures.

Proposition 5.12 (Absolute continuity and spectral measures)**.** *Let* $U \in \mathscr{L}(H)$ *be a unitary operator.*

(a) *For each* $x \in H$ *and each* $f \in \mathrm{L}^2(\sigma_x)$
$$\sigma_{f(U)x} = |f|^2 \cdot \sigma_x.$$
(b) *For each* $x \in H$ *and for each positive measure* $\mu \in \mathrm{M}(\mathbb{T})$
$$\mu \ll \sigma_x \quad \Longleftrightarrow \quad \mu = \sigma_y \quad \text{for some } y \in Z(x).$$
(c) *For each* $x, y \in H$
$$\sigma_x \perp \sigma_y \quad \Longrightarrow \quad \sigma_{x,y} = 0 \quad \Longleftrightarrow \quad Z(x) \perp Z(y).$$
If $x, y \in Z(u)$ *for some* $u \in H$*, then* $\sigma_x \perp \sigma_y \Leftrightarrow Z(x) \perp Z(y)$*.*

Proof. (a) For $f \in \mathrm{P}(\mathbb{T})$ the statement is Proposition 5.5(c), the case of general $f \in \mathrm{L}^2(\sigma_x)$ follows by approximation. We leave the details to the reader.

(b) If $y \in Z(x)$, then there is $f \in \mathrm{L}^2(\sigma_x)$ such that $y = f(U)x$ (Proposition 5.8). So $\sigma_y \ll \sigma_x$ by part (a). Conversely, let the positive measure $\mu \in \mathrm{M}(\mathbb{T})$ be absolutely continuous with respect to σ_x. By the Radon–Nikodym theorem (Theorem 3.26) there is a positive $g \in \mathrm{L}^1(\sigma_x)$ such that $\mu = g \cdot \sigma_x$. Then for $f := \sqrt{g}$ we have $f \in \mathrm{L}^2(\sigma_x)$ and $\sigma_{f(U)x} = f^2 \cdot \sigma_x = g \cdot \sigma_x = \mu$.

(c) Since $\sigma_{x,y} \ll \sigma_x$ and $\sigma_{x,y} \ll \sigma_y$ by Proposition 5.5, the first implication follows from Lemma 5.11(c), (d). Suppose $\sigma_{x,y} = 0$. Then for each $p \in \mathrm{P}(\mathbb{T})$ we have $(p(U)y|x) = \langle p, \sigma_{x,y} \rangle = 0$, so that $x \perp Z(y)$, implying $Z(x) \perp Z(y)$. On the other hand, if $Z(x) \perp Z(y)$, then $(p(U)x|y) = 0$ for each $p \in \mathrm{P}(\mathbb{T})$, showing $\sigma_{x,y} = 0$. Finally, suppose that $x, y \in Z(u)$ for some $u \in H$, and $\sigma_{x,y} = 0$. Then by Proposition 5.8 there are $f, g \in \mathrm{L}^2(\sigma_u)$ such that $x = f(U)u$ and $y = g(U)u$. By approximation, we conclude from Proposition 5.5(a) that $f\overline{g} \cdot \sigma_u = \sigma_{x,y} = 0$. Therefore, $f\overline{g} = 0$ in $\mathrm{L}^2(\sigma_u)$, so $\sigma_x \perp \sigma_y$ by part (a). ∎

3. Projections and eigenvalues

In this section we study algebraic properties of the mapping $f \mapsto f(U)$ and introduce spectral projections. Let $\mathrm{BM}(\mathbb{T})$ denote the vector space of bounded and measurable $\mathbb{C}$-valued functions on $\mathbb{T}$, which becomes a Banach space with the supremum norm and a Banach algebra with the pointwise multiplication.

Theorem 5.13 (Bounded Borel functional calculus)**.** *Let* $U \in \mathscr{L}(H)$ *be a unitary operator on a Hilbert space* H*. The mapping*
$$\Psi : \mathrm{BM}(\mathbb{T}) \to \mathscr{L}(H), \quad \Psi(f) := f(U)$$
has the following properties.

(a) $\Psi(\mathbf{z}) = T$, $\Psi(\mathbf{1}) = I$.
(b) *Ψ is linear, contractive, and satisfies* $\Psi(f\overline{g}) = \Psi(f)\Psi(g)^*$ *for each* $f, g \in \mathrm{BM}(\mathbb{T})$.
(c) *If* $(f_n)_{n\in\mathbb{N}}$ *is* $\|\cdot\|_\infty$*-bounded and pointwise convergent to* f*, then* $\Psi(f_n)x \to \Psi(f)x$ *as* $n \to \infty$ *for each* $x \in H$.

Proof. (a) is clear.

(b) Linearity of Ψ is obvious. Note that, for $f \in \mathrm{BM}(\mathbb{T})$, by Propositions 5.5 and 5.7 we have that

$$\|\Psi(f)x\| = \|f\|_{\mathrm{L}^2(\sigma_x)} \leq \|f\|_\infty \|\sigma_x\| = \|f\|_\infty \|x\|$$

for every $x \in H$, showing that Ψ is a contraction. Let $f, g \in \mathrm{BM}(\mathbb{T})$. Then by Proposition 5.7, for every $x, y \in H$

$$(g(U)^* f(U)x|y) = (f(U)x|g(U)y) = ((f\overline{g})(U)x|y),$$

whence the identity $\Psi(f\overline{g}) = \Psi(f)\Psi(g)^*$ follows.

(c) Take $(f_n)_{n\in\mathbb{N}}$ and f as in the assertion. Then for each $x \in H$

$$\|\Psi(f - f_n)x\|^2 = \int_{\mathbb{T}} |f - f_n|^2 \,\mathrm{d}\sigma_x \to 0$$

as $n \to \infty$ by Lebesgue's dominated convergence theorem. ∎

For a Borel set $A \subset \mathbb{T}$ define $P_A := \mathbf{1}_A(U)$, i.e.,

$$\int_{\mathbb{T}} \mathbf{1}_A \,\mathrm{d}\sigma_{x,y} = (P_A x|y) \quad \text{for every } x, y \in H.$$

From the properties of the Borel functional calculus one can easily deduce that each operator $P_A \in \mathscr{L}(H)$ is a projection, i.e., $P_A P_A = P_A$, and that $P_{A^c} = I - P_A$. It follows that P_A is an orthogonal projection and $x = P_A x$ holds if and only of $\sigma_x(A^c) = 0$. The mapping

$$\mathcal{B}(\mathbb{T}) \to \mathscr{L}(H), \quad A \mapsto P_A$$

is called the **projection valued spectral measure** of the operator U, see Exercise 5.9 for some further properties.

Before considering the important case when A consists of just one point, we first need the following characterization.

Proposition 5.14 (Characterization of eigenvalues via scalar spectral measures)**.** *Let* $U \in \mathscr{L}(H)$ *be a unitary operator with scalar spectral measures* $(\sigma_{x,y})_{x,y\in H}$*, and let* $x \in H$ *and* $\lambda \in \mathbb{C}$ *be fixed. The following assertions are equivalent.*

(i) $x \in \ker(\lambda - U)$.
(ii) $\sigma_{x,y} = (x|y)\delta_\lambda$ *for each* $y \in H$.
(iii) $\sigma_x = \|x\|^2 \delta_\lambda$.

Proof. (i)⇒(ii): If $Ux = \lambda x$, then $U^n x = \lambda^n x$ for each $n \in \mathbb{Z}$. We then have for each $n \in \mathbb{Z}$ and each $y \in H$ that $\lambda^{-n}(x|y) = (U^{-n}x|y) = (x|U^n y) = \hat{\sigma}_{x,y}(n)$. It follows, by the injectivity of the Fourier transform, that $\sigma_{x,y} = (x|y)\delta_\lambda$. The implication (ii)⇒(iii) is trivial.

(iii)⇒(i): If $\sigma_x = \|x\|^2\delta_\lambda$, then for each $y \in H$ we have that

$$|(\lambda x - Ux|y)|^2 \le \|(\lambda - U)x\|^2\|y\|^2 = \int_{\mathbb{T}} |\lambda - \mathbf{z}|^2 \,\mathrm{d}\delta_\lambda \cdot \|x\|^2\|y\|^2 = 0.$$

This implies $\lambda x - Ux = 0$. ∎

For each $\lambda \in \mathbb{T}$ we set $P_\lambda := \mathbf{1}_{\{\lambda\}}(U)$.

Proposition 5.15. *The operator P_λ is the orthogonal projection onto $\ker(\lambda - U)$.*

Proof. We have $(\lambda - \mathbf{z})\mathbf{1}_{\{\lambda\}} = 0$, so that $(\lambda - U)P_\lambda = 0$, meaning $\mathrm{rg}(P_\lambda) \subset \ker(\lambda - U)$. If $x \in \ker(\lambda - U)$, then by Proposition 5.14

$$(P_\lambda x|x) = \int_{\mathbb{T}} \mathbf{1}_{\{\lambda\}} \,\mathrm{d}\sigma_x = \|x\|^2.$$

It follows $\|P_\lambda x - x\|^2 = (P_\lambda x - x|P_\lambda x - x) = (P_\lambda x - x|P_\lambda x) = 0$, i.e., $P_\lambda x = x$. The equality $\mathrm{rg}(P_\lambda) = \ker(\lambda - U)$ follows. ∎

The following shows the connection between eigenvalues, scalar spectral measures and the spectral projections P_λ.

Proposition 5.16. *Let $U \in \mathscr{L}(H)$ be a unitary operator with scalar spectral measures $(\sigma_{x,y})_{x,y\in H}$, and let $x \in H$ and $\lambda \in \mathbb{C}$ be fixed. The following assertions are equivalent.*

(i) $\sigma_x\{\lambda\} > 0$.
(ii) $\|P_\lambda x\| > 0$.
(iii) $Z(x) \cap \ker(\lambda - U) \ne \{0\}$.
(iv) $(x|y) \ne 0$ *for some* $y \in \ker(\lambda - U)$.

Proof. The equivalence (i)⇔(ii) follows from $\|P_\lambda x\|^2 = \sigma_x\{\lambda\}$. For the equivalence (ii)⇔(iii) note that $P_\lambda x = \mathbf{1}_{\{\lambda\}}(U)x \in Z(x) \cap \ker(\lambda - U)$ (by Propositions 5.8 and 5.15). Suppose y is as in (iv). Then $0 \ne (x|y) = (P_\lambda x|y)$, so that $P_\lambda x \ne 0$, and that is (ii). To see the implication (iii)⇒(iv), notice that if $x \perp \ker(\lambda - U)$, then $Z(x) \perp \ker(\lambda - U)$. ∎

4. Ideal decompositions

We shall show that certain decompositions of $\mathrm{M}(\mathbb{T})$ lead to orthogonal decompositions of H into U- and U^*-invariant subspaces. Before that we need another short excursion to measure theory.

Let $\mathcal{X}$ be a σ-algebra over the set X. A subspace $I \subset \mathfrak{M}(X, \mathcal{X})$ is called an (order) **ideal**, if $\mu \in I$, $\nu \in \mathfrak{M}(X, \mathcal{X})$ and $|\nu| \le |\mu|$ imply $\nu \in I$.

Proposition 5.17 (Properties of ideals)**.** (a) *For a closed subspace $I \subset \mathfrak{M}(X, \mathcal{X})$ the following assertions are equivalent.*

(i) *I is an ideal.*
(ii) *For each $\mu, \nu \in \mathfrak{M}(X, \mathcal{X})$ with $\nu \ll \mu$ and $\mu \in I$ also $\nu \in I$ holds.*

(b) *Let I and J be closed ideals of $\mathfrak{M}(X, \mathcal{X})$. Then $I \cap J$ is a closed ideal and*

$$I \cap J = \{0\} \quad \Longleftrightarrow \quad \nu \perp \mu \quad \text{for each } \nu \in I \text{ and } \mu \in J.$$

(c) *Let I and J be closed ideals of $\mathfrak{M}(X, \mathcal{X})$ with $I \cap J = \{0\}$. Then $I + J$ is a closed ideal.*

Proof. (a) Suppose (ii) and that $\mu \in I$, $|\nu| \le |\mu|$. Then $|\mu| \in I$, and $\nu \ll |\mu|$, so that $\nu \in I$ by the hypothesis.

Suppose (i), let $\mu \in I$, and let $\nu \ll \mu$. Then by the Radon–Nikodym theorem there is $f \in \mathrm{L}^1(|\mu|)$ such that $\nu = f \cdot |\mu|$. Consider $f_n := \operatorname{sign}(f)\min(|f|, n\mathbf{1})$, then $|f_n \cdot \mu| = |f_n| \cdot |\mu| \le n|\mu|$, hence $f_n|\mu| \in I$ by the hypothesis. By Lebesgue's dominated convergence theorem $f_n \to f$ in $\mathrm{L}^1(|\mu|)$, so $f_n \cdot |\mu| \to f \cdot |\mu| = \nu$, and $\nu \in I$ follows, since I is closed.

(b) Suppose $I \cap J = \{0\}$, and let $\mu \in I$ and $\nu \in J$. Then by the Lebesgue decomposition (Theorem 3.26(a)), $\mu = \mu_a + \mu_s$ with $\mu_a \ll |\nu| \in J$ and $\mu_s \perp \nu$. Since also $\mu_a \ll \mu$, we obtain $\mu_a \in I \cap J$, hence $\mu_a = 0$ and $\mu = \mu_s \perp \nu$. The converse implication is even easier.

(c) Obviously $I+J$ is a linear subspace of $\mathfrak{M}(X, \mathcal{X})$. By part (b) and Lemma 5.11(e) for $\mu \in I$ and $\nu \in J$ we have $\|\mu + \nu\| = |\mu + \nu|(X) = |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|$. Therefore, the projection

$$P_I : I + J \to I, \qquad \mu + \nu \mapsto \mu$$

is bounded, and by Proposition 1.5 the subspace $I + J$ is closed. To prove that $I + J$ is an ideal, take $\mu \in I$, $\nu \in J$ and let $\rho \in \mathfrak{M}(X, \mathcal{X})$ be such that $\rho \ll \mu + \nu$. Then by Lemma 5.11(f) there are $\mu', \nu' \in \mathfrak{M}(X, \mathcal{X})$ with $\mu' \ll \mu$, $\nu' \ll \nu$ and $\rho = \mu' + \nu' \in I + J$. The rest follows from (a). ∎

We see that even though $\mathfrak{M}(X, \mathcal{X})$ is far from being a Hilbert space, it carries a structure that is very similar to that of a Hilbert space.

Example 5.18. (1) The subspaces $\mathrm{M}_d(\mathbb{T})$ and $\mathrm{M}_c(\mathbb{T})$ are closed ideals, see Exercise 5.5.

(2) The subspace $\mathrm{M}_r(\mathbb{T})$ of Rajchman measures is a closed ideal, see Exercise 4.2 and Proposition 4.6.

(3) For a given $\mu \in \mathrm{M}(\mathbb{T})$ the subspaces

$$I(\mu) := \{\nu \in \mathrm{M}(\mathbb{T}) : \nu \ll \mu\} \quad \text{and} \quad I(\mu)^{\perp} := \{\nu \in \mathrm{M}(\mathbb{T}) : \nu \perp \mu\}$$

are closed ideals, see Lemma 5.11 and Exercise 5.5. In particular, $\mathrm{L}^1(\mathbb{T}, \mathrm{m})$ is the smallest ideal that contains m (we use the identification from Proposition 4.3.)

Let $U \in \mathscr{L}(H)$ be a unitary operator on a Hilbert space H. We associate to each closed ideal I in $\mathrm{M}(\mathbb{T})$ a subspace in H by defining

$$H(I) := \{x \in H : \sigma_{x,y} \in I \text{ for all } y \in H\}.$$

Proposition 5.19. *For a closed ideal I in* $\mathrm{M}(\mathbb{T})$

$$H(I) = \big\{x \in H : \sigma_x \in I\big\}. \tag{5.2}$$

Moreover, $H(I)$ is a closed, U- and U^*-invariant subspace of H. If J is another closed ideal in $\mathrm{M}(\mathbb{T})$ *with $I \cap J = \{0\}$, then*

$$H(I + J) = H(I) \oplus H(J)$$

as an orthogonal direct sum.

Proof. To show (5.2) note that the inclusion "$\subset$" is trivial. For the converse inclusion take $x \in H$ such that $\sigma_x \in I$. Since $\sigma_{x,y} \ll \sigma_x$ for each $y \in H$, by Proposition 5.17 we obtain $\sigma_{x,y} \in I$, and the asserted equality is proven. Since $\sigma_{Ux,y} = \sigma_{x,U^*y}$ and $\sigma_{U^*x,y} = \sigma_{x,Uy}$ for every $x, y \in H$, the invariance of $H(I)$ follows immediately. By Proposition 5.5(e) the mapping $x \mapsto \sigma_x$ is continuous, so that $H(I)$ is closed by the closedness of I.

Let J be as in the assertion, then, by Proposition 5.17, $I + J$ is a closed ideal. Let $x \in H(I)$ and $y \in H(J)$. Then $\sigma_{x,y} \in I$ and $\sigma_{x,y} = \overline{\sigma_{y,x}} \in J$. From the hypothesis we obtain $\sigma_{x,y} = 0$, which yields $x \perp y$, by Proposition 5.12(c). Hence, $H(I) \perp H(J)$.

We finally prove that $H(I + J) = H(I) \oplus H(J)$, where only the inclusion "$\subset$" requires a proof. Take $z \in H(I + J)$, so that $\sigma_z = \mu + \nu$ for some measures $\mu \in I$ and $\nu \in J$. We then have $\sigma_z = |\sigma_z| = |\mu + \nu| = |\mu| + |\nu|$ (since $\mu \perp \nu$), whence we conclude $\mu = |\mu| \geq 0$ and $\nu = |\nu| \geq 0$. Both measures μ and ν are absolutely continuous with respect to σ_z. By Proposition 5.12 there are $x, y \in Z(z)$ with $\sigma_x = \mu$ and $\sigma_y = \nu$. Take $f, g \in \mathrm{L}^2(\sigma_z)$ with $x = f(U)z$ and $y = g(U)z$. Then $\sigma_x = |f|^2 \cdot \sigma_z$ and $\sigma_y = |g|^2 \cdot \sigma_z$, and hence

$$\sigma_z = \mu + \nu = \sigma_x + \sigma_y = |f|^2 \cdot \sigma_z + |g|^2 \sigma_z = (|f|^2 + |g|^2) \cdot \sigma_z.$$

This yields that $|f|^2 + |g|^2 = \mathbf{1}$ holds σ_z-almost everywhere. Since $\sigma_x \perp \sigma_y$, we have $|f| \cdot |g| = 0$ and thus $|f| + |g| = \mathbf{1}$. Define $x' := |f|(U)z$ and $y' := |g|(U)z$. Then $\sigma_{x'} = |f|^2 \sigma_z = \sigma_x = \mu \in I$, so $x' \in H(I)$ (and the same arguments show $y' \in H(J)$). We conclude $z = \mathbf{1}(U)z = (|f| + |g|)(U)z = x' + y' \in H(I) + H(J)$. ∎

In the next lecture we study concrete versions of such abstract decompositions.

Exercises

Exercise 5.1 (Multiplication operators)**.** Let (X, μ) be a σ-finite measure space, and let $m \in \mathrm{L}^0(X, \mu)$ be fixed. Consider the multiplication operator

$$M_m : \mathrm{L}^0(X, \mu) \to \mathrm{L}^0(X, \mu), \quad f \mapsto mf.$$

(a) Prove the following assertions. The operator M_m leaves $\mathrm{L}^2(X, \mu)$ invariant if and only if $m \in \mathrm{L}^\infty(X, \mu)$. In this case M_m is bounded and $\|M_m\| = \|m\|_\infty$.
(b) Prove that M_m is unitary if and only if $|m| = \mathbf{1}$ holds μ-almost everywhere.
(c) Show that $\sigma(M_m) = \operatorname{ess\,rg}(m)$, where the essential range of m is

$$\operatorname{ess\,rg}(m) := \{c \in \mathbb{C} : \mu(m^{-1}(U)) > 0 \text{ for every open set } U \subset \mathbb{C} \text{ with } c \in U\}.$$

(d) For unitary M_m determine the scalar spectral measures and the spectral projections P_A.
(e) Determine the bounded Borel calculus, i.e., the mapping Ψ from Theorem 5.13, in the case of unitary M_m.

Exercise 5.2 (Shift)**.** Consider the Hilbert space $\ell^2(\mathbb{Z})$, and the shift operator $S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $(x_n)_{n\in\mathbb{Z}} \to (x_{n+1})_{n\in\mathbb{Z}}$ thereon. Prove that S is a unitary operator and determine the scalar spectral measures.

Exercise 5.3 (Cyclic vectors)**.** Give an example of a unitary operator with (and without) a cyclic vector.

Exercise 5.4. Prove Lemma 5.11.

Exercise 5.5 (Ideals in $\mathrm{M}(\mathbb{T})$)**.** (a) Prove that $\mathrm{M}_c(\mathbb{T})$ and $\mathrm{M}_d(\mathbb{T})$ are closed ideals in $\mathrm{M}(\mathbb{T})$.
(b) Prove that $I(\mu)$ and $I(\mu)^\perp$ are closed ideals in $\mathrm{M}(\mathbb{T})$ for every $\mu \in \mathrm{M}(\mathbb{T})$.
(c) Let $M \subset \mathrm{M}(\mathbb{T})$ be a subset in $\mathrm{M}(\mathbb{T})$. Prove that

$$M^\perp := \{\nu : \nu \perp \mu \text{ for every } \mu \in M\}$$

is a closed ideal.

Exercise 5.6 (Orthogonal decomposition into a direct sum of cyclic subspaces)**.** Let $U \in \mathscr{L}(H)$ be a unitary operator on a separable Hilbert space H. Prove that there is an at most countable set N of unit vectors such that

$$H = \bigoplus_{x \in N} Z(x)$$

as an orthogonal direct sum.

Exercise 5.7 (Spectral theorem, multiplication form)**.** Let $U \in \mathscr{L}(H)$ be a unitary operator on a separable Hilbert space H. Using the result of the foregoing exercise, prove that there is a complete, separable, metric space Ω, a σ-finite, regular, positive, Borel measure μ on Ω, and a measurable function $m : \Omega \to \mathbb{T}$ such that U is unitarily equivalent to the multiplication operator $M_m : \mathrm{L}^2(\Omega, \mu) \to \mathrm{L}^2(\Omega, \mu)$, $f \mapsto mf$.

Exercise 5.8 (Commutation and the Borel functional calculus)**.** Let U be a unitary operator on a Hilbert space H. Prove that for every bounded and measurable function $f : \mathbb{T} \to \mathbb{C}$ the operator $f(U)$ commutes with every bounded operator on H which commutes with U (and U^*).

Exercise 5.9 (Spectral projections)**.** Let U be a unitary operator on a Hilbert space H. Given a Borel set $B \in \mathcal{B}(\mathbb{T})$ we define $E(B) := P_B = \mathbf{1}_B(U)$. Prove the following assertions.
(a) Each operator $E(B)$, $B \in \mathcal{B}(\mathbb{T})$, is an orthogonal projection.
(b) $E(\mathbb{T}) = I$ and $E(\emptyset) = 0$.
(c) For every Borel set $B \subset \mathbb{T}$ with $B \cap \sigma(U) = \emptyset$ we have $E(B) = 0$.
(d) For a pairwise disjoint sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{T})$,

$$E\Big(\bigcup_{n\in\mathbb{N}} B_n\Big)x = \sum_{n\in\mathbb{N}} E(B_n)x \quad \text{for all } x \in H.$$

(e) For $B, C \in \mathcal{B}(\mathbb{T})$ one has $E(B \cap C) = E(B)E(C)$.
(f) An operator $S \in \mathscr{L}(H)$ commutes with U (and U^*) if and only if

$$E(B)S = SE(B) \quad \text{for every } B \in \mathcal{B}(\mathbb{T}).$$

(g) For $x, y \in H$ we have

$$(Ux|y) = \int_{\sigma(U)} \mathbf{z} \, \mathrm{d}(E(\cdot)x|y).$$

(h) If $B \in \mathcal{B}(\sigma(U))$ has non-empty relative interior in $\sigma(U)$, then $E(B) \neq 0$.

LECTURE 6

Decompositions in Hilbert spaces

In this lecture we discuss three fundamental decompositions for linear contractions on Hilbert spaces: the von Neumann decomposition, the rational spectrum decomposition and the Jacobs–de Leeuw–Glicksberg decomposition. These all share the common feature that in one of the components we collect eigenvectors of the operator. The idea behind this is that the action of the operator on eigenvectors is very simple, so we like to think of these components as *structured*. On our way to present these decompositions we encounter one more, and that is the one of Szőkefalvi-Nagy and Foiaş.

1. Von Neumann's decomposition

Before we present the famous von Neumann decomposition, we briefly discuss some elementary properties of contractions on Hilbert spaces.

Proposition 6.1 (Eigenvectors of contractions on Hilbert spaces)**.** *Let H be a Hilbert space, let $S \in \mathscr{L}(H)$ be a contraction and let $x \in H$. The following assertions are equivalent.*

(i) $Sx = x$.
(ii) $S^*x = x$.
(iii) $(Sx|x) = \|x\|^2$.

In particular, $\mathrm{Fix}(S) = \mathrm{Fix}(S^*)$. *In addition,* $\ker(\lambda - S) = \ker(\overline{\lambda} - S^*)$ *holds for each* $\lambda \in \mathbb{T}$. *Moreover, for different* $\mu, \lambda \in \mathbb{T}$ *we have* $\ker(\mu - S) \perp \ker(\lambda - S)$.

Proof. It is clear that (i) or (ii) implies (iii). Supposing (iii) and using that S is a contraction we obtain

$$\|x - Sx\|^2 = \|x\|^2 - 2\mathrm{Re}(x|Sx) + \|Sx\|^2 = \|Sx\|^2 - \|x\|^2 \leq 0.$$

This proves (i), while (ii) follows by symmetry. The equivalence of the three assertions has been proved. For $\lambda \in \mathbb{T}$ the operator $\frac{1}{\lambda}S = \overline{\lambda}S$ is again a contraction. So by the first part $\ker(\lambda - S) = \mathrm{Fix}(\overline{\lambda}S) = \mathrm{Fix}(\lambda S^*) = \ker(\overline{\lambda} - S^*)$.

Let $\mu, \nu \in \mathbb{T}$, let $x \in \ker(\lambda - S)$ and $y \in \ker(\mu - S) = \ker(\overline{\mu} - S^*)$. Then

$$(\lambda - \mu)(x|y) = (Sx|y) - (x|S^*y) = (Sx|y) - (Sx|y) = 0.$$

If $\mu \neq \lambda$, then $(x|y) = 0$ must hold. ∎

Corollary 6.2. *Let H, K be Hilbert spaces, $S \in \mathscr{L}(H, K)$ be a contraction and let $x \in H$ be fixed. The following assertions are equivalent.*

(i) $\|Sx\| = \|x\|$.
(ii) $(Sx|Sy) = (x|y)$ *for every* $y \in H$.
(iii) $S^*Sx = x$.

Proof. Only the implication (i)⇒(iii) requires a proof. If $\|Sx\| = \|x\|$, then $\|x\|^2 = (S^*Sx|x)$. Proposition 6.1, applied to the contraction S^*S, yields $S^*Sx = x$. ∎

The following is an elementary result, recorded here for later reference.

Proposition 6.3 (Kernel and range)**.** *Let H, K be Hilbert spaces. For each $S \in \mathscr{L}(H, K)$ we have*

$$\ker(S) = \operatorname{rg}(S^*)^\perp \quad \textit{and} \quad \ker(S)^\perp = \overline{\operatorname{rg}(S^*)}.$$

Proof. We have $x \in \operatorname{rg}(S^*)^\perp$ if and only if $0 = (x|S^*y) = (Sx|y) = 0$ for every $y \in H$. But this holds if and only if $Sx = 0$. The first equality is proven, and the second one follows directly from the fact that $\overline{\operatorname{rg}(S^*)} = \operatorname{rg}(S^*)^{\perp\perp}$. ∎

Recall from Lecture 5 that a closed subspace is **reducing** for S, or S-reducing, if it is S- and S^*-invariant.

Proposition 6.4. *Let $S \in \mathscr{L}(H)$ and let $F \subset H$ be a closed subspace. Then F is S^*-invariant if and only if $F^\perp$ is S-invariant.*

Proof. Suppose F is S^*-invariant, and let $x \in F^\perp$. We have to prove $Sx \perp F$, so take $y \in F$. Then we have $S^*y \in F$ and hence $(Sx|y) = (x|S^*y) = 0$. It follows that $Sx \perp F$. The converse implication can be proved analogously. ∎

Remark 6.5. Thus, for a closed subspace $F \subset H$

$$F \text{ is reducing} \iff F^\perp \text{ is reducing} \iff F \text{ and } F^\perp \text{ are } S\text{-invariant.}$$

Theorem 6.6 (Von Neumann's decomposition for contractions)**.** *Let S be a contraction on a Hilbert space H. Then the orthogonal decomposition*

$$H = \operatorname{Fix}(S) \oplus \overline{\operatorname{rg}(I - S)} \tag{6.1}$$

into closed reducing subspaces holds.

Proof. We have $\operatorname{Fix}(S) = \operatorname{Fix}(S^*)$ by Proposition 6.1 and therefore $\operatorname{Fix}(S)$ is S- and S^*-invariant and hence reducing, and so is its orthogonal complement by Remark 6.5. By Proposition 6.3, applied to the operator $I - S$, we obtain

$$H = \operatorname{Fix}(S) \oplus \operatorname{Fix}(S)^\perp = \operatorname{Fix}(S) \oplus \operatorname{Fix}(S^*)^\perp = \operatorname{Fix}(S) \oplus \overline{\operatorname{rg}(I - S)}.$$ ∎

Note that if $S \in \mathscr{L}(H)$ is a contraction, then so is S^* having the same von Neumann decomposition as S.

2. The rational spectrum decomposition

Let $S \in \mathscr{L}(H)$ be a contraction on a Hilbert space H. One part in the von Neumann decomposition was $\operatorname{Fix}(S)$ consisting of eigenvectors corresponding to the eigenvalue 1. We now make this part larger by adding eigenvectors corresponding to all other eigenvalues which are roots of unity.

We call $\lambda \in \mathbb{T}$ **rational** if there is $q \in \mathbb{N}$ with $\lambda^q = 1$, i.e., if λ is a root of unity, and otherwise **irrational**. Clearly, $\lambda \in \mathbb{T}$ is rational if and only if its argument is a rational multiple of π.

Definition 6.7. Let H be a Hilbert space and let $S \in \mathscr{L}(H)$. The subspace

$$H_{\mathrm{rat}} := \overline{\operatorname{lin}}\{x \in H : Sx = \lambda x \text{ for some rational } \lambda \in \mathbb{T}\}$$

is called the **rational spectrum component** of S.

Clearly, H_{rat} is a closed S-invariant subspace of H. Moreover, it has the following representation.

Lemma 6.8 (Representation of the rational spectrum component)**.** *Let H be a Hilbert space and let $S \in \mathscr{L}(H)$.*

(a) *We have* $\mathrm{Fix}(S) \subset \mathrm{Fix}(S^2) \subset \mathrm{Fix}(S^{2\cdot 3}) \subset \cdots \subset \mathrm{Fix}(S^{n!}) \subset \cdots$.

(b) *If S is a contraction, then $H_{\mathrm{rat}} = \overline{\bigcup_{k\in\mathbb{N}} \mathrm{Fix}(S^k)}$.*

Proof. (a) follows from the fact that $k|l$ (k divides l) implies $\mathrm{Fix}(S^k) \subset \mathrm{Fix}(S^l)$.

(b) Observe first that by (a), $\bigcup_{k\in\mathbb{N}} \mathrm{Fix}(S^k)$ is an S-invariant linear subspace of H. We will show

$$\mathrm{lin}\{x \in H : Sx = \lambda x \text{ for some rational } \lambda \in \mathbb{T}\} = \bigcup_{k\in\mathbb{N}} \mathrm{Fix}(S^k). \tag{6.2}$$

Assume first that $x \in H$ satisfies $Sx = \lambda x$ with $\lambda^q = 1$. Then $S^q x = \lambda^q x = x$, i.e., $x \in \mathrm{Fix}(S^q)$. The inclusion "$\subset$" follows by linearity.

To see the inclusion "$\supset$" let $k \in \mathbb{N}$ and consider $S_k := S|_{\mathrm{Fix}(S^k)}$. We first check that S_k is unitary. Indeed, by $S_k^k = I$ it follows immediately that S_k is surjective. Moreover, since S (and hence S_k) is contractive, one has for every $x \in \mathrm{Fix}(S^k)$

$$\|x\| = \|S_k^k x\| \le \|S_k x\| \le \|x\|,$$

i.e., S_k isometry (and surjective). Take $x \in \mathrm{Fix}(S^k)$ and restrict S_k to the subspace $\overline{\mathrm{lin}}\{S^n x : n \in \mathbb{N}_0\}$. This restriction is still denoted by S_k. Then x is a cyclic vector for S_k. By Theorem 5.10 we can assume without loss of generality that $H = \mathrm{L}^2(\mathbb{T}, \mu)$ for some probability measure μ on $\mathbb{T}$, $x = \mathbf{1}$ and S_k is of the form $S_k = M_{\mathbf{z}}$, the multiplication by $\mathbf{z}$. Since $S_k^k = I$, we obtain that $z^k f(z) = f(z)$ must be valid for every $f \in \mathrm{L}^2(\mathbb{T}, \mu)$ and for μ-almost every $z \in \mathbb{T}$. It follows that μ is supported in the set $\{\lambda \in \mathbb{T} : \lambda^k = 1\} =: \{\lambda_1, \dots, \lambda_k\}$ of kth roots of unity. This implies

$$x = \mathbf{1} = \mathbf{1}_{\{\lambda_1\}} + \mathbf{1}_{\{\lambda_2\}} + \cdots + \mathbf{1}_{\{\lambda_k\}}.$$

Since $S_k \mathbf{1}_{\{\lambda_j\}} = \lambda_j \mathbf{1}_{\lambda_j}$, we see that x belongs to the left-hand side of (6.2). ∎

The corresponding decomposition is the following.

Proposition 6.9 (Rational spectrum decomposition for contractions)**.** *Let $S \in \mathscr{L}(H)$ be a contraction on a Hilbert space H. Then the orthogonal decomposition*

$$H = H_{\mathrm{rat}} \oplus \bigcap_{k\in\mathbb{N}} \overline{\mathrm{rg}(I - S^k)}$$

into closed S-reducing subspaces holds.

Proof. The von Neumann decomposition (Theorem 6.6) applied to the powers of S implies the orthogonal decompositions

$$H = \mathrm{Fix}(S^k) \oplus \overline{\mathrm{rg}(I - S^k)} \quad \text{for all } k \in \mathbb{N}. \tag{6.3}$$

Thus by Lemma 6.8(a) we obtain the orthogonal decomposition

$$H = \overline{\bigcup_{k\in\mathbb{N}} \mathrm{Fix}(S^k)} \oplus \bigcap_{k\in\mathbb{N}} \overline{\mathrm{rg}(I - S^k)},$$

where the components are S-reducing by Proposition 6.1. The rest follows from Lemma 6.8(b). ∎

3. The Szőkefalvi-Nagy–Foiaş and Wold decompositions

In this section we present a technique that allows us to apply the spectral theorem to study contractions.

Proposition 6.10 (Szőkefalvi-Nagy–Foiaş[1] decomposition)**.** *For a contraction $S \in \mathscr{L}(H)$ on a Hilbert space H define*

$$H_{\mathrm{uni}} := \big\{x \in H : \|S^n x\| = \|S^{*n} x\| = \|x\| \text{ for all } n \in \mathbb{N}\big\}.$$

Then H_{uni} is a closed, S-reducing subspace and the restriction of S to H_{uni} is unitary. Furthermore, H_{uni} is the largest closed, S-reducing subspace of H such that the restriction of S becomes unitary.

Proof. Let $F \subset H$ be a closed, S-reducing subspace of H such that $S|_F$ is unitary. Then $S^*|_F = (S|_F)^* = (S|_F)^{-1}$ is also unitary. Therefore, the operators S^n and S^{*n} are isometries on F for each $n \in \mathbb{N}$. It follows that $F \subset H_{\mathrm{uni}}$.

By Corollary 6.2 for each $n \in \mathbb{N}$ the identity $\|S^n x\| = \|x\|$ holds if and only if $S^{*n}S^n x = x$, and $\|S^{*n}x\| = \|x\|$ holds if and only if $S^n S^{*n} x = x$. Whence we obtain

$$H_{\mathrm{uni}} = \{x \in H : S^{*n}S^n x = x = S^n S^{*n} x \text{ for every } n \in \mathbb{N}\},$$

or in other words

$$H_{\mathrm{uni}} = \bigcap_{n\in\mathbb{N}} \Big(\mathrm{Fix}(S^{*n}S^n) \cap \mathrm{Fix}(S^n S^{*n})\Big). \tag{6.4}$$

Now it is evident that H_{uni} is a closed subspace of H.

Next we show that H_{uni} is a reducing subspace. For $x \in H_{\mathrm{uni}}$ and $n \in \mathbb{N}$ we have $\|S^n Sx\| = \|S^{n+1}x\| = \|x\| = \|Sx\|$. On the other hand $\|S^{*n}Sx\| = \|S^{*(n-1)}S^*Sx\| = \|S^{*(n-1)}x\| = \|x\| = \|Sx\|$. Altogether we conclude that $Sx \in H_{\mathrm{uni}}$. By symmetry we also obtain $S^*x \in H_{\mathrm{uni}}$. By $SS^*x = x = S^*Sx$ for $x \in H_{\mathrm{uni}}$, both operators S and S^* are unitary. ∎

For a given contraction $S \in \mathscr{L}(H)$ the subspace H_{uni} in the foregoing proposition is called the **unitary part** of H with respect to S. If we want to emphasize the corresponding operator, we write $H_{\mathrm{uni}}(S)$. Its orthogonal complement

$$H_{\mathrm{cnu}} := H_{\mathrm{cnu}}(S) := H_{\mathrm{uni}}(S)^{\perp}$$

is called the **completely non-unitary part** of H with respect to S. It is an S-reducing subspace of H and contains no non-trivial, closed, S-reducing subspace of H on which S acts as a unitary operator. The next proposition is due to Foguel[2] and yields that on the completely non-unitary part the powers S^n converge weakly to 0, i.e., $S|_{H_{\mathrm{cnu}}}$ is **weakly stable**.

Proposition 6.11 (Weak stability on the completely non-unitary part)**.** *Let S be a contraction on a Hilbert space H. For every $x, y \in H_{\mathrm{cnu}}$*

$$(S^n x|y) \to 0 \quad \text{as } n \to \infty.$$

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[1] B. Sz.-Nagy and C. Foiaş, *Sur les contractions de l'espace de Hilbert. IV*, Acta Sci. Math. Szeged 21 (1960), 251–259.

[2] S. R. Foguel, *Powers of a contraction in Hilbert space*, Pacific J. Math. 13 (1963), 551–562.

Proof. Let $u \in H$ be arbitrary and notice that $\lim_{n\to\infty} \|S^n u\|$ exists, since S is a contraction. We have for $k \in \mathbb{N}$ that

$$\begin{aligned}\|S^{*k}S^kS^nu - S^nu\|^2 &= \|S^{*k}S^{n+k}u\|^2 - 2\|S^{n+k}u\|^2 + \|S^nu\|^2\\ &\le \|S^{n+k}u\|^2 - 2\|S^{n+k}u\|^2 + \|S^nu\|^2\\ &= \|S^nu\|^2 - \|S^{n+k}u\|^2 \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Therefore for each $k \in \mathbb{N}_0$

$$\big((I - S^{*k}S^k)S^nu\big|v\big) \to 0 \quad \text{for every } u, v \in H \text{ as } n \to \infty,$$

hence

$$\big(S^nx\big|y\big) \to 0 \quad \text{for every } x, y \in \operatorname{rg}(I - S^{*k}S^k) \text{ as } n \to \infty.$$

By symmetry we also have

$$\big(S^nx\big|y\big) = \big(x\big|S^{*n}y\big) \to 0 \quad \text{for every } x, y \in \operatorname{rg}(I - S^kS^{*k}) \text{ as } n \to \infty.$$

Therefore, for each

$$x, y \in \overline{\operatorname{lin} \bigcup_{k\in\mathbb{N}} \Big(\operatorname{rg}(I - S^kS^{*k}) \cup \operatorname{rg}(I - S^kS^{*k})\Big)}$$

we have $(S^nx|y) \to 0$ as $n \to \infty$. Since, by Theorem 6.3 and by (6.4),

$$\begin{aligned}&\overline{\operatorname{lin} \bigcup_{k\in\mathbb{N}} \Big(\operatorname{rg}(I - S^kS^{*k}) \cup \operatorname{rg}(I - S^{*k}S^k)\Big)}\\ &\qquad = \Big(\bigcap_{k\in\mathbb{N}} \big(\operatorname{Fix}(S^kS^{*k}) \cap \operatorname{Fix}(S^{*k}S^k)\big)\Big)^{\perp} = H_{\mathrm{uni}}^{\perp} = H_{\mathrm{cnu}},\end{aligned}$$

the proof is complete. ∎

For isometries more detailed information and a finer decomposition are available. For this, a basic building block is the **right shift** R on $\ell^2(\mathbb{N}_0)$ defined on the elements of the standard orthonormal basis $(e_n)_{n\in\mathbb{N}_0}$ by

$$Re_n = e_{n+1}.$$

Theorem 6.12 (Wold decomposition)**.** *Let S be an isometry on a Hilbert space H. Then*

$$H_{\mathrm{uni}} = \bigcap_{n\in\mathbb{N}_0} \operatorname{rg}(S^n),$$

and H_{cnu} can be written as an orthogonal sum

$$H_{\mathrm{cnu}} = \bigoplus_{\alpha\in A} H_\alpha$$

for some index set A, where each H_α is S-invariant and $S : H_\alpha \to H_\alpha$ is unitarily equivalent to the right shift R on $\ell^2(\mathbb{N}_0)$.

Proof. Recall that, since S^n is an isometry, we have $S^{*n}S^n = I$ and the subspace $\operatorname{rg}(S^n)$ is closed. Then $F := \bigcap_{n\in\mathbb{N}_0} \operatorname{rg}(S^n)$ is an S-invariant closed subspace and $S|_F : F \to F$ is surjective, and by construction F is the largest subspace of H with these properties. So we have $H_{\mathrm{uni}} \subset F$, and since $S|_F$ is unitary, $F = H_{\mathrm{uni}}$ follows.

For the proof of the decomposition of H_{cnu}, define for $n \in \mathbb{N}$ the subspace F_n as the orthogonal complement of $\operatorname{rg}(S^n)$ in $\operatorname{rg}(S^{n-1})$. By construction

$$\operatorname{rg}(S^{n+1}) \oplus F_{n+1} = \operatorname{rg}(S^n) = S\operatorname{rg}(S^{n-1}) = S(\operatorname{rg}(S^n) \oplus F_n).$$

Since S is an isometry, it preserves scalar products, and we obtain for $n \in \mathbb{N}$ that $S(\operatorname{rg}(S^n) \oplus F_n) = \operatorname{rg}(S^{n+1}) \oplus SF_n$ and $\operatorname{rg}(S^{n+1}) \perp SF_n$, hence we must have $SF_n = F_{n+1}$.

The subspaces F_k for $k \in \mathbb{N}$ are pairwise orthogonal, and clearly

$$H_{\mathrm{uni}} = \bigcap_{n\in\mathbb{N}} \operatorname{rg}(S^n) \subset \Bigl(\bigoplus_{k\in\mathbb{N}} F_k\Bigr)^{\perp}.$$

On the other hand, if $x \perp F_k$ for all $k \in \mathbb{N}$ then $x \in \operatorname{rg}(S^n)$ for all $n \in \mathbb{N}$. Therefore

$$\Bigl(\bigoplus_{k\in\mathbb{N}} F_k\Bigr)^{\perp} \subset H_{\mathrm{uni}}.$$

Altogether it follows that

$$H_{\mathrm{cnu}} = \bigoplus_{k\in\mathbb{N}} F_k.$$

Finally, take an orthonormal basis $(e_\alpha)_{\alpha\in A}$ in F_1. Then $(S^k e_\alpha)_{\alpha\in A}$ is an orthonormal basis in $S^k F_1 = F_{k+1}$ for each $k \in \mathbb{N}$, and therefore with $H_\alpha := \overline{\operatorname{lin}}\{S^k e_\alpha : k \in \mathbb{N}_0\}$ we clearly have

$$\bigoplus_{\alpha\in A} H_\alpha = \bigoplus_{k\in\mathbb{N}} F_k = H_{\mathrm{cnu}},$$

and $(S^k e_\alpha)_{k\in\mathbb{N}}$ is an orthonormal basis in H_α. The last statement about unitary equivalence follows at once. ∎

4. The Jacobs–de Leeuw–Glicksberg decomposition

For a given contraction $S \in \mathscr{L}(H)$ on a Hilbert space we now enlarge the structured part H_{rat} by collecting all eigenvectors corresponding to unimodular eigenvalues. We first restrict ourselves to the case of unitary operators $U \in \mathscr{L}(H)$ with scalar spectral measures $(\sigma_{x,y})_{x,y\in H}$. According to Proposition 4.11, the space $\mathrm{M}(\mathbb{T})$ decomposes into the direct sum of the closed ideals $\mathrm{M}_{\mathrm{c}}(\mathbb{T})$ and $\mathrm{M}_{\mathrm{d}}(\mathbb{T})$, i.e.,

$$\mathrm{M}(\mathbb{T}) = \mathrm{M}_{\mathrm{d}}(\mathbb{T}) \oplus \mathrm{M}_{\mathrm{c}}(\mathbb{T}). \tag{6.5}$$

This induces, by Proposition 5.19, an orthogonal decomposition of the Hilbert space

$$H = H_{\mathrm{d}} \oplus H_{\mathrm{c}}$$

into the closed U- and U^*-invariant subspaces

$$H_{\mathrm{d}}(U) = \bigl\{x \in H : \sigma_x \text{ is discrete}\bigr\}$$

and

$$H_{\mathrm{c}}(U) = \bigl\{x \in H : \sigma_x \text{ is continuous}\bigr\}.$$

Our next purpose is to give different descriptions of these parts.

Proposition 6.13 (Structured part for unitary operators)**.** *For a unitary operator $U \in \mathscr{L}(H)$ we have*

$$H_{\mathrm{d}} = \overline{\operatorname{lin}}\bigl\{x \in H : x \text{ is an eigenvector of } U\bigr\},$$

and for the orthogonal projection P_{d} onto H_{d} we have

$$P_{\mathrm{d}} x = \sum_{a\in\mathbb{T}} P_a x \quad \text{for every } x \in H,$$

where the summands are pairwise orthogonal and at most countably many of them are non-zero. Moreover, we have for each $x, y \in H$ that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |(U^n x|y)|^2 = \sum_{a\in\mathbb{T}} |(P_a x|y)|^2 = |(P_{\mathrm{d}} x|y)|^2.$$

Proof. If $x \in H$ is an eigenvector of U, then, by Proposition 5.14 we have $\sigma_x = \|x\|^2\delta_\lambda$, so that $\sigma_x \in \mathrm{M_d}(\mathbb{T})$. On the other hand, if $\sigma_x \in \mathrm{M_d}(\mathbb{T})$, then there is a countable set A such that $\sigma_x(A^c) = 0$. Since

$$(P_A x|x) = \int_{\mathbb{T}} \mathbf{1}_A \,\mathrm{d}\sigma_x = \int_{\mathbb{T}} \mathbf{1} \,\mathrm{d}\sigma_x = (x|x),$$

we obtain $P_A x = x$. By Proposition 5.15, $\mathbf{1}_{\{a\}}(U)x \in \ker(a-U)$. By Theorem 5.13

$$x = P_A x = \sum_{a\in A} \mathbf{1}_{\{a\}}(U)x \in \overline{\mathrm{lin}}\{x \in H : x \text{ is an eigenvector of } U\}.$$

The description of H_{d} is proven. The statement about the orthogonal projection P_{d} follows also directly.

For the last statement take $x, y \in H$. By Wiener's formula in Proposition 4.12

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |(U^n x|y)|^2 &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}_{x,y}(n)|^2 \\ &= \sum_{a\in\mathbb{T}} |\sigma_{x,y}\{a\}|^2 = \sum_{a\in\mathbb{T}} |(P_a x|y)|^2,\end{aligned}$$

where we have used Proposition 5.14 for the last equality. ∎

Next we turn to the component H_{c} and first present an important lemma. The **(upper/lower) density of a subsequence** $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ is the (upper/lower) density of the set $\{n_k : k \in \mathbb{N}\}$.

Lemma 6.14. *Let S be an isometry on a Hilbert space H and let $x \in H$. The following assertions are equivalent.*

(i) $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |(S^n x|x)| = 0.$

(ii) $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |(S^n x|y)| = 0$ *for every $y \in H$.*

(iii) *There is a subsequence $(n_j)_{j\in\mathbb{N}}$ of density* 1 *such that $S^{n_j} x \to 0$ weakly as $j \to \infty$.*

Proof. The implication (ii)⇒(i) is trivial. To see the converse implication suppose that x satisfies (i). For a given $m \in \mathbb{N}$ consider $y = S^m x$. Then for $n \geq m$ we have

$$|(S^n x|y)| = |(S^n x|S^m y)| = |(S^{n-m} x|y)|,$$

and therefore

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |(S^n x|y)| = \lim_{N\to\infty} \frac{1}{N} \sum_{n=m}^{N+m-1} |(S^n x|y)| = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |(S^n x|x)| = 0.$$

Thus, by linearity, the assertion in (ii) holds for every $y \in \mathrm{lin}\{S^m x : m \in \mathbb{N}_0\} =: Y$. Assume that $y \in \overline{Y}$ and take $\varepsilon > 0$ and $z \in Y$ with $\|y-z\| < \varepsilon$. From the inequality

$$|(S^n x|y)| \leq |(S^n x|z)| + |(S^n x|y-z)| < |(S^n x|z)| + \varepsilon\|x\|$$

and from the above we conclude

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |(S^n x|y)| \le \varepsilon\|x\| \quad \text{for all } \varepsilon > 0.$$

So the assertion in (ii) holds for every $y \in \overline{Y}$. Since for $y \in Y^\perp$ the definition of Y implies that $(S^n x|y) = 0$ for every $n \in \mathbb{N}$, (ii) follows by linearity.

The implication (iii)⇒(ii) follows from the Koopman–von Neumann Lemma 4.18.

(ii)⇒(iii): Consider the closed subspace $Y = \overline{\operatorname{lin}}\{S^n x : n \in \mathbb{N}_0\}$ and a dense subset $D = \{y_k : k \in \mathbb{N}\} \subset Y$. For $k \in \mathbb{N}$ let $A_k \subset \mathbb{N}$ be a subset with $\mathrm{d}(A_k) = 1$ and $\lim_{n\to\infty,\, n\in A_k}(S^n x|y_k) = 0$ (use the Koopman–von Neumann Lemma 4.18). Take $A \subset \mathbb{N}$ such that $\mathrm{d}(A) = 1$ and $A\setminus A_k$ is finite for every $k \in \mathbb{N}$ (use Proposition 4.17). Then clearly $\lim_{n\to\infty,\, n\in A}(S^n x|y_k) = 0$ for every $k \in \mathbb{N}$. By an approximation argument we conclude $\lim_{n\to\infty,\, n\in A}(S^n x|y) = 0$ for every $y \in Y$. If $y \in Y^\perp$, then $(S^n x|y) = 0$ holds for every $n \in \mathbb{N}$. Therefore we obtain $\mathrm{D}\text{-}\lim_{n\to\infty}(S^n x|y) = 0$ for every $y \in H$, and that was to be proved. ∎

Proposition 6.15 (Almost weakly stable part for unitary operators)**.** *Let $U \in \mathscr{L}(H)$ be a unitary operator on a Hilbert space. A vector $x \in H$ belongs to H_c if and only if*

$$\mathrm{D}\text{-}\lim_{n\to\infty} S^n x = 0 \quad \textit{weakly.}$$

Proof. We have $x \in H_\mathrm{c}$ if and only if for each $y \in H$ the scalar spectral measure $\sigma_{x,y}$ is continuous. By Wiener's lemma (Proposition 4.12) and by the Koopman–von Neumann Lemma 4.18 this happens if and only if

$$0 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\hat{\sigma}_{x,y}(n)| = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |(U^n x|y)|.$$

Lemma 6.14 finishes the proof. ∎

We now come to the most involved result of this lecture.

Theorem 6.16 (Jacobs–de Leeuw–Glicksberg decomposition for contractions)**.** *Let $S \in \mathscr{L}(H)$ be a contraction on a Hilbert space H. Then the orthogonal decomposition $H = H_\mathrm{kr} \oplus H_\mathrm{aws}$ into two closed S-invariant subspaces holds, where*

$$H_\mathrm{kr} := H_\mathrm{kr}(S) := \overline{\operatorname{lin}}\{x \in H : Sx = \lambda x \textit{ for some } \lambda \in \mathbb{T}\},$$
$$H_\mathrm{aws} := H_\mathrm{aws}(S) := \big\{x \in H : \mathrm{D}\text{-}\lim_{n\to\infty} S^n x = 0 \textit{ weakly}\big\}.$$

The subspace H_kr is called **Kronecker** (or **reversible**) **part** and the subspace H_aws is called **almost weakly stable part**. Vectors belonging to H_aws are called **almost weakly stable** (or **flight**) **vectors**.

Proof. Consider the Szőkefalvi-Nagy–Foiaş decomposition $H = H_\mathrm{uni} \oplus H_\mathrm{cnu}$ for the contraction S. We evidently have $H_\mathrm{kr} \subset H_\mathrm{uni}$. For the unitary operator $U := S|_{H_\mathrm{uni}}$ consider the discrete–continuous decomposition $H_\mathrm{uni} = H_\mathrm{d} \oplus H_\mathrm{c}$. By Proposition 6.13 we have $H_\mathrm{kr} = H_\mathrm{d}$. We set $H_0 := H_\mathrm{c} \oplus H_\mathrm{cnu}$. By construction $H_\mathrm{kr} \perp H_0$ and $H_\mathrm{kr} \oplus H_0 = H$. It remains to prove that $H_0 = H_\mathrm{aws}$. Decompose a given $x \in H_0$ as $x = x_\mathrm{cnu} + x_\mathrm{c}$ with $x_\mathrm{cnu} \in H_\mathrm{cnu}$, $x_\mathrm{c} \in H_\mathrm{c}$. For $y \in H$ we have

$$|(S^n x|y)| \le |(S^n x_\mathrm{c}|y)| + |(S^n x_\mathrm{cnu}|y)|,$$

where the last term tends to 0 for $n \to \infty$ by Foguel's result (Proposition 6.11). For the first term we have by Proposition 6.15

$$|(S^n x_{\mathrm{c}}|y)| = |(S^n x_{\mathrm{c}}|y_{\mathrm{c}})| \xrightarrow{\mathrm{D}} 0 \text{ as } n \to \infty,$$

where the density one sequence is independent of y. This proves $H_0 \subset H_{\mathrm{aws}}$.

Conversely, suppose $x \in H_{\mathrm{aws}}$, and decompose $x = x_{\mathrm{kr}} + x_0$ with $x_{\mathrm{kr}} \in H_{\mathrm{kr}}$ and $x_0 \in H_0$. Then we have

$$(S^n x|x) = (S^n x_{\mathrm{kr}}|x) + (S^n x_0|x),$$

where the first and the last terms converge to 0 in density (for the last one we use the already proved inclusion $H_0 \subset H_{\mathrm{aws}}$). So we must have

$$0 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |(S^n x_{\mathrm{kr}}|x_{\mathrm{kr}})|^2 = \sum_{a\in\mathbb{T}} \|P_a x_{\mathrm{kr}}\|^4,$$

where the last equality follows from Proposition 6.13 and the fact that orthogonal projections are self-adjoint. We conclude $\|x_{\mathrm{kr}}\|^2 = \sum_{a\in\mathbb{T}} \|P_a x_{\mathrm{kr}}\|^2 = 0$, and obtain $x = x_0$. The equality $H_0 = H_{\mathrm{aws}}$ is proven. ∎

Proposition 6.17 (Characterization of the almost weakly stable part)**.** *For a contraction $S \in \mathscr{L}(H)$ on a Hilbert space H the following assertions are equivalent.*

(i) *$x \in H_{\mathrm{aws}}$, i.e., there is a subsequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ of density 1 such that $\lim_{j\to\infty} S^{n_j} x = 0$ weakly.*
(ii) *There is a subsequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that $\lim_{j\to\infty} S^{n_j} x = 0$ weakly.*

Proof. We only have to show the implication (ii) ⇒(i). Suppose (ii) and let z be an eigenvector corresponding to a unimodular eigenvalue $\lambda \in \mathbb{T}$ and write $x = x_{\mathrm{kr}} + x_{\mathrm{aws}}$ with $x_{\mathrm{kr}} \in H_{\mathrm{kr}}$ and $x_{\mathrm{aws}} \in H_{\mathrm{aws}}$. Since

$$\big|(S^{n_j} x|z)\big| = \big|(x|S^{*n_j} z)\big| = \big|\lambda^{n_j}(x|z)\big| = \big|(x_{\mathrm{kr}}|z)\big|$$

holds for every $j \in \mathbb{N}$, (ii) implies $(x_{\mathrm{kr}}|z) = 0$. Thus $x_{\mathrm{kr}} \perp H_{\mathrm{kr}}$, i.e., $x_{\mathrm{kr}} = 0$ and therefore $x = x_{\mathrm{aws}} \in H_{\mathrm{aws}}$. ∎

We close this lecture with another characterization of the Kronecker part. A vector $x \in H$ with $S^{\mathbb{N}_0}x := \{S^n x : n \in \mathbb{N}_0\} \subset H$ relatively compact is called **asymptotically almost periodic**. Define

$$H_{\mathrm{aap}} := \{x \in H : \text{ is asymptotically almost periodic}\}.$$

It is Exercise 6.5 to show that for a contraction $S \in \mathscr{L}(H)$ the set H_{aap} is a closed subspace of H.

Proposition 6.18 (Characterization of the Kronecker part for isometries)**.** *Let $S \in \mathscr{L}(H)$ be an isometry on a Hilbert space H. Then $H_{\mathrm{aap}} = H_{\mathrm{kr}}$, the Kronecker part. Moreover, for every $x \in H_{\mathrm{kr}}$ even the set*

$$S^{\mathbb{Z}}x := \{S^n x : n \in \mathbb{Z}\} \subset H$$

is relatively compact (note that S is unitary on H_{kr}.)

Proof. If $x \in \ker(a - S)$ for some $a \in \mathbb{T}$, then $S^{\mathbb{Z}}x = \{a^n x : n \in \mathbb{Z}\}$ is relatively compact. It follows that $H_{\mathrm{kr}} \subset H_{\mathrm{aap}}$, and also the last assertion follows analogously if we show $H_{\mathrm{aap}} \subset H_{\mathrm{kr}}$.

Let $x \in H_{\mathrm{aap}}$, and define $K := \overline{S^{\mathbb{N}_0}x}$, which is a compact set which is invariant under S. We thus obtain a topological system $(K;S)$. Let $z \in K$ be an almost periodic point† (see Theorem 3.10), take $\varepsilon > 0$, and consider the open ball $U = \mathrm{B}(z,\varepsilon)$. Then the set of return times $R_U(z) = \{n_1 < n_2 < \cdots\}$ of z to U is syndetic (see Lecture 3). Let $m \in \mathbb{N}$ be such that $S^m x \in \mathrm{B}(z,\varepsilon)$. Since S is an isometry, we can write for each $k \in \mathbb{N}$ that

$$\begin{aligned}&\|S^{n_k}x - x\| = \|S^{n_k+m}x - S^m x\|\\ &\le \|S^{n_k+m}x - S^{n_k}z\| + \|S^{n_k}z - z\| + \|z - S^m x\| = 2\|S^m x - z\| + \varepsilon < 3\varepsilon.\end{aligned}$$

Write $x = x_{\mathrm{kr}} + x_{\mathrm{aws}}$ with $x_{\mathrm{kr}} \in H_{\mathrm{kr}}$ and $x_{\mathrm{aws}} \in H_{\mathrm{aws}}$. By Proposition 6.17 there is a set J with density 1 such that $\lim_{n\to\infty, n\in J} S^n x_{\mathrm{aws}} = 0$ weakly. By Proposition 4.17 and Exercise 4.7 the set $J \cap R_U(z)$ has positive lower density, in particular $J \cap R_U(z) = \{m_1 < m_2 < \cdots\}$ is infinite. We conclude

$$\begin{aligned}9\varepsilon^2 &> \|S^{m_j}x - x\|^2 = \|S^{m_j}x_{\mathrm{kr}} - x_{\mathrm{kr}}\|^2 + \|S^{m_j}x_{\mathrm{aws}} - x_{\mathrm{aws}}\|^2\\ &\ge \|S^{m_j}x_{\mathrm{aws}}\|^2 + \|x_{\mathrm{aws}}\|^2 - 2\mathrm{Re}(S^{m_j}x_{\mathrm{aws}}|x_{\mathrm{aws}}) \to 2\|x_{\mathrm{aws}}\|^2\end{aligned}$$

as $j \to \infty$. Since $\varepsilon > 0$ was arbitrary, we obtain $x_{\mathrm{aws}} = 0$, i.e., $x = x_{\mathrm{kr}} \in H_{\mathrm{kr}}$. ∎

For contractions an additional component can appear in H_{aap}.

Theorem 6.19 (Relative compactness of orbits of contractions). *Let $S \in \mathscr{L}(H)$ be a contraction on a Hilbert space H. Then one has the orthogonal decomposition*

$$H_{\mathrm{aap}} = H_{\mathrm{kr}} \oplus \big\{x : \lim_{n\to\infty} \|S^n x\| = 0\big\}. \tag{6.6}$$

Proof. To show that the two subsets are orthogonal, let $x, y \in H$ satisfy $Sx = ax$ for some $a \in \mathbb{T}$ and $\lim_{n\to\infty} S^n y = 0$. Then we have by Proposition 6.1 that $|(x|y)| = |a^{-n}(x|y)| = |(S^{*n}x|y)| = |(x|S^n y)| \to 0$ as $n \to \infty$. The orthogonality of the subspaces follows now by linearity and by the continuity of the scalar product. The inclusion "$\supset$" in (6.6) is clear. For the converse inclusion let $x \in H_{\mathrm{aap}}$ with $x \perp H_{\mathrm{kr}}$. Let P_{uni} be the orthogonal projection onto H_{uni}. Since $S^n P_{\mathrm{uni}}x = P_{\mathrm{uni}}S^n x$, we see that $P_{\mathrm{uni}}x \in H_{\mathrm{aap}}$ and hence $P_{\mathrm{uni}}x \in H_{\mathrm{kr}}$ by Proposition 6.18, leading to $P_{\mathrm{uni}}x = 0$. We have shown $x \in H_{\mathrm{cnu}}$. Thus by Proposition 6.11, $\lim_{n\to\infty} S^n x = 0$ weakly. Since $x \in H_{\mathrm{aap}}$, there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ an $z \in H$ such that $\lim_{j\to\infty} S^{n_j}x = z$. Uniqueness of the weak limit implies $z = 0$. Since this holds for every convergent subsequence of the orbit $(S^n x)_{n\in\mathbb{N}_0}$, we conclude $\lim_{n\to\infty} S^n x = 0$. ∎

The Jacobs–de Leeuw–Glicksberg decomposition can be elaborated in the far more general setting of weakly compact operator semigroups on Banach spaces. Some elements of this theory, with much less elementary proofs than in this lecture, can be found, for instance, in [EFHN, Ch. 16] or [E, Sec. I.1]. The original papers are due to Jacobs[3], Glicksberg and de Leeuw[4][5].

†From the proof it will follow that every point is almost periodic in $(K;S)$.

[3] K. Jacobs, *Ergodentheorie und fastperiodische Funktionen auf Halbgruppen*, Math. Z. 64 (1956), 298–338.

[4] K. de Leeuw and I. Glicksberg, *Almost periodic compactifications*, Bull. Amer. Math. Soc. 65 (1959), 134–139.

[5] K. de Leeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. 105 (1961), 63–97.

5. Exercises

Exercise 6.1 (Weakly stable part)**.** Let $S \in \mathscr{L}(H)$ be a contraction on a Hilbert space. Define

$$H_{\mathrm{ws}} := \{x \in H : S^n x \to 0 \text{ weakly}\}.$$

Prove that H_{ws} is a closed S-invariant subspace of H.

Exercise 6.2 (Norm stable part)**.** Let $S \in \mathscr{L}(H)$ be a contraction on a Hilbert space. Define

$$H_{\mathrm{s}} := \{x \in H : S^n x \to 0 \text{ in norm}\}.$$

Prove that H_{s} is a closed S-invariant subspace of H.

Exercise 6.3 (Weak stability)**.** Let $S \in \mathscr{L}(H)$ be a unitary operator on a Hilbert space, and let $x \in H$. Prove that $S^n x \to 0$ weakly if and only if $S^{*n} x \to 0$ weakly. Is this equivalence true for contractions or isometries?

Exercise 6.4 (Weak stability)**.** Let $D, D' \subset H$ be subsets in a Hilbert space H such that $\overline{\mathrm{lin}}(D) = H$ and $\overline{D'} = H$. For $n \in \mathbb{N}$ let $S_n \in \mathscr{L}(H)$ be a contraction. Show that the following assertions are equivalent.

(i) $S_n x \to 0$ weakly for every $x \in D$.
(ii) $S_n x \to 0$ weakly for every $x \in H$.
(iii) $(S_n x|y) \to 0$ for every $x \in D$, $y \in D'$.

Exercise 6.5 (Compact $\mathbb{N}$-orbits)**.** Let H be a Hilbert space and let $S \in \mathscr{L}(H)$ be a contraction. Prove that

$$H_{\mathrm{aap}}(S) = \bigl\{x \in H : \{S^n x : n \in \mathbb{N}_0\} \subset H \text{ is relatively compact}\bigr\}$$

is a closed, S-invariant subspace of H. Is this subspace S-reducing?

Exercise 6.6 (Compact $\mathbb{Z}$-orbits)**.** Let H be a Hilbert space and let $S \in \mathscr{L}(H)$ be a unitary operator. Prove that

$$H_{\mathrm{ap}}(S) = \bigl\{x \in H : \{S^n x : n \in \mathbb{Z}\} \subset H \text{ is relatively compact}\bigr\}$$

is a closed, S-reducing subspace of H. Prove that in general $H_{\mathrm{ap}} \subsetneq H_{\mathrm{aap}}$, and give an example with equality here. (The vectors $x \in H_{\mathrm{ap}}(S)$ are called **almost periodic**.)

Exercise 6.7 (Weakly stable part)**.** Let H be a Hilbert space and let $S \in \mathscr{L}(H)$ be contraction. Prove that $H_{\mathrm{aws}}(S)$ is a closed S-invariant subspace of H. Show furthermore that for each $k \in \mathbb{N}$ we have

$$H_{\mathrm{aws}}(S) = H_{\mathrm{aws}}(S^k).$$

Exercise 6.8 (Unitary multiplication operators)**.** Let $\mu \in \mathrm{M}(\mathbb{T})$ be a positive measure. Consider the multiplication operator $M_{\mathbf{z}} : \mathrm{L}^2(\mathbb{T}, \mu) \to \mathrm{L}^2(\mathbb{T}, \mu)$. Determine the Kronecker and the almost weakly stable parts, and the corresponding orthogonal projections.

Exercise 6.9 (Multiplication operators)**.** Let (X, μ) be a finite measure space and let $m : X \to \overline{\mathbb{D}}$ be a measurable function, where $\overline{\mathbb{D}}$ is the closed unit disc in $\mathbb{C}$. Consider the multiplication operator $M_m : \mathrm{L}^2(X, \mu) \to \mathrm{L}^2(X, \mu)$, $f \mapsto mf$. Determine the unitary and the completely non-unitary parts, and the corresponding orthogonal projections.

Exercise 6.10 (Two-sided shift)**.** Consider the left shift on $\ell^2(\mathbb{Z})$. Determine the Kronecker and the almost weakly stable parts.

Exercise 6.11 (One-sided shifts)**.** Consider the left and right shifts on $\ell^2(\mathbb{N}_0)$. Determine the Kronecker and the almost weakly stable parts.

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LECTURE 7

Classical ergodic theorems and more

In this lecture we study convergence of ergodic averages and present the classical ergodic theorems due to von Neumann[1] and Birkhoff[2] from 1933, along with some of their extensions. Let S be a linear operator on a Banach space. For every $N \in \mathbb{N}$, the operator

$$A_N := \frac{1}{N} \sum_{n=0}^{N-1} S^n$$

is called the Nth **ergodic average** or **Cesàro average** of S. We are interested in the convergence (in various senses) of these ergodic averages as $N \to \infty$ and in a description of the limit.

1. The mean ergodic theorem

Let us begin with a simple observation. Every $\lambda \in \mathbb{T}$ with $\lambda \neq 1$ satisfies

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n \right| = \left| \frac{\lambda^N - 1}{N(\lambda - 1)} \right| \leq \frac{2}{N(\lambda - 1)}.$$

On the other hand, $\frac{1}{N} \sum_{n=0}^{N-1} 1^n = 1$ for every $N \in \mathbb{N}$. Thus we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n = \begin{cases} 1, & \text{if } \lambda = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(What is the geometric meaning of $\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n$?) Thus convergence holds in both cases $\lambda = 1$ and $\lambda \neq 1$, but to different values and by different reasons. Such a phenomenon we shall see several times later and even twice (or more) in this lecture.

Let us recall from Lecture 6 von Neumann's decomposition for a Hilbert space contraction $S \in \mathscr{L}(H)$. Theorem 6.6 yields the orthogonal decomposition

$$H = \operatorname{Fix}(S) \oplus \overline{\operatorname{rg}(I - S)}.$$

Evidently, for $x \in \operatorname{Fix}(S)$ the ergodic averages satisfy $A_N x = x$. As a first step we characterize the elements of $\overline{\operatorname{rg}(I - S)}$.

Proposition 7.1 (Characterization of the complement of the fixed component)**.** *Let $S \in \mathscr{L}(H)$ be a contraction on a Hilbert space. Then $x \perp \operatorname{Fix}(S)$, i.e., $x \in$*

[1] J. von Neumann, *Proof of the quasi-ergodic hypothesis.*, Proc. Natl. Acad. Sci. USA 18 (1932), 70–82 (English).

[2] G. D. Birkhoff, *Proof of the ergodic theorem.*, Proc. Natl. Acad. Sci. USA 17 (1931), 656–660 (English).

$\overline{\mathrm{rg}(I-S)}$ *holds if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} S^n x = 0. \tag{7.1}$$

Proof. For each $N \in \mathbb{N}$ we have

$$(I-S)\sum_{n=0}^{N-1} S^n = \sum_{n=0}^{N-1} S^n(I-S) = \sum_{n=0}^{N-1}(S^n - S^{n+1}) = I - S^N. \tag{7.2}$$

Since $\|S^N\| \le 1$ for each $N \in \mathbb{N}$, we conclude that for $x = (I-S)y \in \mathrm{rg}(I-S)$

$$\Big\| \frac{1}{N}\sum_{n=0}^{N-1} S^n x \Big\| \le \frac{2\|y\|}{N},$$

and the validity of (7.1) follows. Since all ergodic averages are contractions, (7.1) holds also for all $x \in \overline{\mathrm{rg}(I-S)}$ (see Exercise 1.3).

From (7.2) we conclude that

$$I - \frac{1}{N}\sum_{n=0}^{N-1} S^n = \frac{1}{N}\sum_{n=0}^{N-1}(I - S^n) = \frac{1}{N}\sum_{n=0}^{N-1}(I-S)\sum_{j=0}^{n-1} S^j.$$

Hence for each $x \in H$ we have

$$x = \frac{1}{N}\sum_{n=0}^{N-1} S^n x + \frac{1}{N}(I-S)\sum_{n=0}^{N-1}\sum_{j=0}^{n-1} S^j x.$$

Now if x satisfies (7.1), then

$$x = \lim_{N\to\infty}(I-S)\frac{1}{N}\sum_{n=0}^{N-1}\sum_{j=0}^{n-1} S^j x,$$

therefore $x \in \overline{\mathrm{rg}(I-S)}$. ∎

Thus, we see that only the part $\mathrm{Fix}(S)$ contributes to the limit of ergodic averages with non-zero values, and the limit does not change if we project the function onto $\mathrm{Fix}(S)$. For this reason $\mathrm{Fix}(S)$ is called **characteristic** for the ergodic averages $(A_N)_{N\in\mathbb{N}}$. We immediately obtain the first ergodic theorem.

Theorem 7.2 (Von Neumann's mean ergodic theorem for contractions)**.** *For a contraction $S \in \mathscr{L}(H)$ on a Hilbert space H*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} S^n x = P_{\mathrm{Fix}(S)}x \quad \textit{for each } x \in H, \tag{7.3}$$

where $P_{\mathrm{Fix}(S)}$ is the orthogonal projection onto the fixed space $\mathrm{Fix}(S)$.

Proof. Convergence in (7.3) clearly holds for every $x \in \mathrm{Fix}(S)$. By von Neumann's decomposition and linearity, it remains to show that (7.3) holds for every $x \in \overline{\mathrm{rg}(I-S)}$ with limit zero. But this is exactly Proposition 7.1. ∎

This assertion can be strengthened slightly, see Exercise 7.2.

Corollary 7.3. *For a contraction $S \in \mathscr{L}(H)$ on a Hilbert space H and for each $x \in H$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} S^{n+M} x = P_{\mathrm{Fix}(S)} x \quad \textit{uniformly in } M \in \mathbb{N}.$$

Another corollary characterizes ergodicity of a measure-preserving system in terms of the limit of ergodic averages.

Corollary 7.4 (Characterization of ergodicity)**.** *Let (X, μ, T) be a measure-preserving system, let T denote the corresponding Koopman operator on $\mathrm{L}^2(X, \mu)$. Then the following assertions are equivalent.*

(i) *The measure-preserving system (X, μ, T) is ergodic.*
(ii) *For every $f \in \mathrm{L}^2(X, \mu)$ one has*

$$\frac{1}{N} \sum_{n=0}^{N-1} T^n f = \int_X f \, \mathrm{d}\mu \cdot \mathbf{1} \quad \textit{in } \mathrm{L}^2(X, \mu).$$

Proof. If (ii) holds, then by the mean ergodic theorem every T-invariant function in $\mathrm{L}^2(X, \mu)$ is constant, therefore the system is ergodic.

Assume now that the system is ergodic and let $f \in \mathrm{L}^2(X, \mu)$. By the mean ergodic theorem, the limit of $\frac{1}{N} \sum_{n=0}^{N-1} T^n f$ exists and equals $P_{\mathrm{Fix}}(T) f$ which is a constant function by ergodicity. Moreover, this constant function satisfies

$$(P_{\mathrm{Fix}(T)} f | \mathbf{1}) = (f | \mathbf{1}) = \int_X f \, \mathrm{d}\mu,$$

implying (ii). ∎

An operator $S \in \mathscr{L}(E)$ on a Banach space E is called **mean ergodic** if the limit

$$Px := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} S^n x \quad \text{exists for all } x \in E.$$

In this case, P is a projection onto $\mathrm{Fix}(S)$ called the **mean ergodic projection** of S (see Exercise 7.5). In fact, many more operators are mean ergodic than just contractions on Hilbert spaces, see, e.g., [E, Sec. I.2.1] or [EFHN, Ch. 8]. An important class is that of contractions on reflexive Banach spaces. We mention here, without proof, only the following characterization.

Proposition 7.5 (Characterization of mean ergodicity)**.** *Let $S \in \mathscr{L}(E)$ be a contraction on a Banach space E. Then the following assertions are equivalent.*

(i) *S is mean ergodic.*
(ii) *$E = \mathrm{Fix}(S) \oplus \overline{\mathrm{rg}(I - S)}$.*
(iii) *$\mathrm{Fix}(S)$ separates $\mathrm{Fix}(S')$.*

Remark 7.6. Since the L^1-norm is dominated by the L^2-norm by the Cauchy–Schwarz inequality, the mean ergodic theorem implies that the Koopman operator T for a measure-preserving system (X, μ, T) is mean ergodic on $\mathrm{L}^1(X, \mu)$. By similar methods one can directly show that T is mean ergodic on every $\mathrm{L}^p(X, \mu)$, $p \in [1, \infty)$, see Exercise 7.3.

2. Uniform convergence for uniquely ergodic systems

Recall from Lecture 3 that a topological system (K, T) is called uniquely ergodic if it has a unique invariant probability measure, which is then a fortiori ergodic. For such systems one has a stronger convergence of ergodic averages. Before discussing this we introduce a new notion.

Definition 7.7. Let (K, T) be a topological system. A sequence $(\mu_N)_{N\in\mathbb{N}}$ in $\mathrm{M}(K)$ is called **asymptotically T-invariant** if

$$\lim_{N\to\infty} |\langle f\circ T, \mu_N\rangle - \langle f, \mu_N\rangle| = 0$$

holds for every $f \in \mathrm{C}(K)$.

The following easy but important lemma explains the use of such sequences.

Lemma 7.8. *Every accumulation point of an asymptotically T-invariant sequence of measures is an invariant measure.*

The proof is left as Exercise 7.6.

A crucial step in the proof of the Krylov–Bogolyubov theorem (see Theorem 2.26) was to show that $\frac{1}{N}\sum_{n=0}^{N-1}\delta_{T^n x}$ defines an asymptotically T-invariant sequence of measures. The following is a slightly more general statement, which helps to produce invariant measures with particular properties.

Lemma 7.9 (Examples of asymptotically invariant sequences)**.** *Let (K, T) be a topological system and let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures in $\mathrm{M}(K)$. Then the measures μ_N, $N\in\mathbb{N}$, defined by*

$$\mu_N := \frac{1}{N}\sum_{n=0}^{N-1} T_*^n \nu_N$$

form an asymptotically T-invariant sequence.

The proof is left as Exercise 7.7. The following is a version of the pointwise ergodic theorem (coming later) for uniquely ergodic systems and continuous functions.

Theorem 7.10 (Uniform convergence for uniquely ergodic systems)**.** *Let (K, T) be a uniquely ergodic topological system with (unique) invariant probability measure $\mu \in \mathrm{M}(K)$. Then for every $f \in \mathrm{C}(K)$*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} T^n f = \int_K f\,\mathrm{d}\mu\cdot\mathbf{1} \quad \textit{in the supremum norm.}$$

Proof. Observe that for $x \in K$ with

$$\mu_N := \frac{1}{N}\sum_{n=0}^{N-1}\delta_{T^n x} = \frac{1}{N}\sum_{n=0}^{N-1} T_*^n\delta_x$$

we have

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) = \frac{1}{N}\sum_{n=0}^{N-1}\langle f, \delta_{T^n x}\rangle = \langle f, \mu_N\rangle.$$

By Lemma 7.9 the sequence $(\mu_N)_{N\in\mathbb{N}}$ of probability measures is asymptotically T-invariant. Thus every accumulation point of $(\mu_N)_{N\in\mathbb{N}}$ is an invariant probability measure (Lemma 7.8), which has to be equal to μ by unique ergodicity. Thus we have

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) = \frac{1}{N}\sum_{n=0}^{N-1} \langle f, \delta_{T^n x}\rangle \to \langle f, \mu\rangle \quad \text{as } N \to \infty$$

for every $x \in K$.

It remains to show that this convergence is uniform in x. Assume the contrary, i.e., that there exists $f \in \mathrm{C}(K)$ and $\varepsilon > 0$ such that for every $N_0 \in \mathbb{N}$ there exist $N > N_0$ and $x_N \in K$ with

$$\Big|\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x_N) - \int_K f \,\mathrm{d}\mu\Big| = \big|\langle f, \mu_N - \mu\rangle\big| \geq \varepsilon, \tag{7.4}$$

where $\mu_N := \frac{1}{N}\sum_{n=0}^{N-1} \delta_{T^n x_N}$. However, the sequence $(\mu_N)_{N\in\mathbb{N}}$ of probability measures is asymptotically invariant by Lemma 7.9 with the unique accumulation point μ, contradicting (7.4). ∎

In fact, pointwise convergence of ergodic averages characterizes unique ergodicity.

Proposition 7.11. *Let (K, T) be a topological system and suppose that for each $f \in \mathrm{C}(K)$ there is a number $C_f \in \mathbb{C}$ such that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) = C_f \quad \text{for each } x \in K.$$

Then (K, T) is uniquely ergodic.

Proof. For $\mu \in \mathrm{M}_1(K, T)$, by the dominated convergence theorem, we conclude that

$$C_f = \int_K \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)\,\mathrm{d}\mu = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int_K f(T^n x)\,\mathrm{d}\mu = \langle f, \mu\rangle$$

for every $f \in \mathrm{C}(K)$. Thus any two T-invariant probability measures are equal, i.e., (K, T) is uniquely ergodic. ∎

Corollary 7.12 (Characterization of unique ergodicity)**.** *Let (K, T) be a topological system.*

(a) *(K, T) is uniquely ergodic if and only if the Koopman operator T is mean ergodic on $\mathrm{C}(K)$ with $\mathrm{Fix}(T) = \mathbb{C}\mathbf{1}$.*

(b) *(K, T) is uniquely ergodic and the unique invariant probability measure μ has full support, i.e., satisfies $\mathrm{supp}(\mu) = K$ if and only if (K, T) is minimal and the Koopman operator is mean ergodic on $\mathrm{C}(K)$.*

The proof of this corollary is left as Exercise 7.9.

3. The polynomial mean ergodic theorem

In this section we study the L^2-mean convergence of the so-called polynomial ergodic averages and prove the following result.

Theorem 7.13 (Polynomial mean ergodic theorem for isometries)**.** *Let P be a polynomial with integer coefficients such that $P(\mathbb{N}_0) \subset \mathbb{N}_0$. For an isometry $S \in \mathscr{L}(H)$ on a Hilbert space H the* ***polynomial ergodic averages***

$$\frac{1}{N}\sum_{n=0}^{N-1} S^{P(n)}x \tag{7.5}$$

converge for every $x \in H$. Moreover, non-constant polynomials P satisfy

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} S^{P(n)}x = 0 \quad \textit{if } x \perp H_{\mathrm{rat}}. \tag{7.6}$$

For convenience, we simply call polynomials P as in the statement **integer polynomials**. Of course, the assertion remains true if the condition $P(\mathbb{N}_0) \subset \mathbb{N}_0$ is relaxed to $P(\mathbb{N}_0 + m) \subset P(\mathbb{N}_0)$ for some fixed $m \in \mathbb{N}$; in this case one has to start averaging from m and not from 0.

Remark 7.14. Similarly to how the fixed space is characteristic for the averages in the mean ergodic theorem, we say that the rational spectrum component is *characteristic* for polynomial ergodic averages. This is justified by the fact the limit of (7.5) remains unchanged when replacing x by the projection $P_{H_{\mathrm{rat}}}x$ onto the rational spectrum component H_{rat}.

We need the following technical but very useful result which helps to control the norm of the averages of a sequence via the averages of the correlation between the members of the sequence.

Lemma 7.15 (Van der Corput's inequality, finitary version)**.** *Let H be a Hilbert space, let $N \in \mathbb{N}_0$ and let $u_0, \dots, u_{N-1} \in H$. Then for each $J \in \{0, \dots, N-1\}$*

$$\Bigl\|\sum_{n=0}^{N-1} u_n\Bigr\|^2 \le \frac{N+J}{J}\sum_{n=0}^{N-1}\|u_n\|^2 + \frac{2(N+J)}{J^2}\sum_{h=1}^{J-1}(J-h)\,\mathrm{Re}\sum_{n=0}^{N-h-1}(u_{n+h}|u_n).$$

Proof. For the sake of convenience set $u_n := 0$ for $n \in \mathbb{Z}\setminus\{0,\dots,N-1\}$. We have for each $h \in \{0, \dots, J-1\}$ that

$$\sum_{n=0}^{N-1} u_n = \sum_{n=0}^{N+J-1} u_{n-h}.$$

Summing up these identities for $h = 0, \dots, J-1$ we conclude

$$J\sum_{n=0}^{N-1} u_n = \sum_{h=0}^{J-1}\sum_{n=0}^{N+J-1} u_{n-h} = \sum_{n=0}^{N+J-1}\sum_{h=0}^{J-1} u_{n-h}.$$

Therefore, by the Cauchy–Schwarz inequality, we can write

$$J^2\Bigl\|\sum_{n=0}^{N-1} u_n\Bigr\|^2 \le \Bigl(\sum_{n=0}^{N+J-1}\Bigl\|\sum_{h=0}^{J-1} u_{n-h}\Bigr\|\Bigr)^2 \le (N+J)\sum_{n=0}^{N+J-1}\Bigl\|\sum_{h=0}^{J-1} u_{n-h}\Bigr\|^2. \tag{7.7}$$

Next, we make the following computations:

$$\begin{aligned}\sum_{n=0}^{N+J-1}\left\|\sum_{h=0}^{J-1}u_{n-h}\right\|^2 &= \sum_{n=0}^{N+J-1}\sum_{h=0}^{J-1}\|u_{n-h}\|^2+2\mathrm{Re}\sum_{i=0}^{J-1}\sum_{j=0}^{i-1}\sum_{n=0}^{N+J-1}(u_{n-i}|u_{n-j})\\ &= J\sum_{n=0}^{N-1}\|u_n\|^2+2\mathrm{Re}\sum_{i=0}^{J-1}\sum_{j=0}^{i-1}\sum_{n=i}^{N+i-1}(u_{n-i}|u_{n-j})\\ &= J\sum_{n=0}^{N-1}\|u_n\|^2+2\mathrm{Re}\sum_{i=0}^{J-1}\sum_{n=0}^{N-1}\sum_{j=0}^{i-1}(u_n|u_{n+i-j}).\end{aligned}$$

After substituting $h := i-j$ and collecting the formally identical terms (of which, for every $h \in \{1,\dots,J-1\}$ there are as many as $J-h$) we can continue as follows:

$$\begin{aligned}\sum_{n=0}^{N+J-1}\left\|\sum_{h=0}^{J-1}u_{n-h}\right\|^2 &= J\sum_{n=0}^{N-1}\|u_n\|^2+2\mathrm{Re}\sum_{h=1}^{J-1}(J-h)\sum_{n=0}^{N-1}(u_n|u_{n+h})\\ &= J\sum_{n=0}^{N-1}\|u_n\|^2+2\mathrm{Re}\sum_{h=1}^{J-1}(J-h)\sum_{n=0}^{N-h-1}(u_n|u_{n+h}).\end{aligned}$$

Inserting this back into (7.7) and dividing by J^2 yield the assertion. ∎

By passing to limits in van der Corput's inequality we directly obtain the following powerful result.

Lemma 7.16 (Van der Corput's lemma)**.** *For a Hilbert space H and a bounded sequence $(u_n)_{n\in\mathbb{N}}$ in H we have the inequality*

$$\limsup_{N\to\infty}\left\|\frac{1}{N}\sum_{n=0}^{N-1}u_n\right\|^2 \le 2\liminf_{J\to\infty}\frac{1}{J}\sum_{h=0}^{J-1}\limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=0}^{N-1}(u_n|u_{n+h})\right|.$$

In particular,

$$\liminf_{J\to\infty}\frac{1}{J}\sum_{h=0}^{J-1}\limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=0}^{N-1}(u_n|u_{n+h})\right| = 0 \quad\Longrightarrow\quad \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}u_n = 0.$$

Proof. By van der Corput's inequality we can write

$$(7.8)\quad \frac{1}{N^2}\left\|\sum_{n=0}^{N-1}u_n\right\|^2 \le \frac{N+J}{JN^2}\sum_{n=0}^{N-1}\|u_n\|^2+\frac{2(N+J)}{J^2N^2}\sum_{h=1}^{J-1}(J-h)\left|\sum_{n=0}^{N-h-1}(u_{n+h}|u_n)\right|.$$

Let $h \in \mathbb{N}$ be fixed. Then, since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded,

$$\gamma_h := \limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=0}^{N-h-1}(u_n|u_{n+h})\right| = \limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=0}^{N-1}(u_n|u_{n+h})\right|.$$

Now, if $J \in \mathbb{N}$ is fixed, then for the second term on the right-hand side of (7.8)

$$\limsup_{N\to\infty}\frac{2(N+J)}{NJ^2}\sum_{h=1}^{J-1}(J-h)\frac{1}{N}\left|\sum_{n=0}^{N-h-1}(u_{n+h}|u_n)\right| \le \frac{2}{J^2}\sum_{h=1}^{J-1}(J-h)\gamma_h \le \frac{2}{J}\sum_{h=0}^{J-1}\gamma_h,$$

and for the first term on the right-hand side of (7.8)

$$\limsup_{N\to\infty} \frac{N+J}{JN^2} \sum_{n=0}^{N-1} \|u_n\|^2 \leq \frac{1}{J} \sup_{n\in\mathbb{N}_0} \|u_n\|^2.$$

Take now $\limsup_{N\to\infty}$ in the inequality (7.8), and take the previous two estimates, being true for every $J \in \mathbb{N}$, into account to conclude

$$\limsup_{N\to\infty} \frac{1}{N^2} \Bigg\| \sum_{n=0}^{N-1} u_n \Bigg\|^2 \leq \lim_{J\to\infty} \frac{1}{J} \sup_{n\in\mathbb{N}_0} \|u_n\|^2 + 2 \liminf_{J\to\infty} \frac{1}{J} \sum_{h=0}^{J-1} \gamma_h = 2 \liminf_{J\to\infty} \frac{1}{J} \sum_{h=0}^{J-1} \gamma_h.$$

The proof is complete. ∎

Remark 7.17. The constant 2 before lim inf is not optimal. It can be proved that

$$\limsup_{N\to\infty} \Bigg\| \frac{1}{N} \sum_{n=0}^{N-1} u_n \Bigg\|^2 \leq \liminf_{J\to\infty} \frac{1}{J} \sum_{h=0}^{J-1} \limsup_{N\to\infty} \frac{1}{N} \mathrm{Re} \sum_{n=0}^{N-1} (u_n | u_{n+h}).$$

We have now all the tools to prove the announced polynomial mean ergodic theorem for isometries.

Proof of Theorem 7.13. We use the rational spectrum decomposition from Chapter 6 and define $A_N := \frac{1}{N}\sum_{n=0}^{N-1} S^{P(n)}$. Let $\lambda \in \mathbb{T}$ be rational, i.e., $\lambda^k = 1$ for some $k \in \mathbb{N}$, and let $x \in H$ satisfy $Sx = \lambda x$. Observe that

$$P(n+k) = P(n) \bmod k$$

holds for every $n \in \mathbb{N}$ (check this first for monomials). By the equality $S^{P(n)}x = \lambda^{P(n)}x$, the sequence $(S^{P(n)}x)_{n\in\mathbb{N}}$ is k-periodic and hence Cesàro convergent by Proposition 4.13(d). Since for each $N \in \mathbb{N}$ the operators A_N are contractions, by linearity we obtain that the sequence $(A_N x)_{N\in\mathbb{N}}$ converges for every $x \in H_{\mathrm{rat}}$.

It remains to show (7.6). We prove this by induction on the degree $d := \deg(P)$ of the polynomial P. For $d = 1$, i.e., when $P(n) = an + b$ for some $a, b \in \mathbb{N}_0$, $a \neq 0$ we have

$$\frac{1}{N} \sum_{n=0}^{N-1} S^{P(n)} x = S^b \Big(\frac{1}{N} \sum_{n=0}^{N-1} (S^a)^n x \Big).$$

Let $x \in H$ be such that $x \perp H_{\mathrm{rat}}$. Then, by Lemma 6.8, we have $x \perp \mathrm{Fix}(S^a)$. Now the validity of (7.6) follows from Proposition 7.1 applied to S^a.

Let $d \geq 2$ and assume that (7.6) holds for all integer polynomials of degree smaller than d. Let $x \perp H_{\mathrm{rat}}$ and let P be as in the assertion with $\deg(P) = d$. Consider $u_n := S^{P(n)}x$ for $n \in \mathbb{N}$, which form a bounded sequence in H. Define $Q_h(\cdot) := P(\cdot + h) - P(\cdot)$. Then Q_h is a polynomial with integer coefficients and satisfies $\deg(Q_h) < \deg(P)$. Observe that for large enough h we have $Q_h(\mathbb{N}_0) \subset \mathbb{N}_0$. Since S is an isometry, it follows for large enough h that

$$(u_{n+h}|u_n) = (S^{P(n+h)}x | S^{P(n)}x) = (S^{P(n+h)-P(n)}x | x) = (S^{Q_h(n)}x | x).$$

By the induction hypothesis we conclude that

$$\gamma_h := \limsup_{N\to\infty} \Big| \frac{1}{N} \sum_{n=0}^{N-1} (u_{n+h}|u_n) \Big| = \limsup_{N\to\infty} \Big| \frac{1}{N} \sum_{n=0}^{N-1} (S^{Q_h(n)}x | x) \Big| = 0.$$

Therefore, (7.6) follows from the van der Corput Lemma 7.16. ∎

Remark 7.18. Using the theory of unitary dilations (which we have finally decided not to include into these lectures) one can easily show that the assertions of Theorem 7.13 holds for all contractions on Hilbert spaces. A recent result of ter Elst and Müller [11] extends this to all power bounded operators on Hilbert spaces. The case of contractions on reflexive spaces (in particular on L^p-spaces) is still open.

4. The pointwise ergodic theorem

We now return to measure-preserving systems and prove the famous pointwise ergodic theorem.

Theorem 7.19 (Birkhoff's pointwise ergodic theorem)**.** *Let (X, μ, T) be a measure-preserving system. Then for every $f \in \mathrm{L}^1(X, \mu)$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \tag{7.9}$$

exists for μ-almost every $x \in X$. Moreover, the system (X, μ, T) is ergodic if and only if for every $f \in \mathrm{L}^1(X, \mu)$ the above limit equals $\int_X f \,\mathrm{d}\mu$ μ-almost everywhere.

The proof of the last part (characterization of ergodicity) is left as Exercise 7.12.

Remark 7.20. Specializing $f := \mathbf{1}_A$ provides that for ergodic systems

$$\lim_{N\to\infty} \frac{|\{n \le N : T^n x \in A\}|}{N} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_A(T^n x) = \int\limits_X \mathbf{1}_A \,\mathrm{d}\mu = \mu(A),$$

i.e., time mean (on the left) equals space mean (on the right) for a.e. $x \in X$, providing the proof of the quasi-ergodic hypothesis for ergodic systems and almost every initial state.

There are several, substantially different, proofs of the pointwise ergodic theorem. We present one which is relatively long but very structured and, most importantly, the developed techniques will be useful later. To show the main part of Theorem 7.19, we begin with the following observation. Define for the Koopman operator T and for $N \in \mathbb{N}$

$$A_N := \frac{1}{N} \sum_{n=0}^{N-1} T^n,$$

and consider the subspace

$$D := \mathrm{Fix}(T) \oplus (I - T)\mathrm{L}^\infty(X, \mu) \subset \mathrm{L}^1(X, \mu).$$

Then D is dense in $\mathrm{L}^2(X, \mu)$ and hence in $\mathrm{L}^1(X, \mu)$ by von Neumann's decomposition (Theorem 6.6). By carefully inspecting the proof of the mean ergodic theorem, we see that for every $f \in D$ the ergodic averages converge in the essential supremum norm and hence almost everywhere. Indeed, by the identity in (7.2) we have for each $f \in \mathrm{Fix}(T)$, each $g = (I - T)h$ with $h \in \mathrm{L}^\infty(X, \mu)$ and almost every $x \in X$ that

$$\begin{aligned} \Big|\frac{1}{N} \sum_{n=0}^{N-1} (T^n(f+g))(x) - f(x)\Big| &= \Big|\frac{1}{N} \sum_{n=0}^{N-1} (T^n g)(x)\Big| \\ &= \frac{1}{N}\big|h(x) - (T^N h)(x)\big| \le \frac{2\|h\|_\infty}{N}. \end{aligned}$$

(Note that one has to argue with a function $f \in \mathscr{L}^1(X,\mu)$ representing the corresponding equivalence class in $\mathrm{L}^1(X,\mu)$.) Thus it remains to show that the set

$$F := \Big\{ f \in \mathrm{L}^1(X,\mu) : \lim_{N\to\infty} A_N f \text{ exists a.e.} \Big\} \tag{7.10}$$

is closed in $\mathrm{L}^1(X,\mu)$.

To show this approximation result for the almost everywhere convergence, we introduce the following notion. For a sequence $(S_N)_{N\in\mathbb{N}}$ of linear operators on $\mathrm{L}^1(X,\mu)$ the corresponding **maximal operator** is defined by

$$S^* : \mathrm{L}^1(X,\mu) \to \mathrm{L}^0(X,\mu), \quad S^* f := \sup_{N\in\mathbb{N}} |S_N f|, \tag{7.11}$$

where the supremum is defined pointwise. Although this operator is not linear (why?), it has two important properties:

(1) $S^*(\alpha f) = |\alpha| \cdot S^* f$ for each $\alpha \in \mathbb{C}$ and $f \in \mathrm{L}^1(X,\mu)$,
(2) $S^*(f+g) \le S^* f + S^* g$ for each $f, g \in \mathrm{L}^1(X,\mu)$,

see Exercise 7.11.

Definition 7.21. We say that a sequence of operators $(S_N)_{N\in\mathbb{N}}$ on $\mathrm{L}^1(X,\mu)$ with maximal operator S^* satisfies **a maximal inequality** if there exists a function $c : (0,\infty) \to [0,\infty)$ with $\lim_{\lambda\to\infty} c(\lambda) = 0$ and

$$\mu[S^* f > \lambda] \le c(\lambda) \quad \text{for each } \lambda > 0 \text{ and } f \in \mathrm{L}^1(X,\mu) \text{ with } \|f\|_1 \le 1. \tag{7.12}$$

The following shows the importance of maximal inequalities in the study of almost everywhere convergence of ergodic averages.

Proposition 7.22 (Banach's principle)**.** *Let $(S_N)_{N\in\mathbb{N}}$ be a sequence of linear operators on $\mathrm{L}^1(X,\mu)$ for some probability space (X,μ). If the corresponding maximal operator satisfies a maximal inequality, then the set*

$$F := \Big\{ f \in \mathrm{L}^1(X,\mu) : \lim_{N\to\infty} S_N f \text{ exists a.e.} \Big\}$$

is a closed linear subspace of $\mathrm{L}^1(X,\mu)$.

Proof. It is clear that F is a linear subspace of $\mathrm{L}^1(X,\mu)$. To show the closedness, take $f \in \overline{F}$ (we may suppose $f \notin F$) and a sequence $(f_n)_{n\in NN}$ in F with $f_n \to f$ in $\mathrm{L}^1(X,\mu)$. We need to show that $(S_N f)_{N\in\mathbb{N}}$ is a.e. a Cauchy sequence in $\mathbb{C}$, i.e., that

$$h := \limsup_{k,l\to\infty} |S_k f - S_l f| = 0 \quad \text{almost everywhere.}$$

For $k, l, n \in \mathbb{N}$ we have

$$\begin{aligned} |S_k f - S_l f| &\le |S_k(f - f_n)| + |S_k f_n - S_l f_n| + |S_l(f_n - f)| \\ &\le 2S^*(f - f_n) + |S_k f_n - S_l f_n|. \end{aligned}$$

By the hypothesis that $f_n \in F$ we conclude

$$h = \limsup_{k,l\to\infty} |S_k f - S_l f| \le 2S^*(f - f_n) + \limsup_{k,l\to\infty} |S_k f_n - S_l f_n| = 2S^*(f - f_n).$$

Let $\lambda > 0$. By the above and the maximal inequality it follows that

$$\mu[h > 2\lambda] \le \mu[S^*(f - f_n) > \lambda] \le c\Big(\tfrac{\lambda}{\|f - f_n\|_1}\Big) \to 0 \quad \text{as } n \to \infty.$$

Thus $\mu[h > 2\lambda] = 0$ holds for every $\lambda > 0$, i.e., $h = 0$ almost everywhere. ∎

We now come back to the measure-preserving system (X, μ, T) with the ergodic averages $(A_N)_{N\in\mathbb{N}}$ of the Koopman operator T and shall prove that the corresponding maximal operator satisfies the maximal inequality with $c(\lambda) = \frac{1}{\lambda}$. The following is the key result to achieve this.

Theorem 7.23 (Maximal ergodic theorem)**.** *Let (X, μ, T) be a measure-preserving system with Koopman operator T in $\mathrm{L}^1(X, \mu)$. Define for $n \in \mathbb{N}$*

$$S_n := \sum_{j=0}^{n-1} T^j.$$

Let $f \in \mathrm{L}^1(X, \mu)$ be real valued and set $E_N := [S_n f > 0$ for some $n \leq N]$. Then for every $N \in \mathbb{N}$

$$\int_{E_N} f \,\mathrm{d}\mu \geq 0.$$

In particular, for $E := [S_n f > 0$ for some $n \in \mathbb{N}]$

$$\int_{E} f \,\mathrm{d}\mu \geq 0. \tag{7.13}$$

Proof. The last assertion follows from the firs one and the facts that $E_N \subset E_{N+1}$ for each $N \in \mathbb{N}$ and $E = \bigcup_{N\in\mathbb{N}} E_N$.

To show the first assertion, set $f_0 := 0$, $f_n := S_{n-1} f$ for $n \in \mathbb{N}$, and for $N \in \mathbb{N}$ define the function

$$F_N := \max\{f_0, f_1, \ldots, f_N\}.$$

The functions F_N satisfy $TF_N + f \geq Tf_n + f = f_{n+1}$ for every $n \leq N$. This implies

$$TF_N + f \geq \max\{f_1, \ldots, f_N\}.$$

If $x \in E_N$, i.e., $F_N(x) > 0$, then

$$\max\{f_1(x), \ldots, f_N(x)\} = \max\{f_0(x), f_1(x), \ldots, f_N(x)\}$$

and therefore it follows that

$$TF_N + f \geq F_N \quad \text{on } E_N.$$

Since $F_N \geq 0$, and hence $TF_N \geq 0$, and since $F_N = 0$ on $X \setminus E_N$, we conclude

$$\begin{aligned}\int_{E_N} f \,\mathrm{d}\mu &\geq \int_{E_N} F_N \,\mathrm{d}\mu - \int_{E_N} TF_N \,\mathrm{d}\mu = \int_X F_N \,\mathrm{d}\mu - \int_{E_N} TF_N \,\mathrm{d}\mu \\ &\geq \int_X F_N \,\mathrm{d}\mu - \int_X TF_N \,\mathrm{d}\mu = 0,\end{aligned}$$

proving the assertion. ∎

We now can prove the desired maximal inequality.

Corollary 7.24 (Maximal inequality)**.** *Let (X, μ, T) be a measure-preserving system with ergodic averages $(A_N)_{N\in\mathbb{N}}$ and the corresponding maximal operator A^* and $f \in \mathrm{L}^1(X, \mu)$. Then the following inequality holds:*

$$\mu[A^* f > \lambda] \leq \frac{\|f\|_1}{\lambda} \quad \text{for all } \lambda > 0.$$

In particular, A^* satisfies the maximal inequality with $c(\lambda) := \frac{1}{\lambda}$.

Proof. Let $\lambda > 0$ and consider the real-valued function $g := |f| - \lambda$. Observe that

$$g + \cdots + T^{n-1}g > 0 \iff |f| + \cdots + T^{n-1}|f| > n\lambda \iff \frac{1}{n}\sum_{j=0}^{n-1} T^j|f| > \lambda.$$

Thus the maximal ergodic theorem (Theorem 7.23) implies for the set

$$E := \Big[\frac{1}{n}\sum_{j=0}^{n-1} T^j|f| > \lambda \text{ for some } n \in \mathbb{N}\Big]$$

that

$$\int_E (|f| - \lambda)\,\mathrm{d}\mu \geq 0.$$

Therefore we obtain

$$\begin{aligned}\|f\|_1 \geq \int_E |f|\,\mathrm{d}\mu &\geq \lambda \cdot \mu\Big[\frac{1}{n}\sum_{j=0}^{n-1} T^j|f| > \lambda \text{ for some } n \in \mathbb{N}\Big]\\ &\geq \lambda \cdot \mu\Big[\Big|\frac{1}{n}\sum_{j=0}^{n-1} T^j f\Big| > \lambda \text{ for some } n \in \mathbb{N}\Big]\\ &= \lambda \cdot \mu[A^* f > \lambda].\end{aligned}$$

The proof is complete. ■

We summarize the proof of the pointwise ergodic theorem.

Proof of Theorem 7.19. By von Neumann's decomposition (Theorem 6.6) and by the proof of von Neumann's mean ergodic theorem, the ergodic averages converge a.e. on the dense subset

$$D := \mathrm{Fix}(T) \oplus (I - T)\mathrm{L}^\infty(X, \mu)$$

of $\mathrm{L}^1(X, \mu)$. By the maximal inequality (Corollary 7.24) and Banach's principle (Proposition 7.22), the ergodic averages converge a.e. on the closure of D in $\mathrm{L}^1(X, \mu)$, i.e., on the whole $\mathrm{L}^1(X, \mu)$. ■

We remark that neither the previous nor other proofs of the pointwise ergodic theorem give a description of the points for which the convergence actually holds. For ergodic systems such points got their own name.

Definition 7.25. Let (X, μ, T) be an ergodic measure-preserving system and $f \in \mathrm{L}^1(X, \mu)$. We call $x \in X$ **generic** for f with respect to T if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) = \int_X f\,\mathrm{d}\mu.$$

Thus, the pointwise ergodic theorem says in particular that for ergodic systems and for a function $f \in \mathrm{L}^1(X, \mu)$, the generic points form a set of full measure. (Note that this set depends on the representative of the equivalence class.) Moreover, we saw in Section 2 that a topological system (K, T) is uniquely ergodic if and only if for each $f \in \mathrm{C}(K)$ and each $\mu \in \mathrm{M}_1(K, T)$ every point is generic.

Remark 7.26 (Pointwise convergence of polynomial ergodic averages)**.** (a) It is a good exercise to go through the proof of Theorem 7.10 to make sure that it does not work for polynomial ergodic averages. In fact, we do not know any result on uniform (or even everywhere) convergence of polynomial ergodic averages for good general topological systems (such as uniquely ergodic systems) in the spirit of Theorem 7.10. Although for some classes of systems such as rotations or, more generally, so-called nilrotations such results exist, see Leibman [10, 9].

(b) Also the proof of the pointwise ergodic theorem presented in Section 4 does not go through for polynomial ergodic averages. Indeed, already the first step of finding a good dense subset of functions for which a.e. convergence holds fails (why?). Using much more complicated arguments using harmonic analysis, Bourgain [3, 4, 5] showed in 1988 (using a maximal inequality) that the polynomial ergodic averages converge a.e. for every $f \in \mathrm{L}^p(X,\mu)$ where $p \in (1,\infty)$ is arbitrary. The case $p = 1$ was open until Buczolich and Mauldin [7] showed in 2007 that the a.e. convergence fails in general (even for ergodic measure-preserving systems) for $p = 1$ and $P(n) = n^2$, see also Buczolich, Mauldin [6] and LaVictoire [8].

Exercises

Exercise 7.1 (Mean ergodic theorem, proof via the spectral theorem)**.** Prove von Neumann's mean ergodic theorem for unitary operators by means of the spectral theorem. Restrict the investigation to a cyclic subspace.

Exercise 7.2 (Mean ergodic theorem)**.** Prove Corollary 7.3.

Exercise 7.3 (Mean ergodicity of Koopman operators in L^p)**.** Prove the assertion of Remark 7.6. (Hint: For $p > 2$ estimate the L^p-norm by L^2-norm for bounded functions.)

Exercise 7.4. Which of the following operators is mean ergodic on the given Banach space? Determine, where applicable, the mean ergodic projection.

(a) Left/right shift on $\ell^1(\mathbb{N})$.
(b) Left/right shift on the space $\mathrm{c}_0(\mathbb{N})$ of null-sequences (with supremum norm).
(c) An isometric multiplication operator on $\ell^1(\mathbb{N})$.
(d) An isometric multiplication operator on $\mathrm{c}_0(\mathbb{N})$ (with supremum norm).

Exercise 7.5 (Mean ergodic projection)**.** Let $S \in \mathscr{L}(E)$ be a mean ergodic contraction on a Banach space E. Prove that

$$Px := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} S^n x$$

defines a bounded, linear projection with $\mathrm{rg}(P) = \mathrm{Fix}(S)$ and $\ker(P) = \overline{\mathrm{rg}(I - S)}$.

Exercise 7.6 (Asymptotically invariant sequences of measures)**.** (a) Prove that a topological system is uniquely ergodic if and only if every asymptotically T-invariant sequence of probability measures converges to an invariant probability measure.

(b) Prove the assertion of Lemma 7.8.

Exercise 7.7 (Asymptotic invariance). Let (K, T) be a topological system and $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures in $\mathrm{M}(K)$. Then the sequence $(\mu_N)_{N\in\mathbb{N}}$ of measures μ_N, $N \in \mathbb{N}$, defined by

$$\mu_N := \frac{1}{N}\sum_{n=0}^{N-1} T_*^n \nu_N,$$

is asymptotically T-invariant.

Exercise 7.8 (Uniform convergence of ergodic averages). (a) Does the statement of Proposition 7.11 remain true without the assumption that the everywhere limit of ergodic averages should be constant?

(b) Show that the assertion of Proposition 7.11 remains true if the ergodic averages converge everywhere for every f in a dense subset of $\mathrm{C}(K)$.

Exercise 7.9 (Characterization of unique ergodicity). Prove Corollary 7.12.

Exercise 7.10 (Polynomial mean ergodic theorem). Consider the Hilbert space $H = \ell^2(\mathbb{N})$ and sequence $m \in \ell^\infty(\mathbb{N})$ with $|m_n| = 1$ for each $n \in \mathbb{N}$. Let P be an integer polynomial and let $S = M_m$ be the multiplication operator by m. Determine the limit in the polynomial mean ergodic theorem for the isometry S.

Exercise 7.11 (Maximal operator). Let (X, μ) be a probability space and $(S_N)_{N\in\mathbb{N}}$ be a sequence of linear operators on $\mathrm{L}^1(X, \mu)$. Show that the maximal operator S^* defined by (7.11) has the following two properties:

(1) $S^*(\alpha f) = |\alpha| \cdot S^* f$ for each $\alpha \in \mathbb{C}$ and $f \in \mathrm{L}^1(X, \mu)$,

(2) $S^*(f + g) \le S^* f + S^* g$ for each $f, g \in \mathrm{L}^1(X, \mu)$.

Exercise 7.12 (Ergodicity and pointwise a.e. convergence). Deduce von Neumann's mean ergodic theorem for the Koopman operator of a measure-preserving system from Birkhoff's pointwise ergodic theorem. Prove that a measure-preserving system (X, μ, T) is ergodic if and only if the limit of ergodic averages equals $\int_X f \,\mathrm{d}\mu$ μ-almost everywhere for every $f \in \mathrm{L}^1(X, \mu)$.

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LECTURE 8

Factors of measure-preserving systems

In this lecture we discuss factors of measure-preserving systems. A fundamental tool is provided by Markov operators, which are briefly touched upon here, while a more systematic treatment can be found in [EFHN, Ch. 13]. We also present some important examples of factors: the fixed factor, the rational spectrum factor, the Kronecker factor and the Abramov factor.

1. Factors

We begin with motivating examples and some essential constructions.

Example 8.1 (Products)**.** Let (X, μ, T) and (Y, ν, S) be two measure-preserving systems and consider the product probability space $(X \times Y, \mu \otimes \nu)$. The mapping

$$X \times Y \to X \times Y, \quad (x, y) \to (Tx, Sy)$$

is measure-preserving and yields the **product system** $(X \times Y, \mu \otimes \nu, T \times S)$ (see Lemma 2.5). Analogously, one defines the product of finitely many measure-preserving systems. Obviously, $(\mathbb{T}^d, \mathrm{m}^d, a)$ is a product of d one-dimensional torus rotations.

Example 8.2 (Skew product)**.** Let (X, μ, T) be a measure-preserving system, let (Y, ν) be a probability space and let $\rho : X \times Y \to Y$ be measurable (where $X \times Y$ is equipped with the product σ-algebra) such that $y \mapsto \rho(x, y)$ preserves ν for each $x \in X$. Define

$$S : X \times Y \to X \times Y, \quad S(x, y) := (Tx, \rho(x, y)).$$

Then $(X \times Y, \mu \otimes \nu, S)$ is a measure-preserving system, called the **ρ-skew-product**. If $Y = G$ is a compact group, $\nu = \mathrm{m}_G$, the Haar measure, and $\rho(x, \cdot)$ is a left rotation for each $x \in X$, then the skew-product is called a **group extension**. A concrete example of a group extension is the skew product $(\mathbb{T}^2, \mathrm{m}^2, T_a)$ from Example 2.13 (recall: $T_a(x, y) := (ax, xy)$ for a fixed $a \in \mathbb{T}$).

The previous two examples share the common feature that the projection onto the first component respect the dynamics. We give a name to this phenomenon.

Definition 8.3. Let (X, μ, T) and (Y, ν, S) be two measure-preserving systems. The system (Y, ν, S) is called a (point) **factor** of (X, μ, T) (and (X, μ, T) is called a (point) **extension** of (Y, ν, S)) if there exists a measure-preserving map $\pi : X \to Y$ such that $S \circ \pi = \pi \circ T$ almost everywhere, i.e., such that the following diagram is commutative (π is then called a (point) **factor map**).

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "S"] & Y \end{tikzcd}$$

Example 8.4 (Products, skew rotation and group extensions)**.** Each of the systems (X, μ, T) and (Y, ν, S) is a factor of the product system $(X \times Y, \mu \otimes \nu, T \times S)$, the factor map is given by the corresponding coordinate projections $\pi_1(x, y) := x$ or $\pi_2(x, y) := y$, respectively.

Analogously, the rotation system $(\mathbb{T}, \mathrm{m}, a)$ is a factor of the skew rotation presented in Example 2.13. Generally, a group extension from Example 8.2 is indeed an extension of the original system (X, μ, T), explaining the name "group extension".

The following proposition summarizes the properties of Koopman operators induced by point factor maps. Recall the notation $\langle f, g\rangle = \int_X fg \,\mathrm{d}\mu$ for $f \in \mathrm{L}^1(X, \mu)$, $g \in \mathrm{L}^\infty(X, \mu)$.

Proposition 8.5 (Koopman operators of point factor maps)**.** *Let $\pi : (X, \mu, T) \to (Y, \nu, S)$ be a point factor map. The Koopman operator $S_\pi : \mathrm{L}^0(Y, \nu) \to \mathrm{L}^0(X, \mu)$ has the following properties.*

(a) $S_\pi f \geq 0$ *for each positive* $f \in \mathrm{L}^0(Y, \nu)$.
(b) $S_\pi(fg) = S_\pi(f)S_\pi(g)$ *for each pair* $f, g \in \mathrm{L}^0(Y, \nu)$.
(c) $S_\pi|f| = |S_\pi f|$ *for every* $f \in \mathrm{L}^0(Y, \nu)$.
(d) $S_\pi : \mathrm{L}^1(Y, \nu) \to \mathrm{L}^1(X, \nu)$ *is an isometry.*
(e) $S_\pi \mathbf{1} = \mathbf{1}$.
(f) $\langle S_\pi f, \mathbf{1}\rangle = \langle f, \mathbf{1}\rangle$ *for each* $f \in \mathrm{L}^1(Y, \nu)$*, i.e.,* $S_\pi' \mathbf{1} = \mathbf{1}$.
(g) S_π *intertwines the Koopman operators T and S.*

We make some operator theoretic definitions out of these properties.

Definition 8.6. (a) Let (X, μ), (Y, ν) be probability spaces. A linear operator $S : \mathrm{L}^1(Y, \nu) \to \mathrm{L}^1(X, \mu)$ is called a **Markov operator** if
(1) S is **positive**, i.e., $Sf \geq 0$ whenever $f \geq 0$.
(2) $S\mathbf{1} = \mathbf{1}$.
(3) $S'\mathbf{1} = \mathbf{1}$, i.e., $\langle Sf, \mathbf{1}\rangle = \langle f, \mathbf{1}\rangle$ for every $f \in \mathrm{L}^1(Y, \nu)$.
A Markov operator S is called a **Markov homomorphism** (or **Markov embedding**) if $|Sf| = S|f|$, and a bijective Markov homomorphism is called a **Markov isomorphism**.

(b) We call a measure-preserving system (Y, ν, S) a **Markov factor** (or sometimes simply a factor) of (X, μ, T), if there is a Markov homomorphism $J : \mathrm{L}^1(Y, \nu) \to \mathrm{L}^1(X, \mu)$ intertwining the Koopman operators T and S, i.e., such that the following diagram is commutative. In this case, we also say that (X, μ, T) is a (Markov) **extension** of (Y, ν, S).

$$\begin{tikzcd} \mathrm{L}^1(Y, \nu) \arrow[r, "S"] \arrow[d, "J"'] & \mathrm{L}^1(Y, \nu) \arrow[d, "J"] \\ \mathrm{L}^1(X, \mu) \arrow[r, "T"] & \mathrm{L}^1(X, \mu) \end{tikzcd}$$

We denote this by writing $J : (\mathrm{L}^1(Y,\nu),T) \to (\mathrm{L}^1(X,\mu),S)$ and say that J is a Markov homomorphism.

(c) Two measure-preserving systems (X,μ,T) and (Y,ν,S) are called **Markov isomorphic** if there is a Markov isomorphism $J : \mathrm{L}^1(Y,\nu) \to \mathrm{L}^1(X,\mu)$ intertwining the Koopman operators.

By the above the Koopman operator S_π of a point factor map π is a Markov homomorphism, and a point factor (Y,ν,S) is a Markov factor. Isomorphic systems (see Lecture 2) are Markov isomorphic.

Remark 8.7. A measure-preserving system (X,μ,T) is called **standard** if it is (point) isomorphic to (Z,ν,S), where Z is a complete metric space and ν is a Borel measure on Z. One can prove that for standards systems (X,μ,T) and (Y,ν,S) each Markov homomorphism $J : (\mathrm{L}^1(Y,\nu),S) \to (\mathrm{L}^1(X,\mu),T)$ is induced by a point factor map $\pi : X \to Y$. Thus the two notions of factors coincide for standard systems. This is yet another theorem of von Neumann, see [EFHN, App. F] for a proof due to M. Haase.

Proposition 8.8. *A Markov factor (a point factor) of an ergodic measure-preserving system is ergodic.*

The proof is left as Exercise 8.7. The following classical result shows that skew products from Example 8.2 represent a typical form of extensions.

Theorem 8.9 (Rokhlin skew product theorem)**.** *Let (X,μ,T) be an ergodic measure-preserving system, let (Y,ν,S) be a factor of (X,μ,T), and suppose that both systems are standard. Then (X,μ,T) is a ρ-skew-product of (Y,ν,S) and a standard probability space along an appropriate ρ.*

A proof can be found in [GETJ, Thm. 3.18].

2. Markov operators and a characterization of factors

Our next aim is to give different characterizations of Markov homomorphisms, and hence of factors. First we collect some general properties of Markov operators.

Proposition 8.10 (Basic properties of Markov operators)**.** *Let (X,μ) and (Y,ν) be probability spaces and let $S : \mathrm{L}^1(Y,\nu) \to \mathrm{L}^1(X,\mu)$ be a positive operator, i.e., satisfying $Sf \geq 0$ for each $f \geq 0$.*

(a) *S maps real-valued functions to real-valued functions, and $\mathrm{Re}Sf = S\mathrm{Re}f$.*
(b) *S respects conjugation, i.e., $S\overline{f} = \overline{Sf}$.*
(c) *If $g \leq f$, then $Sg \leq Sf$.*
(d) *$|Sf| \leq S|f|$ for every $f \in \mathrm{L}^1(Y,\nu)$.*
(e) *If $S\mathbf{1} = \mathbf{1}$, then it restricts to $S : \mathrm{L}^\infty(Y,\nu) \to \mathrm{L}^\infty(X,\mu)$ and this restriction is contractive with respect to the L^∞-norms.*
(f) *If $S'\mathbf{1} = \mathbf{1}$, then S is contractive with respect to the L^1-norms.*
(g) *If S is Markov operator, then for each $p \in [1,\infty]$ and $f \in \mathrm{L}^p(Y,\nu)$ we have $\|Sf\|_p \leq \|f\|_p$.*

Proof. The proof of the assertions (a)–(f) is left as Exercise 8.1, while (g) is handled in Exercise 8.3. ∎

It will be important to relate the different algebraic structures on $\mathrm{L}^0(X,\mu)$ which are respected by Koopman operators. A subspace $E \subset \mathrm{L}^0(X,\mu)$ is called a (vector) **sublattice** if $f \in E$ implies $|f| \in E$, while a subspace E is a **subalgebra** in $\mathrm{L}^0(X,\mu)$ if it is closed under multiplication, and E is **conjugation invariant** if $\overline{f} \in E$ for every $f \in E$. For example, $\mathrm{L}^\infty(X,\mu)$ is a subalgebra and for each $p \in [1,\infty]$ the space $\mathrm{L}^p(X,\mu)$ is a sublattice in $\mathrm{L}^0(X,\mu)$. Note that a subspace E is conjugation invariant if and only if $\mathrm{Re}f \in E$ for every $f \in E$. Moreover, if $f, g \in E$ are real-valued and E is a sublattice, then

$$\max\{f,g\} = \tfrac{1}{2}(f+g+|f-g|) \in E \quad \text{and} \quad \min\{f,g\} = \tfrac{1}{2}(f+g-|f-g|) \in E.$$

The following result connects these two types of structure, a proof can be found, e.g., in [EFHN, Thm. 7.23].

Proposition 8.11 (Sublattices vs. subalgebras)**.** (a) *For a conjugation invariant, closed subspace $E \subset \mathrm{L}^\infty(X,\mu)$ with $\mathbf{1} \in E$ the following are equivalent.*

  (i) *E is a subalgebra of $\mathrm{L}^\infty(X,\mu)$.*

  (ii) *E is a sublattice of $\mathrm{L}^\infty(X,\mu)$.*

(b) *Let E be a subalgebra as in (a), and let $J : E \to \mathrm{L}^\infty(Y,\nu)$ be a conjugation preserving, linear operator with $J\mathbf{1} = \mathbf{1}$. The following assertions are equivalent.*

  (i) *J is **multiplicative**, i.e., $J(fg) = Jf \cdot Jg$ for each pair $f, g \in E$.*

  (ii) *J is a **lattice homomorphism**, i.e., $J|f| = |Jf|$ for each $f \in E$.*

Let $S : \mathrm{L}^1(Y,\nu) \to \mathrm{L}^1(X,\mu)$ be a Markov operator. The adjoint operator $S' : \mathrm{L}^\infty(X,\mu) \to \mathrm{L}^\infty(Y,\nu)$ has the following properties: S' is a positive operator, $S'\mathbf{1} = \mathbf{1}$ and $\langle \mathbf{1}, S'g\rangle = \langle \mathbf{1}, g\rangle$ for every $g \in \mathrm{L}^\infty(X,\mu)$. It follows that S' is contractive for the L^1-norms (see Proposition 8.10), hence can be extended to the entire $\mathrm{L}^1(X,\mu)$ by density, and the extension, still denoted by S', is a Markov operator. We call $S' : \mathrm{L}^1(X,\mu) \to \mathrm{L}^1(Y,\nu)$ the **Markov adjoint** of S. Some basic properties are summarized in Exercise 8.4.

Proposition 8.12 (Characterization of Markov homomorphisms)**.** *For a Markov operator $S : \mathrm{L}^1(Y,\nu) \to \mathrm{L}^1(X,\mu)$ the following assertions are equivalent.*

(i) *S is multiplicative on $\mathrm{L}^\infty(Y,\nu)$.*

(ii) *$S'S = I$, the identity on $\mathrm{L}^1(Y,\nu)$.*

(iii) *S is an isometry with respect to the L^1-norms.*

(iv) *S is a Markov homomorphism.*

Proof. (i)⇒(ii): For $f, g \in \mathrm{L}^\infty(Y,\nu)$ we have $\langle Sf, Sg\rangle = \langle S(fg), \mathbf{1}\rangle = \langle fg, \mathbf{1}\rangle = \langle f, g\rangle$. This implies that $S'Sf = f$ for $f \in \mathrm{L}^\infty(Y,\nu)$, and (ii) follows by density.

The implication (ii)⇒(iii) follows from the facts that the Markov operators S and S' are contractive for the L^1-norm. Indeed, $\|f\|_1 = \|S'Sf\|_1 \le \|Sf\|_1 \le \|f\|_1$.

(iii)⇒(iv): Let $f \in \mathrm{L}^1(Y,\nu)$. Since $|Sf| \le |f|$ and S is isometric, we must have $|Sf| = |f|$.

(iv)⇒(i) has been discussed in Proposition 8.11(b). ∎

Remark 8.13. (a) Let S be a Markov homomorphism. Then S is isometric for the L^p-norms for each $p \in [1,\infty]$.

(b) If S is a Markov operator which is isometric for the L^2-norms (or for the L^p-norms for some $p \in [1,\infty)$), then S is a Markov homomorphism, see [EFHN, Thm. 13.9].

Proposition 8.14 (Sublattices in L^1 and subalgebras of L^∞)**.** (a) *Let E be a closed, conjugation invariant sublattice in $\mathrm{L}^1(X,\mu)$ with $\mathbf{1}\in E$. Then*

$$F := \mathrm{L}^\infty(X,\mu)\cap E$$

is a closed, conjugation invariant subalgebra in $\mathrm{L}^\infty(X,\mu)$ and it is dense in E.

(b) *Let $F\subset \mathrm{L}^\infty(X,\mu)$ be a conjugation invariant, closed subalgebra with $\mathbf{1}\in E$. Then the L^1-closure E of F is a closed sublattice of $\mathrm{L}^1(X,\mu)$.*

(c) *Let E be a closed, conjugation invariant sublattice in $\mathrm{L}^1(X,\mathcal{X},\mu)$ with $\mathbf{1}\in E$. Then there is a sub-σ-algebra $\mathcal{X}'$ of $\mathcal{X}$ such that $E=\mathrm{L}^1(X,\mathcal{X}',\mu)$.*

Proof. (a) The subspace F is clearly a conjugation invariant sublattice, and it is closed in $\mathrm{L}^\infty(X,\mu)$. Indeed, if $\|f_n-f\|_\infty\to 0$ as $n\to\infty$ and $f_n\in F$, then $\|f_n-f\|_1\le\|f_n-f\|_\infty\to 0$ and $f\in E$, since E is closed. Altogether we conclude that $f\in F$. Let $f\in E$, by conjugation invariance we may suppose that f is real-valued, and even that $f\ge 0$. Then $f_n:=\min\{n\mathbf{1},f\}\in F$ converges to f by Lebesgue's dominated convergence theorem, which proves the density of F in E.

(b) is clear by Proposition 8.11 (note that the closure of a sublattice is a sublattice).

(c) Set $F:=\mathrm{L}^\infty(X,\mu)\cap E$, which is then a closed subalgebra of $\mathrm{L}^\infty(X,\mu)$ by part (a). Define $\mathcal{X}':=\{A\in\mathcal{X}:\mathbf{1}_A\in E\}=\{A\in\mathcal{X}:\mathbf{1}_A\in F\}$. It is easy to see that $\mathcal{X}'$ is a sub-σ-algebra (exercise). The $\mathcal{X}'$-measurable simple functions belong to $F\subset E$, and are dense in $\mathrm{L}^1(X,\mathcal{X}',\mu)$. We conclude $\mathrm{L}^1(X,\mathcal{X}',\mu)\subset E$. For the converse inclusion take $f\in E$. Since E is conjugation invariant we may suppose that f is real-valued. Let $\alpha\in\mathbb{R}$, and define $g:=\max\{f-\alpha\mathbf{1},0\}$ and $f_n:=\min\{ng,\mathbf{1}\}$. Since E is a sublattice, we have $g,f_n\in E$. By the monotone convergence theorem we obtain that $\mathbf{1}_{[f>\alpha]}=\lim_{n\to\infty}f_n$ belongs to E. This, being true for each $\alpha\in\mathbb{R}$, implies that f is $\mathcal{X}'$-measurable. ∎

Remark 8.15. Of course, part (a) and (b) in the previous statement remain true if one replaces the space $\mathrm{L}^1(X,\mu)$ by any of the spaces $\mathrm{L}^p(X,\mu)$ with $p\in[1,\infty]$.

The main result of this section is the following characterization.

Proposition 8.16 (Characterization of factors)**.** *For two measure-preserving systems (X,μ,T) and (Y,ν,S) the following assertions are equivalent.*

(i) *(Y,ν,S) is a factor of (X,μ,T).*

(ii) *There exists a T-invariant sub-σ-algebra $\mathcal{X}'$ of $\mathcal{X}$ such that $(Y,\mathcal{Y},\nu,S)$ and $(X,\mathcal{X}',\mu,T)$ are Markov isomorphic.*

In particular, if both systems are standard, then (Y,ν,S) is a factor of (X,μ,T) if and only if $(Y,\mathcal{Y},\nu,S)$ is isomorphic to $(X,\mathcal{X}',\mu,T)$ for some T-invariant sub-σ-algebra $\mathcal{X}'$ of $\mathcal{X}$.

Proof. (i)⇒(ii): Let $J:\mathrm{L}^1(Y,\nu)\to\mathrm{L}^1(X,\mu)$ be a Markov homomorphism. Then $E=\mathrm{rg}(J)$ is a closed, conjugation invariant sublattice of $\mathrm{L}^1(X,\mu)$. Since $JS=TJ$, we obtain that E is invariant under T. By Proposition 8.14 there is a sub-σ-algebra $\mathcal{X}'$ of $\mathcal{X}$ such that $E=\mathrm{L}^1(X,\mathcal{X}',\mu)$. Since E is T-invariant and $T\mathbf{1}_A=\mathbf{1}_{T^{-1}A}$ for each $A\in\mathcal{X}'$, we conclude that $\mathcal{X}'$ is T-invariant. Finally, $J:\mathrm{L}^1(Y,\mathcal{Y},\nu)\to\mathrm{L}^1(X,\mathcal{X}',\mu)$ is a Markov isomorphism.

The implication (ii)⇒(i) is trivial. ∎

Proposition 8.17. *Let (X, μ, T) be a measure-preserving system, and let $F \subset \mathrm{L}^\infty(X, \mu)$ be a closed, conjugation invariant and T-invariant subalgebra with $\mathbf{1} \in F$. Then there is a T-invariant sub-σ-algebra $\mathcal{X}'$ of $\mathcal{X}$ such that $(X, \mathcal{X}', \mu, T)$ is a Markov factor of $(X, \mathcal{X}, \mu, T)$ such that F is dense in $\mathrm{L}^1(X, \mathcal{X}', \mu)$.*

Proof. The L^1-closure E of F is a closed sublattice by Proposition 8.14, and there is a T-invariant sub-σ-algebra $\mathcal{X}'$ such that $E = \mathrm{L}^1(X, \mathcal{X}', \mu)$. Since F is T-invariant, so is E, and hence the identity $J : \mathrm{L}^1(X, \mathcal{X}', \mu) \to \mathrm{L}^1(X, \mathcal{X}, \mu)$ provides a Markov homomorphism from $(X, \mathcal{X}', \mu, T)$ to $(X, \mathcal{X}, \mu, T)$. ∎

Note, however, that different subalgebras may give rise to the same factor, actually what matters is the L^1-closure of the subalgebra.

Definition 8.18. Let $J : (\mathrm{L}^1(Y, \nu), S) \to (\mathrm{L}^1(X, \mu), T)$ be a Markov homomorphism between the measure-preserving systems (Y, ν, S) and (X, μ, T). The operator $P := JJ' : \mathrm{L}^1(X, \mu) \to \mathrm{L}^1(X, \mu)$ is called the **projection onto the factor**, or **conditional expectation operator**.

These names are explained by the following proposition.

Proposition 8.19 (Projection onto a factor)**.** *Let $J : (\mathrm{L}^1(Y, \nu), S) \to (\mathrm{L}^1(X, \mu), T)$ be a Markov homomorphism, and let $P = JJ' : \mathrm{L}^1(X, \mu) \to \mathrm{L}^1(X, \mu)$. Then the following assertions hold.*

(a) *P is a Markov operator.*
(b) *P is a projection.*
(c) $\mathrm{rg}(P) = \mathrm{rg}(J)$.
(d) *P leaves $\mathrm{L}^2(X, \mu)$ invariant and restricts to the orthogonal projection onto the L^2-closure of $\mathrm{L}^\infty(Y, \nu)$.*
(e) *For each $f, g \in \mathrm{L}^\infty(X, \mu)$ we have $P(Pf \cdot g) = Pf \cdot Pg$.*

Proof. (a) P is a product of two Markov operators, hence has this property itself.

(b) We have $P^2 = JJ'JJ' = JIJ' = JJ' = P$ by Proposition 8.12.

(c) The inclusion $\mathrm{rg}(P) \subset \mathrm{rg}(J)$ is clear. Since $PJ = JJ'J = J$ by Proposition 8.12, we obtain the inclusion $\mathrm{rg}(J) \subset \mathrm{rg}(P)$.

(d) By Proposition 8.10 the operator P is L^2-contractive and by (a) it is a projection, hence it is an orthogonal projection.

(e) For $f, g, h \in \mathrm{L}^\infty(Y, \nu)$ we have that $\langle J'(Jf \cdot g), h\rangle = \langle Jf \cdot g, Jh\rangle = \langle J(fh), g\rangle = \langle fJ'g, h\rangle$. We thus obtain $J'(Jf \cdot g) = f \cdot J'g$. Now we can write $P(Pf \cdot g) = JJ'(JJ'f \cdot g) = J(J'f \cdot J'g) = JJ'f \cdot JJ'g = Pf \cdot Pg$. ∎

3. Examples

We will now use the above characterization to discuss the important factors mentioned at the beginning. We start by showing a relevant property of eigenfunctions of Koopman operators.

Proposition 8.20 (Eigenfunctions of Koopman operators for general systems)**.** *Let (X, μ, T) be a measure-preserving system, $p \in [1, \infty)$ and let T denote the Koopman operator on $\mathrm{L}^p(X, \mu)$.*

(a) *For every $\lambda \in P\sigma(T)$, the set $\ker(\lambda - T) \cap \mathrm{L}^\infty(X, \mu)$ is dense in $\ker(\lambda - T)$, i.e., bounded eigenfunctions are dense in the space of all L^p-eigenfunctions. In particular, for every eigenvalue there is a corresponding bounded eigenfunction.*

(b) $P\sigma(T) \subset \mathbb{T}$ *is a union of subgroups of* $\mathbb{T}$ *and is independent of* p.

Proof. Since T is an isometry on $\mathrm{L}^p(X,\mu)$, every eigenvalue of T belongs to $\mathbb{T}$.

(a) We first observe that for every $f \in \mathrm{fix}(T)$ and every $n \in \mathbb{N}$, one has $\mathbf{1}_{[|f|\le n]} \in \mathrm{fix}(T)$. Indeed, the T-invariance of f implies

$$(T\mathbf{1}_{[|f|\le n]})(x) = \mathbf{1}_{[|f|\le n]}(Tx) = \mathbf{1}_{[|f|\le n]}(x) \quad \text{for a.e. } x \in X.$$

Take now $f \in \ker(\lambda - T)$, i.e., $Tf = \lambda f$. By the algebraic properties of T, we have $T|f| = |Tf| = |\lambda||f| = |f|$, i.e., $|f| \in \mathrm{fix}(T)$. For $n \in \mathbb{N}$ consider $f_n := f\mathbf{1}_{[|f|\le n]}$. These functions are bounded and converge to f. Moreover, each f_n is an eigenfunction w.r.t. λ since by the above observation

$$Tf_n = Tf \cdot T\mathbf{1}_{[|f|\le n]} = \lambda f \cdot \mathbf{1}_{[|f|\le n]} = \lambda f_n.$$

(b) follows as in the ergodic case, see Proposition 3.23, whereas the independence from p follows from (a). ∎

Example 8.21 (Factors arising from eigenfunctions)**.** For a subgroup $G \subset \mathbb{T}$ the set

$$F_G := \overline{\mathrm{lin}}\{f \in \mathrm{L}^\infty(X,\mu) : Tf = \lambda f \text{ for some } \lambda \in G\}$$

is a T-invariant, closed, conjugation invariant subalgebra of $\mathrm{L}^\infty(X,\mu)$, hence a factor of (X,μ,T). The following already familiar factors are precisely of this type.

(a) The **fixed factor** $\mathrm{fix}(X,\mu,T)$ with $G = \{1\}$. The projection onto this factor is the mean ergodic projection $P_{\mathrm{fix}} : \mathrm{L}^1(X,\mu) \to \mathrm{L}^1(X,\mu)$ (with range $\mathrm{fix}(T)$). The system is ergodic if and only if the fixed factor $\mathrm{fix}(X,\mu,T)$ is trivial, i.e., contains only constants. In this case the corresponding T-invariant sub-σ-algebra $\mathcal{X}_{\mathrm{fix}}$ contains only sets of measure 1 and 0.

(b) The **rational spectrum factor** (or component) $\mathrm{rat}(X,\mu,T)$ with $G = \{a \in \mathbb{T} : a \text{ is rational}\}$. The L^2-closure of F_G yields the rational spectrum component H_{rat} of the Koopman operator on $H = \mathrm{L}^2(X,\mu)$ (see Lecture 6).

(c) The **Kronecker factor** $\mathrm{kr}(X,\mu,T)$ with $G = \mathbb{T}$. In this case, the Kronecker part H_{kr} associated to the Koopman operator T (see Lecture 6) is the closure of $F_{\mathbb{T}}$ in $H = \mathrm{L}^2(X,\mu)$. The extremal case when the Kronecker factor is trivial will be discussed later.

The importance of the Kronecker factor is emphasized by the following classical theorem, representing the extremal case when $(X,\mu,T) = \mathrm{kr}(X,\mu,T)$. Such systems are called systems with **discrete spectrum**.

Theorem 8.22 (Halmos–von Neumann)**.** (a) *A compact Abelian group rotation system has discrete spectrum.*

(b) *An ergodic measure-preserving system with discrete spectrum is* Markov isomorphic *to an ergodic (necessarily Abelian), compact group rotation system.*

The proof can be found, e.g., in [EW, Sec. 6.4] or [EFHN, Ch. 17, Thm. 17.11], while the original paper, for the metrizable case, is due to Halmos and von Neumann [4]. Next, we present factors of a more general type, due to Abramov [1].

Example 8.23 (Abramov factors)**.** Let (X,μ,T) be an ergodic measure-preserving system. We first define generalized eigenfunctions inductively. We call eigenfunctions of the Koopman operator T with modulus 1 generalized eigenfunction of order 1. We further call $f \in \mathrm{L}^\infty(X,\mu)$ a **generalized eigenfunction of order** k for

$k \geq 2$ if $|f| = 1$ a.e. and $Tf = gf$ holds for some generalized eigenfunction g of order $k-1$. Then for any given $k \in \mathbb{N}$ we define

$$A_k := A_k(X, \mu, T) := \overline{\mathrm{lin}}\{\text{generalized eigenfunctions of } T \text{ of order} \leq k\},$$

which is a T-invariant, closed subalgebra of $\mathrm{L}^\infty(X, \mu)$ with $\mathbf{1} \in A_k$ (see Exercise 8.8), hence defines a factor of (X, μ, T), called the **kth Abramov factor** of (X, μ, T). The **Abramov** factor is

$$A := A(X, \mu, T) := \overline{\mathrm{lin}}\{\text{generalized eigenfunctions of } T \text{ of some order}\}.$$

Obviously, $A_1(X, \mu, T) = \mathrm{kr}(X, \mu, T)$.

If (X, μ, T) is ergodic and $A(X, \mu, T) = (X, \mu, T)$ holds, then the system (X, μ, T) has, similarly to the statement of the Halmos–von Neumann theorem, some group-theoretic structure, see the papers [1] and [3]. Systems satisfying $A(X, \mu, T) = (X, \mu, T)$ are said to have **quasi-discrete spectrum**.

We now turn to more concrete examples.

Example 8.24 (Skew rotation)**.** Consider the skew rotation system $(\mathbb{T}^2, \mathrm{m}^2, T_a)$ from Example 2.13, where $a \in \mathbb{T}$ and

$$T_a : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (ax, xy).$$

The rotation system $(\mathbb{T}, \mathrm{m}, a)$ is a factor of the skew shift $(\mathbb{T}^2, \mathrm{m}^2, T_a)$ as discussed in Example 8.4. Next we study the fixed, Kronecker and Abramov factors. Let T denote the corresponding Koopman operator on $\mathrm{L}^2(\mathbb{T}^2)$, and consider the rotation system $(\mathbb{T}, \mathrm{m}, a)$ with Koopman operator L_a on $\mathrm{L}^2(\mathbb{T})$. Write

$$f = \sum_{n \in \mathbb{Z}^2} a_n \mathbf{z}^n$$

as an $\mathrm{L}^2(\mathbb{T}^2)$-convergent series, where $\mathbf{z}^n = \mathbf{z}_1^{n_1} \mathbf{z}_2^{n_2}$, $n = (n_1, n_2) \in \mathbb{Z}^2$. Then we have $T\mathbf{z}^n = (L_a \mathbf{z}_1^{n_1}) \mathbf{z}_1^{n_2} \mathbf{z}_2^{n_2}$ and

$$Tf = \sum_{n \in \mathbb{Z}^2} a_n (L_a \mathbf{z}_1^{n_1}) \mathbf{z}_1^{n_2} \mathbf{z}_2^{n_2}.$$

If $\lambda f = Tf$ for some $\lambda \in \mathbb{T}$, then

$$\sum_{n \in \mathbb{Z}^2} \lambda a_n \mathbf{z}_1^{n_1} \mathbf{z}_2^{n_2} = \lambda f = Tf = \sum_{n \in \mathbb{Z}^2} a_n (L_a \mathbf{z}_1^{n_1}) \mathbf{z}_1^{n_2} \mathbf{z}_2^{n_2}.$$

Hence we must have $a_n a^{n_1} \mathbf{z}_1^{n_1+n_2} \mathbf{z}_2^{n_2} = \lambda a_n \mathbf{z}_1^{n_1} \mathbf{z}_2^{n_2}$ for each $n \in \mathbb{Z}^2$. For each fixed $n_2 \in \mathbb{Z}$ we obtain that

$$a_{(j,n_2)} a^j = \lambda a_{(j+n_2,n_2)} \quad \text{for every } j \in \mathbb{Z}.$$

Since $\sum_{j \in \mathbb{Z}} |a_{(j,n_2)}|^2 < \infty$, it follows that $a_{(j,n_2)} = 0$ for every $j \in \mathbb{Z}$ provided $n_2 \neq 0$. We thus conclude that

$$f = \sum_{m \in \mathbb{Z}} a_{(m,0)} \mathbf{z}_1^m.$$

We also obtain that $\mathrm{Fix}(T) = \mathbb{C}\mathbf{1}$ if a is irrational, hence the system is ergodic in this case. For the Kronecker factor we conclude that it is isomorphic to the rotation factor $(\mathbb{T}, \mathrm{m}, a)$.

Observe that while functions $\mathbf{z}_1^n$, $n \in \mathbb{Z}$, are eigenfunctions of T corresponding to the eigenvalues a^n, the monomials $\mathbf{z}_2^n$, $n \in \mathbb{Z}$, are generalized eigenfunctions of order 2, since

$$(T\mathbf{z}_2)(x, y) = \mathbf{z}_2(ax, xy) = \mathbf{z}_1\mathbf{z}_2(x, y).$$

Thus the Abramov factor in the ergodic case coincides with the whole system, i.e., the system has quasi-discrete spectrum.

Example 8.25 (Heisenberg systems)**.** Consider the set G of upper triangular real matrices with ones on the diagonal, i.e.,

$$G = \left\{ \left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix} \right) : x, y, z \in \mathbb{R} \right\},$$

which is a group with the matrix multiplication, called the **Heisenberg group**. For $x, y, z \in \mathbb{R}$ we introduce the abbreviation

$$[x, y, z] := \left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix} \right),$$

and identify $[x, y, z] \in G$ with $(x, y, z) \in \mathbb{R}^3$. With the Euclidean topology of $\mathbb{R}^3$ the group G becomes a topological group. We can write

$$[x_0, y_0, z_0] \cdot [x, y, z] = [x + x_0, y + y_0, z + z_0 + x_0 y].$$

Consider the subgroup $H := \{[x, y, z] : x, y, z \in \mathbb{Z}\}$. Then H is closed and discrete in G. The set G/H of *left cosets* becomes a Hausdorff topological space if endowed with the quotient topology under the canonical map

$$q(g) := gH, \quad G \to G/H,$$

i.e., $A \subset G/H$ is open if and only if $q^{-1}(A)$ is open in G (so that q is continuous). The subset $A = \{[x, y, z] : x, y, z \in [0, 1)\}$ of G is a complete set of representatives (a so-called **fundamental domain**), i.e., contains from every left coset precisely one element. So $G = AH = \bigcup_{h \in H} Ah$ as a union of pairwise disjoint sets. Since $\overline{A}$ is compact in G, q is continuous and $q(\overline{A}) = G/H$, this latter set is compact. For a given $a = [\alpha, \beta, \gamma] \in G$ the left rotation by a on G/H is defined by $gH \mapsto agH$. In this way we obtain an invertible topological system, denoted by $(\mathbb{H}, a) = (\mathbb{H}, [\alpha, \beta, \gamma])$ and called the **Heisenberg system**.

Next we present a probability measure on $\mathbb{H}$ which is invariant under each the left rotation by $g \in G$. After identifying G with $\mathbb{R}^3$ as above we can consider the three-dimensional Lebesgue measure λ on G, and multiplication by g clearly preserves λ. Let $A, B \subset \mathbb{R}^3$ be two Borel measurable complete sets of representatives of the left cosets in G/H. Note that $M \in \mathcal{B}(G/H)$ if and only if $q^{-1}(M)$ is Borel in G. We prove that for each Borel set $M \subset G/H$ we have $\lambda\big(A \cap q^{-1}(M)\big) = \lambda\big(B \cap q^{-1}(M)\big)$. Since $q^{-1}(M)h = q^{-1}(M)$ for every $h \in H$ and since λ is invariant under the left rotations by $g \in G$, we have the following chain of equalities

$$\begin{aligned}
\lambda\big(A \cap q^{-1}(M)\big) &= \lambda\Big(A \cap q^{-1}(M) \cap \bigcup_{h \in H} Bh\Big) = \sum_{h \in H} \lambda\big(A \cap q^{-1}(M) \cap Bh\big) \\
&= \sum_{h \in H} \lambda\big(Ah^{-1} \cap q^{-1}(M)h^{-1} \cap B\big) = \sum_{h \in H} \lambda\big(Ah^{-1} \cap q^{-1}(M) \cap B\big) \\
&= \lambda\Big(\bigcup_{h \in H} Ah^{-1} \cap q^{-1}(M) \cap B\Big) = \lambda\big(q^{-1}(M) \cap B\big).
\end{aligned}$$

For a Borel set $M \subset G/H$ define

$$\mathrm{m}_{\mathbb{H}}(M) := \lambda\big(A \cap q^{-1}(M)\big),$$

(note that, by the above, it is immaterial which complete set of representatives one chooses here). Then $\mathrm{m}_{\mathbb{H}}$ is a probability measure on $\mathcal{B}(G/H)$, and we claim that $\mathrm{m}_{\mathbb{H}}$ is invariant under every left rotation by $g \in G$. Indeed, for $g \in G$ the set $g^{-1}A$ is again a measurable complete set of representatives. Hence, by the invariance of λ and by the previously established fact we obtain that

$$\lambda\big(A \cap q^{-1}(gM)\big) = \lambda\big(A \cap gq^{-1}(M)\big) = \lambda\big(g^{-1}A \cap q^{-1}(M)\big) = \lambda\big(A \cap q^{-1}(M)\big).$$

The measure $\mathrm{m}_{\mathbb{H}}$ is also called **Haar measure** on $\mathbb{H}$. For each $a \in G$ the measure-preserving system $(\mathbb{H}, \mathrm{m}_{\mathbb{H}}, a)$ is a measure-preserving system, called a **Heisenberg system**.

Example 8.26 (Rotation factors of Heisenberg systems)**.** For every $\alpha, \beta, \gamma \in \mathbb{R}$ the two dimensional shift $([0,1)^2, \lambda^2, (\alpha, \beta))$ is a factor of the Heisenberg system $(\mathbb{H}, \mathrm{m}_{\mathbb{H}}, [\alpha, \beta, \gamma])$ under the point factor map

$$[x, y, z]H \mapsto (x \bmod 1, y \bmod 1),$$

see Exercise 8.9.

Proposition 8.27 (The Kronecker factor of Heisenberg systems)**.** *For $a = [\alpha, \beta, \gamma]$ with $\alpha = \beta = 0$, the Kronecker factor of $(\mathbb{H}, \mathrm{m}_{\mathbb{H}}, a)$ is the entire system. If $\alpha \neq 0$ or $\beta \neq 0$, then the Kronecker factor is the rotation factor described above.*

Proof. The center of the Heisenberg group G is

$$Z(G) := \big\{g \in G : gh = hg \text{ for all } h \in G\big\} = \big\{[0, 0, r] : r \in \mathbb{R}\big\},$$

see Exercise 8.9. For $g \in G$ let L_g be the Koopman operator on $\mathrm{L}^2(\mathbb{H})$ of the left rotation by g, and let $T = L_a$. It is easy to see that $F = \bigcap_{g \in Z(G)} \mathrm{fix}(L_g)$ consists precisely of functions $f \in \mathrm{L}^2(\mathbb{H})$ which depend on the first two coordinates only, i.e., F corresponds to the rotation factor from Example 8.26.

If $\alpha = \beta = 0$, then $a \in Z(G)$ and it is Exercise 8.9 to show that the Kronecker factor coincides with the system itself. In the following suppose that $(\alpha, \beta) \neq 0$. We start with the case $\beta \neq 0$.

For every $g, h \in G$ we have $L_{hg} = L_g L_h$ and $L_g^* = L_g^{-1} = L_{g^{-1}}$. To determine the Kronecker factor let $f \in \mathrm{L}^2(\mathbb{H})$ and $\lambda \in \mathbb{T}$ such that $Tf = \lambda f$. We shall prove that for every $h \in Z(G)$ one has $L_h f = f$, and this implies that $f \in F$, i.e., f belongs to the rotation factor by what has been said at the beginning of the proof.

For $x \in \mathbb{R}$ we set $g_x := [x, 0, 0] \in G$, and prove by induction that

$$a^n g_x a^{-n} = [x, 0, n\beta x] \quad \text{for } n \in \mathbb{N}_0.$$

For a given $r \in \mathbb{R}$ and $n \in \mathbb{N}$ set $r_n := \frac{r}{\beta n}$. Then $g_{r_n} \to [0, 0, 0]$ and $a^n g_x a^{-n} \to [0, 0, r]$ as $n \to \infty$. For each $g \in \mathrm{C}(\mathbb{H})$ we obtain

$$L_{g_{r_n}} g - T^{n*} L_{g_{r_n}} T^n g \to g - L_{[0,0,r]} g \quad \text{as } n \to \infty,$$

and by density this convergence extends to $g \in \mathrm{L}^2(\mathbb{H})$. For f as above we can write

$$\begin{aligned}(f - L_{[0,0,r]} f|f) &= \lim_{n\to\infty} (L_{g_{r_n}} f - T^{n*} L_{g_{r_n}} T^n f|f) \\ &= \lim_{n\to\infty} \big((L_{g_{r_n}} f|f) - (L_{g_{r_n}} T^n f|T^n f)\big) \\ &= \lim_{n\to\infty} \big((L_{g_{r_n}} f|f) - |\lambda^n|^2 (L_{g_{r_n}} f|f)\big) = 0.\end{aligned}$$

By Proposition 6.1 the equality $f - L_{[0,0,r]} f = 0$ follows. The case when $\alpha \neq 0$ can be established analogously with the choice $g_x = [0, x, 0]$. ∎

Remark 8.28 (Abramov factors of Heisenberg systems)**.** Furstenberg showed in [2] that the Heisenberg system in the ergodic case for some choice of α, β, γ does not have any generalized eigenfunctions apart from the eigenfunctions themselves, i.e., the Abramov and the Kronecker factors coincide.

Exercises

Exercise 8.1 (Positive operators)**.** Prove (a)–(f) of Proposition 8.10.

Exercise 8.2 (Jensen's inequality)**.** Let $S : \mathrm{L}^1(Y,\nu) \to \mathrm{L}^1(X,\mu)$ be a Markov operator, and let $g : [0,\infty) \to \mathbb{R}$ be a convex function. Prove that for every positive $f \in \mathrm{L}^1(Y,\nu)$ such that $g \circ f \in \mathrm{L}^1(Y,\nu)$ we have $g \circ Sf \leq S(g \circ f)$.

Exercise 8.3 (L^p-contractivity of Markov operators)**.** Use the result of the foregoing exercise to prove that a Markov operator $S : \mathrm{L}^1(X,\nu) \to \mathrm{L}^1(X,\mu)$ respects the L^p-spaces and is contractive for the L^p-norms .

Exercise 8.4 (Markov adjoint)**.** Let $S : \mathrm{L}^1(Y,\nu) \to \mathrm{L}^1(X,\mu)$ and $R : \mathrm{L}^1(X,\mu) \to \mathrm{L}^1(Z,\lambda)$ be Markov operators. Prove the following assertions.

(a) $S'' = S$.
(b) RS is a Markov operator.
(c) $(RS)' = S'R'$.
(d) $S'|_{\mathrm{L}^2(X,\mu)} = (S|_{\mathrm{L}^2(Y,\nu)})^*$.

Exercise 8.5 (Projection onto a factor)**.** Let $J : (\mathrm{L}^1(Y,\nu), S) \to (\mathrm{L}^1(X,\mu), T)$ be a Markov homomorphism, and let $P = JJ'$. Prove the following assertions.

(a) $P'' = P$.
(b) $PTP = TP$.
(c) If $SJ' = J'T$, then $PT = TP$.

Exercise 8.6 (The compact factor)**.** Let (X, μ, T) be a measure-preserving system with Koopman operator T. A function $f \in \mathrm{L}^2(X,\mu)$ is called **compact** (or asymptotically almost periodic with respect to T), if

$$\{T^n f : n \in \mathbb{N}\}$$

is relatively compact in $\mathrm{L}^2(X,\mu)$. Prove that

$$E := \{f \in \mathrm{L}^2(X,\mu) : f \text{ is a compact function}\}$$

is a closed, T-invariant and conjugation invariant sublattice in $\mathrm{L}^2(X,\mu)$ with $\mathbf{1} \in E$, and the set $F := \mathrm{L}^\infty(X,\mu) \cap E$ is a closed subalgebra yielding the Kronecker factor (cf. Exercise 6.5).

Exercise 8.7 (Factors of ergodic systems)**.** Prove that Markov (and point) factors of ergodic systems are ergodic.

Exercise 8.8 (Abramov factors)**.** Let (X, μ, T) be a measure-preserving system. Prove the following assertion for the sets A and A_k from Example 8.23.

(a) The set A and for each $k \in \mathbb{N}$ the sets A_k are closed, conjugation invariant and T-invariant subalgebras of $\mathrm{L}^\infty(X, \mu)$.
(b) $A_k \subset A_{k+1} \subset A$ for each $k \in \mathbb{N}$.

Exercise 8.9 (Heisenberg systems)**.** Consider the Heisenberg system $(\mathbb{H}, \mathrm{m}_{\mathbb{H}}, a)$ with $a = [\alpha, \beta, \gamma]$, $\alpha, \beta, \gamma \in \mathbb{R}$.

(a) Prove that as in Example 8.26
$$[x, y, z]H \mapsto (x \bmod 1, y \bmod 1)$$
is a point factor map onto the two-dimensional shift $([0,1)^2, \lambda^2, (\alpha, \beta))$.
(b) Determine the center of the Heisenberg group.
(c) Prove that in the case $\alpha = \beta = 0$, the system $(\mathbb{H}, \mu, a)$ coincides with its Kronecker factor, i.e., has discrete spectrum. (Hint: Exercise 8.6.)

Exercise 8.10 (Affine endomorphisms on $\mathbb{T}$)**.** For $a \in \mathbb{T}$ and $k \in \mathbb{N}$ define $T : \mathbb{T} \to \mathbb{T}$ by $Tx = ax^k$. Prove the following assertions.

(a) $(\mathbb{T}, \mathrm{m}, T)$ is a measure-preserving system.
(b) $(\mathbb{T}, \mathrm{m}, T)$ is ergodic if and only if $k \geq 2$ or a is irrational.
(c) The Kronecker factor of $(\mathbb{T}, \mathrm{m}, T)$ is trivial, i.e., consists of constants, if $k \geq 2$.

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LECTURE 9

First applications of ergodic theorems

In this lecture we present some applications of ergodic theorems and related techniques in different areas of mathematics.

1. Applications in stochastics

Let $(\Omega, \mathbb{P})$ be a probability space and for $j \in \mathbb{N}$ let $f_j : \Omega \to \mathbb{R}$ be measurable functions, i.e., random variables. Recall that the random variables f_j, $j \in \mathbb{N}$, are called **independent** if for every $m \in \mathbb{N}$, every $n_1, \ldots, n_m \in \mathbb{Z}$, and all Borel sets $B_1, \ldots, B_m \subset \mathbb{R}$ one has

$$\mathbb{P}\Big(\bigcap_{j=1}^{m} f_{n_j}^{-1} B_j\Big) = \prod_{j=1}^{m} \mathbb{P}(f_{n_j}^{-1} B_j).$$

This will be assumed in the following and provides a special case of stationary stochastic processes from Example 2.17. Suppose that the random variables are **identically distributed**, i.e., that $f_{j*}\mathbb{P} = f_{i*}\mathbb{P}$ holds for each $i, j \in \mathbb{N}$ (where $f_{j*}\mathbb{P}$ is the push-forward measure defined by $f_{j*}\mathbb{P}(B) = \mathbb{P}(f_j^{-1}B)$). We set $\nu := f_{1*}\mathbb{P}$, the common distribution of the random variables.

Consider the one-sided shift system with state space $(\mathbb{R}, \nu)$, i.e., the measure-preserving system $(X, \mathcal{X}, \mu, T)$ with $X = \mathbb{R}^{\mathbb{N}}$, the product σ-algebra $\mathcal{X} = \mathcal{B}(\mathbb{R}^{\mathbb{N}})$, the product measure μ on $\mathcal{X}$ with ν in each component, and the shift T defined as $T(x_n)_{n\in\mathbb{N}} = (x_{n+1})_{n\in\mathbb{N}}$. This system is ergodic by Example 3.17. As in Example 2.17 consider the mapping

$$\theta : \Omega \to X, \quad \theta(\omega) := (f_j(\omega))_{j\in\mathbb{N}},$$

satisfying $\theta_*\mathbb{P} = \mu$ since the random variables are independent. The Koopman operator $S_\theta : \mathrm{L}^1(X, \mu) \to \mathrm{L}^1(\Omega, \mathbb{P})$ is a Markov homomorphism, so that we have $\mathbb{E}(S_\theta g) = \int_X g \,\mathrm{d}\mu$ for every $g \in \mathrm{L}^1(X, \mu)$, where $\mathbb{E}(h) = \int_\Omega h \,\mathrm{d}\mathbb{P}$ is the **expectation** of the random variable h. We first show how the mean ergodic theorem can be used to show the weak law of large numbers.

Theorem 9.1 (Weak law of large numbers)**.** *Let f_n, $n \in \mathbb{N}$, be identically distributed, independent, real random variables. Then for every $\varepsilon > 0$*

$$\mathbb{P}\Big[\Big|\frac{1}{N}\sum_{n=1}^{N} f_n - \mathbb{E}(f_1)\Big| > \varepsilon\Big] \to 0 \quad \textit{as } N \to \infty.$$

Proof. Let $g : X \to \mathbb{R}$ be the projection onto the first component, then $g \in \mathrm{L}^1(X,\mu)$. Since T is mean ergodic on $\mathrm{L}^1(X,\mu)$ by Remark 7.6 and a density argument, from ergodicity we conclude

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n g = \int_X g \,\mathrm{d}\mu \cdot \mathbf{1} \quad \text{in } \mathrm{L}^1(X,\mu).$$

This yields

$$\lim_{N\to\infty} \mathbb{E}\Big|\frac{1}{N}\sum_{n=1}^{N} f_n - \mathbb{E}(f_1)\Big| = \lim_{N\to\infty} \Big\|S_\theta\Big(\frac{1}{N}\sum_{n=1}^{N} T^n g - \int_X g\,\mathrm{d}\mu\cdot\mathbf{1}\Big)\Big\|_{\mathrm{L}^1(\Omega,\mathbb{P})} = 0.$$

It is self-evident (see Exercise 9.1) that

$$\mathbb{P}\Big[\Big|\frac{1}{N}\sum_{n=1}^{N} f_n - \mathbb{E}(f_1)\Big| > \varepsilon\Big] \leq \frac{1}{\varepsilon}\mathbb{E}\Big|\frac{1}{N}\sum_{n=1}^{N} f_n - \mathbb{E}(f_1)\Big|.$$

The proof is hence complete. ∎

With structurally the same proof it is possible to strengthen this result. We first need a lemma for that.

Lemma 9.2. *Let (X,μ), (Y,ν) be probability spaces and let $\theta : X \to Y$ be a measure-preserving mapping. Suppose that the functions $g_n \in \mathrm{L}^0(Y,\nu)$, $n \in \mathbb{N}$, converge ν-almost everywhere to $g \in \mathrm{L}^0(Y,\nu)$. Then $S_\theta g_n \to S_\theta g$ almost everywhere with respect to μ.*

Proof. We can suppose without loss of generality that the functions g_n are real-valued. Moreover, we note that for each sequence of real-valued functions $(h_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^0(Y,\nu)$ we have

$$S_\theta \sup_{n\in\mathbb{N}} h_n = \sup_{n\in\mathbb{N}} S_\theta h_n \quad \text{and} \quad S_\theta \inf_{n\in\mathbb{N}} h_n = \inf_{n\in\mathbb{N}} S_\theta h_n.$$

As a consequence

$$S_\theta \limsup_{n\to\infty} g_n = \limsup_{n\to\infty} S_\theta g_n \quad \text{and} \quad S_\theta \liminf_{n\to\infty} g_n = \liminf_{n\to\infty} S_\theta g_n.$$

The proof is complete. ∎

We are now ready to deduce the strong law of large numbers from the pointwise ergodic theorem.

Theorem 9.3 (Kolmogorov's strong law of large numbers)**.** *For $n \in \mathbb{N}$ let f_n be identically distributed, independent real random variables. Then*

$$\frac{1}{N}\sum_{n=1}^{N} f_n \to \mathbb{E}(f_1) \quad \mathbb{P}\text{-almost surely as } N\to\infty.$$

Proof. The proof is almost the same as for the weak law of large numbers, we only use the pointwise ergodic theorem. With the notations of that proof we obtain that

$$\frac{1}{N}\sum_{n=1}^{N} T^n g \to \int_X g\,\mathrm{d}\mu\cdot\mathbf{1} \quad \mu\text{-almost everywhere as } N\to\infty.$$

By Lemma 9.2 applied to S_θ we conclude

$$\frac{1}{N}\sum_{n=1}^{N} f_n = S_\theta \frac{1}{N}\sum_{n=1}^{N} T^n g \to \mathbb{E}(f_1) \quad \mathbb{P}\text{-almost everywhere as } N \to \infty. \qquad \blacksquare$$

2. Applications in number theory

Next we come to more surprising applications in number theory. The following property of real numbers was introduced by Borel in 1909.

Definition 9.4. Let $x \in \mathbb{R}$ and $b \in \{2, 3, \ldots\}$. Denote by $N_b(a, n)$ the number of appearances of the digit a in the first n positions after the "decimal point*" of x in base b. The number x is called **simply normal in base** b if

$$\lim_{n\to\infty} \frac{N_b(a,n)}{n} = \frac{1}{b} \quad \text{for each } a \in \{0, \ldots, b-1\},$$

and **simply normal** if x is simply normal in every base. Analogously, x is called **normal in base** b if for every finite word $S := a_1 \ldots a_d \in \{0, 1, \ldots, b-1\}^d$, the number $N_b(S, n)$ of appearances of S among the first n positions after the decimal point of x in base b satisfies

$$\lim_{n\to\infty} \frac{N_b(S,n)}{n} = \frac{1}{b^d},$$

and **normal** if x is normal in every base.

For example, the number $0.01234567890123456789\ldots$ is simply normal in base 10 but not normal in base 2 (Exercise 9.2). The so-called Champernowne number $0.01234567891011121314\ldots$ is normal in base 10 but it is unknown whether it is normal in other bases.

Only a few examples of normal numbers are known. Even for the numbers $\sqrt{2}$, π and e it is not known whether they are normal. Nevertheless, the following result shows that normal numbers are the rule and not the exception.

Theorem 9.5 (Borel)**.** *Lebesgue almost every number is normal.*

We present here an ergodic theoretic proof.

Proof. Since the set of possible bases is countable, it is enough to show that for a fixed base b, almost every real number is normal in base b. Similarly, as there are countably many finite words, it suffices to prove that every fixed finite word $S = a_1 \ldots a_d \in \{0, \ldots, b-1\}^d$ has the asymptotic frequency $1/b^d$ of occurrence in almost every $y \in [0, 1]$. Consider the Bernoulli shift $B(1/b, \ldots, 1/b)$ on $X := \{0, \ldots, b-1\}^{\mathbb{N}}$. By Proposition 3.17, it is ergodic with respect to the product measure μ. Let $a_1 \ldots a_d$ be a finite word in base b. Applying the pointwise ergodic theorem to the cylinder set $A := \{(x_1, x_2, \ldots) : x_1 = a_1, \ldots, x_d = a_d\}$ we obtain

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_A(T^n x) = \mu(A) = \frac{1}{b^d}$$

*Only in base $b = 10$ this is a *decimal* point, we keep however this terminology.

for almost every $x \in X$. The term $\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_A(T^n x)$ on left-hand side equals the number of appearances of S in the first N positions of x divided by N. Now, the mapping

$$\theta : X \to [0,1], \quad (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} \frac{x_n}{b^n}$$

is measure-preserving (if $[0,1]$ is endowed with the Lebesgue measure) and even essentially invertible, with essential inverse $y \mapsto (y_n)_{n\in\mathbb{N}}$ the expansion of y in base b. An application of Lemma 9.2 finishes the proof. ∎

We now turn to a property of sequences of numbers which we will explore in more detail in the next lecture. At this point our aim is not to present applications in number theory, but rather to describe a relation between a number theoretic and an ergodic theoretic property.

Recall that the rotation system $(\mathbb{T}, a)$ is uniquely ergodic for irrational $a \in \mathbb{T}$ (see Theorem 3.31). Endow $[0,1)$ with the topology that makes $\theta : t \mapsto \mathrm{e}^{2\pi \mathrm{i} t}$ a homeomorphism between $[0,1)$ and $\mathbb{T}$ (i.e., consider the quotient topology on $[0,1]$ for which 0 and 1 are identified with each other). If $a = \mathrm{e}^{2\pi\mathrm{i}\alpha}$, we obtain a topological system $([0,1), \alpha)$, with the continuous mapping $t \mapsto \{t+\alpha\}$. Recall the notation $\{x\} := x - \lfloor x \rfloor$ for the **fractional part** of $x \in \mathbb{R}$. The Lebesgue measure λ is an invariant measure for this system (see Example 2.11). The translation system $([0,1), \lambda, \alpha)$ is isomorphic to $(\mathbb{T}, \mathrm{m}, a)$ via the previous *homeomorphism*. It is Exercise 9.7 to show that $([0,1), \alpha)$ is uniquely ergodic if and only if $(\mathbb{T}, a)$ is uniquely ergodic.

Remark 9.6. We note that the condition that the isomorphism θ is a homeomorphism is vital. If (K, μ, T) and (L, ν, S) are isomorphic measure-preserving systems with (K, T) and (L, S) being topological systems, it is not true in general that (K, T) and (L, S) are simultaneously uniquely ergodic or not. In fact, one has the following result, called the Jewett–Krieger theorem [4], [5]: For every ergodic invertible, standard *measure-preserving system* (X, μ, T) there is a uniquely ergodic topological system (L, S) with unique invariant measure ν such that (X, μ, T) and (L, ν, S) are isomorphic, and ν satisfies $\mathrm{supp}(\nu) = L$.

Nevertheless, by the considerations before the remark, the translation system $([0,1), \lambda, \alpha)$ is uniquely ergodic if α is irrational. By Theorem 7.10 we immediately obtain the following proposition.

Proposition 9.7. *For a given irrational number $\alpha \in \mathbb{R}$ consider the sequence $(\{n\alpha\})_{n\in\mathbb{N}_0}$. For each continuous, 1-periodic function $f : \mathbb{R} \to \mathbb{C}$ one has*

$$\int_0^1 f(s)\,\mathrm{d}s = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\{x\}).$$

We now replace the rotation on $\mathbb{T}$ by the skew rotation on $\mathbb{T}^2$ given by $T_a(x,y) := (ax, xy)$, see Example 2.13. Recall also the additive variant, the skew shift, defined on $[0,1)^2$ by

$$\widetilde{T}_\alpha : [0,1)^2 \to [0,1)^2, \quad (x,y) \mapsto (\{x+\alpha\}, \{x+y\}).$$

By endowing $[0,1)$ with the topology as above, this mapping becomes continuous, giving rise to the **topological skew shift** system $([0,1)^2, \widetilde{T}_\alpha))$, which is uniquely ergodic if and only if the corresponding skew shift $(\mathbb{T}^2, T_a)$, $a = \mathrm{e}^{2\pi\mathrm{i}\alpha}$, is uniquely ergodic, see Exercise 9.7.

Proposition 9.8. *The skew rotation* $(\mathbb{T}^2, \mathrm{m}^2, T_a)$ *is uniquely ergodic if* $a \in \mathbb{T}$ *is irrational (i.e., not a root of unity). Equivalently, the skew shift* $([0,1)^2, \lambda^2, \widetilde{T}_\alpha)$ *is uniquely ergodic if* α *is irrational.*

One can prove this statement by ergodic theoretic argumentations in a far more general situation, and the previous statement becomes a special case of the next result of Furstenberg [1], see also [EFHN, Sec. 10.4 and Ch. 15, Supp.] for a proof.

Theorem 9.9 (Furstenberg's skew product theorem)**.** *Let* (K, T) *be a uniquely ergodic topological system, let* G *be a compact group, and let* $\alpha : K \to G$ *be a continuous mapping. Consider the topological group extension system* $(K \times G, \widetilde{S}_a)$ *with* $\widetilde{S}_a(x, g) = (Tx, \alpha(x)g)$*, which has* $\mu \otimes \mathrm{m}_G$ *as an invariant measure. If* $\mu \otimes \mathrm{m}_G$ *is an ergodic measure, then* $(K \times G, \widetilde{S}_a)$ *is uniquely ergodic.*

This directly implies unique ergodicity of $(\mathbb{T}^2, \mathrm{m}^2, T_a)$ if a is irrational (why?). However, we have chosen a number theoretic way of proving this, which will be presented in the next lecture. Here we only describe what this unique ergodicity property would give us in number theory in the spirit of the previous example. The next is a preparatory result with the proof left as Exercise 9.5.

Proposition 9.10 (Cocycle for the skew shift)**.** *For each* $(x, y) \in \mathbb{T}^2$ *we have*

$$T_a^n(x, y) = \big(a^n x, a^{\frac{n(n-1)}{2}} x^n y\big)$$

and for the skew shift $([0,1)^2, \lambda^2, \widetilde{T}_\alpha)$ *and for every* $(s, t) \in [0,1)^2$

$$\widetilde{T}_\alpha^n(s, t) = \big(n\alpha + s, \tfrac{n(n-1)}{2}\alpha + ns + t\big).$$

Thus Proposition 9.8 implies the following result.

Proposition 9.11. *Let* $\alpha \in \mathbb{R}$ *be an irrational number and consider the sequence* $(\{\frac{n(n-1)}{2}\alpha\})_{n\in\mathbb{N}_0}$*. For each continuous,* 1*-periodic function* $f : [0,1] \to \mathbb{C}$ *one has*

$$\int_0^1 f(s)\,\mathrm{d}s = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f\big(\{\tfrac{n(n-1)}{2}\alpha\}\big).$$

Proof. Apply Propositions 9.8, 9.10 and Theorem 7.10 to the continuous function $g : (x, y) \mapsto x$. ■

The property, as described in Propositions 9.7 and 9.11, of the sequences $(\{n\alpha\})_{n\in\mathbb{N}_0}$ and $(\{\frac{n(n-1)}{2}\alpha\})_{n\in\mathbb{N}_0}$ means that they are equidistributed in $[0, 1)$. A more detailed study of equidistribution will be carried out in the next lecture.

3. Applications in combinatorial number theory

We now show how ergodic theory can help to solve combinatorial problems in number theory. We will encounter the first instance of the following phenomenon: If a subset $E \subset \mathbb{N}$ is large (in some appropriate sense), then it contains some structured subsets. In this lecture "largeness" will be quantified by the upper density. Recall, from Lecture 6, that the upper density of a set $E \subset \mathbb{N}$ is

$$\overline{\mathrm{d}}(E) = \limsup_{N\to\infty} \frac{|E \cap \{1, \dots, N\}|}{N}.$$

Polynomial configurations in large sets. In the first example, structure means a polynomial progression, and we show how ergodic theorems help to show existence of such polynomial progressions in large sets of integers.

Theorem 9.12 (Furstenberg–Sarközy)**.** *Let $E \subset \mathbb{N}$ have positive upper density and let P be an integer polynomial with $P(0) = 0$. Then there exist $a, n \in \mathbb{N}$ with $a, a + P(n) \in E$.*

Remark 9.13. The following easy example shows that the condition $P(0) = 0$ cannot be dropped. Let $P(x) := 2x + 1$ and $E := 2\mathbb{N} + 1$ be the set of all odd numbers. It is clear that E has no progressions of the form $a, a + 2n + 1$.

For the polynomial $P(n) = n^2$ Sárközy [7] proved this with number theoretic arguments, while Furstenberg [2] gave an ergodic theoretic proof by first showing the following property of measure-theoretic dynamical systems. The general case is settled in [8] and [3].

Theorem 9.14 (Polynomial recurrence)**.** *Let (X, μ, T) be a measure-preserving system, let $f \in \mathrm{L}^\infty(X, \mu)$ satisfy $f > 0$ (meaning $f \geq 0$ and $f \neq 0$) and take an integer polynomial P with $P(0) = 0$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_X f \cdot T^{P(n)} f \,\mathrm{d}\mu = \Big(f \,\Big|\, \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^{P(n)} f \Big) > 0. \tag{9.1}$$

In particular, for every $A \subset X$ with $\mu(A) > 0$ one has

$$\mu\big(A \cap T^{-P(n)} A\big) > 0 \quad \textit{for some } n \in \mathbb{N}.$$

Remark 9.15. Recall that the limit in the middle of (9.1) exists in $\mathrm{L}^2(X, \mu)$ by the polynomial mean ergodic theorem (Theorem 7.13). Thus, (9.1) means that this limit has non-trivial correlation with f.

The following bridge between ergodic theory and combinatorics will finish the proof of the Furstenberg–Sárközy theorem.

Theorem 9.16 (Furstenberg's correspondence principle)**.** *If for every measure-preserving system and for every $f > 0$ the assertion in* (9.1) *holds, then for every subset $E \subset \mathbb{N}$ with $\overline{\mathrm{d}}(E) > 0$ for every integer polynomial P with $P(0) = 0$, there exist $a, n \in \mathbb{N}$ such that $a, a + P(n) \in E$.*

In other words, the statement in Theorem 9.14 implies Theorem 9.12.

Proof. Let $E \subset \mathbb{N}$ have upper density $\overline{\mathrm{d}}(E) > 0$. We construct a corresponding measure-preserving system (X, μ, T) and a set $A \subset X$ with $\mu(A) > 0$ that transfer the combinatorial problem into the realm of ergodic theory.

Let $K := \{0, 1\}^{\mathbb{Z}}$ and consider the shift (K, T) from Example 2.24. Then of course $\mathbf{1}_E \in K$. Define

$$X := \overline{\mathrm{orb}}(\mathbf{1}_E) = \overline{\{T^k \mathbf{1}_E : k \in \mathbb{Z}\}},$$

which is a T- and T^{-1}-invariant, compact set, so that $T : X \to X$ is bijective. Consider the cylinder set $A \subset X$ given by

$$A := \{(t_j)_{j\in\mathbb{Z}} \in X : t_0 = 1\}$$

and let P be an integer polynomial with $P(0) = 0$. The set A has the following properties, whose proof we leave as Exercise 9.8.

(1) A is closed and open in X.
(2) The equivalence
$$T^m \mathbf{1}_E \in A \quad \Longleftrightarrow \quad m \in E$$
holds for each $m \in \mathbb{N}$.
(3) Let $n \in \mathbb{N}$ be fixed. The existence of $a \in \mathbb{N}$ with $a, a + P(n) \in E$ is equivalent to $A \cap T^{-P(n)} A \neq \emptyset$.

As a consequence, the proof will be complete if we find a measure μ on X with $\mu(A \cap T^{-P(n)} A) > 0$ for some $n \in \mathbb{N}$. Thus, assuming Theorem 9.14, it suffices to find a T-invariant probability measure μ on X with $\mu(A) > 0$.

The construction of such a measure μ is similar to the one in the proof of the Krylov–Bogolyubov Theorem 2.26. This is also the first place where we use the assumption on E. Let $(N_k)_{k\in\mathbb{N}}$ be a subsequence of $\mathbb{N}$ such that
$$\lim_{k\to\infty} \frac{|E \cap \{1, \dots, N_k\}|}{N_k} = \overline{\mathrm{d}}(E) > 0.$$
We let $\mu_k := \frac{1}{N_k} \sum_{n=1}^{N_k} \delta_{T^n \mathbf{1}_E}$. Then by (2) from the above we conclude that
$$\begin{aligned}\frac{|E \cap \{1, \dots, N_k\}|}{N_k} &= \frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_E(n) = \frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_A(T^n \mathbf{1}_E)\\ &= \frac{1}{N_k} \sum_{n=1}^{N_k} \delta_{T^n \mathbf{1}_E}(A) = \mu_k(A).\end{aligned}$$
Since $(\mu_k)_{k\in\mathbb{N}}$ is a sequence of probability measures on the compact (metric) space X, by the Banach–Alaoglu theorem it has a weak* convergent subsequence with limit μ, a probability measure. The proof that μ is a T-invariant measure on X with $\mu(A) = \overline{\mathrm{d}}(E) > 0$ is again left as Exercise 9.8. The proof is complete. ∎

Remark 9.17. By inspecting the above proof one sees that already Theorem 9.14 for ergodic systems implies Theorem 9.12. In fact, with a little bit more effort in the previous proof one can find an ergodic measure μ with $\mu(A) > 0$.

Thus to prove Theorem 9.12 it suffices to prove Theorem 9.14.

Proof of Theorem 9.14. We can assume without loss of generality that $\|f\|_\infty = 1$. Decompose f as $f = g + h$ according to the rational spectrum decomposition (Proposition 6.9) applied to $H := \mathrm{L}^2(X, \mu)$ and the Koopman operator T on H with $g \in H_{\mathrm{rat}}$ and $h \perp H_{\mathrm{rat}}$. By the polynomial mean ergodic theorem for contractions (Theorem 7.13), one has
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^{P(n)} h = 0$$
and therefore by orthogonality
$$\Big(f \,\Big|\, \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{P(n)} f\Big) = \Big(g \,\Big|\, \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{P(n)} g\Big).$$
Since the projection P_{rat} onto the rational spectrum component is a restriction of a Markov operator (see Proposition 8.19 and Example 8.21(b)), we obtain that

$g = P_{\mathrm{rat}} f \geq 0$, $g \in \mathrm{L}^\infty(X,\mu)$ and $\int_X g \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu > 0$. Thus we can assume without loss of generality that $f = g \in H_{\mathrm{rat}}$. Recall from Lecture 6 that

$$H_{\mathrm{rat}} = \overline{\bigcup_{k\in\mathbb{N}} \mathrm{Fix}(T^k)} \tag{9.2}$$

Suppose first that $f \in \mathrm{Fix}(T^k)$. Since $P(0) = 0$, we conclude that k divides $P(k), P(2k), \ldots$. Thus $(T^{P(kn)} f \,|\, f) = \|f\|_2^2$ for every $n \in \mathbb{N}$ and therefore

$$\lim_{N\to\infty} \frac{1}{kN+1} \sum_{n=0}^{kN} (T^{P(n)} f \,|\, f) \geq \limsup_{N\to\infty} \frac{1}{kN+1} \sum_{n=1}^{N} \|f\|_2^2 = \frac{1}{k}\|f\|_2^2 > 0. \tag{9.3}$$

Let now $f \in H_{\mathrm{rat}}$ be arbitrary and set $d := \int_X f \,\mathrm{d}\mu > 0$. By (9.2), there exists $k \in \mathbb{N}$ such that $f_k := P_{\mathrm{Fix}(T^k)} f$ satisfies $\|f - f_k\|_2 < d^2/4$. Recall that $P_{\mathrm{Fix}(T^k)}$ is also a Markov projection, so that $\|f_k\|_\infty \leq \|f\|_\infty \leq 1$. Observe that

$$|(T^n f \,|\, f) - (T^n f_k \,|\, f_k)| = |(T^n f_k \,|\, f - f_k) + (T^n(f - f_k) \,|\, f)| \leq 2\|f - f_k\|_2 < \frac{d^2}{2}$$

foreach $n \in \mathbb{N}$. In particular, we have

$$|(T^{P(kn)} f \,|\, f) - (T^{P(kn)} f_k \,|\, f_k)| \leq 2\|f - f_k\|_2 < \frac{d^2}{2}.$$

Since $P(0) = 0$ implies $T^{P(kn)} f_k = f_k$, the Cauchy–Schwarz inequality implies

$$(T^{P(kn)} f \,|\, f) \geq \|f_k\|_2^2 - \frac{d^2}{2} \geq (f_k \,|\, \mathbf{1})^2 - \frac{d^2}{2} = d^2 - \frac{d^2}{2} = \frac{d^2}{2} \quad \text{for each } n \in \mathbb{N}.$$

Whence we conclude

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (T^{P(n)} f \,|\, f) &= \lim_{N\to\infty} \frac{1}{kN} \sum_{n=1}^{kN} (T^{P(n)} f \,|\, f) \\ &\geq \frac{1}{k} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (T^{P(nk)} f \,|\, f) \geq \frac{d^2}{2k}. \end{aligned}$$

∎

Arithmetic progressions of length 3. We continue with another application to combinatorial number theory. Now structure means an arithmetic progression of length 3, and the goal of this section is to give an ergodic theoretic proof due to Furstenberg of the following result.

Theorem 9.18 (Roth [6])**.** *Let $E \subset \mathbb{N}$ have positive upper density. Then there exist $a, n \in \mathbb{N}$ such that $a, a+n, a+2n \in E$.*

We begin with some observations. Consider the Jacob–de Leeuw–Glicksberg decomposition $H = H_{\mathrm{kr}} \oplus H_{\mathrm{aws}}$ of the Hilbert space $H = \mathrm{L}^2(X,\mu)$ with respect to the Koopman operator. Let P_{kr} be the orthogonal projection onto the Kronecker part H_{kr}, and let P_{aws} be the complementary projection onto H_{aws}. By Example 8.21 P_{kr} is the (restriction of the) Markov projection onto the Kronecker factor. The Kronecker factor of an ergodic system has the following property.

Proposition 9.19 (Kronecker factor is characteristic for double convergence)**.** *Let (X,μ,T) be an ergodic measure-preserving system and let $f, g \in \mathrm{L}^\infty(X,\mu)$. If f or g belongs to H_{aws}, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n f \cdot T^{2n} g = 0. \tag{9.4}$$

As a consequence, we have for each $f, g \in \mathrm{L}^\infty(X,\mu)$ that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n f \cdot T^{2n} g = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n P_{\mathrm{kr}} f \cdot T^{2n} P_{\mathrm{kr}} g.$$

Proof. Set $u_n := T^n f \cdot T^{2n} g$ for $n \in \mathbb{N}$ and let $h \in \mathbb{N}$. Since T is measure-preserving, we have that

$$\begin{aligned}(u_n \,|\, u_{n+h}) &= \int_X T^n f \cdot T^{2n} g \cdot T^{n+h}\overline{f} \cdot T^{2n+2h}\overline{g}\,\mathrm{d}\mu\\ &= \int_X f \cdot T^h\overline{f} \cdot T^n(g \cdot T^{2h}\overline{g})\,\mathrm{d}\mu.\end{aligned}$$

We thus obtain

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (u_n \,|\, u_{n+h}) &= \int_X f \cdot T^h\overline{f} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n(g \cdot T^{2h}\overline{g})\,\mathrm{d}\mu\\ &= \int_X f \cdot T^h\overline{f}\,\mathrm{d}\mu \cdot \int_X g \cdot T^{2h}\overline{g}\,\mathrm{d}\mu\end{aligned}$$

by ergodicity of T (see Proposition 7.4). Therefore, if $g \in H_{\mathrm{aws}}(T)$, we can write

$$\begin{aligned}0 &\le \limsup_{J\to\infty} \frac{1}{J} \sum_{h=0}^{J-1} \lim_{N\to\infty} \frac{1}{N} \Big| \sum_{n=1}^{N} (u_n \,|\, u_{n+h}) \Big|\\ &\le \limsup_{J\to\infty} \frac{1}{J} \sum_{h=0}^{J-1} |(g|T^{2h}g)| \cdot \|f\|_\infty^2.\end{aligned}$$

By Exercise 6.7 $H_{\mathrm{aws}}(T^2) = H_{\mathrm{aws}}(T)$, so that $g \in H_{\mathrm{aws}}(T^2)$. By Proposition 6.15 and Lemma 6.14 we conclude

$$0 \le \limsup_{J\to\infty} \frac{1}{J} \sum_{h=0}^{J-1} |(g|T^{2h}g)| \cdot \|f\|_\infty^2 = 0.$$

Van der Corput's Lemma 7.16 implies that the limit in (9.4) is 0, provided $g \in H_{\mathrm{aws}}(T)$. The case when $f \in H_{\mathrm{aws}}$ can be proved analogously.

For arbitrary $f, g \in \mathrm{L}^\infty(X,\mu)$ we have

$$\begin{aligned}T^n f \cdot T^{2n} g = {}& T^n P_{\mathrm{kr}} f \cdot T^{2n} P_{\mathrm{kr}} g + T^n P_{\mathrm{aws}} f \cdot T^{2n} P_{\mathrm{kr}} g\\ &+ T^n P_{\mathrm{kr}} f \cdot T^{2n} P_{\mathrm{aws}} g + T^n P_{\mathrm{aws}} f \cdot T^{2n} P_{\mathrm{aws}} g.\end{aligned}$$

The Cesàro averages of the last three terms converge here to 0, by what has been discussed above. The proof is complete. ∎

We have the following analogue of the polynomial convergence presented in Theorem 7.13.

Theorem 9.20 (Double convergence)**.** *Let (X,μ,T) be an ergodic measure-preserving system and let $f, g \in \mathrm{L}^\infty(X,\mu)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n f \cdot T^{2n} g \,\mathrm{d}\mu \quad \textit{exists in the } \mathrm{L}^2\textit{-norm.} \tag{9.5}$$

Proof. Let $g \in \mathrm{L}^\infty(X,\mu)$ be fixed and consider for $N \in \mathbb{N}$ the linear operators

$$S_N f := \frac{1}{N}\sum_{n=0}^{N-1} T^n f \cdot T^{2n} g$$

on $\mathrm{L}^2(X,\mu)$, whose norms are uniformly bounded by $\|g\|_\infty$. If $f \in \ker(\lambda - T)$ for some $\lambda \in \mathbb{T}$, then

$$S_N f = \frac{1}{N}\sum_{n=0}^{N-1} (\lambda T^2)^n g \cdot f.$$

Since λT^2 is a contraction, it is mean ergodic (see Theorem 7.2), and the sequence $(S_N f)_{N\in\mathbb{N}}$ is convergent. By linearity we obtain the convergence on $E := \mathrm{lin}\{f : Tf = \lambda f \text{ for some } \lambda \in \mathbb{T}\}$, and by Exercise 1.3, all over the closed linear hull H_{kr} of E. The rest follows from Proposition 9.19 and linearity. ∎

The following gives more information on the limit in (9.5).

Theorem 9.21 (Double recurrence). *Let (X,μ,T) be an ergodic measure-preserving system and $f \in \mathrm{L}^\infty(X,\mu)$ satisfy $f > 0$. Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int_X f \cdot T^n f \cdot T^{2n} f \,\mathrm{d}\mu = \Big(f \,\Big|\, \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n f \cdot T^{2n} f\Big) > 0.$$

In particular, for every $A \subset X$ with $\mu(A) > 0$

$$\mu(A \cap T^{-n}A \cap T^{-2n}A) > 0 \quad \textit{for some } n \in \mathbb{N}.$$

We remark that the convergence in (9.5) and double recurrence hold even without the assumption of ergodicity.

The following builds a bridge between Roth's Theorem 9.18 and double recurrence. The proof is left as Exercise 9.9

Theorem 9.22 (Furstenberg's correspondence principle). *If for every ergodic measure-preserving system (X,μ,T) and $f \in \mathrm{L}^\infty(X,\mu)$ with $f > 0$ one has*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int_X f \cdot T^n f \cdot T^{2n} f \,\mathrm{d}\mu = \Big(f \,\Big|\, \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n f \cdot T^{2n} f\Big) > 0, \tag{9.6}$$

then every set $E \subset \mathbb{N}$ with positive upper density contains arithmetic progressions of length 3. That is to say the statement in Theorem 9.21 implies Theorem 9.18.

Thus to show Roth's Theorem 9.18 it remains to prove Theorem 9.21.

Proof of Theorem 9.21. By Theorem 9.20 we know that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f \cdot T^{2n} f$$

exists, implying the existence of the limit in (9.6). Since P_{kr} is (the restriction of) the Markov projection onto the Kronecker factor, it is a positive operator by Example 8.21(c), so $P_{\mathrm{kr}} f \geq 0$. Since $\int_X P_{\mathrm{kr}} f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu > 0$, we obtain that $P_{\mathrm{kr}} f > 0$. By the orthogonality of H_{aws} and H_{kr}, and by Proposition 9.19 we obtain

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int_X f \cdot T^n f \cdot T^{2n} f \,\mathrm{d}\mu = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int_X P_{\mathrm{kr}} f \cdot T^n P_{\mathrm{kr}} f \cdot T^{2n} P_{\mathrm{kr}} f \,\mathrm{d}\mu.$$

Hence we may assume that $f \in \mathrm{L}^\infty(X,\mu) \cap H_{\mathrm{kr}}$, and without loss of generality also that $\|f\|_\infty \le 1$. Let $0 < \varepsilon < \int_X f^3 \,\mathrm{d}\mu$. Define $K := \overline{T^{\mathbb{N}_0} f}$, which is a compact subset of $\mathrm{L}^2(X,\mu)$ by Proposition 6.18, and invariant under T. We thus obtain a forward transitive topological system (K,T), which is "easily" seen to be minimal, see Exercise 9.3. By Proposition 3.6 every point, in particular $f \in K$, is almost periodic. For $U := \mathrm{B}(f, \varepsilon/3)$ the set of return times $R_U(f)$ is syndetic (see Lecture 3). We have for each $n \in R_U(f)$ that

$$\|T^n f - f\|_2 < \frac{\varepsilon}{3}, \quad \text{and} \quad \|T^{2n} f - f\|_2 \le \|T^{2n} f - T^n f\|_2 + \|T^n f - f\|_2 < \frac{2\varepsilon}{3}.$$

From this we conclude that

$$\begin{aligned}
&\Big|\int_X f \cdot T^n f \cdot T^{2n} f \,\mathrm{d}\mu - \int_X f^3 \,\mathrm{d}\mu\Big| \\
&\quad\le \Big|\int_X f \cdot T^n f \cdot T^{2n} f \,\mathrm{d}\mu - \int_X f^2 \cdot T^{2n} f \,\mathrm{d}\mu\Big| + \Big|\int_X f^2 \cdot T^{2n} f \,\mathrm{d}\mu - \int_X f^3 \,\mathrm{d}\mu\Big| \\
&\quad\le \int_X f \cdot |T^n f - f| \cdot T^{2n} f \,\mathrm{d}\mu + \int_X f^2 \cdot |T^{2n} f - f| \,\mathrm{d}\mu \\
&\quad\le \|f\|_\infty^2 \|T^n f - f\|_2 + \|f\|_\infty^2 \|T^{2n} f - f\|_2 < \varepsilon
\end{aligned}$$

for each $n \in R_U(f)$. As a consequence

$$\int_X f \cdot T^n f \cdot T^{2n} f \,\mathrm{d}\mu > \int_X f^3 \,\mathrm{d}\mu - \varepsilon$$

for every $n \in R_U(f)$. Since a syndetic set has positive lower density $d > 0$ (if the length of the gaps is bounded by ℓ, then the lower density is $\ge \frac{1}{\ell}$, see Exercise 4.7), we conclude (see Exercise 9.4) that

$$\liminf_{N\to\infty} \int_X f \cdot T^n f \cdot T^{2n} f \,\mathrm{d}\mu > d \cdot \Big(\int_X f^3 \,\mathrm{d}\mu - \varepsilon\Big) > 0. \qquad \blacksquare$$

The following generalization of Roth's theorem due to Szemerédi [9] was first proved by using complex and ingenious argumentations from combinatorics. Furstenberg [2] discovered an ergodic theoretic proof, and he established his correspondence principle precisely for this purpose. This result fully explains our leitmotif that a large set contains structured parts.

Theorem 9.23 (Szemerédi). *A subset of the natural numbers with positive upper density contains arbitrarily long arithmetic progressions.*

We have proved the existence of three-terms arithmetic progressions, i.e., the result of Roth. The existence of longer progressions is much more difficult and will be briefly discussed in one of the next lectures.

4. Exercises

Exercise 9.1 (Markov's inequality)**.** Let (X, μ) be a measure space and $f \in \mathrm{L}^0(X, \mu)$, $f \geq 0$. Prove that for every $\varepsilon > 0$ the following inequality holds

$$\mu([f \geq \varepsilon]) \leq \frac{1}{\varepsilon} \int_X f \, \mathrm{d}\mu.$$

Exercise 9.2 (Normal numbers)**.** Prove that the number

$$0.01234567890123456789 \ldots 0123456789 \ldots$$

(repeated blocks) is simply normal in base 10 but not normal in base 2.

Exercise 9.3 (Isometric systems)**.** A topological system (K, T) is called **isometric** if the topology of K comes from a metric d satisfying $d(Tx, Ty) = d(x, y)$ for every pair $x, y \in K$. Prove that in an isometric system the notions of forward topological transitivity and minimality coincide.

Exercise 9.4 ($\liminf$ and lower density)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that for some $\varepsilon > 0$ the set $A := \{n : a_n > \varepsilon\}$ has positive lower density. Prove that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n \geq \varepsilon \cdot \underline{\mathrm{d}}(A).$$

Exercise 9.5. Prove Proposition 9.10.

Exercise 9.6 (Totally ergodic systems)**.** A measure-preserving system is called **totally ergodic** if for each $k \in \mathbb{N}$ the system (X, μ, T^k) is ergodic, here $T^k = T \circ \cdots \circ T$ is the kth iterate of T. Let P be an integer polynomial with $P(0) = 0$. Prove that the fixed factor (with Markov projection P_{fix}) is characteristic for the convergence of polynomial averages, meaning that for every $f \in \mathrm{L}^2(X, \mu)$ one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^{P(n)} f = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^{P(n)} P_{\mathrm{fix}} f.$$

Exercise 9.7. (a) Let (K, T) and (L, S) be topological systems such that there is a homeomorphism $\theta : K \to L$ with $\theta \circ T = S \circ \theta$. In this case the two systems are called **topologically isomorphic**. Prove that (K, T) is uniquely ergodic if and only if (L, S) is uniquely ergodic.

(b) Prove that the shift $([0, 1), \alpha)$ and the rotation $(\mathbb{T}, a)$, $a = \mathrm{e}^{2\pi \mathrm{i}\alpha}$, are topologically isomorphic.

(c) Prove that the skew shift $([0, 1)^2, \widetilde{T}_\alpha)$ and the skew rotation $(\mathbb{T}^2, T_a)$, $a = \mathrm{e}^{2\pi \mathrm{i}\alpha}$, are topologically isomorphic, and therefore simultaneously uniquely ergodic or not.

Exercise 9.8 (Furstenberg's correspondence principle)**.** (a) Show that the set A in the proof of Theorem 9.16 has the properties (1)–(3) given there.

(b) Show that the measure μ constructed in the proof of Theorem 9.16 is a T-invariant probability measure on X with $\mu(A) = \overline{\mathrm{d}}(E) > 0$.

Exercise 9.9 (Furstenberg's correspondence principle for arithmetic progressions of length 3)**.** Prove Theorem 9.22. (Hint: Mimic the proof of Theorem 9.16, but be careful because one needs to find an *ergodic* measure.)

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LECTURE 10

Equidistribution

As a preparation for the later lectures we study a classical number theoretic property which is related to ergodic theorems. A sequence $(s_n)_{n\in\mathbb{N}}$ in $[0,1)$ is called **equidistributed in** $[0,1)$ if for every interval $(a,b)\subset[0,1)$,

$$\lim_{N\to\infty}\frac{|\{n\le N:\ s_n\in(a,b)\}|}{N}=b-a. \tag{10.1}$$

One can replace open intervals (a,b) in the above by closed, or other types of intervals, still obtaining the same notion (why?). A sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is called **equidistributed modulo** 1 if the sequence $(\{s_n\})_{n\in\mathbb{N}}$ is equidistributed in $[0,1)$. (Recall that $\{x\}=x-\lfloor x\rfloor$ is the fractional part of x.)

Next we define equidistributed sequences in $\mathbb{T}$. We call a connected subset $I\subset\mathbb{T}$ an **interval**. Consider the continuous mapping

$$e:\mathbb{R}\to\mathbb{T},\quad t\mapsto \mathrm{e}^{2\pi \mathrm{i}t}.$$

Then $I\subset\mathbb{T}$ is an interval if and only if $e^{-1}(I)=J+2\pi\mathbb{Z}$ for some interval $J\subset\mathbb{R}$. Now, a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{T}$ is called **equidistributed** (or **uniformly distributed**) **in** $\mathbb{T}$ if for every interval $I\subset\mathbb{T}$

$$\lim_{N\to\infty}\frac{|\{n\le N: a_n\in I\}|}{N}=\mathrm{m}(I),$$

where m denotes the Haar measure on $\mathbb{T}$.

Remark 10.1. It is easy to see that a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is equidistributed modulo 1 if and only if the sequence $(e(s_n))_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$, see Exercise 10.1. Furthermore, we note that an equidistributed sequence is dense in $\mathbb{T}$, $[0,1)$ or in $\mathbb{R}$ mod 1, respectively.

Remark 10.2. Obviously, a sequence $(a_n)_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$ if and only if for each interval $I\subset\mathbb{T}$ we have

$$\mathrm{m}(I)=\int_{\mathbb{T}}\mathbf{1}_I\,\mathrm{dm}=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_I(a_n).$$

1. Weyl's equidistribution criterion

We present a very useful criterion for equidistribution of sequences due to H. Weyl[1]. His original formulation was for equidistribution mod 1. For the equidistribution on the torus we need the following considerations. A function $f:\mathbb{T}\to\mathbb{C}$

[1] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins.*, Math. Ann. 77 (1916), 313–352 (German).

is called **Riemann integrable** if $f \circ e$ is Riemann integrable on $[0,1]$. In this case we call

$$\int_0^1 f(e(t))\,\mathrm{d}t,$$

the Riemann integral of f on $\mathbb{T}$, and we have $\int_{\mathbb{T}} f\,\mathrm{dm} = \int_0^1 f(e(t))\,\mathrm{d}t$.

Theorem 10.3 (Weyl's equidistribution criterion)**.** *For a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{T}$ the following assertions are equivalent.*

(i) *The sequence $(a_n)_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$.*
(ii) *For every Riemann integrable function $f : \mathbb{T} \to \mathbb{C}$ one has*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(a_n) = \int_{\mathbb{T}} f\,\mathrm{dm}. \tag{10.2}$$

(iii) *Every $f \in \mathrm{C}(\mathbb{T})$ satisfies (10.2); in other words, the measures $\frac{1}{N}\sum_{n=1}^{N}\delta_{a_n}$ converge to the Haar measure* m *in the weak* topology.*
(iv) *For every $h \in \mathbb{Z}\setminus\{0\}$ we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n^h = 0.$$

Analogous assertions hold for equidistribution in $[0,1)$ or modulo 1, respectively. In particular, a sequence $(s_n)_{n\in\mathbb{N}}$ is equidistributed modulo 1, if and only if for every $h \in \mathbb{Z}\setminus\{0\}$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(hs_n) = 0.$$

Proof. We only prove the statement concerning equidistribution on $\mathbb{T}$, the rest is left as Exercise 10.2.

The implications (ii)⇒(iii)⇒(iv) are evident (the last one relies on the fact that the functions $\mathbf{z}^h : z \mapsto z^h$ have mean zero for $h \in \mathbb{Z}\setminus\{0\}$). Thus it remains to show the implications (iv)⇒(iii)⇒(i)⇒(ii).

(iv)⇒(iii): By assumption, (10.2) holds for all trigonometric monomials $f = \mathbf{z}^h$ with $h \in \mathbb{Z}\setminus\{0\}$, and it is immediate that it also holds for $f = \mathbf{1} = \mathbf{z}^0$. Thus, by linearity, (10.2) holds for every trigonometric polynomial. Since these are dense in $\mathrm{C}(\mathbb{T})$ (see Proposition 1.9) and since the operators $S_N : f \mapsto \frac{1}{N}\sum_{n=1}^{N} f(a_n)$, $N \in \mathbb{N}$, are uniformly bounded, we obtain (iii) by an application of the assertion in Exercise 1.3.

(iii)⇒(i): Let $I \subset \mathbb{T}$ be an interval. We have to prove (10.2) for the function $f = \mathbf{1}_I$ (see Remark 10.2). Let $\varepsilon > 0$. It is easy to construct $g, h \in \mathrm{C}(\mathbb{T})$ with $g \le \mathbf{1}_I \le h$ and $\int_{\mathbb{T}} h - g\,\mathrm{dm} < \varepsilon$ (take, for instance, suitable piecewise linear approximations of

$\mathbf{1}_I$). Observe that $0 \le \int_{\mathbb{T}} h - \mathbf{1}_I \,\mathrm{dm} < \varepsilon$ and $0 \le \int_{\mathbb{T}} \mathbf{1}_I - g \,\mathrm{dm} < \varepsilon$, and therefore

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} g(a_n) - \int_{\mathbb{T}} g \,\mathrm{dm} - \varepsilon &< \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_I(a_n) - \int_{\mathbb{T}} \mathbf{1}_I \,\mathrm{dm} \\ &< \frac{1}{N}\sum_{n=1}^{N} h(a_n) - \int_{\mathbb{T}} h \,\mathrm{dm} + \varepsilon.\end{aligned}$$

Now, (iii) applied to g and h yield that for sufficiently large N

$$-2\varepsilon < \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_I(a_n) - \int_{\mathbb{T}} \mathbf{1}_I \,\mathrm{dm} < 2\varepsilon$$

proving the validity of (10.2) for $f = \mathbf{1}_I$.

(i)$\Rightarrow$(ii): By assumption, (10.2) holds for functions of the form $\mathbf{1}_I$ and hence for all step functions being (finite) linear combinations of such characteristic functions. Since for every Riemann integrable function $f : \mathbb{T} \to \mathbb{R}$ and every $\varepsilon > 0$ there are step functions g, h on $\mathbb{T}$ with $g \le f \le h$ and $\int_{\mathbb{T}} h - g \,\mathrm{dm} < \varepsilon$, the same argument as in the proof of (iii)$\Rightarrow$(i) above finishes the proof (it suffices to prove (ii) for real-valued functions). ■

The first example of an equidistributed sequence is already familiar from Lectures 1 and 9.

Example 10.4 (Kronecker)**.** For every irrational $\lambda \in \mathbb{T}$ the sequence $(\lambda^n)_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$. This is essentially contained in Proposition 9.7, but also follows easily from Weyl's equidistribution criterion, Theorem 10.3. Indeed, take $h \in \mathbb{Z} \setminus \{0\}$ and observe

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda^{hn} = \lim_{N\to\infty} \frac{\lambda^h}{N} \cdot \frac{1-\lambda^{hN}}{1-\lambda^h} = 0,$$

implying the equidistribution property. In the additive setting this means that the sequence $(n\alpha)_{n\in\mathbb{N}}$ is equidistributed modulo 1 for every irrational $\alpha \in \mathbb{R}$.

The following gives a (theoretical) way to produce equidistributed sequences. We follow here the exposition by Rosenblatt and Wierdl [2, Thm. 2.22], see also [KN, Thm. 1.4.1].

Proposition 10.5. *Let $(k_n)_{n\in\mathbb{N}}$ be a strictly monotone sequence in $\mathbb{Z}$. Then for Haar almost every $\lambda \in \mathbb{T}$ the sequence $(\lambda^{k_n})_{n\in\mathbb{N}}$ is equidistributed.*

Proof. The functions $\mathbf{z}^{k_n}$, $n \in \mathbb{N}$, form an orthonormal family, which implies that

$$\int_{\mathbb{T}} \Big|\frac{1}{N}\sum_{n=1}^{N} \mathbf{z}^{k_n}\Big|^2 \,\mathrm{dm} = \Big\|\frac{1}{N}\sum_{n=1}^{N} \mathbf{z}^{k_n}\Big\|_2^2 = \frac{1}{N^2} \cdot N = \frac{1}{N}.$$

Whence we conclude

$$\int_{\mathbb{T}} \sum_{N=1}^{\infty} \Big|\frac{1}{N^2}\sum_{n=1}^{N^2} \mathbf{z}^{k_n}\Big|^2 \,\mathrm{dm} = \sum_{N=1}^{\infty} \int_{\mathbb{T}} \Big|\frac{1}{N^2}\sum_{n=1}^{N^2} \mathbf{z}^{k_n}\Big|^2 \,\mathrm{dm} = \sum_{N=1}^{\infty} \frac{1}{N^2} < \infty.$$

As a consequence, we have

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{n=1}^{N^2}\lambda^{k_n}=0 \quad \text{for m-almost every } \lambda\in\mathbb{T}. \tag{10.3}$$

Next we show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda^{k_n}=0 \quad \text{for m-almost every } \lambda\in\mathbb{T}. \tag{10.4}$$

Let $N\in\mathbb{N}$ and let $k\in\mathbb{N}$ be such that $k^2\le N<(k+1)^2$ holds. Observe that $N-k^2\le 2k+1\le 3\sqrt{N}$ and therefore

$$\begin{aligned}\Big|\frac{1}{N}\sum_{n=1}^{N}\lambda^{k_n}\Big| &\le \frac{k^2}{N}\cdot\Big|\frac{1}{k^2}\sum_{n=1}^{k^2}\lambda^{k_n}\Big|+\frac{1}{N}\Big|\sum_{n=k^2+1}^{N}\lambda^{k_n}\Big|\\ &\le \Big|\frac{1}{k^2}\sum_{n=1}^{k^2}\lambda^{k_n}\Big|+\frac{N-k^2}{N}\le\Big|\frac{1}{k^2}\sum_{n=1}^{k^2}\lambda^{k_n}\Big|+\frac{3}{\sqrt{N}}.\end{aligned}$$

Letting $N\to\infty$ and using (10.3) proves (10.4).

To show the equidistribution property, take $h\in\mathbb{Z}\setminus\{0\}$. Since $(hk_n)_{n\in\mathbb{N}}$ is strictly monotone, we can replace k_n by hk_n in (10.4) and obtain that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda^{hk_n}=0 \quad \text{for m-almost every } \lambda\in\mathbb{T}.$$

Since a countable intersection of full measure sets still has full measure, Weyl's equidistribution criterion, Theorem 10.3, finishes the proof. ∎

More (and rather involved) examples of equidistributed sequences will be discussed in Section 4 below. On the other hand, it is easy to construct artificial examples of sequences which are not equidistributed, even if dense. The following is a non-trivial and *natural* example of such a sequence involving the natural logarithm log.

Example 10.6. Let $a_n:=e(\log(n))$. Then $(a_n)_{n\in\mathbb{N}}$ is not equidistributed in $\mathbb{T}$ or, equivalently, the sequence $(\log(n))_{n\in\mathbb{N}}$ is not equidistributed modulo 1. To see this latter property we take the interval $I=(0,1/2)$ and compute how often $\{\log(n)\}$ belongs to it as n ranges over $\mathbb{N}$. Let $N\in\mathbb{N}$ and $m\in\{0,\dots,N-1\}$. We have

$$\log(n)\in(m,m+1/2) \iff n\in(\mathrm{e}^m,\mathrm{e}^{1/2}\mathrm{e}^m).$$

Thus, for every m there exists $\varepsilon(m)\in[0,1)$ such that

$$\begin{aligned}\big|\big\{n\le\lfloor \mathrm{e}^N\rfloor:\{\log(n)\}\in I\big\}\big| &= \sum_{m=0}^{N-1}\Big((\mathrm{e}^{1/2}-1)\mathrm{e}^m+\varepsilon(m)\Big)\\ &=\frac{(\mathrm{e}^{1/2}-1)(\mathrm{e}^N-1)}{\mathrm{e}-1}+\sum_{m=0}^{N-1}\varepsilon(m).\end{aligned}$$

This yields that

$$\lim_{N\to\infty}\frac{\big|\big\{n\le\lfloor \mathrm{e}^N\rfloor:\{\log(n)\}\in I\big\}\big|}{\lfloor \mathrm{e}^N\rfloor}=\frac{\mathrm{e}^{1/2}-1}{\mathrm{e}-1}=\frac{1}{\mathrm{e}^{1/2}+1}<\frac{1}{2}.$$

Thus (10.1) fails and $(\log(n))_{n\in\mathbb{N}}$ is not equidistributed modulo 1. On the other hand, it is easy to see that $(a_n)_{n\in\mathbb{N}}$ is dense in $\mathbb{T}$ using that the sequence $\log(n)$ goes monotonically to infinity with $\log(n+1)-\log(n)$ tending to zero, see Exercise 10.3.

Remark 10.7. The function log is an example of a logarithmico-exponential function (i.e., a real function constructed by using finitely many combinations of usual arithmetic symbols and the function symbols exp and log) of subpolynomial growth. For such logarithmico-exponential functions f there is a complete characterization of the equidistribution property of the sequence $(e(f(n)))_{n\in\mathbb{N}}$ in terms of the growth of f, see, e.g., Boshernitzan's paper [3]. As an example we mention that $(e(n^\alpha))_{n\in\mathbb{N}}$ is equidistributed for every $\alpha > 0$.

2. Multidimensional equidistribution

Condition (iii) in Theorem 10.3 motivates the following generalization of equidistribution of sequences to compact metric spaces.

Definition 10.8. Let K be a compact metric space with a Borel probability measure μ. Then a sequence $(a_n)_{n\in\mathbb{N}}$ in K is called **equidistributed** in K (with respect to the reference measure μ) if for every $f \in \mathrm{C}(K)$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(a_n) = \int_{\mathbb{T}} f \,\mathrm{d}\mu,$$

i.e., if the measures $\frac{1}{N}\sum_{n=1}^{N}\delta_{a_n}$ converge to μ in the weak* topology.

Remark 10.9. If μ has full support, then every equidistributed sequence $(a_n)_{n\in\mathbb{N}}$ in K with respect to μ is dense in K, see Exercise 10.4.

For $K = \mathbb{T}^d$ we take $\mu = \mathrm{m}^d$ the normalised Haar measure as the reference measure. The following criterion is a multidimensional analogue of Theorem 10.3 which we state multiplicatively only and leave the proof as Exercise 10.5.

Theorem 10.10 (Weyl's equidistribution criterion for $\mathbb{T}^d$)**.** *For a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{T}^d$ the following assertions are equivalent.*

(i) *The sequence $(a_n)_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}^d$.*
(ii) *For every $h \in \mathbb{Z}^d \setminus \{0\}$*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n^h = 0,$$

where the multi-index notation $(x_1,\dots,x_d)^h := x_1^{h_1}\cdot\ldots\cdot x_d^{h_d}$ is used.

The following example is a multidimensional version of Example 10.4.

Example 10.11 (Kronecker)**.** For $\lambda \in \mathbb{T}^d$ and $n \in \mathbb{N}$ consider $a_n := \lambda^n = (\lambda_1^n,\dots,\lambda_d^n)$. Then $(a_n)_{n\in\mathbb{N}}$ is equidistributed if and only if the corresponding topological system $(\mathbb{T}^d,\lambda)$ is minimal, i.e., if and only if $\lambda \in \mathbb{T}^d$ is **irrational** in the sense that for every $h \in \mathbb{Z}^d$

$$\lambda^h = (1,\dots,1) \quad \Longrightarrow \quad h = (0,\dots,0) \tag{10.5}$$

(cf. Theorem 3.31). The proof is left as Exercise 10.6.

Remark 10.12. The above property of irrationality of λ implies irrationality of every coordinate of λ (why?), but the converse implication is not true (cf. Theorem 3.31). Take for example λ whose coordinates are irrational and all equal. Although, by Example 10.11, the sequence $(\lambda^n)_{n\in\mathbb{N}}$ for such λ is not equidistributed in $\mathbb{T}^d$ (if $d \geq 2$), it is equidistributed in the diagonal $\{(z,\ldots,z): \ z \in \mathbb{T}\}$ with respect to the induced (1-dimensional normalized Lebesgue, or Haar) measure on it (by Example 10.4). For more on this phenomenon see, e.g., [5]

3. Equidistribution of polynomial sequences

In Lecture 7 we have discussed a powerful tool, developed by the number theorist J. van der Corput to estimate exponential sums. We recall it in a special case.

Lemma 10.13 (Van der Corput's lemma)**.** *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$. Then*

$$\liminf_{J\to\infty} \frac{1}{J}\sum_{h=1}^{J} \limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=1}^{N} u_{n+k}\overline{u_n}\right| = 0 \quad \Longrightarrow \quad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} u_n = 0.$$

Based on this we can immediately deduce the following important corollary.

Corollary 10.14. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{T}^d$ be such that for every $k \in \mathbb{N}$ the sequence $(a_{n+k}\overline{a_n})_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}^d$. Then $(a_n)_{n\in\mathbb{N}}$ itself is equidistributed in $\mathbb{T}^d$.*

Proof. By Theorem 10.10 it suffices to show that for every $h \in \mathbb{Z}^d \setminus \{0\}$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} a_n^h = 0. \tag{10.6}$$

Take $h \in \mathbb{Z}^d\setminus\{0\}$ and define $u_n := a_n^h$. For $k \in \mathbb{N}$ we observe that, by the hypothesis and by Theorem 10.10

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} u_{n+k}\overline{u_n} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} (a_{n+k}\overline{a_n})^h = 0.$$

Thus (10.6) follows from Lemma 10.13. ∎

Remark 10.15. The sequences $(a_{n+k}\overline{a_n})_{n\in\mathbb{N}}$ are often called **(multiplicative) discrete derivatives** of $(a_n)_{n\in\mathbb{N}}$. This terminology becomes apparent when writing a_n as $e(s_n)$, so that $a_{n+k}\overline{a_n} = e(s_{n+k} - s_n)$. The additive version of Corollary 10.14, where the terms $a_{n+k}\overline{a_n}$ are replaced by $s_{n+k} - s_n$, is called **van der Corput's difference theorem**.

Recall that by Proposition 10.5, for every subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ the sequence $(\lambda^{k_n})_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$ for (Lebesgue) almost every $\lambda \in \mathbb{T}$. However, it is sometimes important to know for which λ exactly equidistribution occurs. Our aim in the rest of this lecture is to show that whenever $(k_n)_{n\in\mathbb{N}}$ is produced by polynomials, by primes or by polynomials of primes, one has equidistribution for every irrational λ. This will be important for later lectures.

We first need a simple fact whose proof we leave as Exercise 10.7.

Lemma 10.16. *Let $m \in \mathbb{N}$, let $a \in \mathbb{C}$ and let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$. Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n = a \quad \Longleftrightarrow \quad \lim_{N\to\infty} \frac{1}{mN}\sum_{n=1}^{mN} a_n = a.$$

The following is a polynomial version of Example 10.4.

Theorem 10.17 (Weyl, equidistribution of polynomial sequences)**.** *Let P be a real polynomial, $P(x) = a_d x^d + \dots + a_1 x + a_0$, with at least one among the coefficients $a_1, \dots, a_d$ being irrational. Then the sequence $(e(P(n)))_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$.*

Proof. We proceed by induction on the degree d of P. If $d = 1$, the assertion follows from Example 10.4. Suppose that $d \geq 2$ and the assertion holds for polynomials with degree smaller than d.

We first consider the case when the leading coefficient a_d is rational and take $m \in \mathbb{N}$ such that $ma_d \in \mathbb{Z}$. Observe that for $h \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{mN} e(hP(n)) = \frac{1}{N}\sum_{n=1}^{N} e(hP(mn)) + \frac{1}{N}\sum_{n=1}^{N} e(hP(mn-1)) + \dots \\ + \frac{1}{N}\sum_{n=1}^{N} e(hP(mn-m+1)).\end{aligned}$$

The polynomials $Q_{h,j}$ defined as $Q_{h,j}(x) := hP(mx-j) - hm^d a_d x^d$, $j \in \{0, \dots, m-1\}$, are of degree at most $d-1$. Since $ma_d \in \mathbb{Z}$, they have at least one irrational coefficient (different from the constant term) and $e(Q_{h,j}(n)) = e(hP(mn-j))$ for each $j \in \{0, \dots, m-1\}$ and $n \in \mathbb{N}$. Hence the equidistribution of $(e(P(n)))_{n\in\mathbb{N}}$ follows from the induction hypothesis, Lemma 10.16 and Theorem 10.3.

Suppose now that the leading coefficient of P is irrational. Define $u_n := e(P(n))$ and observe that

$$u_{n+k}\overline{u_n} = e(P(n+k) - P(n)) = e(Q_k(n))$$

for every $k \in \mathbb{N}$, where Q_k is the polynomial defined as $Q_k(x) := P(x+k) - P(x)$. It is easy to see that $\deg(Q_k) = \deg(P) - 1 = d - 1$ and that the leading coefficient of Q_k is irrational as well. The induction hypothesis together with Corollary 10.14 finishes the argument. ∎

Remark 10.18. It is clear that equidistribution fails if all coefficients of P, with the possible exception of the constant term, are rational, see Exercise 10.8.

Analogously one can prove the multidimensional case (we leave the details to the reader).

Proposition 10.19 (Equidistribution of multivariate polynomials)**.** *Let $P \in \mathbb{R}^d[\cdot]$ be a multivariate polynomial with at least one irrational (in the sense of (10.5)) coefficient different from the constant term. Then the sequence $(e(P(n)))_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}^d$.*

In Lecture 9 we briefly mentioned how equidistribution is connected to unique ergodicity of certain group extensions. Here we illustrate by an easy example how equidistribution can be used in ergodic theory.

Proposition 10.20. *If $a \in \mathbb{T}$ is irrational, the skew shift $(\mathbb{T}^2, T_a)$ is uniquely ergodic.*

Proof. First of all we recall that $(\mathbb{T}^2, \mathrm{m}^2, T_a)$ is ergodic (see Example 8.24). By Corollary 7.12 we only need to prove the mean ergodicity of the Koopman operator T on $\mathrm{C}(\mathbb{T})$, i.e., taking the ergodicity of $(\mathbb{T}^2, \mathrm{m}^2, T_a)$ into account it suffices to prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_a^n f = \int_{\mathbb{T}} f \,\mathrm{dm} \cdot \mathbf{1} \tag{10.7}$$

for every $f \in \mathrm{C}(\mathbb{T}^2)$ (pointwise or uniformly, all the same here, see Section 2). We first consider trigonometric monomials $f = \mathbf{z}^m$, $m = (m_1, m_2) \in \mathbb{Z}^2$. For such a function we have by Proposition 9.10

$$\frac{1}{N} \sum_{n=0}^{N-1} T_a^n \mathbf{z}^m = \frac{1}{N} \sum_{n=0}^{N-1} a^{m_1 n} \mathbf{z}_1^{m_1} \cdot a^{m_2 \frac{n(n-1)}{2}} \mathbf{z}_1^{n m_2} \mathbf{z}_2^{m_2}.$$

For $x, y \in \mathbb{T}$ and $m \neq (0,0)$ we obtain from the equidistribution theorem for polynomials (Theorem 10.17) that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_a^n \mathbf{z}^m(x,y) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a^{m_1 n} a^{m_2 \frac{n(n-1)}{2}} x^{n m_2} x^{m_1} y^{m_2} = 0.$$

Since (10.7) is trivial for $m = (0,0)$, we obtain (10.7) for every trigonometric monomial, and then by linearity for every trigonometric polynomial. A density argument proves (10.7) for every $f \in \mathrm{C}(\mathbb{T}^2)$. ∎

4. Equidistribution of primes and polynomials of primes

Up to now the lectures were more or less self-contained. This changes now dramatically since we turn our attention to prime numbers, and for that we need really deep results from number theory. These will be used as a black box, and we only prepare here some important tools for the next lectures.

Denote by $\mathbb{P}$ the set of primes and list them increasingly in the sequence* $(p_n)_{n\in\mathbb{N}}$, i.e., $p_1 = 2$. Denote by $\pi : \mathbb{R} \to \mathbb{N}_0$ the prime counting function given by

$$\pi(x) := |\{n : p_n \leq x\}|.$$

Theorem 10.21 (Prime number theorem, de la Vallée Poussin)**.** *We have*

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1.$$

To average over primes we shall need the following, so-called, **modified von Mangoldt function** Λ' defined as

$$\Lambda'(n) := \log(n)\mathbf{1}_{\mathbb{P}}(n) = \begin{cases} \log(n), & \text{if } n \in \mathbb{P}, \\ 0, & \text{otherwise.} \end{cases}$$

*Euclid tells us that the sequence is infinite.

It involves a simple computation to show that the prime number theorem is equivalent to the following asymptotics for the function Λ'

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \Lambda'(n) = 1, \tag{10.8}$$

see Exercise 10.10. In what follows, we use p to denote a prime number without writing explicitly $p \in \mathbb{P}$. The following easy but very useful result shows one important property of the function Λ': Averaging along the primes is the same as averaging with the weight Λ' along the natural numbers.

Lemma 10.22. *For a bounded sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ we have*

$$\lim_{N\to\infty} \Big| \frac{1}{N}\sum_{n=1}^{N} \Lambda'(n) b_n - \frac{1}{\pi(N)}\sum_{p\le N} b_p \Big| = 0. \tag{10.9}$$

Proof. Assume without loss of generality that $|b_n| \le 1$ for every $n \in \mathbb{N}$. For $N \in \mathbb{N}$ we have by the prime number theorem, Theorem 10.21 that

$$\begin{aligned}
\limsup_{N\to\infty} \Big| \frac{1}{\pi(N)}\sum_{p\le N} b_p - \frac{1}{N}\sum_{n=1}^{N}\Lambda'(n) b_n \Big| &= \limsup_{N\to\infty} \Big| \frac{1}{N}\sum_{p\le N} (\log(N) - \log(p)) \cdot b_p \Big| \\
&\le \lim_{N\to\infty} \Big(\frac{\log(N)\cdot \pi(N)}{N} - \frac{1}{N}\sum_{p\le N}\log(p) \Big) = 1 - \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\Lambda'(n) = 0,
\end{aligned}$$

where for the last equality we used (10.8). ∎

We now need some further notation. For $\omega \in \mathbb{N}$ define

$$W = W_\omega := \prod_{p\le\omega} p.$$

For $r < W$ coprime to W consider the **modified Λ'-function** defined as

$$\Lambda'_{r,\omega}(n) := \frac{\varphi(W)}{W}\Lambda'(Wn + r),$$

for the Euler totient function φ given by

$$\varphi(n) := |\{a < n : \ a \text{ coprime to } n\}|.$$

We mention the following refinement of the prime number theorem.

Theorem 10.23 (Prime number theorem for arithmetic progressions)**.** *Let $a, q \in \mathbb{N}$ be coprime and denote by $\pi_{a,q}(x)$ the number of primes not exceeding x which are congruent to a modulo q. Then*

$$\lim_{x\to\infty} \frac{\pi_{a,q}(x)}{\frac{x}{\log(x)}} = \frac{1}{\varphi(q)}.$$

Again one can prove that this implies

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\Lambda'_{r,\omega}(n) = 1.$$

The following is a special case of a deep result of Green and Tao [6, Prop. 10.2], see also [4, Cor. 2.2], which we will use as black box. We just mention that to obtain this special case from their version requires some further theory (such as the notion of nilsequences and their properties) which we skip here.

Theorem 10.24 (Green–Tao)**.** *For every integer polynomial P and every $\lambda \in \mathbb{T}$*

$$\lim_{w\to\infty} \limsup_{N\to\infty} \max_{r<W,(r,W)=1} \left| \frac{1}{N} \sum_{n=1}^{N} (\Lambda'_{r,w}(n) - 1)\lambda^{P(n)} \right| = 0. \tag{10.10}$$

We note that the result of Green and Tao in [6] is conditional to the so-called Möbius-Nilsequence-Conjecture which was proved by them later in [7].

The following restricts the sequence $\mathbb{N}$ in Example 10.4 to the primes and polynomials of primes. We will use Theorem 10.24 to reduce equidistribution of polynomials of primes to equidistribution of polynomials.

Theorem 10.25 (Equidistribution of primes and polynomials of primes)**.** *For every integer polynomial P and every irrational $\lambda \in \mathbb{T}$, the sequence $(\lambda^{P(p_n)})_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$. In the additive setting, for every irrational $s \in \mathbb{R}$, the sequence $(sP(p_n))_{n\in\mathbb{N}}$ is equidistributed modulo 1.*

Proof. By Weyl's equidistribution criterion, Theorem 10.3, it is enough to show that for every irrational $\lambda \in \mathbb{T}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda^{P(p_n)} = 0.$$

Define $a_n := \Lambda'(n)\lambda^{P(n)}$. By Lemma 10.22 it suffices to show

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = 0. \tag{10.11}$$

Since for $n > 1$, we have $a_n \neq 0$ if and only if $n \in \mathbb{P}$, Lemma 10.16 implies for every ω and $W = W_\omega$ that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} \frac{1}{WN} \sum_{n=1}^{WN} a_n = \frac{1}{W} \sum_{r<W,\,(r,W)=1} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_{Wn+r}.$$

Thus, taking $\lim_{\omega\to\infty}$ and using Theorem 10.24 leads to

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n &= \lim_{\omega\to\infty} \frac{1}{W} \sum_{r<W,\,(r,W)=1} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_{Wn+r} \\
&= \lim_{\omega\to\infty} \frac{1}{\varphi(W)} \sum_{r<W,\,(r,W)=1} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Lambda'_{r,\omega}(n)\lambda^{P(Wn+r)} \\
&= \lim_{\omega\to\infty} \frac{1}{\varphi(W)} \sum_{r<W,\,(r,W)=1} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda^{P(Wn+r)}. \qquad (10.12)
\end{aligned}$$

Since $P(W\cdot +r)$ are again integer polynomials and λ is irrational, the sequences $(\lambda^{P(Wn+r)})_{n\in\mathbb{N}}$ are all equidistributed in $\mathbb{T}$ by Theorem 10.17 and hence the right-hand side of (10.12) equals zero. Thus (10.11) is proved. ∎

Of course, the above proof is so short because it uses the very deep and powerful Theorem 10.24. The original proof of Theorem 10.25 is due to Vinogradov and uses the classical Hardy–Littlewood method (see, for example, [8]).

Exercises

Exercise 10.1. Prove that a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is equidistributed modulo 1 if and only if the sequence $(e(s_n))_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$.

Exercise 10.2. Formulate analogous conditions to the ones in Theorem 10.3 for equidistribution modulo 1 and prove the equivalence of them.

Exercise 10.3. Show that $\{\log(n) : n \in \mathbb{N}\}$ is dense in $[0, 1)$ (cf. Example 10.6).

Exercise 10.4 (Equidistribution in compact metric spaces)**.** (a) Let K be a compact metric space and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in K. If $(a_n)_{n\in\mathbb{N}}$ is equidistributed with respect to a measure with full support, then $(a_n)_{n\in\mathbb{N}}$ is dense.

(b) Construct a dense sequence in $K := \{0, 1\}$ which is not equidistributed with respect to any probability measure on K.

Exercise 10.5 (Multidimensional Weyl criterion)**.** Prove Theorem 10.10.

Exercise 10.6 (Equidistribution in $\mathbb{T}^d$)**.** Work out the details of Example 10.11.

Exercise 10.7 (Cesàro convergence)**.** Prove Lemma 10.16.

Exercise 10.8 (Equidistribution of polynomials)**.** Show that $(e(P(n)))_{n\in\mathbb{N}}$ is not equidistributed if all coefficients of P are rational, with the possible exception of the constant term. This shows that the irrationality assumption in Theorem 10.17 is necessary.

Exercise 10.9 (Density of polynomial sequences and primes)**.** (a) Let P be an integer polynomial with $\deg(P) \geq 2$. Show that the set $P(\mathbb{N}_0)$ has density 0.

(b) Prove that for each $n \in \mathbb{N}$

$$n^{\pi(2n)-\pi(n)} \leq \binom{2n}{n} < 4^n.$$

Use this and a telescopic summation to estimate $\pi(2^k)$ and show that the density of $\mathbb{P}$ satisfies is 0, i.e.,

$$\lim_{n\to\infty} \frac{\pi(n)}{n} = 0.$$

Exercise 10.10 (Prime number theorem and the modified von Mangoldt function)**.** Prove the equivalence

$$\lim_{n\to\infty} \frac{\pi(n)\log(n)}{n} = 1 \quad \Longleftrightarrow \quad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \Lambda'(n) = 1.$$

(Hint: Use summation by parts.)

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LECTURE 11

Multiple and subsequential ergodic theorems

In this lecture we generalize the mean ergodic theorem in two different ways, both of which have already been mentioned earlier: We discuss the convergence of multiple ergodic averages as a generalizations of the double convergence theorem, Theorem 9.20 from Lecture 9, and the convergence of subsequential ergodic averages, as a generalization of the polynomial mean ergodic theorem, Theorem 7.13 from Lecture 7.

1. Kronecker factor and weak mixing

Before turning to ergodic theorems we discuss two important properties which a measure-preserving system may or may not have. Recall that by Proposition 3.4 for an ergodic system (X, μ, T) and for sets $A, B \subset X$ with $\mu(A) > 0$ and $\mu(B) > 0$ there are infinitely many $n \in \mathbb{N}$ such that $\mu(T^{-n}A \cap B) > 0$. That is to say, for infinitely many times the region A is "mixed" with region B under the dynamics. The following two properties present even stronger requirements.

Definition 11.1 (Mixing and weak mixing)**.** Let (X, μ, T) measure-preserving system. The system, and the transformation T itself, are called **(strongly) mixing** if for each pair $A, B \subset X$ of measurable sets

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B),$$

and **weakly mixing** if for each pair $A, B \subset X$ of measurable sets

$$\text{D-}\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B).$$

Evidently, every strongly mixing system is weakly mixing. Moreover, every weakly mixing system is ergodic. Indeed, if A is an invariant set in a weakly mixing system, then $\mu(A) = \mu(A)^2$, so that $\mu(A) \in \{0, 1\}$, implying ergodicity.

However, there are many ergodic systems that are not weakly mixing, see Example 11.5 or Exercise 11.3. To show that weak mixing does not imply mixing is much more difficult. For a fixed, standard probability space, Halmos[1] showed that set of weakly mixing systems is large in the sense of Baire category (namely a dense G_δ set), while Rokhlin[2] showed that strongly mixing systems form a small (meager) set, if one endows the set of all measure-preserving systems on the given probability space with an appropriate complete metric. These results imply that a

[1] P. R. Halmos, *In general a measure preserving transformation is mixing*, Ann. Math. (2) 45 (1944), 786–792.

[2] V. A. Rokhlin, *A "general" measure-preserving transformation is not mixing*, Doklady Akad. Nauk SSSR (N.S.) 60 (1948), 349–351.

"typical" measure-preserving transformation is weakly mixing but not mixing. The first explicit such example was constructed by Chacon[3] only in 1969.

Proposition 11.2 (Characterization of weak mixing)**.** *Let (X,μ,T) be a measure-preserving system with Koopman operator T on $\mathrm{L}^0(X,\mu)$, let $p,q\in[1,\infty]$ satisfy $\frac{1}{p}+\frac{1}{q}=1$, let $D\subset \mathrm{L}^p(X,\mu)$ and $D'\subset \mathrm{L}^q(X,\mu)$ be subsets whose linear hull is dense in the respective spaces. Let $\mathcal{E}$ and $\mathcal{E}'$ be $\cap$-stable generators* of the underlying σ-algebra. The following assertions are equivalent.*

(i) *The system (X,μ,T) is weakly mixing.*

(ii) *For every $A\in\mathcal{E}$ and $B\in\mathcal{E}'$*

$$\mathrm{D}\text{-}\lim_{n\to\infty}\mu(T^{-n}A\cap B)=\mu(A)\mu(B).$$

(iii) *For every $f\in D$ and every $g\in D'$*

$$\mathrm{D}\text{-}\lim_{n\to\infty}\langle T^n f,g\rangle=\langle f,\mathbf{1}\rangle\langle\mathbf{1},g\rangle. \tag{11.1}$$

(iv) *For every $f\in\mathrm{L}^p(X,\mu)$ and every $g\in\mathrm{L}^q(X,\mu)$*

$$\mathrm{D}\text{-}\lim_{n\to\infty}\langle T^n f,g\rangle=\langle f,\mathbf{1}\rangle\langle\mathbf{1},g\rangle.$$

Proof. The implications (i)⇒(ii) and (iv)⇒(iii) are trivial, while the implication (iv)⇒(i) follows by specializing $f=\mathbf{1}_A$ and $g=\mathbf{1}_B$ in (iv).

(iii)⇒(iv): This is an easy approximation argument, see Exercise 11.1.

(ii)⇒(i): Consider the subsets $F:=\mathrm{lin}\{\mathbf{1}_A : A\in\mathcal{E}\}$ and $F':=\mathrm{lin}\{\mathbf{1}_B : B\in\mathcal{E}'\}$, both of which are dense in $\mathrm{L}^2(X,\mu)$ (why?). Hypothesis (ii) yields the equality (11.1) for $f=\mathbf{1}_A$, $g=\mathbf{1}_B$ with $A\in\mathcal{E}$, $B\in\mathcal{E}'$. By linearity we obtain equality (11.1) for $f\in F$ and $g\in F'$. By the implication (iii)⇒(iv) we obtain the validity of (iv) for $p=q=2$, and then (i) follows.

(i)⇒(iv): Apply the the already proven implication (iii)⇒(iv) to the space $D=D'$ of simple functions, for which (11.1) holds by linearity and hypothesis (i). ∎

Example 11.3 (Shifts are weakly mixing)**.** Each (one- or two-sided) shift with arbitrary state space (Y,ν), in particular, each Bernoulli shift $B(p_0,\dots,p_{k-1})$ is mixing and hence weakly mixing. A proof is based on the equivalence (i)⇔(ii) in the previous characterization, see Exercise 11.2.

Simple examples of not weakly mixing systems are presented in Exercise 11.3. Before the next important, operator theoretic characterization of weak mixing, we recall the following facts from Lectures 2 and 8. The Koopman operator T leaves each L^p-space invariant, T is an isometry on each L^p-space so $P\sigma(T)\subset\mathbb{T}$, and the set $P\sigma(T)$ of eigenvalues is independent of $p\in[1,\infty]$. For given $\lambda\in\mathbb{T}$ the space of L^∞-eigenfunctions of T is dense in the space of L^p-eigenfunctions (for every $p\in[1,\infty]$).

Theorem 11.4 (Characterization of weak mixing)**.** *For a measure-preserving system (X,μ,T) with Koopman operator T on $\mathrm{L}^p(X,\mu)$, $p\in[1,\infty]$, the following assertions are equivalent.*

(i) *(X,μ,T) is weakly mixing.*

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[3] R. V. Chacon, *Weakly mixing transformations which are not strongly mixing*, Proc. Amer. Math. Soc. 22 (1969), 559–562.

*$\mathcal{E}$ is $\cap$-stable if $A,B\in\mathcal{E}$ implies $A\cap B\in\mathcal{E}$.

(ii) *The Kronecker factor of* (X, μ, T) *is trivial, i.e.,* $H_{\mathrm{kr}} = \mathbb{C}\mathbf{1}$.
(iii) (X, μ, T) *is ergodic and the point spectrum of the Koopman operator* T *satisfies* $P\sigma(T) = \{1\}$.

By what is said above it suffices to consider $p = 2$ only.

Proof. (ii)$\Leftrightarrow$(iii): We have $H_{\mathrm{kr}} = \mathrm{Fix}(T)$ (i.e., $\mathrm{kr}(X, \mu, T) = \mathrm{fix}(X, \mu, T)$) if and only if $P\sigma(T) = \{1\}$, and, by Proposition 3.21, the system is ergodic if and only if $\mathrm{Fix}(T) = \mathbb{C}\mathbf{1}$. The required equivalence follows.

(ii)$\Rightarrow$(i): We prove assertion (iv) in Proposition 11.2 for every $f, g \in \mathrm{L}^2(X, \mu)$. Write an arbitrary $f \in \mathrm{L}^2(X, \mu)$ as $f = P_{\mathrm{kr}} f + (I - P_{\mathrm{kr}}) f$. By hypothesis $P_{\mathrm{kr}} f = \langle f, \mathbf{1}\rangle \mathbf{1}$, so that $f - \langle f, \mathbf{1}\rangle \mathbf{1} \in H_{\mathrm{aws}}$. By Theorem 6.16 applied to the Koopman operator T on the Hilbert space $H = \mathrm{L}^2(X, \mu)$ we obtain

$$\operatorname*{D-lim}_{n\to\infty} (\langle T^n f, g\rangle - \langle f, \mathbf{1}\rangle\langle \mathbf{1}, g\rangle) = 0$$

for each $g \in \mathrm{L}^2(X, \mu)$. By Proposition 11.2 this is equivalent to the weak mixing property of (X, μ, T).

(i)$\Rightarrow$(iii): Assume (i). The ergodicity of (X, μ, T) was discussed above. Let now $\lambda \in P\sigma(T) \subset \mathbb{T}$. By Proposition 3.23 there exists a bounded eigenfunction $f \in \mathrm{L}^\infty(X, \mu)$ with $Tf = \lambda f$. Since $\langle T^n f, \overline{f}\rangle = \lambda^n \|f\|_2^2$, Proposition 11.2(iv) applied to $g := \overline{f}$ yields that $\mathrm{D\text{-}lim}_{n\to\infty} \lambda^n$ exists. This implies $\lambda = 1$ (why?). ∎

Example 11.5 (Group rotations are not weakly mixing)**.** Since for the torus rotation $(\mathbb{T}^d, \mathrm{m}^d, a)$ the Kronecker factor is the whole system, these systems are not weakly mixing. More generally, each compact group rotation system (G, m_G, a) with G having at least two elements is not weakly mixing (see Theorem 8.22).

It is easy to see that Koopman isomorphic systems are simultaneously weakly mixing or not. As a consequence a non-trivial system with discrete spectrum is not weakly mixing (see again Theorem 8.22).

For a further remarkable characterization of weak mixing we need the following.

Remark 11.6. Let (X, μ, T), (Y, ν, S) be measure-preserving systems. For $f \in \mathrm{L}^2(X, \mu)$ and $g \in \mathrm{L}^2(Y, \nu)$ define $f \otimes g : (x, y) \mapsto f(x)g(y)$. Then $f \otimes g \in \mathrm{L}^2(X \times Y, \mu \otimes \nu)$ and the linear hull of such functions is dense in $\mathrm{L}^2(X \times Y, \mu \otimes \nu)$. We have

$$(f \otimes g \,|\, u \otimes v)_{\mathrm{L}^2(X\times Y, \mu\otimes\nu)} = (f\,|\,u)_{\mathrm{L}^2(X,\mu)} (g\,|\,v)_{\mathrm{L}^2(Y,\nu)}.$$

The Koopman operator of the product system is denoted by $T \otimes S$ and we have $(T \otimes S)(f \otimes g) = Tf \otimes Sg$ for $f \in \mathrm{L}^2(X, \mu)$, $g \in \mathrm{L}^2(Y, \nu)$.

Proposition 11.7 (Product characterization of weak mixing)**.** *For a measure-preserving system* (X, μ, T) *the following are equivalent.*

(i) (X, μ, T) *is weakly mixing.*
(ii) $(X \times Y, \mu \otimes \nu, T \times S)$ *is ergodic for every ergodic system* (Y, ν, S).
(iii) $(X \times X, \mu \otimes \mu, T \times T)$ *is ergodic.*
(iv) $(X \times Y, \mu \otimes \nu, T \times S)$ *is weakly mixing for every weakly mixing system* (Y, ν, S).
(v) $(X \times X, \mu \otimes \mu, T \times T)$ *is weakly mixing.*

Proof. We denote by T and S the Koopman operators of (X, μ, T) and (Y, ν, S), respectively.

(i)⇒(ii): Let (Y, ν, S) be an ergodic system, and let $f, u \in \mathrm{L}^2(X, \mu)$ and $g, v \in \mathrm{L}^2(Y, \nu)$. Then for the Koopman operator $T \otimes S$ we have

$$((T \otimes S)^n (f \otimes g) \,|\, u \otimes v) = (T^n f \,|\, u) \cdot (S^n g \,|\, v).$$

Since $(T^n f \,|\, u) \xrightarrow{\mathrm{D}} (f \,|\, \mathbf{1}_X)(\mathbf{1}_X \,|\, u)$ by Proposition 11.2, and since $\frac{1}{N}\sum_{n=0}^{N-1}(S^n g \,|\, v) \to (g \,|\, \mathbf{1}_Y)(\mathbf{1}_Y \,|\, v)$ by ergodicity and by the mean ergodic theorem (Proposition 7.4), we conclude that

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}((T \otimes S)^n (f \otimes g) \,|\, u \otimes v) &\to (f \,|\, \mathbf{1}_X)(\mathbf{1}_X \,|\, u)(g \,|\, \mathbf{1}_Y)(\mathbf{1}_Y \,|\, v) \\ &= (f \otimes g \,|\, \mathbf{1}_X \otimes \mathbf{1}_Y)(\mathbf{1}_X \otimes \mathbf{1}_Y \,|\, u \otimes v)\end{aligned}$$

(cf. Proposition 4.13). By Remark 11.6 this implies

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}(T \otimes S)^n h = \int_{X\times Y} h \,\mathrm{d}\mu \otimes \nu \cdot \mathbf{1}_{X\times Y}$$

for every $h \in \mathrm{L}^2(X \times Y, \mu \otimes \nu)$, so Proposition 7.4 establishes the ergodicity of the product system.

(ii)⇒(iii): The one-point system $(\{0\}, \delta_0, \mathrm{id})$ (with trivial dynamics) is ergodic and the product $(X \times \{0\}, \mu \otimes \delta_0, T \times \mathrm{id})$ is isomorphic to (X, μ, T). We obtain from the hypothesis that the latter system is ergodic. Now the validity of (iii) follows also from the hypothesis (ii).

(iii)⇒(i): For $f, g \in \mathrm{L}^2(X, \mu)$ we have $(T^n f \,|\, g)\overline{(T^n f \,|\, g)} = ((T \otimes T)^n (f \otimes \overline{f}) \,|\, g \otimes \overline{g})$. By ergodicity and the mean ergodic theorem (see Proposition 7.4), the Cesàro averages of these expressions converge to $(f \otimes \overline{f} \,|\, \mathbf{1} \otimes \mathbf{1})(\mathbf{1} \otimes \mathbf{1} \,|\, g \otimes \overline{g}) = |(f \,|\, \mathbf{1})|^2 |(\mathbf{1} \,|\, g)|^2$. Moreover, by ergodicity we have that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}(T^n f \,|\, g) = (f \,|\, \mathbf{1})(\mathbf{1} \,|\, g)$. Thus we conclude that

$$\begin{aligned}&\frac{1}{N}\sum_{n=0}^{N-1} |(T^n f \,|\, g) - (f \,|\, \mathbf{1})(\mathbf{1} \,|\, g)|^2 \\ &\quad = \frac{1}{N}\sum_{n=0}^{N-1} |(T^n f \,|\, g)|^2 - 2\mathrm{Re}(T^n f \,|\, g)(f \,|\, \mathbf{1})(\mathbf{1} \,|\, g) + |(f \,|\, \mathbf{1})(\mathbf{1} \,|\, g)|^2 \\ &\quad \to |(f \,|\, \mathbf{1})|^2 |(\mathbf{1} \,|\, g)|^2 - 2\mathrm{Re}|(f \,|\, \mathbf{1})|^2 |(\mathbf{1} \,|\, g)|^2 + |(f \,|\, \mathbf{1})|^2 |(\mathbf{1} \,|\, g)|^2 = 0 \quad \text{as } n \to \infty.\end{aligned}$$

The Koopman–von Neumann lemma (Lemma 4.18) finishes the proof of (i).

(ii)⇒(iv): Let (Y, ν, S) be a weakly mixing system, and let (Z, κ, R) be an ergodic system. Then by the already proven equivalence we have that $(Y \times Z, \nu \otimes \kappa, S \times R)$ is ergodic, and similarly that $((X \times (Y \times Z), \mu \otimes (\nu \otimes \kappa), T \times (S \times R))$ is ergodic. Hence $(X \times Y, \mu \otimes \mu, T \times S)$ is weakly mixing.

(iv)⇒(i): Since the one-point system $(\{0\}, \delta_0, \mathrm{id})$ is weakly mixing, the assertion is clear.

If (i) holds, then (iv) holds, and (v) is clear. The implication (v)⇒(iii) is trivial. ∎

2. Multiple ergodic theorems

Next we turn to the following generalization of the double convergence theorem (Theorem 9.20) from Lecture 9.

Theorem 11.8 (Host–Kra, multiple ergodic theorem, 2005)**.** *Let (X, μ, T) be an ergodic measure-preserving system, let $k \geq 2$, and let $f_1, \dots, f_{k-1} \in \mathrm{L}^\infty(X, \mu)$. Then the limit of the multiple ergodic averages*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \big(T^n f_1\big) \cdot \big(T^{2n} f_2\big) \cdots \big(T^{(k-1)n} f_{k-1}\big) \tag{11.2}$$

exists in $\mathrm{L}^2(X, \mu)$.

We shall not prove this deep result for general ergodic measure-preserving systems, but discuss two special classes, for which the proof is easy. The first such particular case is when the system is completely structured, meaning that it has **discrete spectrum**, i.e., the Kronecker factor coincides with the whole system. Recall that by definition this means that the linear hull of the L^∞-eigenfunctions is dense in L^2 (and L^1), see Lecture 8.

Proposition 11.9. *Let (X, μ, T) be an ergodic measure-preserving system with discrete spectrum. Then for every $k \geq 2$ and every $f_1, \dots, f_{k-1} \in \mathrm{L}^\infty(X, \mu)$ the limit in* (11.2) *exists in the* L^2*-norm .*

Proof. First, let $f_1, \dots, f_{k-1} \in \mathrm{L}^\infty(X, \mu)$ be eigenfunctions of the Koopman operator T. Then we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \big(T^n f_1\big) \cdot \big(T^{2n} f_2\big) \cdots \big(T^{(k-1)n} f_{k-1}\big) = \frac{1}{N} \sum_{n=0}^{N-1} \big(\lambda_1 \lambda_2^2 \cdots \lambda_{k-1}^{k-1}\big)^n \cdot f_1 \cdots f_{k-1},$$

and we have convergence in $\mathrm{L}^2(X, \mu)$ for $N \to \infty$ as asserted. By linearity we obtain the existence of the limit whenever $f_1, \dots, f_{k-1}$ are linear combinations of L^∞-eigenfunctions of T, and these form a dense subspace of $\mathrm{L}^2(X, \mu)$ by the hypothesis. The assertion follows now by an elementary approximation argument. ∎

The other special case is when the system is as unstructured as it can be, meaning that the Kronecker factor is trivial, i.e., contains only constants.

Proposition 11.10. *Let (X, μ, T) be a weakly mixing measure-preserving system. Then for every $k \geq 2$ and every $f_1, \dots, f_{k-1} \in \mathrm{L}^\infty(X, \mu)$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \big(T^n f_1\big) \cdot \big(T^{2n} f_2\big) \cdots \big(T^{(k-1)n} f_{k-1}\big) = \Big(\int_X f_1 \,\mathrm{d}\mu \cdots \int_X f_{k-1} \,\mathrm{d}\mu\Big) \cdot \mathbf{1}$$

holds in $\mathrm{L}^2(X, \mu)$.

Proof. The proof is by induction on k. For $k = 2$ the assertion is just the mean ergodic theorem for ergodic systems (recall that every weakly mixing system is ergodic). Suppose now that the assertion is proved for some $k \geq 2$, and let $f_1, \dots, f_k \in \mathrm{L}^\infty(X, \mu)$. Decompose $f_k = P_{\mathrm{kr}} f_k + (I - P_{\mathrm{kr}}) f_k$. By linearity it suffices to handle the two cases $f_k = P_{\mathrm{kr}} f_k$ or $f_k = (I - P_{\mathrm{kr}}) f_k$ separately. Since the system is weakly mixing, P_{kr} is the orthogonal projection onto $\mathbb{C}\mathbf{1}$ by Proposition 11.2. So if $f_k = P_{\mathrm{kr}} f \in H_{\mathrm{kr}}$, then

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}\bigl(T^n f_1\bigr)\cdot\bigl(T^{2n}f_2\bigr)\cdots\bigl(T^{(k-1)n}f_{k-1}\bigr)\bigl(T^{kn}f_k\bigr)\\ = \frac{1}{N}\sum_{n=0}^{N-1}\bigl(T^n f_1\bigr)\cdot\bigl(T^{2n}f_2\bigr)\cdots\bigl(T^{(k-1)n}f_{k-1}\bigr)f_k,\end{aligned}$$

and by the induction hypothesis we obtain that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\bigl(T^n f_1\bigr)\cdot\bigl(T^{2n}f_2\bigr)\cdots\bigl(T^{kn}f_k\bigr) = \Bigl(\int_X f_1\,\mathrm{d}\mu\cdots\int_X f_{k-1}\,\mathrm{d}\mu\Bigr)\cdot\int_X f_k\,\mathrm{d}\mu\cdot\mathbf{1}.$$

Next, suppose that $f_k = (I - P_{\mathrm{kr}})f_k$, i.e., $f \in H_{\mathrm{aws}}$ (see Theorem 6.16). We apply the van der Corput trick (Lemma 7.16) to the functions $u_n := \bigl(T^n f_1\bigr)\cdot\bigl(T^{2n}f_2\bigr)\cdots\bigl(T^{kn}f_k\bigr)$. We obtain by the T-invariance of μ that

$$\begin{aligned}(u_n\,|\,u_{n+h}) &= \int_X \bigl(T^n f_1\cdot T^{2n}f_2\cdots T^{kn}f_k\bigr)\cdot\bigl(T^{n+h}\overline{f_1}\cdot T^{2(n+h)}\overline{f_2}\cdots T^{k(n+h)}\overline{f_k}\bigr)\,\mathrm{d}\mu\\ &= \int_X \bigl(f_1\cdot T^n f_2\cdots T^{(k-1)n}f_k\bigr)\cdot\bigl(T^h\overline{f_1}\cdot T^{n+2h}\overline{f_2}\cdots T^{(k-1)n+kh)}\overline{f_k}\bigr)\,\mathrm{d}\mu\\ &= \int_X \bigl(f_1T^h\overline{f_1}\bigr)\cdot T^n\bigl(f_2T^{2h}\overline{f_2}\bigr)\cdots T^{(k-1)n}\bigl(f_kT^{kh}\overline{f_k}\bigr)\,\mathrm{d}\mu.\end{aligned}$$

By the continuity of the scalar product and by the induction hypothesis

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(u_n\,|\,u_{n+h}) = \int_X f_1T^h\overline{f_1}\,\mathrm{d}\mu\cdot\int_X f_2T^{2h}\overline{f_2}\,\mathrm{d}\mu\cdots\int_X f_kT^{kh}\overline{f_k}\,\mathrm{d}\mu.$$

Hence we conclude that

$$\begin{aligned}&\limsup_{J\to\infty}\frac{1}{J}\sum_{h=0}^{J-1}\Bigl|\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(u_n\,|\,u_n+h)\Bigr|\\ &= \limsup_{J\to\infty}\frac{1}{J}\sum_{h=0}^{J-1}\Bigl|\int_X f_1T^h\overline{f_1}\,\mathrm{d}\mu\cdot\int_X f_2T^{2h}\overline{f_2}\,\mathrm{d}\mu\cdots\int_X f_kT^{kh}\overline{f_k}\,\mathrm{d}\mu\Bigr|\\ &\le \|f_1\|_\infty^2\cdots\|f_{k-1}\|_\infty^2\limsup_{J\to\infty}\frac{1}{J}\sum_{h=0}^{J-1}\int_X|f_kT^{kh}\overline{f_k}|\,\mathrm{d}\mu = 0,\end{aligned}$$

since $f_k \in H_{\mathrm{aws}}$ (use Theorem 6.16 and Lemma 4.18). ∎

The general case of the multiple ergodic theorem of Host and Kra, see [11], requires much more sophisticated techniques than what we have developed in these lectures so far. If one is interested in recurrence only, see the result right below, the proof is little bit more tractable but still beyond the scope of these lectures, see, e.g., [9].

Theorem 11.11 (Furstenberg, multiple recurrence, 1977)**.** *For a measure-preserving system and $k \in \mathbb{N}$ with $k \ge 2$ we have the following.*

(a) *For $f \in \mathrm{L}^\infty(X,\mu)$ with $f > 0$*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_X f \cdot T^n f \cdot T^{2n} f \cdots T^{(k-1)n} f \, \mathrm{d}\mu > 0.$$

(b) *In particular, for every $A \subset X$ with $\mu(A) > 0$*

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-(k-1)n}A) > 0 \quad \textit{for some } n \in \mathbb{N}.$$

As a matter of fact, Furstenberg proved the equivalence of (a) and (b) in the previous theorem, and of the celebrated result of Szemerédi which we recall here from Lecture 9, see also [EFHN, Ch. 20]. Then he proved the multiple recurrence theorem, thus presenting an ergodic theoretic proof of Szemerédi's following theorem.

Theorem 11.12 (Szemerédi, 1975). *Every subset of the natural numbers with positive upper density contains arbitrarily long arithmetic progressions.*

We do not give the proof of the previous results here, but present a remarkable property of weakly mixing systems that was discovered by Furstenberg on the way to his proof of the multiple recurrence theorem, Theorem 11.11.

A measure-preserving system is called **weakly mixing of order** $k \in \mathbb{N}$, if for every choice $A_0, \dots, A_{k-1} \subset X$ of measurable subsets

$$\text{D-}\lim_{n\to\infty} \mu(A_0 \cap T^{-1}A_1 \cap \dots T^{-(k-1)n}A) = \mu(A_0)\mu(A_1)\cdots\mu(A_{k-1}).$$

Clearly, weak mixing of order $k = 2$ is precisely weak mixing, and a kth-order weakly mixing system is weakly mixing of any order m with $m \le k$, in particular, it is weak mixing. The following is a surprising result, whose proof we leave as Exercise 11.5.

Proposition 11.13 (Furstenberg). *Every weakly mixing measure-preserving system is weakly mixing of all orders.*

From this it is easy to deduce the multiple recurrence theorem for weakly mixing systems. For the case of systems with discrete spectrum we refer to Exercise 11.4.

Analogously to weak mixing of higher orders one can make the following definition. A measure preserving system is called **(strongly) mixing of order** $k \in \mathbb{N}$, if for every choice $A_0, \dots, A_{k-1} \subset X$ of measurable subsets

$$\lim_{n\to\infty} \mu(A_0 \cap T^{-1}A_1 \cap \dots T^{-(k-1)n}A) = \mu(A_0)\mu(A_1)\cdots\mu(A_{k-1}).$$

A still open problem of Rokhlin is whether strong mixing implies strong mixing of order $k = 3$, or even strong mixing of all orders. A particular case has been settled by Host [10], see also Nadkarni [NSTDS, Ch. 10].

Theorem 11.14 (Host). *Let (X,μ,T) be an invertible, standard system. For every $f \in \mathrm{L}^2(X,\mu)$ consider the scalar spectral measure σ_f of the (unitary) Koopman operator on $\mathrm{L}^2(X,\mu)$, and suppose that σ_f and the Haar measure on $\mathbb{T}$ are mutually singular. If the system is strongly mixing, then it is strongly mixing of all orders.*

3. Subsequential ergodic theorems

Suppose that, for a given subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$, we are interested in how systems evolve along the times $(k_n)_{n\in\mathbb{N}}$, and we ignore all other times. For example, we could be interested what the system does on Wednesdays only (that would mean a periodic sequence $(k_n)_{n\in\mathbb{N}}$), or what an orbit does along the days that are numbered by the primes.

This motivates the following definition. Here and later, we understand under a subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ an *eventually* strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$.

Definition 11.15. A subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ is called a **good subsequence for the mean ergodic theorem** (or **mean good**) if for every invertible measure-preserving system (X, μ, T) and every $f \in \mathrm{L}^2(X, \mu)$ the averages along $(k_n)_{n\in\mathbb{N}}$

$$\frac{1}{N}\sum_{n=1}^{N} T^{k_n} f \tag{11.3}$$

converge in $\mathrm{L}^2(X, \mu)$. A subsequence which is not mean good is called **mean bad** (or **bad for the mean ergodic theorem**).

Example 11.16 (Affine sequences)**.** A first easy example of a mean good subsequence is an affine subsequence of the form $k_n := an + b$ for a fixed $a \in \mathbb{N}$, $b \in \mathbb{Z}$. This follows from the identity $T^{k_n} f = (T^a)^n (T^b f)$ and the mean ergodic theorem.

It is easy to characterize good subsequences for the mean ergodic theorem.

Proposition 11.17 (Characterization of mean good sequences)**.** *For a subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ the following assertions are equivalent.*

(i) *$(k_n)_{n\in\mathbb{N}}$ is a good subsequence for the mean ergodic theorem.*

(ii) *For every unitary operator S on a Hilbert space H and every $x \in H$ the averages*

$$\frac{1}{N}\sum_{n=1}^{N} S^{k_n} x \quad \textit{converge.} \tag{11.4}$$

(iii) *For every $\lambda \in \mathbb{T}$ the limit*

$$d(\lambda) := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda^{k_n} \quad \textit{exists.}$$

In this case, for every unitary operator S on a Hilbert space H and every $x \in H$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} S^{k_n} x = \sum_{\lambda\in\mathbb{T}} d(\lambda) P_\lambda x, \tag{11.5}$$

where $P_\lambda = P_{\ker(\lambda - S)}$ denotes the orthogonal projection onto $\ker(\lambda - S)$.

Remark 11.18. Using the theory of dilations, one can add the following assertion to the previous list of equivalent conditions:

(ii') *For every contraction S on a Hilbert space H and every $x \in H$, the averages in (11.4) converge.*

The formula in (11.5) remains also true, see [EFHN, Ch. 21]. Note that for $\lambda_1, \lambda_2 \in \mathbb{T}$ with $\lambda_1 \neq \lambda_2$ we have $\mathrm{rg}(P_{\lambda_1}) \perp \mathrm{rg}(P_{\lambda_2})$ (Proposition 6.1), so that the sum in (11.5) contains at most countably many non-zero terms (use Bessel's inequality).

Proof. The implication (ii)⇒(i) is trivial since the Koopman operator $S = S_T$ of an invertible measure-preserving system is unitary.

(i)⇒(iii): Let $\lambda \in \mathbb{T}$ and consider the measure-preserving system $(\mathbb{T}, \mathrm{m}, \lambda)$, and the monomial $\mathbf{z}$. Then $T^n\mathbf{z} = \lambda^n\mathbf{z}$ and the convergence of (11.3) in $\mathrm{L}^2(\mathbb{T}, \mathrm{m})$ implies the existence of $d(\lambda)$.

(iii)⇒(ii): Let S be a unitary operator on a Hilbert space H and take $x \in H$. By restricting to the cyclic subspace $Z(x)$ we can assume that x is cyclic vector. By the spectral theorem, Theorem 5.10, we can assume that T is the multiplication $M_{\mathbf{z}} : f \mapsto \mathbf{z} \cdot f$ on $\mathrm{L}^2(\mathbb{T}, \mu)$ for some probability measure μ on $\mathbb{T}$ and $x = \mathbf{1}$. Now, the identity $(M_{\mathbf{z}}^n \mathbf{1})(\lambda) = \lambda^n$ and the hypothesis that $(k_n)_{n\in\mathbb{N}}$ is mean good imply, by Lebesgue's dominated convergence theorem, the L^2-convergence of

$$\frac{1}{N}\sum_{n=1}^{N} M_{\mathbf{z}}^{k_n}\mathbf{1}. \tag{11.6}$$

The equivalence of the assertions is hence proved.

We now turn to the last assertion and stay in the setting of $T = M_{\mathbf{z}}$ as above. The limit of (11.6) at the point λ equals $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda^{k_n} = d(\lambda)$. Thus

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} M_{\mathbf{z}}^{k_n}\mathbf{1} = \sum_{\lambda \text{ atom of } \mu} d(\lambda)\mathbf{1}_{\{\lambda\}} = \sum_{\lambda \in P\sigma(M_{\mathbf{z}})} d(\lambda)P_\lambda\mathbf{1}.$$

By Lebesgue's dominated convergence theorem we obtain the assertion in (11.5). ∎

Remark 11.19. (a) By the equidistribution result, see Proposition 10.5, for every $(k_n)_{n\in\mathbb{N}}$ one has $d(\lambda) = 0$ for m-almost every $\lambda \in \mathbb{T}$.

(b) If $(k_n)_{n\in\mathbb{N}}$ is good sequence for the mean ergodic theorem, then $d : \mathbb{T} \to \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ is a Borel measurable function, as a pointwise limit of a sequence continuous functions.

Before turning to examples let us mention the following property of the function d, see [8] for further properties.

Proposition 11.20. *Let $(k_n)_{n\in\mathbb{N}}$ be a good sequence for the mean ergodic theorem. Then $G = \{\lambda : |d(\lambda)| = 1\}$ is a finite (cyclic) subgroup of $\mathbb{T}$.*

Proof. We clearly have $d(1) = 1$ and $d(\overline{\lambda}) = \overline{d(\lambda)}$. If $\lambda \in G$, then $\text{D-}\lim_{n\to\infty} \lambda^{k_n} = 1$. Indeed, take $b_n := \mathrm{Re}(\overline{d(\lambda)}\lambda^{k_n})$ and apply he Koopman–von Neumann lemma, Proposition 4.18(b), to conclude $\text{D-}\lim_{n\to\infty} \mathrm{Re}(\overline{d(\lambda)}\lambda^{k_n}) = 1$. Since $|\overline{d(\lambda)}\lambda^{k_n}| \leq 1$, we also obtain $\text{D-}\lim_{n\to\infty} \overline{d(\lambda)}\lambda^{k_n} = 1$, and that was to be proved. Altogether we conclude that G is a subgroup of $\mathbb{T}$, and as such, it is either finite or dense in $\mathbb{T}$ (why?). Assume, with the aim of arriving at a contradiction, that G is dense. By Remark 11.19 $d = 0$ almost everywhere with respect to m, and we conclude that d is nowhere continuous. This is a contradiction, since d is a pointwise limit of continuous functions, hence has many continuity points. ∎

Next we turn to more sophisticated examples.

Example 11.21 (Polynomial sequences)**.** We saw in Proposition 7.13 that for an integer polynomial P and $(k_n)_{n\in\mathbb{N}} = (P(n))_{n\in\mathbb{N}}$ one has the convergence of the polynomial averages as in (11.4), i.e., $(P(n))_{n\in\mathbb{N}}$ is a good sequence for the mean ergodic theorem. We give here another proof and calculate the limit. Let $P \in \mathbb{Q}[\cdot]$ be a polynomial of degree at least 1 and such that $P(\mathbb{N}_0) \subset \mathbb{N}_0$.

Let $\lambda \in \mathbb{T}$ be irrational. By Weyl's equidistribution theorem for polynomial sequences, Theorem 10.17, we have

$$d(\lambda) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda^{P(n)} = 0.$$

Next, let $\lambda \in \mathbb{T}$ be a root of unity, i.e., $\lambda^M = 1$ for some $M \in \mathbb{N}$. Let $\ell \in \mathbb{N}$ be such that the coefficients of ℓP are integers and set $q := \ell M$. Observe that for every $n \in \mathbb{N}$

$$P(n+q) \equiv P(n) \bmod q.$$

Take $N \in \mathbb{N}$ and write $N = mq + r$ with $r \in \{0, \dots, q-1\}$. We then have

$$\begin{aligned} \frac{1}{N}\sum_{n=1}^{N} \lambda^{P(n)} &= \frac{1}{N}\sum_{n=1}^{mq} \lambda^{P(n)} + \frac{1}{N}\sum_{n=mq+1}^{N} \lambda^{P(n)} \\ &= \frac{mq}{N}\frac{1}{q}\sum_{n=1}^{q} \lambda^{P(n)} + \frac{1}{N}\sum_{n=mq+1}^{N} \lambda^{P(n)} \to \frac{1}{q}\sum_{n=1}^{q} \lambda^{P(n)} \quad \text{as } N\to\infty. \end{aligned}$$

Thus we see that

$$d(\lambda) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda^{k_n} = \begin{cases} \frac{1}{q}\sum_{n=1}^{q} \lambda^{P(n)}, & \lambda^M = 1, \\ 0, & \lambda \text{ not a root of unity.} \end{cases} \tag{11.7}$$

This shows by Proposition 11.17 that the subsequence $(P(n))_{n\in\mathbb{N}}$ is good for the mean ergodic theorem with the limit of subsequential ergodic averages given by (11.5) and (11.7). See [12] and [13] for more information on the limit.

Example 11.22 (Primes and polynomials of primes)**.** As in Lecture 10 we denote by p_n the nth prime number. Let P be an integer polynomial. By using Proposition 11.17 we show that the sequence $(P(p_n))_{n\in\mathbb{N}}$ is good for the mean ergodic theorem. Let $\lambda \in \mathbb{T}$ be irrational. Then by Theorem 10.25, more precisely by its proof, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda^{P(p_n)} = 0$ (the sequence $(\lambda^{P(p_n)})_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$ by Weyl's criterion, Theorem 10.3). Let now λ be a root of unity, say $\lambda^q = 1$ for some $q \in \mathbb{N}$. It remains to show that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda^{P(p_n)}$ exists.

In what follows we will again denote prime numbers by p without saying it explicitly every time. Recall that by the prime number theorem in arithmetic progressions (Theorem 10.23), for every $a \in \{1, \dots, q\}$ with $(a,q) = 1$ (i.e., a and q coprime)

$$\lim_{N\to\infty} \frac{|\{p \le N : p \equiv a \bmod q\}|}{N/\log(N)} = \frac{1}{\varphi(q)}. \tag{11.8}$$

Since $P(n+q) \equiv P(n) \bmod q$ holds for every $n \in \mathbb{N}$ and $\lambda^q = 1$, we obtain for $N \in \mathbb{N}$ that

$$\begin{aligned} \sum_{p\le N} \lambda^{P(p)} &= \sum_{a<q,\,(a,q)=1}\ \sum_{p\le N,\, p\equiv a \bmod q} \lambda^{P(p)} = \sum_{a<q,(a,q)=1}\ \sum_{p\le N,\, p\equiv a \bmod q} \lambda^{P(a)} \\ &= \sum_{a<q,(a,q)=1} \lambda^{P(a)} |\{p \le N\ p \equiv a \bmod q\}|. \end{aligned}$$

Dividing by $\pi(N)$ and using (11.8) and the prime number theorem leads to

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\le N} \lambda^{P(p)} = \frac{1}{\varphi(q)} \sum_{a<q,\,(a,q)=1} \lambda^{P(a)}.$$

We have shown that $(P(p_n))_{n\in\mathbb{N}}$ is a mean good sequence.

It is not hard to construct a bad subsequence for the mean ergodic theorem. Indeed, take for instance $k_n := 2n + a_n$ with $a_n \in \{0,1\}$ such that the sequence $((-1)^{k_n})_{n\in\mathbb{N}} = ((-1)^{a_n})_{n\in\mathbb{N}}$ is Cesàro divergent. In such a way, it is easy to destroy goodness for many good sequences by the addition of a_n with $a_n \in \{0,1\}$.

This shows that it is not the growth alone that determines that a subsequence is good. However, growth can determine whether a subsequence is *bad*, as the following example shows.

Example 11.23 (Lacunary sequences)**.** A subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ is called **lacunary** if there is $q > 1$ such that $k_{n+1} \ge qk_n$ holds for every $n \in \mathbb{N}$. In [4] Bellow showed that every lacunary sequence is mean bad. Following Rosenblatt, Wierdl [16] we present here a proof of this fact for lacunary sequences provided $q \ge 8$.

Let the subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ satisfy $k_{n+1} \ge 8k_n$ for every $n \in \mathbb{N}$. By Proposition 11.17 we need to find $\lambda \in \mathbb{T}$ such that the sequence $(\frac{1}{N}\sum_{n=1}^{N} \lambda^{k_n})_{n\in\mathbb{N}}$ diverges.

We first show that for every given sequence $(I_n)_{n\in\mathbb{N}}$ of closed intervals in $[0,1]$ with each of them having length at least $1/4$ there exists $\alpha \in [0,1]$ with $\{k_n\alpha\} \in I_n$ for every $n \in \mathbb{N}$ or, equivalently,

$$\alpha \in \bigcap_{n=1}^{\infty} (I_n + \mathbb{Z})/k_n. \tag{11.9}$$

We recursively define a new sequence $(J_n)_{n\in\mathbb{N}}$ of closed intervals and a sequence $(m_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$, and start with $J_1 := I_1$ and $m_1 := 1$. Since $k_2 \ge 8$ and since the length of I_1 is at least $1/4$, there exists $m_2 \in \mathbb{Z}$ such that $[\frac{m_2}{k_2}, \frac{m_2+1}{k_2}] \subset I_1$. Thus we have

$$J_2 := (I_2 + m_2)/k_2 \subset \left[\tfrac{m_1}{k_2}, \tfrac{m_2+1}{k_2}\right] \subset J_1.$$

Since the length of J_2 is at least $\frac{1}{4k_2}$ and $k_3 \ge 8k_2$, there exists $m_3 \in \mathbb{N}$ with $[\frac{m_3}{k_3}, \frac{m_3+1}{k_3}] \subset J_2$, and we have $J_3 := (I_3 + m_3)/k_3 \subset J_2$, etc. Inductively, we obtain a sequence of closed, nested subintervals $(J_n)_{n\in\mathbb{N}}$ of $[0,1]$. Denote by α the unique number in $[0,1]$ with $\{\alpha\} := \bigcap_{n=1}^{\infty} J_n$. By construction we have $\alpha \in (I_n + m_n)/k_n$ for every $n \in \mathbb{N}$, proving (11.9).

Now the proof that $(k_n)_{n\in\mathbb{N}}$ is mean bad is completed by carefully choosing the intervals $I_1, I_2, \ldots$, each of them with length not less than $1/4$. The details are left as Exercise 11.6.

Remark 11.24 (Pointwise good sequences)**.** For $p \in [1,\infty)$, a subsequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ is called a **good subsequence for the pointwise ergodic theorem in** L^p if for every measure-preserving system (X,μ,T) and every $f \in \mathrm{L}^p(X,\mu)$ the subsequential averages (11.3) converge almost everywhere. Birkhoff's pointwise ergodic theorem tells that the sequence $(n)_{n\in\mathbb{N}}$ has this property. Using deep and complicated harmonic analysis arguments, Bourgain [5, 6] showed that polynomial sequences are pointwise good in L^p for every $p > 1$ and the primes are pointwise good in in L^p for every $p > \frac{1+\sqrt{3}}{2}$. In the case of primes Wierdl [17] completed this for all $p > 1$. There are generalizations to polynomials of primes by Wierdl [18] and Nair [15]. The case $p = 1$ is, however, dramatically different. Both monomial sequences and primes are **universally bad** for $p = 1$ and the pointwise ergodic theorem, meaning that for every ergodic system with non-atomic measure one can find an L^1-function f whose ergodic averages (along the subsequence) diverge, see Buczolich, Mauldin [7] and LaVictoire [14]. For more information on good and bad sequences for the pointwise ergodic theorem and further references see the extensive survey by Rosenblatt, Wierdl [16].

Exercises

Exercise 11.1 (Weak mixing)**.** Prove the implication (iii)⇒(iv) in Proposition 11.2.

Exercise 11.2 (Shifts are strongly mixing)**.** Prove that a shift (X, μ, T) with state space (Y, ν) is strongly mixing.

Exercise 11.3 (Not weakly mixing systems)**.** Show by using the definition only, that rotations on finite groups and on $\mathbb{T}$ are not weakly mixing.

Exercise 11.4 (Multiple recurrence and discrete spectrum)**.** Prove Theorem 11.11 for systems with discrete spectrum.

Exercise 11.5 (Weak mixing of all orders)**.** Prove that every weakly mixing system (X, μ, T) is weakly mixing of all orders. (Hint: Use Proposition 11.10 for the product system $(X \times X, \mu \otimes \mu, T \times T)$.)

Exercise 11.6 (Lacunary sequences)**.** (a) Show that the Fibonacci numbers form a lacunary sequence, and hence are bad for the mean ergodic theorem.
(b) Work out the details of the construction at the end of Example 11.23.

Exercise 11.7. Prove that every subsequence $(k_n)_{n\in\mathbb{N}}$ with density 1 is good.

Exercise 11.8 (Wiener's lemma along subsequences)**.** Let $(k_n)_{n\in\mathbb{N}}$ be a good sequence in $\mathbb{N}$. Prove that every complex Borel measure $\mu \in \mathrm{M}(\mathbb{T})$ satisfies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(k_n)|^2 = \int_{\mathbb{T}^2} d(\lambda_1 \overline{\lambda_2}) \, \mathrm{d}(\mu \otimes \overline{\mu})(\lambda_1, \lambda_2).$$

Suppose further that $\Lambda := [d \neq 0]$ is at most countable (which is true, i.e., for polynomial sequences, primes and polynomials of primes as shown in the previous lecture). Show that in this case every $\mu \in \mathrm{M}(\mathbb{T})$ satisfies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(k_n)|^2 = \sum_{\lambda\in\Lambda} d(\lambda) \sum_{a \text{ atom}} \overline{\mu}(\{a\overline{\lambda}\})\mu(\{a\}).$$

Conclude that if μ is continuous, then

$$\mathrm{D}\text{-}\lim_{n\to\infty} \hat{\mu}(k_n) = 0.$$

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LECTURE 12

Weighted ergodic theorems

In this lecture we discuss yet another variant of the classical ergodic theorems. Given a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers and a measure-preserving system (X,μ,T) with Koopman operator T we are interested in the convergence of the following **weighted ergodic averages**

$$\frac{1}{N}\sum_{n=1}^{N} a_n T^n f. \tag{12.1}$$

For the sake of simplicity, we shall consider invertible measure-preserving systems only. This assumption makes the Koopman operator unitary on L^2, and we can apply the spectral theorem. This is, however, just a technical restriction, and the results are true for general measure-preserving systems.

1. Mean good weights

We begin with the study of weights that are good for L^2-mean convergence.

Definition 12.1. A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is called a **good weight for the mean ergodic theorem** (or a **mean good weight**) if for every (invertible) measure-preserving system (X,μ,T) and every $f\in\mathrm{L}^2(X,\mu)$ the weighted averages in (12.1) converge in L^2.

Thus, the sequence $(1)_{n\in\mathbb{N}}$ is a mean good weight by von Neumann's ergodic theorem, Theorem 7.2. The first result is a characterization of mean good weights.

Proposition 12.2 (Characterization of mean good weights)**.** *Let $(a_n)_{n\in\mathbb{N}}$ satisfy $\sup_{N\in\mathbb{N}}\frac{1}{N}\sum_{n=1}^{N}|a_n|<\infty$. Then the following assertions are equivalent.*

(i) *$(a_n)_{n\in\mathbb{N}}$ is a good weight for the mean ergodic theorem.*
(ii) *For every $\lambda\in\mathbb{T}$ the limit*

$$c(\lambda):=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_n\lambda^n\in\mathbb{C}\quad \textit{exists.}$$

(iii) *For every unitary operator $S\in\mathscr{L}(H)$ on a Hilbert space H and every $x\in H$ the averages*

$$\frac{1}{N}\sum_{n=1}^{N}a_n S^n x \tag{12.2}$$

converge.

In this case, for every Hilbert space H, every unitary operator $S \in \mathscr{L}(H)$ and every $x \in H$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n S^n x = \sum_{\lambda\in\mathbb{T}} c(\lambda) P_\lambda x, \tag{12.3}$$

where $P_\lambda = P_{\ker(\lambda - S)}$ denotes the orthogonal projection onto $\ker(\lambda - S)$.

We leave the proof as Exercise 12.1 (a glimpse at the proof the characterization of mean good subsequences might be helpful, see Proposition 11.17).

Remark 12.3. (a) Note that if $c(\lambda)$ exists, then $|c(\lambda)| \le \sup_{N\in\mathbb{N}} \frac{1}{N}\sum_{n=1}^{N} |a_n|$. So for mean good weights the function $c : \mathbb{T} \to \mathbb{C}$ is a bounded (and measurable) function. Similarly to Remark 11.18 the series on the right-hand side of (12.3) is unconditionally convergent.

(b) Moreover, one can add the following assertion to the list of equivalent conditions in Proposition 12.2:

(iii') *For every contraction $S \in \mathscr{L}(H)$ on a Hilbert space H (with* $\dim(H) \ge 1$*) and every $x \in H$ the averages* (12.2) *converge.*

In this case, the identity in (12.3) holds. This implies also that the assumption of invertibility of the measure-preserving systems in the definition of mean good sequences can be dropped.

Now, the mean ergodic theorem, Theorem 7.2, applied to the operator $\lambda_0 T$ and Proposition 12.2, yield that for given $\lambda_0 \in \mathbb{T}$ the sequence $(\lambda_0^n)_{n\in\mathbb{N}}$ is a mean good weight. Moreover, for a given a unitary operator $S \in \mathscr{L}(H)$ and $x \in H$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} S^n x = P_{\mathrm{Fix}(\lambda S)} x = P_{\ker(\overline{\lambda} - S) f}.$$

The following example generalizes this and shows that subsequential and weighted averages are connected with each other. See Exercise 12.5 for another connection, and recall the notation $e(s) := \mathrm{e}^{2\pi \mathrm{i} s}$ for $s \in \mathbb{R}$.

Example 12.4 (Polynomial sequences)**.** We show that for every polynomial $P \in \mathbb{R}[\cdot]$ the sequence $(e(P(n)))_{n\in\mathbb{N}}$ is a good weight for the mean ergodic theorem. Let $\lambda \in \mathbb{T}$ and take $s \in \mathbb{R}$ such that $\lambda = e(s)$. For the polynomial Q given by $Q(x) := P(x) + sx$ we have

$$\frac{1}{N}\sum_{n=1}^{N} e(P(n))\lambda^n = \frac{1}{N}\sum_{n=1}^{N} e(P(n) + sn) = \frac{1}{N}\sum_{n=1}^{N} e(Q(n)).$$

Since, by Example 11.21, the right-hand side here converges in $\mathbb{C}$, Proposition 12.2 yields that the sequence $(e(P(n)))_{n\in\mathbb{N}}$ is indeed a mean good weight.

2. Pointwise good weights

Next we turn to the study of pointwise convergence of ergodic averages.

Definition 12.5. A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is called a **good weight for the pointwise ergodic theorem in** L^1 (**pointwise good** for short) if for every measure-preserving system (X, μ, T) and every $f \in \mathrm{L}^1(X, \mu)$ the weighted ergodic averages (12.1) converge almost everywhere.

Thus, the sequence $(1)_{n\in\mathbb{N}}$ is a pointwise good weight in L^1 by Birkhoff's ergodic theorem, Theorem 7.19. The following is a more general example of pointwise good weights.

Example 12.6. We show that the sequence $(\lambda_0^n)_{n\in\mathbb{N}}$ is a pointwise good weight in L^1. Let (X,μ,T) be a measure-preserving system and $f\in\mathrm{L}^1(X,\mu)$. Consider the product $Y:=X\times\mathbb{T}$ with the product measure $\nu:=\mu\otimes\mathrm{m}$, the transformation $S:Y\to Y$ given by $S(x,z):=(Tx,\lambda_0 z)$, and let $g\in\mathrm{L}^1(Y,\nu)$ be defined by $g(x,z):=zf(x)$. Then we have

$$g(S^n(x,z))=g(T^nx,\lambda_0^n z)=\lambda_0^n f(T^nx)z,$$

which implies

$$\frac{1}{N}\sum_{n=1}^{N}\lambda_0^n f(T^nx)=\frac{1}{N}\sum_{n=1}^{N}g(S^n(x,z))\overline{z}.$$

Thus almost everywhere convergence of the left-hand side follows from Birkhoff's ergodic theorem applied to the product system $(X\times Y,\mu\otimes\mathrm{m},S)$ and g.

The following sufficient condition for a sequence to be a pointwise good weight is due to El Abdalaoui, Kułaga-Przymus, Lemańczyk and de la Rue, see [4, Prop. 3.1 and its proof], and will be important in the next (and last) lecture.

Theorem 12.7 (A sufficient condition for pointwise good weights in L^1)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence such that for some $A>1$ and $C\geq 0$ one has*

$$\sup_{\lambda\in\mathbb{T}}\Big|\frac{1}{N}\sum_{n=1}^{N}a_n\lambda^n\Big|\leq\frac{C}{\log^A(N)}.$$

Then for every measure-preserving system (X,μ,T) and every $f\in\mathrm{L}^1(X,\mu)$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_n f(T^nx)=0\quad\text{for almost every } x\in X. \tag{12.4}$$

In particular, $(a_n)_{n\in\mathbb{N}}$ is a good weight for the pointwise ergodic theorem in L^1.

For the proof we use the strategy that we employed in the proof of Birkhoff's ergodic theorem. We prove a maximal inequality and the convergence on a dense subset, then we deduce the statement from Banach's principle. The first ingredient is hence the corresponding maximal inequality.

Proposition 12.8 (Maximal inequality for weighted averages)**.** *Let (X,ν,T) be a measure-preserving system with Koopman operator T, and let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$. For $N\in\mathbb{N}$ and $f\in\mathrm{L}^1(X,\mu)$ define*

$$S_Nf:=\frac{1}{N}\sum_{n=1}^{N}a_nT^nf\quad\text{and}\quad S^*f:=\sup_{N\in\mathbb{N}}|S_Nf|.$$

Then for every $f\in\mathrm{L}^1(X,\mu)$ we have

$$\mu[S^*f>\lambda]\leq\sup_{n\in\mathbb{N}}|a_n|\cdot\frac{\|f\|_1}{\lambda}\quad\text{for all }\lambda>0.$$

We leave the proof of this proposition as Exercise 12.4.

Proof of Theorem 12.7. Consider the Hilbert space $H = \mathrm{L}^2(X,\mu)$ and take $f \in \mathrm{L}^\infty(X,\mu) \subset H$ with $\|f\|_2 = 1$ (this latter property means no loss of generality). By restricting to the cyclic subspace $Z(f)$ we may assume that f is a cyclic vector, i.e., $Z(f) = H$. By the spectral theorem, Theorem 5.10, we conclude that

$$\Big\|\frac{1}{N}\sum_{n=1}^{N} a_n T^n f\Big\|_2 = \Big\|\frac{1}{N}\sum_{n=1}^{N} a_n \mathbf{z}^n\Big\|_{\mathrm{L}^2(\mathbb{T},\sigma_f)} \le \Big\|\frac{1}{N}\sum_{n=1}^{N} a_n \mathbf{z}^n\Big\|_\infty \le \frac{C}{\log^A(N)},$$

where σ_f is the corresponding spectral measure.

Take $\rho > 1$ and consider $N \in \mathbb{N}$ of the form $N := \lfloor \rho^m \rfloor$ with $m \in \mathbb{N}$. The previous considerations imply

$$\Big\|\frac{1}{\lfloor \rho^m \rfloor}\sum_{n=1}^{\lfloor \rho^m \rfloor} a_n T^n f\Big\|_2 \le \frac{C'_A}{(m\log(\rho))^A}.$$

From this we obtain that

$$\sum_{m=1}^{\infty}\Big\|\frac{1}{\lfloor \rho^m \rfloor}\sum_{n=1}^{\lfloor \rho^m \rfloor} a_n T^n f\Big\|_2 < \infty,$$

and hence the function

$$\sum_{m=1}^{\infty}\Big|\frac{1}{\lfloor \rho^m \rfloor}\sum_{n=1}^{\lfloor \rho^m \rfloor} a_n T^n f\Big|$$

belongs to $\mathrm{L}^2(X,\mu)$. Whence we conclude that

$$\lim_{m\to\infty} \frac{1}{\lfloor \rho^m \rfloor}\sum_{n=1}^{\lfloor \rho^m \rfloor} a_n T^n f = 0 \quad \text{almost everywhere.}$$

Let now $N \in \mathbb{N}$ be arbitrary and take $m \in \mathbb{N}$ with $\lfloor \rho^m \rfloor \le N < \lfloor \rho^{m+1} \rfloor$. We have

$$\begin{aligned}\Big|\frac{1}{N}\sum_{n=1}^{N} a_n T^n f\Big| &\le \Big|\frac{1}{N}\sum_{n=1}^{\lfloor \rho^m \rfloor} a_n T^n f\Big| + \Big|\frac{1}{N}\sum_{n=\lfloor \rho^m \rfloor+1}^{N} a_n T^n f\Big| \\ &\le \Big|\frac{1}{\lfloor \rho^m \rfloor}\sum_{n=1}^{\lfloor \rho^m \rfloor} a_n T^n f\Big| + \sup_{n\in\mathbb{N}} |a_n| \frac{\lfloor \rho^{m+1} \rfloor - \lfloor \rho^m \rfloor}{\lfloor \rho^m \rfloor}\|f\|_\infty.\end{aligned}$$

Since $\lim_{m\to\infty} \frac{\lfloor \rho^{m+1} \rfloor - \lfloor \rho^m \rfloor}{\lfloor \rho^m \rfloor} = \rho - 1$, we see by taking ρ arbitrarily close to 1 that

$$\lim_{N\to\infty}\Big|\frac{1}{N}\sum_{n=1}^{N} a_n T^n f\Big| = 0 \quad \text{almost everywhere}$$

(why?). We have proved (12.4) for every $f \in \mathrm{L}^\infty(X,\mu)$. The assertion follows now from Banach's principle, Proposition 7.22 and the maximal inequality, Proposition 12.8, since $\mathrm{L}^\infty(X,\mu)$ is dense in $\mathrm{L}^2(X,\mu)$. ∎

3. The Wiener–Wintner theorem

We now strengthen the assertion in Example 12.6 and show that the weights $(\lambda^n)_{n\in\mathbb{N}}$, as λ varies in $\mathbb{T}$, are simultaneously good for pointwise ergodic theorem in L^1 with the same set of pointwise convergence a having full measure. This is the statement of the Wiener–Wintner theorem.

Theorem 12.9 (Wiener–Wintner, 1941)**.** *Let (X,μ,T) be an ergodic measure-preserving system and let $f\in\mathrm{L}^1(X,\mu)$. Then there exists $X'\subset X$ with $\mu(X')=1$ such that*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda^n f(T^n x) \tag{12.5}$$

converges for every $x\in X'$ and every $\lambda\in\mathbb{T}$. For $f\in\mathrm{L}^2(X,\mu)$ the limit is almost everywhere equal to $P_{\overline{\lambda}}f$.

The idea of the proof of the Wiener–Wintner theorem is to establish the statement for f from a dense subset of $\mathrm{L}^1(X,\mu)$, and then to make an approximation argument. This last step is taken care of by the following lemma, whose proof is left as Exercise 12.7. Recall that for an ergodic measure-preserving system and for $f\in\mathscr{L}^1(X,\mu)$ a point $x\in X$ is called **generic** for f with respect to T if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(T^n x)=\int_X f\,\mathrm{d}\mu.$$

Lemma 12.10. *Let (X,μ,T) be an ergodic measure-preserving system, let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$, and for $j\in\mathbb{N}$ let $f,f_j\in\mathscr{L}^1(X,\mu)$ be such that $\lim_{j\to\infty}\|f-f_j\|_1=0$. Suppose that $x\in X$ is a generic point for $|f_j|$ and for $|f-f_j|$ for every $j\in\mathbb{N}$, and also that*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_n f_j(T^n x)=:b_j$$

exists for every $j\in\mathbb{N}$. Then the limit $\lim_{j\to\infty}b_j$ exists and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_n f(T^n x)=\lim_{j\to\infty}b_j.$$

Before the proof of the Wiener–Wintner theorem some further remarks are in order. In what follows we will work with fixed representatives of functions from $\mathscr{L}^1(X,\mu)$ and the generic points are always to be understood with respect to these. Moreover, in the proof, for a given $f\in\mathrm{L}^1(X,\mu)$, we will work with only countably many functions from L^1 (e.g., in the approximation argument), so that we can take a subset $X'\subset X$ with $\mu(X')=1$ such that *every* $x\in X'$ is generic for *each* occurring function (this is so by Birkhoff's ergodic theorem and by taking intersection of countably many sets of full measure).

Proof of Theorem 12.9. By Lemma 12.10, it is enough to prove the assertion for $f\in\mathrm{L}^2(X,\mu)$. By the Jacobs–de Leeuw–Glicksberg decomposition, see Theorem 6.16, it suffices to consider the cases when $f\in H_{\mathrm{kr}}$ and $f\in H_{\mathrm{aws}}$, separately.

To handle the case $f \in H_{\mathrm{kr}}$ we can suppose by linearity and by Lemma 12.10 that f is an eigenfunction of T with $Tf = \lambda_0 f$. Then $T^n f = \lambda_0^n f$ and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda^n T^n f = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} (\lambda_0\lambda)^n f = \mathbf{1}_{\{\overline{\lambda}\}}(\lambda_0) f = P_{\overline{\lambda}} f,$$

see also Proposition 6.1; the "convergence" here is almost everywhere, and the additional statement follows also at once.

Suppose now that $f \in H_{\mathrm{aws}}$ and define for $x \in X$ and for $n \in \mathbb{N}$

$$u_n(x) := \lambda^n (T^n f)(x) \in \mathbb{C}.$$

For $h \in \mathbb{N}$ we have by Birkhoff's ergodic theorem, Theorem 7.19, that there is a set $X_h \subset X$ with $\mu(X_h) = 1$ such that

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} u_{n+h}(x)\overline{u_n}(x) &= \frac{1}{N}\sum_{n=1}^{N} \lambda^{n+h}(T^{n+h}f)(x)\cdot \lambda^{-n}(T^n\overline{f})(x)\\ &= \frac{1}{N}\sum_{n=1}^{N} \lambda^h (T^n(T^h f\cdot \overline{f}))(x)\end{aligned}$$

converges for every $x \in X_h$ to

$$\lambda^h \int_X T^h f \cdot \overline{f}\,\mathrm{d}\mu = \lambda^h (T^h f\,|\,f).$$

This implies

$$\gamma_h(x) := \lim_{N\to\infty}\Bigl|\frac{1}{N}\sum_{n=1}^{N} u_{n+h}(x)\overline{u_n}(x)\Bigr| = |\lambda^h (T^h f\,|\,f)| = |(T^h f\,|\,f)|.$$

Define $X' := \bigcap_{h\in\mathbb{N}} X_h$. Then $\mu(X') = 1$ and, since $f \in H_{\mathrm{aws}}$ by assumption, for every $x \in \bigcap_{h\in\mathbb{N}} X_h$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{h=1}^{N}\gamma_h(x) = 0.$$

Van der Corput's Lemma 7.16 implies

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} u_n(x) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda^n (T^n f)(x) = 0$$

for every $x \in X'$. The proof is complete. ∎

Remark 12.11. If we use the van der Corput *inequality*, see Remark 7.17, we obtain the upper bound of the Wiener–Wintner averages for all functions $f \in \mathrm{L}^2(X,\mu)$. In fact, for almost every $x \in X$, we have the estimate

$$\sup_{\lambda\in\mathbb{T}}\limsup_{N\to\infty}\Bigl|\frac{1}{N}\sum_{n=1}^{N}\lambda^n f(T^n x)\Bigr| \le \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|(T^n f\,|\,f)|. \tag{12.6}$$

Bourgain observed that if one uses the finitary version of the van der Corput inequality, see Lemma 7.15, one obtains an even stronger assertion for almost weakly stable functions: For almost every $x \in X$ we have

$$\limsup_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \Big| \frac{1}{N} \sum_{n=1}^{N} \lambda^n f(T^k x) \Big| \le \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |(T^n f \,|\, f)|.$$

The interested reader can check this (or see [1, Sec. 2.4.2].) This leads to *uniform* convergence (w.r.t. λ) to zero in (12.5) for each almost weakly stable function f.

Remark 12.12. Another careful inspection of the proof and the uniform convergence of ergodic averages for uniquely ergodic systems, see Theorem 7.10, yield the following: If (X, μ, T) is ergodic with (X, T) a uniquely ergodic topological system, then one can take $X' = X$ in the Wiener–Wintner Theorem 12.9 and the convergence is uniform for $x \in X$ and $f \in \mathrm{C}(X)$. Combining this with the previous remark, we even obtain

$$\lim_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \Big\| \frac{1}{N} \sum_{n=1}^{N} \lambda^n T^n f \Big\|_\infty = 0$$

whenever the system is uniquely ergodic and $f \in \mathrm{C}(X)$ is an almost weakly stable function. More information on Wiener–Wintner ergodic theorems, can be found in the book [1] of Assani.

The following result of Lesigne [5, 6], which we state without proof here, generalizes the Wiener–Wintner theorem to polynomial weights. (Recall the notation $e(s) := \mathrm{e}^{2\pi i s}$.)

Theorem 12.13 (Lesigne)**.** *Let (X, μ, T) be an ergodic measure-preserving system. Then for every $f \in \mathrm{L}^1(X, \mu)$ there exists a set $X' \subset X$ with $\mu(X') = 1$ such that the averages*

$$\frac{1}{N} \sum_{n=1}^{N} e(P(n))(T^n f)(x)$$

converge for every $x \in X'$ and every polynomial $P \in \mathbb{R}[\cdot]$.

The Wiener–Wintner theorem in combination with Proposition 12.2 can be used to produce good weights for the mean ergodic theorem:

Proposition 12.14. *Let (X, μ, T) be an ergodic measure-preserving system and let $f \in \mathscr{L}^\infty(X, \mu)$. Then for almost every $x \in X$ the sequence $(f(T^n x))_{n\in\mathbb{N}}$ is a good weight for the mean ergodic theorem.*

The next example connects the previous proposition to the mean good subsequences from Lecture 11.

Example 12.15 (Most return time subsequences are mean good)**.** Let (X, μ, T) be an ergodic measure-preserving system. Let $A \subset X$ satisfy$\mu(A) > 0$ and for $x \in X$ consider the set $R_A(x) := \{n \in \mathbb{N} : T^n x \in A\}$ of return times. Birkhoff's ergodic theorem, Theorem 7.19, and the ergodicity assumption imply that for almost every $x \in X$

$$\lim_{N\to\infty} \frac{|R_A(x) \cap \{1, \dots, N\}|}{N} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (T^n \mathbf{1}_A)(x) = \mu(A),$$

i.e., the density of the set $R_A(x)$ equals $\mu(A)$ for almost every $x \in X$. Sort the elements of the $R_A(x)$ increasingly into the subsequence $(r_{A,x}(n))_{n\in\mathbb{N}}$ of $\mathbb{N}$. In fact,

$r_{A,x}(n)$ is time of nth visit of x in A under dynamics. By Proposition 12.14 the sequence $(T^n\mathbf{1}_A)_{n\in\mathbb{N}}$ is a good weight for almost every $x \in X$, so by Exercise 12.5 and since the density of $R_A(x)$ is $\mu(A) > 0$ for almost every $x \in X$, we obtain that $(r_{A,x}(n))_{n\in\mathbb{N}}$ is mean good subsequence for almost every $x \in X$.

The following result of Bourgain is much harder to prove, and yields good subsequences and good weights for the pointwise ergodic theorem, see [2], [3].

Theorem 12.16 (Bourgain, return times theorem)**.** *Let (X, μ, T) be an ergodic measure-preserving system, and let $f \in \mathrm{L}^\infty(X, \mu)$. Then for almost every $x \in X$ the sequence $(f(T^n x))_{n\in\mathbb{N}}$ is a good weight for the pointwise ergodic theorem in L^1.*

From this it is easy to deduce that for every ergodic measure-preserving system (X, μ, T), for every $A \subset X$ with $\mu(A) > 0$ and for almost every $x \in X$ the set of return times $R_A(x)$ yields (as in Example 12.15), a pointwise good subsequence.

4. Wiener–Wintner theorem for uniquely ergodic systems

Following Robinson [7] we study the uniform convergence of Wiener–Wintner ergodic averages for uniquely ergodic metric, systems.

For a topological system (K, T) we call $\lambda \in \mathbb{T}$ a **continuous eigenvalue** of T if there exists $g \in \mathrm{C}(K)$ with $g \neq 0$ such that

$$g(Tx) = \lambda g(x) \tag{12.7}$$

holds for every $x \in K$, i.e., λ is an eigenvalue of the Koopman operator T on $\mathrm{C}(K)$. We denote the set of all continuous eigenvalues of T by C_T.

Suppose now that (K, T) is uniquely ergodic with the unique, and hence ergodic, invariant measure μ (see Lecture 3). We call $\lambda \in \mathbb{T}$ a **measurable eigenvalue** of T if there exists $g \in \mathrm{L}^\infty(K, \mu)$ with $g \neq 0$ such that (12.7) holds for μ-almost every $x \in K$. The set of all measurable eigenvalues of T will be denoted by M_T. It is clear that the inclusion $C_T \subset M_T$ holds. Examples show that one can have strict inclusion here, see, e.g., Robinson [7, Sec. 3].

The following lemma is a topological analogue of the upper bound of the Wiener–Wintner ergodic averages (12.6), but with a varying point x.

Lemma 12.17. *Let (K, T) be uniquely ergodic system with invariant measure μ, let $(x_N)_{N\in\mathbb{N}}$ be a sequence in K and let $f \in \mathrm{C}(K)$. Then for $\lambda \in \mathbb{T}$ we have*

$$\limsup_{N\to\infty} \Big| \frac{1}{N} \sum_{n=1}^{N} \lambda^n (T^n f)(x_N) \Big| \leq \|P_{\overline{\lambda}} f\|_{\mathrm{L}^2(K,\mu)},$$

where $P_{\overline{\lambda}}$ is the orthogonal projection onto $\ker(\overline{\lambda} - T)$.

Proof. For $\lambda \in \mathbb{T}$ consider the rotation system $(\mathbb{T}, \mathrm{m}, \lambda)$ and the product system $(K \times \mathbb{T}, \mu \otimes \mathrm{m}, S)$, i.e., $S(x, z) = (Tx, \lambda z)$. For the function $g \in \mathrm{C}(K \times \mathbb{T})$, $(x, z) \mapsto zf(x)$, as in Example 12.6 we have

$$(S^n g)(x, z) = \lambda^n (T^n f)(x) z.$$

For $N \in \mathbb{N}$ define

$$\nu_N := \frac{1}{N} \sum_{n=1}^{N} \delta_{S^n(x_N, 1)} \in \mathrm{M}(K \times \mathbb{T}).$$

By Lemma 7.9 the sequence $(\nu_N)_{N\in\mathbb{N}}$ of probability measures is asymptotically invariant. Let $(N_j)_{j\in\mathbb{N}}$ be a subsequence of $\mathbb{N}$ such that

$$\lim_{j\to\infty}\Big|\frac{1}{N_j}\sum_{n=1}^{N_j}\lambda^n(T^nf)(x_{N_j})\Big| = \limsup_{N\to\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\lambda^n(T^nf)(x_N)\Big|.$$

By the Banach–Alaoglu theorem, Theorem 1.3, there exists an accumulation point ν of $(\nu_{N_j})_{j\in\mathbb{N}}$ with respect to the weak*-topology. By Lemma 7.8 this probability measure ν is S-invariant.

Next, observe that

$$\int\limits_{K\times\mathbb{T}} g\,\mathrm{d}\nu_{N_j} = \frac{1}{N_j}\sum_{n=1}^{N_j} g(S^n(x_{N_j},1)) = \frac{1}{N_j}\sum_{n=1}^{N_j}\lambda^n(T^nf)(x_{N_j}).$$

Letting $j\to\infty$ and by the definition of ν and $(N_j)_{j\in\mathbb{N}}$ we obtain that

$$\limsup_{N\to\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\lambda^n(T^nf)(x_N)\Big| = \lim_{j\to\infty}\Big|\frac{1}{N_j}\sum_{n=1}^{N_j}\lambda^n(T^nf)(x_{N_j})\Big| = \Big|\int\limits_{K\times\mathbb{T}} g\,\mathrm{d}\nu\Big|.$$

Thus, it remains to show that

$$\Big|\int\limits_{K\times\mathbb{T}} g\,\mathrm{d}\nu\Big| \le \|P_{\overline{\lambda}}f\|_{\mathrm{L}^2(K,\mu)}. \tag{12.8}$$

Consider the measure $\tilde{\mu}$ on K defined by

$$\tilde{\mu}(E) := \nu(E\times\mathbb{T}).$$

By the S-invariance of ν we have

$$\tilde{\mu}(T^{-1}E) = \nu(S^{-1}(E\times\mathbb{T})) = \nu(E\times\mathbb{T}) = \tilde{\mu}(E),$$

i.e., $\tilde{\mu}$ is a T-invariant probability. By the unique ergodicity this implies $\tilde{\mu}=\mu$. Whence we conclude that

$$\begin{aligned}(S^ng\,|\,g)_{\mathrm{L}^2(K\times\mathbb{T},\nu)} &= \int\limits_{K\times\mathbb{T}} S^ng\cdot\overline{g}\,\mathrm{d}\nu = \int\limits_{K\times\mathbb{T}}\lambda^nT^nf\cdot\overline{f}\mathbf{z}\overline{\mathbf{z}}\,\mathrm{d}\nu \\ &= \int\limits_{K\times\mathbb{T}}\lambda^nT^nf\cdot\overline{f}\,\mathrm{d}\nu = \lambda^n(T^nf\,|\,f)_{\mathrm{L}^2(K,\mu)}.\end{aligned} \tag{12.9}$$

Now, the mean ergodic theorem yields

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(S^ng\,|\,g) = (P_{\mathrm{Fix}(S)}g\,|\,g) = \|P_{\mathrm{Fix}(S)}g\|^2_{\mathrm{L}^2(K\times\mathbb{T},\nu)}.$$

On the other hand, we have $P_{\mathrm{Fix}(\lambda T)} = P_{\overline{\lambda}}$, so that by the mean ergodic theorem

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda^n(T^nf\,|\,f)_{\mathrm{L}^2(K,\mu)} = (P_{\overline{\lambda}}f\,|\,f) = \|P_{\overline{\lambda}}f\|^2_{\mathrm{L}^2(K,\mu)}.$$

This and (12.9) imply

$$\|P_{\mathrm{Fix}(S)}g\| = \|P_{\overline{\lambda}}f\|.$$

Let P be the orthogonal projection onto $\mathbb{C}\mathbf{1}_{K\times\mathbb{T}} \subset \mathrm{L}^2(K\times\mathbb{T},\nu)$, i.e., $Pg = \int_{K\times\mathbb{T}} g\,\mathrm{d}\nu$. Since $\mathrm{rg}(P)\subset \mathrm{rg}(P_{\mathrm{Fix}(S)})$, we conclude that

$$\|Pg\|_{\mathrm{L}^2(K\times\mathbb{T},\nu)} = \|PP_{\mathrm{Fix}(S)}g\|_{\mathrm{L}^2(K\times\mathbb{T},\nu)} \le \|P_{\mathrm{Fix}(S)}g\|_{\mathrm{L}^2(K\times\mathbb{T},\nu)} = \|P_{\overline{\lambda}}f\|_{\mathrm{L}^2(K,\mu)}.$$

Thus (12.8) is proved and the argument is complete. ∎

Theorem 12.18 (Robinson)**.** *Let (K,T) be a uniquely ergodic topological system. Then for every $\lambda\notin M_T\setminus C_T$ and every $f\in\mathrm{C}(K)$ the averages*

$$\frac{1}{N}\sum_{n=1}^{N}\lambda^n T^n f$$

converge uniformly to $P_{\overline{\lambda}}f$ as $N\to\infty$.

Proof. We distinguish two cases.

Case 1. Assume that $\lambda\in C_T$ and take $g\in\mathrm{C}(K)$ with $g\neq 0$ such that $Tg=\lambda g$. We can assume without loss of generality that $|g(x)|=1$ for every $x\in K$ (Exercise 12.8). For $n\in\mathbb{N}$ we thus have $T^n g=\lambda^n g$ or, equivalently, $\lambda^n = T^n g\cdot\overline{g}$. This implies that for every $f\in\mathrm{C}(K)$ the averages

$$\frac{1}{N}\sum_{n=1}^{N}\lambda^n T^n f = \frac{1}{N}\sum_{n=1}^{N}T^n(fg)\cdot\overline{g}$$

converge uniformly by Theorem 7.10 since the system is uniquely ergodic.

Case 2: Suppose that $\lambda\notin M_T$. We need to show that

$$\lim_{N\to\infty}\Big\|\frac{1}{N}\sum_{n=1}^{N}\lambda^n T^n f\Big\|_\infty = 0.$$

Assume the contrary, i.e., that there exist $\varepsilon>0$, a subsequence $(N_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ and a sequence $(y_j)_{j\in\mathbb{N}}$ in K such that

$$\Big|\frac{1}{N_j}\sum_{n=1}^{N_j}\lambda^n(T^n f)(y_j)\Big|\ge\varepsilon \quad \text{for all } j\in\mathbb{N}.$$

For $j\in\mathbb{N}$ set $x_{N_j}:=y_j$ and for $N\notin\{N_j : j\in\mathbb{N}\}$ let $x_N\in K$ be arbitrary. Then

$$\limsup_{N\to\infty}\Big|\frac{1}{N}\sum_{n=1}^{N}\lambda^n(T^n f)(x_N)\Big|\ge\varepsilon.$$

This and Lemma 12.17 imply $P_{\overline{\lambda}}f\neq 0$ meaning that $\overline{\lambda}\in M_T$. By Proposition 3.23 M_T is a group, so $\lambda\in M_T$, a contradiction.

We have seen the uniform convergence as stated. The mean ergodic theorem applied to the operator λT shows that the limit is as asserted. ∎

Exercises

Exercise 12.1. Prove Proposition 12.2.

Exercise 12.2 (Bad weights)**.** Give examples of **bad**, i.e., not good, bounded weights for the mean ergodic theorem.

Exercise 12.3 (Periodic sequences are good weight)**.** Prove that every periodic sequence is a pointwise good weight.

Exercise 12.4 (Maximal inequality for weighted averages)**.** Prove Proposition 12.8.

Exercise 12.5 (Connection between weighted and subsequential ergodic theorems)**.** Suppose that the subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ has positive positive density, i.e., for the set $A = \{k_n : n \in \mathbb{N}\}$ we have

$$\lim_{N\to\infty} \frac{|\{1, 2, \ldots, N\} \cap A|}{N} > 0.$$

Let $a_n := \mathbf{1}_A(n)$. Prove that for every sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ the convergence of the *subsequential averages*

$$\frac{1}{N}\sum_{n=1}^{N} b_{k_n}$$

is equivalent to the convergence of the *weighted averages*

$$\frac{1}{N}\sum_{n=1}^{N} a_n b_n.$$

Thus, $(k_n)_{n\in\mathbb{N}}$ is mean good sequence if and only of $(a_n)_{n\in\mathbb{N}}$ is a mean good weight.

Exercise 12.6 (Weighted averages for generalized eigenfunctions)**.** Let (X, μ, T) be a measure-preserving system, $k \in \mathbb{N}$ and let f be a generalized eigenfunction of T of order k (see Lecture 8). Compute $T^n f$ for every $n \in \mathbb{N}$ and show that the polynomial weighted averages

$$\frac{1}{N}\sum_{n=1}^{N} e(P(n))T^n f$$

converge almost everywhere for every $P \in \mathbb{R}[\cdot]$.

Exercise 12.7. Prove Lemma 12.10. For this take an accumulation point b of the sequence $(b_j)_{j\in\mathbb{N}}$ and prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n f(T^n x) = b.$$

Exercise 12.8 (Eigenfunctions of uniquely ergodic systems)**.** Let (K, T) be a uniquely ergodic topological system and let $g \in \mathrm{C}(K)$ satisfy $Tg = \lambda g$ for some $\lambda \in \mathbb{T}$. Prove that $|g|$ is constant.

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[6] E. Lesigne, *Spectre quasi-discret et théorème ergodique de Wiener-Wintner pour les polynômes*, Ergodic Theory Dynam. Systems **13** (1993), no. 4, 767–784.

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LECTURE 13

Sarnak's conjecture

With his conjecture, P. Sarnak* discovered in 2010 a deep connection between ergodic theory and number theory concerning the behavior of the primes. This conjecture has initiated extensive research in both fields and became a very active and dynamic area. In this lecture we will give a flavor of Sarnak's conjecture and highlight some connections to the previous lectures.

1. Motivation

Sarnak's conjecture connects dynamical systems to the following important number theoretic function introduced by A.F. Möbius in 1832. As before, we write p (sometimes with a subscript) for prime numbers without mentioning this explicitly.

Definition 13.1. The **Möbius function** $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined as $\mu(1) := 1$ and

$$\mu(n) := \begin{cases} 1, & \text{if } n = p_1 \cdots p_{2k} \text{ for some } k \in \mathbb{N} \text{ and distinct } p_1, \dots p_{2k}, \\ -1, & \text{if } n = p_1 \cdots p_{2k-1} \text{ for some } k \in \mathbb{N} \text{ and distinct } p_1, \dots p_{2k-1}, \\ 0, & \text{if } n \text{ is not square free.} \end{cases}$$

Clearly, the function μ is bounded by 1 and **multiplicative**, i.e., satisfies $\mu(nm) = \mu(n)\mu(m)$ whenever n and m are coprime. Here are its first values:

$$1, -1, -1, 0, -1, 1, -1, 0, 0, 1, -1, 0, -1, 1, 1, 0, \dots$$

Intuitively, μ reflects the behavior of the prime numbers. In particular, the generally believed irregularity of the primes corresponds heuristically to the random behavior of the Möbius function. The weakest variant of random behavior, meaning that there are, in the limit, as many values of μ equal to 1 as to -1, i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) = 0, \tag{13.1}$$

is already a deep fact, known to be equivalent to the prime number theorem due to the works of E. Landau. We remark that the stronger convergence rate

$$\frac{1}{N} \sum_{n=1}^{N} \mu(n) \leq \frac{C}{N^{1/2-o(1)}}$$

is equivalent to the Riemann hypothesis, see Titchmarsh [26, Thm. 14.25(C)].

*Peter Sarnak (born 1953) is a South African-American mathematician. He formulated his conjecture in "Three Lectures on the Möbius Function, Randomness and Dynamics" [22].

The following classical result of Davenport[1], building on the methods of Vinogradov developed in the 1930s, will be used as a black box. It is a generalization of the asymptotics in (13.1).

Theorem 13.2 (Davenport's estimate)**.** *For every $A > 0$ there is $C_A \geq 0$ such that*

$$\sup_{\lambda \in \mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n) \lambda^n \right| \leq \frac{C_A}{\log^A(N)}.$$

It follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)\lambda^n = 0 \quad \text{for all } \lambda \in \mathbb{T}. \tag{13.2}$$

Proposition 13.2 means in particular that the Möbius function satisfies the assumptions of Theorem 12.7. So we immediately obtain the following result announced by Sarnak in 2010 (the present proof is due to El Abdalaoui, Kułaga-Przymus, Lemańczyk, de la Rue [9, Sec. 3]).

Theorem 13.3 (Möbius function is a pointwise good weight in L^1 with limit zero)**.** *For every (invertible) measure-preserving system (X, μ, T) and every $f \in \mathrm{L}^1(X, \mu)$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) f(T^n x) = 0 \quad \textit{for almost every } x \in X. \tag{13.3}$$

In particular, $(\mu(n))_{n\in\mathbb{N}}$ is a good weight for the pointwise ergodic theorem in L^1.

In the previous lectures we followed the general philosophy that a statement holding almost everywhere for general systems (and functions), holds *everywhere* for sufficiently good systems (and functions). Thus, the following question is natural.

Question 1. For which systems and functions does (13.3) hold everywhere?

Sarnak's conjecture is an attempt to answer this question.

Since the Möbius function is believed to have random behavior, it is believed to have zero correlation with deterministic sequences, i.e., to satisfy

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) b_n = 0$$

whenever $(b_n)_{n\in\mathbb{N}}$ is deterministic (in some sense to be defined). Sarnak's idea was to define deterministic sequences as sequences coming from deterministic dynamical systems and continuous functions (as in Proposition 12.14). Thus, before presenting Sarnak's conjecture, we have to define deterministic dynamical systems.

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[1] H. Davenport, *On some infinite series involving arithmetical functions. II*, Q. J. Math., Oxf. Ser. 8 (1937), 313–320.

2. Topological entropy and Sarnak's conjecture

We start with a definition of topological entropy, which is due to R. Bowen[2] and E.I. Dinaburg[3]. Let K be a compact† metrizable space. Let d be any metric defining the topology on K. We call a set of pairwise disjoint open balls of radius $\varepsilon > 0$ an **ε-packing**. By compactness, each ε-packing in K is finite. Denote by $s_d(\varepsilon)$ the largest cardinality of ε-packings with respect to the metric d, called the **ε-packing number**, or ε-capacity.

Now, let (K, T) be a topological system. For $n \in \mathbb{N}$ and $x, y \in K$ we define

$$d_n(x, y) := \max\{d(T^j x, T^j y) : j = 0, \dots, n\}.$$

Then d_n is a metric on K, which is (uniformly) equivalent to d, i.e., the identity mapping $\mathrm{id} : (K, d) \to (K, d_n)$ is (uniformly) continuous with (uniformly) continuous inverse. We call $(x, Tx, \dots, T^n x)$ the n-orbit of x. Observe that two points $x, y \in K$ are at least ε-apart with respect to d_n if and only if their n-orbits are ε-apart, meaning that $T^j x$ and $T^j y$ are ε-apart for some $j = 0, \dots, n$. Therefore the packing number $s_{d_n}(\varepsilon)$ is the maximal number of n-orbits which are ε-apart.

The **topological entropy** of (K, T) with respect to the metric d is defined as

$$h_d(K, T) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(s_{d_n}(\varepsilon)).$$

The existence of this, possibly infinite, limit is clear since, for given $n \in \mathbb{N}$, the mapping $\varepsilon \mapsto s_{d_n}(\varepsilon)$ is decreasing. Topological entropy quantifies the asymptotic growth of the number of "distinct n-orbits", and can be considered as a measure of complexity or randomness of the system (K, T). We finally call a compact metrizable space (K, T) **deterministic** if it has topological entropy zero. In particular, by monotonicity for every $\varepsilon > 0$ such systems satisfy

$$\lim_{n \to \infty} \frac{1}{n} \log(s_{d_n}(\varepsilon)) = 0,$$

i.e., the maximal number of ε-distinct n-orbits grows subexponentially, and such systems are of lowest (exponential) complexity.

Proposition 13.4. *The entropy $h_d(K, T)$ is independent of the metric d. More precisely, any two metrics defining the same topology yield the same topological entropy.*

We leave the proof as Exercise 13.1, and write $h(K, T)$ instead of $h_d(K, T)$.

Proposition 13.5. *If T is contractive (or, in other words, non-expansive) with respect to some metric on the compact space K, then (K, T) is deterministic.*

Proof. If T is a contraction, then $d_n = d$ for each $n \in \mathbb{N}$. Therefore $h(K, T) = 0$. ∎

As a consequence, each torus rotation $(\mathbb{T}, a)$, $a \in \mathbb{T}$, has topological entropy 0, and we see that topological entropy also measures how the transformation expands distances. A topological system is called **equicontinuous** if the family $\{T^n : n \in$

[2] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), 401–414.

[3] E. I. Dinaburg, *A correlation between topological entropy and metric entropy*, Dokl. Akad. Nauk SSSR 190 (1970), 19–22.

†Bowen did not assume this, compactness is only needed to shorten things a bit.

$\mathbb{N}_0\}$ of transformations is uniformly equicontinuous, i.e., if for all $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in K$ and $n \in \mathbb{N}_0$ one has

$$d(x,y) < \delta \quad \Longrightarrow \quad d(T^n x, T^n y) < \varepsilon.$$

Evidently, if (K,T) is **contractive** (i.e., T is contractive), then (K,T) is an equicontinuous system. Conversely, an equicontinuous system can be turned into a contractive one by considering an equivalent metric, see Exercise 13.2. As a consequence equicontinuous systems are deterministic.

Proposition 13.6. *Let G be a compact, metric group. Then each group rotation (G,a), $a \in G$, is deterministic.*

Proof. The mapping $G \times G \to G$, $(x,y) \mapsto xy$ is uniformly continuous, which implies that the set $\{\tau_x : x \in G\}$ of left rotations is equicontinuous. As a consequence (G,a) is an equicontinuous systems for every $a \in G$, and therefore has zero topological entropy by the above. ∎

We present here some more examples without proof, and refer the reader to [WET, Ch. 8].

Example 13.7. (a) Even though the skew shift $(\mathbb{T}^2, T_a)$ is not equicontinuous, see Exercise 13.3, it has 0 topological entropy, i.e., the system is deterministic.

(b) The entropy of the topological shift (one or two-sided, see Example 2.24) on k letters is $\log(k)$.

(c) Let $A \subset \mathbb{N}$ be a subset and let $a = (a_n)_{n\in\mathbb{N}}$ be its characteristic sequence, i.e., $a = \mathbf{1}_A$. Consider the orbit closure $K = \overline{\mathrm{orb}}_+(a)$ of a in the shift system $(\{0,1\}^{\mathbb{N}}, T)$. The topological entropy of the subsystem (K,T) equals

$$\lim_{n\to\infty} \frac{1}{n}\theta_n,$$

where θ_n denotes the number of n-tuples $(b_1, b_2, \cdots, b_n)$ such that $K \cap \{x : x_1 = b_1, \ldots, x_n = b_n\} \neq \emptyset$, i.e., the number of finite patterns of length n occurring in A. We call $h(K,T)$ the entropy of the sequence a or of the set A.

Remark 13.8. In Lecture 2 we briefly mentioned *measure-theoretic entropy* as an important isomorphism invariant of measure-preserving systems. Now, the **variational principle**, due to Dinaburg[3] and Goodwyn[4], states the following: If (K,T) is topological system, then the topological entropy is the supremum of the measure-theoretic entropies of (K,μ,T) where μ ranges over the set of T-invariant probability measures, see also [WET, Ch. 8].

Thus, if $h(K,T) = 0$, then no matter which invariant measure we choose, the measure-preserving system (K,μ,T) has zero measure-theoretic entropy. Furthermore, for uniquely ergodic systems the measure-theoretic entropy equals the topological entropy $h(K,T)$.

We finally formulate the Sarnak conjecture. A sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers is called **deterministic** if there is a deterministic topological system (K,T) and a function $f \in \mathrm{C}(K)$ such that $a_n = f(T^n x)$ for every $n \in \mathbb{N}$.

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[3] E. I. Dinaburg, *A correlation between topological entropy and metric entropy*, Dokl. Akad. Nauk SSSR 190 (1970), 19–22.

[4] L. W. Goodwyn, *Comparing topological entropy with measure-theoretic entropy*, Amer. J. Math. 94 (1972), 366–388.

Conjecture 1 (Sarnak, 2010)**.** Let (K,T) be a deterministic topological dynamical system. Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(n) f(T^n x) = 0$$

holds for every $f \in \mathrm{C}(K)$ and *every* $x \in K$. In other words, μ is **asymptotically orthogonal** to every deterministic sequence $(a_n)_{n\in\mathbb{N}}$, meaning that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n \mu(n) = 0.$$

Some first examples are the following.

Example 13.9 (One-point system)**.** It is a direct consequence of (13.1) that Sarnak's conjecture holds for the one-point system.

Example 13.10 (Periodic systems)**.** One can show that (13.2) for rational λ implies

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(n) b_n = 0 \quad \text{for every periodic sequence } (b_n)_{n\in\mathbb{N}} \text{ in } \mathbb{C}. \tag{13.4}$$

Thus Sarnak's conjecture holds for periodic systems (i.e., topological systems (K,T) with $T^q = \mathrm{id}$ for some $q \in \mathbb{N}$). We leave the details as Exercise 13.6.

Example 13.11 (Rotations)**.** Also rotations on $\mathbb{T}$ satisfy Sarnak's conjecture. This is an easy consequence of (13.2) and the approximation argument presented in Exercise 13.5. We leave it as Exercise 13.7 to work out the details. Analogously, one can show that rotations on compact, metric, abelian groups satisfy Sarnak's conjecture.

For more examples see Example 13.17 and Section 4.1 below.

Connection to Chowla's conjecture. Another motivation for Sarnak's conjecture is its close relation to the famous Chowla conjecture [7], which is widely believed to be true.

Conjecture 2 (Chowla, 1965)**.** For every $m \in \mathbb{N}$ and every $a_1, \dots, a_m \in \{0,1,2\}$ which are not all even,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu^{a_1}(n+1)\mu^{a_2}(n+2)\cdots\mu^{a_m}(n+m) = 0.$$

Chowla's conjecture states that the Möbius function has zero multiple correlation with itself and would confirm (a strong version of) randomness of the Möbius function or, equivalently, of primes. Even the special case

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(n)\mu(n+2) = 0$$

is open. The truth of a certain quantitative version of this implies another famous conjecture, namely the *twin primes conjecture.*

The following gives a relation between these fundamental conjectures of Sarnak and Chowla, for a proof we refer the interested reader to Sarnak [22], see also El Abdalaoui, Kułaga-Przymus, Lemańczyk, de la Rue [9, Section 4].

Theorem 13.12 (Sarnak, 2010)**.** *Chowla's conjecture implies Sarnak's conjecture.*

Thus, the property of μ having zero correlation with deterministic systems (in Sarnak's sense) is weaker than to have zero multiple correlations with itself.

3. The Kátai–Bourgain–Sarnak–Ziegler criterion

We present here an extremely useful, sufficient condition for Möbius orthogonality due to Bourgain, Sarnak, Ziegler [6], see also Kátai [19]. To do so, we slightly simplify the statement as well as the (lengthy and technical but elementary and well-structured) proof from [6]. We recommend to skip the proof at the first reading and to jump directly to page 9, where some consequences are discussed.

Theorem 13.13 (Kátai–Bourgain–Sarnak–Ziegler (KBSZ) criterion)**.** *Let $a = (a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$ such that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_{pn}\overline{a_{qn}} = 0 \tag{13.5}$$

for all distinct primes p, q. Then $(a_n)_{n\in\mathbb{N}}$ is asymptotically orthogonal to every bounded multiplicative function $c : \mathbb{N} \to \mathbb{N}$ in the sense that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n c(n) = 0.$$

In particular, it is asymptotically orthogonal to the Möbius function.

Remark 13.14. (a) For the Möbius function, the above sufficient condition is not necessary (take $a_n := 1$ for every $n \in \mathbb{N}$ and use (13.1)).
(b) Theorem 13.13 shows that if a bounded sequence does not correlate with itself in the sense of (13.5), it does not correlate with any multiplicative function. This gives an *internal* sufficient condition for systems to satisfy Sarnak's conjecture without using the Möbius function at all.

Proof. We assume without loss of generality that both $|a|$ and c are bounded by 1. The idea is to decompose $[1, N] := \{1, \ldots, N\}$ into well-factorized sets (up to a small error) to be able to exploit the multiplicativity of c when computing the average over $[1, N]$.

Step 1: A decomposition of the interval $[1, N]$. Let $\varepsilon \in (0, \frac{1}{2})$ and let α be small (depending on ε) to be defined later. (As we will see, for instance $\alpha := \sqrt{\varepsilon}$ will do the job.) Consider

$$j_0 := \tfrac{1}{\alpha}\big(\log \tfrac{1}{\alpha}\big)^3, \quad D_0 := (1+\alpha)^{j_0}, \quad D_1 := (1+\alpha)^{j_0^2}.$$

Take $N \in \mathbb{N}$ and define

$$S := S(N) := \{n \in [1, N] : n \text{ has a prime divisor in } [D_0, D_1)\}.$$

It is Exercise 13.8 to show that

$$\lim_{N\to\infty} \frac{|[1, N] \setminus S(N)|}{N} \le \alpha, \tag{13.6}$$

i.e., $[1, N] = S(N)$ up to a small error. Thus for sufficiently large N we have $|[1, N] \setminus S(N)| \le 2\alpha N$.

In what follows j and m will always denote elements of $\mathbb{N}$. For $j \in [j_0, j_0^2)$ let

$$P_j := [(1+\alpha)^j, (1+\alpha)^{j+1}) \cap \mathbb{P},$$

and define the sets

$$S_j := S_j(N) := \Big\{n \in S : n \text{ has exactly one divisor in } P_j \text{ and no divisor in } \bigcup_{i=1}^{j-1} P_i\Big\},$$

which are by definition pairwise disjoint. We will show that $[1, N] = \bigcup_{j \in [j_0, j_0^2)} S_j$ up to a small error. By definition we have

$$S \setminus \textstyle\bigcup_{j_0 \le j < j_0^2} S_j = \bigcup_{j_0 \le j < j_0^2} \{n \in [1, N] : n \text{ has at least two prime factors in } P_j\}.$$

Therefore, we obtain

$$|S \setminus \textstyle\bigcup_{j_0 \le j < j_0^2} S_j| \le \displaystyle\sum_{j_0 \le j < j_0^2} \sum_{l,k \in P_j} \frac{N}{lk} \le N \sum_{j_0 \le j < j_0^2} \frac{|P_j|^2}{(1+\alpha)^{2j}}. \tag{13.7}$$

We now estimate $|P_j|$ as follows. By the prime number theorem, see Theorem 10.21, using the usual notation $\pi(x) := |\{p \in \mathbb{P} : p \le x\}|$, we have for small enough α (and therefore large enough j_0 and hence j)

$$\begin{aligned}
|P_j| &\le \pi((1+\alpha)^{j+1}) - \pi((1+\alpha)^j) + 1 \\
&\le \frac{(1+\alpha)^{j+1}}{(j+1)\log(1+\alpha)} - \frac{(1+\alpha)^j}{j\log(1+\alpha)} + \varepsilon \frac{(1+\alpha)^j}{j} \\
&\le (1+\alpha)^j \left(\frac{1}{\log(1+\alpha)}\left(\frac{1+\alpha}{j+1} - \frac{1}{j}\right) + \frac{\varepsilon}{j}\right) \\
&\le (1+\alpha)^j \left(\frac{\alpha}{j\log(1+\alpha)} + \frac{\varepsilon}{j}\right) \le (1+\alpha)^j \left(\frac{1+2\varepsilon}{j}\right) \le \frac{2(1+\alpha)^j}{j}. \qquad (13.8)
\end{aligned}$$

Inserting this into (13.7) yields

$$|S \setminus \textstyle\bigcup_{j_0 \le j < j_0^2} S_j| \le 4N \displaystyle\sum_{j_0 \le j < j_0^2} \frac{1}{j^2} \le 4N \int_{j_0-1}^{j_0^2} \frac{\mathrm{d}x}{x^2} \le \frac{4N}{j_0 - 1}$$

which implies for small enough α by the definition of j_0 that

$$|S \setminus \textstyle\bigcup_{j_0 \le j < j_0^2} S_j| \le 4\alpha N.$$

We thus obtain

$$|[1, N] \setminus \textstyle\bigcup_{j_0 \le j < j_0^2} S_j| \le 6\alpha N \qquad \text{for } N \text{ large enough} \tag{13.9}$$

meaning that $[1, N]$ is covered by the sets S_j, $j_0 \le j < j_0^2$ up to a small error.

Step 2: Refined decomposition into well-factorized sets. We now decompose each set S_j as a product set of the form $P_j Q_j$ up to a small error, where each element of Q_j is coprime to each element of P_j.

Define

$$Q_j := \left\{m \le \frac{N}{(1+\alpha)^{j+1}} : m \text{ has no prime factors in } \textstyle\bigcup_{k \le j} P_k\right\}$$

and observe that by definition

$$P_j Q_j \subset S_j.$$

Moreover, we have the inclusion

$$S_j \setminus P_j Q_j \subset P_j \cdot \left[\tfrac{N}{(1+\alpha)^{j+1}}, \tfrac{N}{(1+\alpha)^j}\right],$$

which implies, by the definition of j_0, by the estimate (13.8), and by the inequality $\log(x) \leq x$ that

$$\begin{aligned}
\sum_{j_0 \leq j < j_0^2} |S_j \setminus P_j Q_j| &\leq \sum_{j_0 \leq j < j_0^2} \frac{\alpha N |P_j|}{(1+\alpha)^{j+1}} \\
&\leq 2\alpha N \sum_{j_0 \leq j < j_0^2} \frac{1}{j} \leq 2\alpha \log\left(j_0^2\right) N \\
&\leq 4\alpha \log\left(\tfrac{1}{\alpha}\left(\log \tfrac{1}{\alpha}\right)^3\right) N \leq 16\alpha \log\left(\tfrac{1}{\alpha}\right) N.
\end{aligned}$$

Combining this with (13.9) yields

$$\left|[1, N] \setminus \textstyle\bigcup_{j_0 \leq j < j_0^2} P_j Q_j\right| \leq 22\alpha \log\left(\tfrac{1}{\alpha}\right) N.$$

Thus we have decomposed $[1, N]$ into disjoint sets of the form $P_j Q_j$ up to a small error. Moreover, from the construction it follows that the mapping $P_j \times Q_j \to P_j Q_j$ is injective and that every $p \in P_j$ is coprime to every $q \in Q_j$ for every j between j_0 and j_0^2.

Step 3: The correlation estimate. We now use the above decomposition to estimate the correlation between a and c. The above and the multiplicativity of c imply (recall that $|a|$ and c are bounded by 1) that

$$\begin{aligned}
\left|\sum_{n=1}^{N} a_n c(n)\right| &\leq \sum_{j_0 \leq j < j_0^2} \left|\sum_{p \in P_j, m \in Q_j} a_{pm} c(p) c(m)\right| + 22\alpha \log\left(\tfrac{1}{\alpha}\right) N \\
&\leq \sum_{j_0 \leq j < j_0^2} \sum_{m \in Q_j} \left|\sum_{p \in P_j} a_{pm} c(p)\right| + 22\alpha \log\left(t\frac{1}{\alpha}\right) N.
\end{aligned}$$

The inner sum can be estimated by the Cauchy–Schwarz inequality as follows

$$\begin{aligned}
\sum_{m \in Q_j} \left|\sum_{p \in P_j} a_{pm} c(p)\right| &\leq |Q_j|^{1/2} \left(\sum_{m \in Q_j} \left|\sum_{p \in P_j} a_{pm} c(p)\right|^2\right)^{1/2} \\
&\leq |Q_j|^{1/2} \left(\sum_{m \leq \frac{N}{(1+\alpha)^{j+1}}} \left|\sum_{p \in P_j} a_{pm} c(p)\right|^2\right)^{1/2} \\
&= |Q_j|^{1/2} \left(\sum_{m \leq \frac{N}{(1+\alpha)^{j+1}}} \sum_{p_1, p_2 \in P_j} a_{p_1 m} c(p_1) \overline{a_{p_2 m} c(p_2)}\right)^{1/2} \\
&\leq |Q_j|^{1/2} \left(\sum_{p_1, p_2 \in P_j} \left|\sum_{m \leq \frac{N}{(1+\alpha)^{j+1}}} a_{p_1 m} \overline{a_{p_2 m}}\right|\right)^{1/2} \qquad (13.10) \\
&\leq I_j + II_j,
\end{aligned}$$

where I_j corresponds to the diagonal contribution (if $p_1 = p_2$) of the outer sum in (13.10) and II_j to the non-diagonal one (if $p_1 \neq p_2$). (For the last estimate we have used the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.) Note that the multiplicative function c has already disappeared at this point.
Since a is bounded by 1, we have

$$I_j = |Q_j|^{1/2} \left(\sum_{p \in P_j} \left| \sum_{m \leq \frac{N}{(1+\alpha)^{j+1}}} a_{pm} \overline{a_{pm}} \right| \right)^{1/2} \leq |Q_j|^{1/2} |P_j|^{1/2} \frac{\sqrt{N}}{(1+\alpha)^{(j+1)/2}}.$$

Thus, by $\sum_{j_0 \leq j < j_0^2} |P_j||Q_j| \leq \sum_{j_0 \leq j < j_0^2} |S_j| \leq N$ and by the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned} \sum_{j_0 \leq j < j_0^2} I_j &\leq \sqrt{N} \left(\sum_{j_0 \leq j < j_0^2} |Q_j||P_j| \right)^{1/2} \left(\sum_{j_0 \leq j < j_0^2} \frac{1}{(1+\alpha)^{j+1}} \right)^{1/2} \\ &\leq N \left(\sum_{j_0 \leq j < j_0^2} \frac{1}{(1+\alpha)^{j+1}} \right)^{1/2} \leq N \left(\frac{1}{\alpha(1+\alpha)^{j_0}} \right)^{1/2} \leq \alpha N, \end{aligned}$$

the last inequality being true for α small enough, see Exercise 13.8(b).

The hypothesis of the theorem (which we have not used until now) implies for large enough N (independently of the finitely many $p_1 \neq p_2$ occurring in the definition of II_j) that

$$\left| \sum_{m \leq \frac{N}{(1+\alpha)^{j+1}}} a_{p_1 m} \overline{a_{p_2 m}} \right| \leq \frac{\varepsilon N}{(1+\alpha)^{j+1}}.$$

Therefore, by (13.8) and again the Cauchy–Schwarz inequality

$$\begin{aligned} \sum_{j_0 \leq j < j_0^2} II_j &\leq \sqrt{\varepsilon N} \sum_{j_0 \leq j < j_0^2} |Q_j|^{1/2} |P_j|^{1/2} |P_j|^{1/2} (1+\alpha)^{-(j+1)/2} \\ &\leq \sqrt{\varepsilon N} \left(\sum_{j_0 \leq j < j_0^2} |P_j||Q_j| \right)^{1/2} \left(\sum_{j_0 \leq j < j_0^2} |P_j| (1+\alpha)^{-(j+1)} \right)^{1/2} \\ &\leq \frac{2\sqrt{\varepsilon N}\sqrt{N}}{1+\alpha} \sum_{j_0 \leq j < j_0^2} \frac{1}{j} \leq 2\sqrt{\varepsilon} N \log(j_0^2) \leq 16\sqrt{\varepsilon} N \log\left(\tfrac{1}{\alpha}\right) \end{aligned}$$

by the definition of j_0 and by the inequality $\log(x) \leq x$. Putting everything together gives that for large enough N

$$\left| \frac{1}{N} \sum_{n=1}^{N} a_n c(n) \right| \leq \alpha + 16\sqrt{\varepsilon} \log\left(\tfrac{1}{\alpha}\right) + 22\alpha \log\left(\tfrac{1}{\alpha}\right).$$

Taking $\alpha := \sqrt{\varepsilon}$ finishes the proof. ∎

Remark 13.15. In particular, using

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \lambda^{pn} \overline{\lambda^{qn}} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (\lambda^{p-q})^n = 0$$

for every two distinct primes p, q and every irrational $\lambda \in \mathbb{T}$, the KBSZ criterion provides an alternative way, without using Davenport's estimate, to show (13.2) for such λ.

A polynomial generalization of this is the following.

Corollary 13.16 (Möbius function is orthogonal to polynomial sequences)**.** *Let $P \in \mathbb{R}[\cdot]$ be a real polynomial. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) e(P(n)) = 0.$$

We remark that polynomial sequences as above come from higher dimensional skew shifts, see, e.g., [EFHN, Proof of Prop. 3.18].

Proof. We can assume $P(0) = 0$. If all coefficients of P are rational, then the sequence $(P(n))_{n\in\mathbb{N}}$ is periodic. Thus the assertion follows from (13.4).

Let now P given by $P(x) = a_1 x + \ldots + a_d x^d$ have at least one irrational coefficient and let p, q be two distinct primes. Then

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} e(P(pn))\overline{e(P(qn))} &= \frac{1}{N}\sum_{n=1}^{N} e(P(pn) - P(qn)) \\ &= \frac{1}{N}\sum_{n=1}^{N} e(a_1(p-q)n + \ldots + a_d(p-q)^d n^d)\end{aligned}$$

converges to zero as $N \to \infty$ by Weyl's equidistribution theorem of polynomials (Theorem 10.17) and Weyl's criterion (Theorem 10.3). The KBSZ criterion finishes the proof. ∎

Example 13.17 (Skew shifts)**.** It is now easy to deduce from Corollary 13.16 and the approximation argument given in Exercise 13.5 that the skew rotation system $(\mathbb{T}^2, T_a)$ (which is deterministic as we have mentioned) given by

$$T_a : (x, y) \mapsto (ax, xy)$$

for a given $a \in \mathbb{T}$ satisfies the assertion of Sarnak's conjecture, see Exercise 13.9.

4. What else is known?

In this last section we briefly indicate some of the fascinating recent developments in the direction of Sarnak's conjecture.

4.1. More examples. Sarnak's conjecture has been verified for many (classes of) dynamical systems, often using the KBSZ criterion, see, e.g., Green, Tao [16], Bourgain, Sarnak, Ziegler [6], Bourgain [5], Liu, Sarnak [21], Kułaga-Przymus, Lemańczyk [20], Ferenczi, Mauduit [13], Karagulyan [18], Ferenczi, Kułaga-Przymus, Lemańczyk, Mauduit [11], and the recent survey by Ferenczi, Kułaga-Przymus, Lemańczyk [12] for further references.

Remark 13.18. There are examples of topological systems with arbitrarily small positive entropy, for which the assertion of Sarnak's conjecture fails, see Karagulyan [17]. On the other hand, there also are examples of topological systems with arbitrarily large entropy, for which the assertion of the Sarnak conjecture holds, see Downarowicz, Serafin [8].

4.2. Uniformity in Sarnak's conjecture. It is natural to ask for uniform convergence in Sarnak's conjecture (which is a seemingly stronger property). It occurs the uniform convergence holds automatically if Sarnak's conjecture is true, see el Abdalaoui, Kułaga-Przymus, Lemańczyk, de la Rue [10, Cor. 10].

4.3. Logarithmic Sarnak's conjecture. There have been much progress on a weaker, namely logarithmic, version of Sarnak's conjecture. We first define logarithmic averages as follows. For a sequence $(a_n)_{\in\mathbb{N}}$ in $\mathbb{C}$, we call

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{a_n}{n}$$

the N**th logarithmic average** of $(a_n)_{n\in\mathbb{N}}$. Note that convergence of logarithmic averages is equivalent to the convergence of

$$\frac{1}{\sum_{n=1}^{N}\frac{1}{n}}\sum_{n=1}^{N}\frac{a_n}{n},$$

being just a modification of the Cesàro averages $\frac{1}{\sum_{n=1}^{N}1}\sum_{n=1}^{N}a_n$, giving a_n with small n more weight than the ones with large n. As Exercise 13.10 shows, convergence of logarithmic averages is weaker than Cesàro convergence.

The following surprising results were shown for the logarithmic version of Sarnak's conjecture (where the Cesàro averages are replaced by logarithmic ones) .

(a) Logarithmic versions of Sarnak's and Chowla's conjectures are equivalent, see Tao [24]. Based on this result, Gomilko, Kwietniak, Lemańczyk [15] showed that the logarithmic Sarnak conjecture implies the (classical) Chowla conjecture along a subsequence.
(b) The logarithmic Chowla conjecture is true for $k = 2$, see Tao [23] and for all odd k, see Tao, Teräväinen [25].
(c) The logarithmic Sarnak conjecture holds for uniquely ergodic deterministic systems, see Frantzikinakis, Host [14].

Exercises

Exercise 13.1 (Topological entropy)**.** Prove Proposition 13.4.

Exercise 13.2 (Equicontinuous systems)**.** Let (K, T) be an equicontinuous topological system and let d be a metric inducing the topology on K. Prove that

$$\rho(x, y) := \sup_{n\in\mathbb{N}_0} d(T^n x, T^n y), \quad (x, y \in K)$$

defines a metric on K, which is (uniformly) equivalent to d, and which makes T a contractive transformation.

Exercise 13.3 (Skew shift)**.** Prove that the skew shift $(\mathbb{T}^2, T_a)$ is not equicontinuous.

Exercise 13.4 (Deterministic sequences)**.** Give an example of a non-periodic, deterministic 0-1-sequence.

Exercise 13.5 (Approximation argument for topological systems)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$. Let further (K,T) be a topological system with K metrizable, and let $M \subset \mathrm{C}(T)$ satisfy $\overline{\mathrm{lin}}(M) = \mathrm{C}(K)$. If for every $f \in M$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a(n) f(T^n x) = 0 \quad \text{for every } x \in K,$$

then this holds for every $f \in \mathrm{C}(K)$.

Exercise 13.6 (Periodic systems)**.** Show that every periodic sequence is a finite linear combination of sequences of the form $(\lambda^n)_{n\in\mathbb{N}}$, $\lambda \in \mathbb{T}$ rational. Deduce that periodic systems satisfy Sarnak's conjecture. (Why are they deterministic?)

Exercise 13.7 (Rotations on $\mathbb{T}$)**.** Sarnak's conjecture holds for rotations on the circle. (Hint: Use (13.1) and Exercise 13.5.)

Exercise 13.8 (Some details for the proof of the KBSZ criterion)**.** (a) Prove the inequality (13.6). To do this, first use the Chinese remainder theorem to estimate the left-hand side of (13.6) from above by

$$\prod_{p\in[D_0,D_1)} \left(1 - \frac{1}{p}\right)$$

and then use the following consequence of the prime number theorem, a result of Mertens: There are constants A and C such that

$$\Big|\sum_{p\le x} \frac{1}{p} - \log\log(x) - A\Big| \le \frac{C}{\log(x)} \quad \text{for every } x > 1.$$

(b) Show that for $j_0 := \frac{1}{\alpha}\left(\log\frac{1}{\alpha}\right)^3$ one has $\frac{1}{(1+\alpha)^{j_0}} \le \alpha^3$ whenever α is small enough.

Exercise 13.9 (Skew shifts)**.** Prove the assertion of Example 13.17.

Exercise 13.10 (Logarithmic averages)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$ and $a \in \mathbb{C}$. Then the following implication holds:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = a \quad \Longrightarrow \quad \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{a_n}{n} = a.$$

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