20th Internet Seminar on Linear Parabolic Equations

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February 9, 2017

Lecture 1

Parabolic maximum principle

This lecture is dedicated to the weak and strong parabolic maximum principles for uniformly elliptic second order operators on bounded domains.
On a bounded, not empty and open subset Ω of $\mathbb{R}^d$ we define the second order elliptic operator

$$\begin{aligned}\mathcal{A}\psi(x) =& \sum_{i,j=1}^{d} q_{ij}(x)D_{ij}\psi(x) + \sum_{j=1}^{d} b_j(x)D_j\psi(x) + c(x)\psi(x)\\ =&\mathrm{Tr}(Q(x)D^2\psi(x)) + \langle b(x), \nabla_x\psi(x)\rangle + c(x)\psi(x)\end{aligned}$$

for a smooth function $\psi : \Omega \to \mathbb{R}$, where $Q(x) = (q_{ij}(x))$ and $b(x) = (b_1(x), \dots, b_N(x))$ for $x \in \Omega$. Throughout the lecture, we assume the following conditions on the coefficients of $\mathcal{A}$.

Hypotheses 1.0.1. (i) The coefficients $q_{ij} = q_{ji}$, b_j $(i, j = 1, \dots, d)$ and c are real valued functions and belong to $C(\overline{\Omega})$;

(ii) there exists $\mu > 0$ such that $\langle Q(x)\xi, \xi\rangle \geq \mu|\xi|^2$ for any $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^d$.

For $T > 0$, define $\Omega_T := (0, T] \times \Omega$ and consider $\Gamma_T = \overline{\Omega_T} \setminus \Omega_T$ the so-called *parabolic boundary of* $\overline{\Omega_T}$. It is clear that $\Gamma_T = \{0\} \times \overline{\Omega} \cup (0, T] \times \partial\Omega$. Here and in the sequel we denote by

$$C^{1,2}(\Omega_T) := \{u : \Omega_T \to \mathbb{R} : \exists\, D_t u,\, D_{ij}u \in C(\Omega_T),\, i, j = 1, \dots, d\},$$

where $D_t u := \frac{\partial u}{\partial t}$ and $D_{ij}u := \frac{\partial^2 u}{\partial x_i \partial x_j}$.

1.1 The parabolic weak maximum principle

With the above notations and assumptions we propose to prove first the weak maximum principle for the parabolic operator $D_t - \mathcal{A}$.

Theorem 1.1.1 (Weak maximum principle)**.** *Fix* $u \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$ *and assume that* $c \equiv 0$ *in* Ω*. The following assertions hold:*

(i)
$$\textit{If } D_t u - \mathcal{A}u \leq 0 \textit{ on } \Omega_T, \textit{ then } \max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u.$$

(ii)
$$\textit{If } D_t u - \mathcal{A}u \geq 0 \textit{ on } \Omega_T, \textit{ then } \min_{\overline{\Omega_T}} u = \min_{\Gamma_T} u.$$

In particular,
$$\textit{If } D_t u - \mathcal{A}u = 0 \textit{ on } \Omega_T, \textit{ then } \max_{\overline{\Omega_T}} |u| = \max_{\Gamma_T} |u|.$$

Proof. It is obvious that the assertion (ii) can be deduced from (i) by considering $-u$ instead of u in (i). So we have to prove only (i).
Let us assume first that $D_t u - \mathcal{A}u < 0$ on Ω_T and prove that u cannot reach a maximum at $(t_0, x_0) \in \Omega_T$. In fact, if such point exists then $D_t u(t_0, x_0) \geq 0$ (since t_0 could be equal to T), $D_j u(t_0, x_0) = 0$ for any $j = 1, \ldots, d$, and the matrix $(D_{ij} u(t_0, x_0))$ is negative semi-definite. Thus, see Exercise 1.4.1,

$$D_t u(t_0, x_0) - \mathcal{A}u(t_0, x_0) = D_t u(t_0, x_0) - \sum_{i,j=1}^{d} q_{ij}(x_0) D_{ij} u(t_0, x_0) \geq 0.$$

This contradicts the above assumption and hence

$$\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u \ \text{ if } \ D_t u - \mathcal{A}u < 0 \ \text{ on } \Omega_T. \tag{1.1}$$

To prove the general case, let us fix $\varepsilon > 0$ and consider the function $u_\varepsilon(t,x) = u(t,x) - \varepsilon t$. We have $D_t u_\varepsilon - \mathcal{A}u_\varepsilon = D_t u - \mathcal{A}u - \varepsilon \leq -\varepsilon < 0$. So, by (1.1), we obtain

$$\max_{\overline{\Omega_T}} u_\varepsilon = \max_{\Gamma_T} u_\varepsilon.$$

Now, since u_ε converges uniformly to u on $\overline{\Omega_T}$, the assertion follows by letting $\varepsilon \to 0$. □

The following result concerns the weak maximum principle when $c \leq 0$ on Ω.

Theorem 1.1.2. *Let* $c \leq 0$ *on* Ω *and* $u \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$*. The following hold:*

(i)
$$\textit{If } D_t u - \mathcal{A}u \leq 0 \textit{ on } \Omega_T, \textit{ then } \max_{\overline{\Omega_T}} u \leq \max_{\Gamma_T} u^+.$$

(ii)
$$\textit{If } D_t u - \mathcal{A}u \geq 0 \textit{ on } \Omega_T, \textit{ then } \min_{\overline{\Omega_T}} u \geq -\max_{\Gamma_T} u^-,$$

where $u^+ := \sup(u, 0)$ *and* $u^- = \sup(-u, 0)$*.*

In particular,

$$\textit{If } D_tu - \mathcal{A}u = 0 \textit{ on } \Omega_T, \textit{ then } \max_{\overline{\Omega_T}} |u| = \max_{\Gamma_T} |u|.$$

Proof. Let us prove (i). Assume first, as in the above proof, that $D_tu - \mathcal{A}u < 0$ on Ω_T. Suppose by contradiction that u reaches its maximum at $(t_0, x_0) \in \Omega_T$ with $u(t_0, x_0) \geq 0$. Then, $D_tu(t_0, x_0) \geq 0$, $D_ju(t_0, x_0) = 0$ for any $j = 1, \ldots, d$, and the matrix $D_{ij}u(t_0, x_0))$ is negative semi-definite. Thus, again by Exercise 1.4.1,

$$D_tu(t_0, x_0) - \mathcal{A}u(t_0, x_0) = D_tu(t_0, x_0) - \sum_{i,j=1}^{d} q_{ij}(x_0)D_{ij}u(t_0, x_0) - c(x_0)u(t_0, x_0) \geq 0$$

which contradicts the fact that $D_tu - \mathcal{A}u < 0$ on Ω_T. So, it follows that, if (t_0, x_0) is a maximum point for u in $\overline{\Omega_T}$, then $(t_0, x_0) \in \Gamma_T$ or $u(t_0, x_0) < 0$. So, in both cases we have

$$\max_{\overline{\Omega_T}} u = u(t_0, x_0) \leq \max_{\Gamma_T} u^+.$$

For the general case, set $u_\varepsilon(t, x) = u(t, x) - \varepsilon t$ for $\varepsilon > 0$. We have $D_tu_\varepsilon - \mathcal{A}u_\varepsilon = D_tu - \mathcal{A}u - \varepsilon + \varepsilon ct \leq -\varepsilon(1 - ct) < 0$. So,

$$\max_{\overline{\Omega_T}} u_\varepsilon \leq \max_{\Gamma_T} u_\varepsilon^+.$$

Now, (i) follows by letting $\varepsilon \to 0$, since u_ε converges uniformly to u on $\overline{\Omega_T}$ and u_ε^+ converges uniformly to u^+ on Γ_T.

Assertion (ii) can be obtained by applying (i) to $-u$, since $(-u)^+ = u^-$.

If $D_tu - \mathcal{A}u = 0$ on Ω_T and, for example,

$$\max_{\overline{\Omega_T}} |u| = \max_{\overline{\Omega_T}} u,$$

then

$$\max_{\Gamma_T} |u| \geq \max_{\Gamma_T} u^+ \geq \max_{\overline{\Omega_T}} u.$$

Thus,

$$\max_{\overline{\Omega_T}} |u| = \max_{\Gamma_T} |u|.$$

To conclude one proceedes similarly when

$$\max_{\overline{\Omega_T}} |u| = -\min_{\overline{\Omega_T}} u.$$

□

One of the first applications of the above weak maximum principle is the uniqueness of the classical solution to the parabolic problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + g(t,x), & t\in(0,T], \quad x\in\Omega, \\ u(t,x) = h(t,x), & t\in(0,T], \quad x\in\partial\Omega, \\ u(0,x) = f(x), & \qquad\qquad\;\; x\in\Omega, \end{cases} \tag{1.2}$$

where $g\in C(\Omega_T)$, $h\in C((0,T]\times\partial\Omega)$ and $f\in C(\overline{\Omega})$. Here by classical solution we mean a function $u\in C^{1,2}(\Omega_T)\cap C(\overline{\Omega_T})$ satisfying (1.2).

Corollary 1.1.3. *Let $g\in C(\Omega_T)$, $h\in C((0,T]\times\partial\Omega)$ and $f\in C(\overline{\Omega})$. Then there exists at most one classical solution to* (1.2)*.*

Proof. Since c is bounded, there is a constant $\alpha\in\mathbb{R}$ such $c\le\alpha$ on Ω. Thanks to the linearity of $\mathcal{A}$, it suffices to prove that if u is a classical solution to (1.2) with $g\equiv 0$, $h\equiv 0$ and $f\equiv 0$, then $u\equiv 0$. Set $v=ue^{-\alpha t}$. Then.

$$D_t v = \mathcal{A}v - \alpha v.$$

So, since $c-\alpha\le 0$, we can apply Theorem 1.1.2 to obtain

$$\max_{\overline{\Omega_T}}|v| = \max_{\Gamma_T}|v| = 0.$$

Thus, $u\equiv 0$. □

Another application of the weak maximum principle concerns the asymptotic behaviour of the solution to the Dirichlet parabolic problem (1.2) when $g\equiv 0$ and $h\equiv 0$.

Corollary 1.1.4. *Assume that there is $\lambda_0\in\mathbb{R}$ such that $c\le-\lambda_0$ on Ω. If u is a classical solution to* (1.2) *with $g\equiv 0$ and $h\equiv 0$, then*

$$|u(t,x)|\le e^{-\lambda_0 t}\|f\|_\infty, \quad (t,x)\in\Omega_T.$$

Proof. Take the function $v(t,x):=e^{\lambda_0 t}u(t,x)$. Then v is a classical solution to

$$\begin{cases} D_t v(t,x) = (\mathcal{A}+\lambda_0)v(t,x), & t\in(0,T], \quad x\in\Omega, \\ v(t,x) = 0, & t\in(0,T], \quad x\in\partial\Omega, \\ v(0,x) = f(x), & \qquad\qquad\;\; x\in\Omega. \end{cases}$$

Since $c+\lambda_0\le 0$, it follows from Theorem 1.1.2 that $|v(t,x)|\le\max_{x\in\overline{\Omega}}|f(x)|$ which means that

$$|u(t,x)|\le e^{-\lambda_0 t}\|f\|_\infty, \quad (t,x)\in\Omega_T.$$

□

The comparison principle (which implies the positivity) can be deduced from the weak maximum principle.

Corollary 1.1.5 (Comparison principle). *Assume that* $f : \Omega_T \times \mathbb{R} \to \mathbb{R}$ *is a continuous function of the variables* t, x *and* y *which satisfies the following one-sided uniform Lipschitz condition in* y*:*

$$f(t,x,y) - f(t,x,z) \le \beta(y-z), \quad (t,x) \in \Omega_T,\ y,\ z \in \mathbb{R},$$

for some $\beta \in \mathbb{R}$*. If* u, $v \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$ *satisfy* $D_t u \le \mathcal{A}u + f(t,x,u)$ *and* $D_t v \ge \mathcal{A}v + f(t,x,v)$ *in* Ω_T*, and* $u \le v$ *in* Γ_T*, then* $u \le v$ *in* Ω_T*.*

Proof. From the assumptions we have

$$D_t(u-v) \le \mathcal{A}(u-v) + f(t,x,u) - f(t,x,v) \le \mathcal{A}(u-v) + \beta(u-v).$$

As in the proofs of Corollary 1.1.3 and Corollary 1.1.4, we introduce the function

$$w(t,x) := e^{-(\|c\|_\infty+\beta)t}(u(t,x) - v(t,x)), \quad (t,x) \in \overline{\Omega_T}.$$

Then, $w \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$ and it satisfies

$$D_t w \le \mathcal{A}w - \|c\|_\infty w.$$

Hence, since $c - \|c\|_\infty \le 0$, it follows from Theorem 1.1.2 and the assumption $(u-v)^+ = 0$ in Γ_T that

$$\max_{\overline{\Omega_T}} w \le \max_{\Gamma_T} w^+ = 0.$$

Thus, $u \le v$ in Ω_T. This proves the claim. $\square$

The positivity of solutions can be also deduced.

Remark 1.1.6. Applying Corollary 1.1.5 with $f \equiv 0$, $\beta = 0$ and $u \equiv 0$, we deduce that if $v \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies $D_t v - \mathcal{A}v \ge 0$ in Ω_T and $v \ge 0$ in Γ_T, then $v \ge 0$ in Ω_T.

1.2 The strong maximum principle for the heat equation

In this section we prove the strong maximum principle for the heat operator $D_t - \Delta$.

To this purpose let us consider the *Gauss–Weierstrass* kernel

$$K(t,|x|) := (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0,\ x \in \mathbb{R}^d.$$

We start with the following auxiliary result.

Lemma 1.2.1. *Let* $\Omega = B_r(x_0)$*, the open ball of centre* x_0 *and radius* r *in* $\mathbb{R}^d$*, and let* $u \in C^{1,2}(\Omega_T) \cap C((0,T] \times \overline{\Omega})$ *be such that* $D_t u - \Delta u \ge 0$ *in* Ω_T*,* $u \ge 0$ *in* Ω_T*. Then*

$$\int_\Omega \rho(t, x-x_0) u(t,x)\,dx \le u(T,x_0), \quad \text{if } 0 < T-t \le \frac{r^2}{2d}, \tag{1.3}$$

where

$$\rho(t,x) := K(T-t,|x|) - K(T-t,r), \quad t \in (0,T),\ x \in \Omega.$$

Proof. We assume without loss of generality that $x_0 = 0$. For any $\varphi \in C(\Omega)$ we have, by a change of variables, that

$$\begin{aligned}\int_\Omega K(T-t,|x|)\varphi(x)\,dx &= (4\pi)^{-\frac{d}{2}}\int_{B_{r/\sqrt{T-t}}(0)} e^{-\frac{|y|^2}{4}}\varphi(y\sqrt{T-t})\,dy\\ &= (4\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\chi_{B_{r/\sqrt{T-t}}(0)}(y)e^{-\frac{|y|^2}{4}}\varphi(y\sqrt{T-t})\,dy,\end{aligned}$$

where χ_A denotes the characteristic function of the set $A \subseteq \mathbb{R}^d$. So, by the dominated convergence theorem, we have

$$\lim_{t\to T}\int_\Omega K(T-t,|x|)\varphi(x)\,dx = \varphi(0). \tag{1.4}$$

Since $u(t,\cdot)$ converges to $u(T,\cdot)$ when $t \to T$, and $\lim_{t\to T} K(T-t,r) = 0$, it follows by (1.4) that

$$\lim_{t\to T}\int_\Omega \rho(t,x)u(t,x)\,dx = u(T,0).$$

On the other hand, we observe that $\rho \in C^\infty([0,T)\times\overline{\Omega})$, $\rho(t,x) > 0$ in $[0,T)\times\Omega$, and $\rho(t,x) = 0$ in $\partial\Omega\times[0,T)$. Moreover, one can see that $(D_t+\Delta)K(T-t,|x|) = 0$ (K is the fundamental solution of the heat equation), and

$$(D_t+\Delta)\rho(t,x) = [r^2 - 2d(T-t)]\frac{K(T-t,r)}{4(T-t)^2} \geq 0$$

if $T-t \leq \frac{r^2}{2d}$.
Thus, if $0 < T-t \leq \frac{r^2}{2d}$ we have

$$D_t\int_\Omega \rho(t,x)u(t,x)\,dx = \int_\Omega (uD_t\rho + \rho D_t u)\,dx \geq \int_\Omega(\rho\Delta u - u\Delta\rho)\,dx,$$

where we have used the positivity of u and ρ, and the inequalities $D_t\rho \geq -\Delta\rho$, $D_t u \geq \Delta u$. We recall Green's formula

$$\int_\Omega(\rho\Delta u - u\Delta\rho)\,dx = \int_{\partial\Omega}\left(\rho\frac{\partial u}{\partial\nu} - u\frac{\partial\rho}{\partial\nu}\right)d\sigma.$$

Taking in account that $\rho \equiv 0$ in $\partial\Omega$, $u \geq 0$ and $\frac{\partial\rho}{\partial\nu} \leq 0$, we deduce that

$$D_t\int_\Omega \rho(t,x)u(t,x)\,dx \geq \int_{\partial\Omega}\left(\rho\frac{\partial u}{\partial\nu} - u\frac{\partial\rho}{\partial\nu}\right)d\sigma = -\int_{\partial\Omega} u\frac{\partial\rho}{\partial\nu}d\sigma \geq 0.$$

Therefore, the function $\int_\Omega \rho(\cdot,x)u(\cdot,x)\,dx$ is nondecreasing if $0 < T-t \leq \frac{r^2}{2d}$. This implies that

$$u(T,0) = \lim_{t\to T}\int_\Omega \rho(t,x)u(t,x)\,dx \geq \int_\Omega \rho(t,x)u(t,x)\,dx$$

for any t such that $0 < T-t \leq \frac{r^2}{2d}$. □

Corollary 1.2.2. *Assume the same assumptions as in the above lemma. If* $u \in C(\overline{\Omega_T})$ *and* $T \leq \frac{r^2}{2d}$, *then*

$$\int_\Omega (r^2 - |x - x_0|^2) u(0,x)\, dx \leq \frac{4T}{K(T,r)} u(T, x_0).$$

Proof. Applying the above lemma with $t = 0$ we obtain

$$\int_\Omega \rho(0, x - x_0) u(0,x)\, dx \leq u(T, x_0).$$

This implies the claim, since

$$\begin{aligned} \rho(0,x) &= (4\pi T)^{-\frac{d}{2}} \left(e^{-\frac{|x|^2}{4T}} - e^{-\frac{r^2}{4T}} \right) = K(T,r) \left(e^{\frac{r^2 - |x|^2}{4T}} - 1 \right) \\ &\geq \frac{K(T,r)}{4T} (r^2 - |x|^2). \end{aligned}$$

□

We are now ready to prove the strong maximum principle for the heat equation.

Theorem 1.2.3. *Let* Ω *be a bounded, open and connected subset of* $\mathbb{R}^d$. *Let* $u \in C^{1,2}(\Omega_T)$ *be such that* $D_t u - \Delta u \leq 0$ *(resp.* $D_t u - \Delta u \geq 0$*) in* Ω_T. *If there exists a point* $x_0 \in \Omega$ *such that* $u(T, x_0) = \sup_{\Omega_T} u$ *(resp.* $u(T,x_0) = \inf_{\Omega_T} u$*), then* u *is constant on* Ω_T.

Proof. Set

$$v(t,x) := \sup_{\Omega_T} u - u(t,x), \quad (t,x) \in \Omega_T.$$

Then $v \in C^{1,2}(\Omega_T)$, $v \geq 0$ and $v_t - \Delta v \geq 0$ in Ω_T. Let $(t_1, x_1) \in \Omega_T$ be such that $v(t_1, x_1) = 0$, and let

$$E_{t_1} := \{x \in \Omega : v(t_1, x) = 0\}.$$

It is clear that $E_{t_1} \neq \emptyset$ and E_{t_1} is closed. We prove that E_{t_1} is also open. To this purpose fix $x \in E_{t_1}$ and $B_r(x) \subset \Omega$. Then $v \in C^{1,2}((0,t_1] \times B_r(x)) \cap C((0,t_1] \times \overline{B_r(x)})$ and, by (1.3), we have

$$0 \leq \int_{B_r(x)} \rho(t, y - x) v(t,y)\, dy \leq v(t_1, x) = 0 \quad \text{if } 0 < t_1 - t \leq \frac{r^2}{2d}.$$

Hence,

$$\rho(t, y - x) v(t,y) = 0, \text{ if } y \in B_r(x) \text{ and } 0 < t_1 - t \leq \frac{r^2}{2d}.$$

Thus, since $\rho(t, \cdot) > 0$ in $B_r(x)$, it follows that $v(t,y) = 0$ for any $y \in B_r(x)$ and $0 < t_1 - t \leq \frac{r^2}{2d}$. Using the continuity of v we deduce that $v(t_1, y) = 0$ in $B_r(x)$, which implies that $B_r(x) \subset E_{t_1}$. So, since Ω is connected, we have $E_{t_1} = \Omega$.
Now, since by assumption $v(T, x_0) = 0$, the same argument as above shows that $v(t,x) = 0$

for any $x \in B_{r_0}(x_0)$, $0 < T - t \le \frac{r_0^2}{2d}$ and some $r_0 > 0$. In particular, $v(t, x_0) = 0$ for any $0 < T - t \le \frac{r_0^2}{2d}$. Iterating k-times this procedure, with k is such that $\frac{kr_0^2}{2d} \ge T$, one obtains $v(t, x_0) = 0$ for any $t \in (0, T]$. This implies that $E_t \ne \emptyset$ for any $t \in (0, T]$ and hence $E_t = \Omega$ for any $t \in (0, T]$. This ends the proof of the strong maximum principle for the heat equation. □

Remark 1.2.4. If $u(t_0, x_0) = \sup_{\Omega_T} u$ (or $u(t_0, x_0) = \inf_{\Omega_T} u$) for some $(t_0, x_0) \in \Omega_T$, then we can only conclude that u is constant in $(0, t_0] \times \Omega$ even in the case when $D_t u - \Delta u = 0$ and $u \in C(\overline{\Omega_T})$. For example, assume that $u \in C^{1,2}((0, T] \times (0, 1)) \cap C([0, T] \times [0, 1])$ solves the heat equation

$$\begin{cases} D_t u(t,x) = \Delta u(t,x), & t \in (0,T], \quad x \in (0,1), \\ u(t,1) = \beta(t), & t \in (0,T], \\ u(t,0) = \alpha(t), & t \in (0,T], \\ u(0,x) = 0, & \qquad\qquad\quad x \in (0,1). \end{cases}$$

Assume that $\alpha(t) = \beta(t) = 0$ in $(0, t_0]$ for some $t_0 \in (0, T)$ and $\alpha(t)$, $\beta(t) > 0$ in $(t_0, T]$. Then $u(t, x) = 0$ in $(0, t_0] \times (0, 1)$, by Corollary 1.1.3. We claim that $u(t, x) > 0$ for any $x \in (0, 1)$ and $t \in (t_0, T]$.
In fact, by the weak maximum principle, we know that $u(t, x) \ge 0$ in Ω_T. If there exists $(t_1, x_1) \in (t_0, T] \times (0, 1)$ such that $u(t_1, x_1) = 0\,(= \inf_{\Omega_T} u)$, then by the strong maximum principle we have $u(t, x) = 0$ for any $(t, x) \in (0, t_1] \times (0, 1)$ and this contradicts the fact that $u(t, 0) = \alpha(t) > 0$ in $(t_0, T]$.
We will prove later that a classical solution to the above heat equation exists.

Remark 1.2.5. The strong maximum principle is also satisfied for the more general parabolic operator $D_t - \mathcal{A}$, but the proof is very technical and based on Harnack's inequality for uniformly parabolic operators. It says that if Ω is a bounded, open and connected domain, and $u \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies $D_t u - \mathcal{A}u \le 0$ (resp. $D_t u - \mathcal{A}u \ge 0$), then

- If $c \equiv 0$ in Ω, and there exists $(t, x) \in \Omega_T$ such that $u(t, x) = \max_{\overline{\Omega_T}} u$ (resp. $u(t, x) = \min_{\overline{\Omega_T}} u$), then u is constant on $\Omega_t = (0, t] \times \Omega$.

- If $c \le 0$ in Ω, and there exists $(t, x) \in \Omega_T$ such that $0 \le u(t, x) = \max_{\overline{\Omega_T}} u$ (resp. $u(t, x) = \min_{\overline{\Omega_T}} u \le 0$), then u is constant on Ω_t.

1.3 Notes

For the weak maximum principle we refer to [1] and the proof of the strong maximum principle presented in this lecture is taken from [2]. For the heat equation another proof of the weak maximum principle can be found in Exercise 1.4.5. We also refer the reader to the classical monographs [3] and [4].

1.4 Exercises

1. Let $B = (b_{ij})_{1\le i,j\le d}$, $C = (c_{ij})_{1\le i,j\le d}$ be two real and symmetric matrices. Assume that B is positive semi-definite and C is negative semi-definite. Prove that
$$Tr(BC) = \sum_{i,j=1}^{d} b_{ij}c_{ij} \le 0.$$

2. Consider the function $u(t,x) := 1 - x^2 - 2t$.
   (a) Verify that u is a solution to the heat equation $D_t u(t,x) = \Delta u(t,x)$.
   (b) Find the minimum and the maximum of u on the closed rectangle $\overline{\Omega_T} := [0,T] \times [0,1]$ for a fixed $T > 0$ without using the maximum principle.
   (c) Find the minimum and the maximum of u on $\overline{\Omega_T}$ by using the weak maximum principle.

3. Let $u \in C^{1,2}((0,+\infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})) \cap C([0,+\infty) \times [-\frac{\pi}{2}, \frac{\pi}{2}])$ satisfy the inequality $D_t u(x) - \Delta u(x) \ge \cos x$ on $(0,+\infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover assume that: $u(t, -\frac{\pi}{2}) \ge 0$, $u(t, \frac{\pi}{2}) \ge 0$ for all $t > 0$ and $u(0,x) \ge 2\cos x$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Show that
$$u(t,x) \ge (1 + e^{-t})\cos x \ \text{ on } \ [0,+\infty) \times [-\frac{\pi}{2}, \frac{\pi}{2}].$$

4. Let $u \in C^{1,2}((0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$ be a solution to the heat equation
$$\begin{cases} D_t u(t,x) = \Delta u(t,x), & (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(0,x) = g(x), & x \in \mathbb{R}^d \end{cases}$$
satisfying
$$u(t,x) \le M e^{a|x|^2}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d$$
for some constants M, $a \ge 0$.
   (a) Assume first that $4aT < 1$ which implies that, there exists $\varepsilon > 0$ such that $4a(T+\varepsilon) < 1$. Fix $\nu > 0$ and consider the function
$$v(t,x) = u(t,x) - \frac{\nu}{(T+\varepsilon-t)^{d/2}} \exp\left(\frac{|x|^2}{4(T+\varepsilon-t)}\right), \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$
      (i) Prove that v solves $D_t v - \Delta v = 0$ on $(0,T) \times \mathbb{R}^d$ and $v \in C([0,T] \times \mathbb{R}^d)$.
      (ii) By applying the weak maximum principle to the function v on the cylinder $[0,T] \times B_r(0)$, show that
$$\max_{[0,T]\times\overline{B_r(0)}} v \le \sup_{\mathbb{R}^d} g$$
for sufficiently large r.

(iii) By letting $\nu \to 0$, deduce that

$$\sup_{[0,T]\times\mathbb{R}^d} u = \sup_{\mathbb{R}^d} g. \tag{1.5}$$

(b) Prove (1.5) without the assumption $4aT < 1$.

5. Assume that $u_0 \in C(\overline{\Omega})$ and $u \in C^{1,2}((0,T)\times\Omega) \cap C([0,T]\times\overline{\Omega})$ is a solution to the heat equation with Dirichlet boundary conditions

$$\begin{cases} D_t u(t,x) = \Delta u(t,x), & (t,x) \in (0,T]\times\Omega, \\ u(t,x) = 0, & (t,x) \in (0,T]\times\partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases}$$

Consider the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(s) := \begin{cases} 1 - e^{-s^2}, & s \geq 0, \\ 0, & s < 0. \end{cases}$$

Set

$$H(t) := \int_0^t \Phi(s)\,ds,\ \varphi(t) := \int_\Omega H(u(t,x) - K)\,dx \ \text{ and } \ K := \max\left(\sup_{x\in\Omega} u_0(x), 0\right).$$

(a) Prove that Φ is a C^1-function, increasing and its derivative is bounded by 1.

(b) Prove that $\varphi(0) = 0$, $\varphi \geq 0$ on $[0,T]$ and $\varphi \in C^1((0,T],\mathbb{R}) \cap C([0,T],\mathbb{R})$. Compute φ' and deduce that

$$u(t,x) \leq K, \quad \forall (t,x) \in [0,T]\times\overline{\Omega}.$$

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Lecture 2

Semigroups of bounded operators. Part I. Strongly continuous semigroups

In this and in the next lecture, we introduce the concept of semigroup of bounded linear operators and study two important classes of semigroups: the strongly continuous and the analytic semigroups. Semigroups of bounded operators naturally arise in the study of parabolic equations: as we will see a suitable semigroup governs the dynamics of parabolic equations. Even if we will not enter into details, this being beyond the purpose of this edition of the Internet Seminar, we stress that semigroups have shown to be a very powerful tool in the study of linear (but also nonlinear) parabolic equations.

The concept of semigroups of bounded operators generalizes what is known since the first courses of Calculus: the solutions of the system $D_t u = Au$ of n ordinary differential equations with constant coefficients are given by $u(t) = e^{tA}c$ for any $t \in \mathbb{R}$, where $c \in \mathbb{R}^d$ is an arbitrary vector and

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \qquad t \in \mathbb{R}. \tag{2.1}$$

The previous formula can be straightforwardly extended to the case when the matrix A is replaced by a bounded operator in a Banach space X. Indeed, the series in the right-hand side of (2.1) converges locally uniformly in $\mathbb{R}$. Since

$$\bigg\| \sum_{n=m}^{m+p} \frac{t^n}{n!} A^n \bigg\|_{L(X)} \le \sum_{n=m}^{m+p} \frac{t^n}{n!} \|A\|_{L(X)}^n$$

and the real valued series $\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A\|_{L(X)}^n$ converges locally uniformly in $\mathbb{R}$ (to $e^{t\|A\|_{L(X)}}$), the series $\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ converges in $L(X)$, locally uniformly with respect to $t \in \mathbb{R}$. Set $a_k = t^k A^k / k!$ and $b_k = s^k A^k / k!$ for any $k \in \mathbb{N} \cup \{0\}$. Then,

$$\sum_{k=0}^{n} a_k b_{n-k} = A^n \sum_{k=0}^{n} \frac{t^k s^{n-k}}{k!(n-k)!} = \frac{(t+s)^n}{n!} A^n, \qquad n \in \mathbb{N} \cup \{0\}.$$

Hence, as for the Cauchy product of scalar series, one can see that

$$e^{tA}e^{sA} = \sum_{i=0}^{\infty}\frac{t^iA^i}{i!}\cdot\sum_{j=0}^{\infty}\frac{s^jA^j}{j!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}a_kb_{n-k}$$
$$= \sum_{n=0}^{\infty}\frac{(t+s)^n}{n!}A^n = e^{(t+s)A} = e^{sA}e^{tA}.$$

Based on the above remarks, we can now give the following definition.

Definition 2.0.1. A family $\{T(t) : t \geq 0\}$ of bounded linear operators on a Banach space X is called a semigroup of bounded operators if it satisfies the semigroup property, i.e., $T(0) = I$ and $T(t + s) = T(t)T(s)$ for every $s, t > 0$.

It follows that, for each $A \in L(X)$, $\{e^{tA}\}$ is a semigroup of bounded operators on the Banach space X.

As a matter of fact, this class of semigroups, usually referred as *uniformly continuous semigroups*, is too small and they are not associated to parabolic equations. For this reason, we need to go further in the study of semigroups.

Throughout this lecture, X will denote a complex Banach space, $\|\cdot\|$ its norm and $L(X)$ is the space of all bounded linear operators on X.

2.1 Strongly continuous semigroups

Definition 2.1.1. A family $\{T(t) : t \geq 0\}$ of bounded operators on X, which satisfies the semigroup property, is a strongly continuous semigroup (or C_0-semigroup[1]) if the function $t \mapsto T(t)x$ is continuous in $[0, \infty)$ with values in X, for each $x \in X$.

The following example is crucial and we will use it in the next lecture to show the main differences between analytic and C_0-semigroups.

Example 2.1.2. On $X = BUC(\mathbb{R})$, the space of bounded and uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$, endowed with the sup-norm, consider the family $\{T(t)\}$ of linear operators defined by $(T(t)f)(x) = f(x + t)$ for each $x \in \mathbb{R}$, $t \geq 0$ and $f \in BUC(\mathbb{R})$. This is a C_0-semigroup on X. Indeed, $T(t)f$ tends to f uniformly in $\mathbb{R}$ as $t \to 0^+$ if and only if $\sup_{x\in\mathbb{R}} |f(x + t) - f(x)|$ vanishes as $t \to 0^+$. But this condition, is a rewriting of the definition of uniform continuity. The semigroup property is straightforward to prove.

We stress that this semigroup, called the semigroup of the left translations, cannot be written in the form (2.1) for some bounded operator A. Indeed, if this were the case, then

$$\|e^{tA} - I\|_{L(X)} \leq \sum_{n=1}^{\infty}\frac{t^n}{n!}\|A\|^n_{L(X)} = e^{t\|A\|_{L(X)}} - 1, \qquad t \geq 0,$$

---

[1] C_0 or $(C, 0)$ abbreviates Cesàro summable of order zero, which means the continuity property $\lim_{t\to 0} T(t)x = x$ for every $x \in X$.

and, consequently, $\lim_{t\to 0^+}\|e^{tA}-I\|_{L(X)}=0$. The semigroup of left translations does not satisfy this property since $\|T(t)-I\|_{L(X)}=2$ for every $t>0$. To check this claim, fix $t>0$ and consider the bounded uniformly continuous function $f_t:\mathbb{R}\to\mathbb{R}$, defined by $f_t(x)=\sin(\pi x/t)$ for each $x\in\mathbb{R}$. As it is immediately seen, $\|f_t\|_\infty=1$ and $f_t(x+t)=-f_t(x)$ for every $x\in\mathbb{R}$. Hence,

$$\|T(t)-I\|_{L(X)}\geq\sup_{x\in\mathbb{R}}|f_t(x+t)-f_t(x)|=2\sup_{x\in\mathbb{R}}|f_t(x)|=2.$$

On the other hand, $\|T(t)f\|_\infty=\|f\|_\infty$ for every $f\in BUC(\mathbb{R})$ and, consequently, $\|T(t)-I\|_{L(X)}\leq\|T(t)\|_{L(X)}+1=2$ for every $t\geq 0$.

Actually, $\{T(t)\}$ is a group of bounded operators, since $T(t)$ can be defined, using the same rule, also for negative values of t.

2.1.1 Basic properties

Throughout this subsection by $\{T(t)\}$ we denote a C_0-semigroup on X, without further mentioning it.

To begin with, we prove that the function $t\mapsto\|T(t)\|_{L(X)}$ grows at most exponentially at infinity.

Proposition 2.1.3. *There exist $M\geq 1$ and $\omega\in\mathbb{R}$ such that $\|T(t)\|_{L(X)}\leq Me^{\omega t}$ for every $t\geq 0$.*

Proof. The core of the proof consists in showing that there exists $\delta>0$ such that

$$\sup_{t\in[0,\delta]}\|T(t)\|_{L(X)}<\infty. \tag{2.2}$$

Once this property is established, the semigroup property allows us to complete the proof. Indeed, if $t>\delta$, then there exist $n\in\mathbb{N}$ and $r\in[0,\delta)$ such that $t=n\delta+r$. By applying the semigroup property, we conclude that $T(t)=T(r)(T(\delta))^n$. Hence, denoting by M the supremum in (2.2) and observing that $M\geq 1$, we get

$$\|T(t)\|_{L(X)}\leq\|T(r)\|_{L(X)}\|T(\delta)\|_{L(X)}^n\leq M^{n+1}=M\exp(n\log(M))\leq Me^{\frac{\log(M)}{\delta}t}$$

and the assertion follows with $\omega=\delta^{-1}\log(M)$.

To prove (2.2), we argue by contradiction. If (2.2) does not hold true, then we can find out a positive infinitesimal sequence (t_n) such that $\|T(t_n)\|_{L(X)}$ tends to ∞ as $n\to\infty$. Since $T(t_n)x$ converges to x as $n\to\infty$, for every $x\in X$, the uniform boundedness principle leads us to a contradiction. ☐

In the proof of Proposition 2.1.3 the constant ω is nonnegative. Actually, ω can be also negative. This is, for instance, the case of the C_0-semigroup $\{T_a(t)\}$ on $\mathbb{R}$, defined by $T_a(t)x=e^{-at}x$ for each $x\in\mathbb{R}$ and $t\geq 0$, where a is a positive number.

In view of Proposition 2.1.3 the following definition makes sense.

Definition 2.1.4. The growth bound ω_0 of a C_0-semigroup $\{T(t)\}$ is defined by $\omega_0 := \inf\{\omega \in \mathbb{R} : \exists\, M = M_\omega \geq 1 \text{ such that } \|T(t)\| \leq Me^{\omega t} \text{ for every } t \geq 0\}$.

Remark 2.1.5. (i) We stress that ω_0 could be also equal to $-\infty$ (see Exercise 2.3.4).

(ii) In general, ω_0 is just an infimum and not a minimum. Consider for instance, the semigroup (actually, a group) $\{T(t)\}$ in $\mathbb{R}^2$, defined by

$$T(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad t \geq 0.$$

Here, $\omega_0 = 0$, but clearly the function $t \mapsto T(t)$ is not bounded in $[0, \infty)$ with values in $L(\mathbb{R}^2)$.

2.1.2 The infinitesimal generator

In this subsection, we show that to any C_0-semigroup it is possible to associate a linear operator, called the *infinitesimal generator* of the semigroup, and we study the main properties of this operator.

Definition 2.1.6. The infinitesimal generator A of a C_0-semigroup $\{T(t)\}$ is the operator defined as follows:

$$\begin{cases} D(A) = \left\{x \in X : \exists \displaystyle\lim_{t \to 0^+} \frac{T(t)x - x}{t} \in X\right\}, \\ Ax = \lim_{t \to 0^+} \dfrac{T(t)x - x}{t}, \quad x \in D(A). \end{cases}$$

In the following proposition we exploit the main basic properties of the infinitesimal generator.

Proposition 2.1.7. *The generator A of a C_0-semigroup $\{T(t)\}$ satisfies the following properties.*

(i) *A is a linear operator satisfying $AT(t) = T(t)A$ on $D(A)$ for every $t \geq 0$. Moreover, for each $x \in D(A)$, the function $u = T(\cdot)x$ belongs to $C^1([0, \infty); X) \cap C([0, \infty); D(A))$ and solves the Cauchy problem*

$$\begin{cases} u'(t) = Au(t), \quad t \geq 0, \\ u(0) = x. \end{cases} \tag{2.3}$$

(ii) *For each $t > 0$ and $x \in X$, $\int_0^t T(s)x\,ds \in D(A)$ and*

$$T(t)x - x = A\int_0^t T(s)xds. \tag{2.4}$$

In particular, if $x \in D(A)$, then

$$A\int_0^t T(s)x\,ds = \int_0^t T(s)Ax\,ds. \tag{2.5}$$

(iii) *A is closed and $D(A)$ is dense in X.*

(iv) *The operator A completely characterizes the semigroups $\{T(t)\}$ in the sense that there exist no other semigroups which admit A as infinitesimal generator.*

Proof. (i) The linearity of the operator A is straightforward to check. Note that $D(A) \neq \varnothing$ since $0 \in D(A)$. To complete the proof of property (i), we fix $x \in D(A)$ and $t > 0$. Using the semigroup property and the continuity of the operator $T(t)$, we get

$$\lim_{h\to 0^+} \frac{T(h)T(t)x - T(t)x}{h} = \lim_{h\to 0^+} T(t)\frac{T(h)x-x}{h} = T(t)\lim_{h\to 0^+}\frac{T(h)x-x}{h} = T(t)Ax.$$

Hence, $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$. Since $T(h)T(t) = T(t+h)$, the above computation shows that the function $T(\cdot)x$ is differentiable from the right in $[0,\infty)$ and the right derivative coincides with the function $AT(\cdot)x$. To prove that the previous function is also differentiable from the left in $(0,\infty)$, we fix $t > 0$ and observe that

$$\begin{aligned}\frac{T(t+h)x - T(t)x}{h} - T(t)Ax =& T(t+h)\frac{T(-h)x-x}{-h} - T(t)Ax\\ =& T(t+h)\left(\frac{T(-h)x-x}{-h} - Ax\right) + T(t+h)Ax - T(t)Ax.\end{aligned}$$

Hence, taking Proposition 2.1.3 into account, we obtain

$$\begin{aligned}&\left\|\frac{T(t+h)x-T(t)x}{h} - T(t)Ax\right\|\\ \leq& \|T(t+h)\|_{L(X)}\left\|\frac{T(-h)x-x}{-h} - Ax\right\| + \|T(t+h)Ax - T(t)Ax\|\\ \leq& Me^{\omega(t+h)}\left\|\frac{T(-h)x-x}{-h} - Ax\right\| + \|T(t+h)Ax - T(t)Ax\|.\end{aligned}$$

Letting h tend to 0^- we conclude that the function $T(\cdot)x$ is differentiable from the left at t. Hence, the function $T(\cdot)x$ is differentiable is $[0,\infty)$ and since its derivative is the function $T(\cdot)Ax$, which is continuous in $[0,\infty)$, it follows that $T(\cdot)x \in C^1([0,\infty);X) \cap C([0,\infty);D(A))$ and solves problem (2.3).

(ii) Fix $x \in X$, $t > 0$, set $y = \int_0^t T(s)x\,ds$ and observe that

$$\begin{aligned}\frac{T(h)y-y}{h} &= \frac{1}{h}\left(T(h)\int_0^t T(s)x\,ds - \int_0^t T(s)x\,ds\right) = \frac{1}{h}\left(\int_0^t T(h+s)x\,ds - \int_0^t T(s)x\,ds\right)\\ &= \frac{1}{h}\left(\int_h^{t+h} T(s)x\,ds - \int_0^t T(s)x\,ds\right) = \frac{1}{h}\left(\int_t^{t+h} T(s)x\,ds - \int_0^h T(s)x\,ds\right).\end{aligned}$$

Taking the limit as h tend to 0^+ gives

$$\lim_{h\to 0^+}\frac{T(h)y-y}{h} = T(t)x - x.$$

Hence, $y \in D(A)$ and (2.4) follows. In the particular case when $x \in D(A)$, by property (i), we know that $T(s)Ax = AT(s)x = D_sT(s)x$. Hence, formula (2.5) follows from the fundamental theorem of calculus and (2.4).

(iii) Let $(x_n) \subset D(A)$ be a sequence converging to some $x \in X$ and such that Ax_n converges to some $y \in X$ as n tends to ∞. By property (ii), we know that

$$\frac{T(h)x_n - x_n}{h} = \frac{1}{h}\int_0^h T(s)Ax_n\,ds, \qquad n \in \mathbb{N},\ h > 0. \tag{2.6}$$

Since $\|T(s)Ax_n - T(s)y\| \le Me^{\omega s}\|Ax_n - y\| \le Me^{\omega^+ h}\|Ax_n - y\|$ for every $s \in [0,h]$, $T(\cdot)Ax_n$ converges to $T(\cdot)y$, uniformly in $[0,h]$. Hence, letting n tend to ∞ in both the sides of (2.6), we conclude that

$$\frac{T(h)x - x}{h} = \frac{1}{h}\int_0^h T(s)y\,ds, \qquad h > 0.$$

Since the function $T(\cdot)y$ is continuous in $[0,\infty)$, letting h tend to 0^+ it follows that $x \in D(A)$ and $Ax = y$. Hence, A is a closed.

To prove that $D(A)$ is dense in X, we fix $x \in X$ and set $x_n = n\int_0^{1/n} T(s)xds$ for each $n \in \mathbb{N}$. By property (ii), $x_n \in D(A)$ for every $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty} x_n = x$.

(iv) Suppose that $\{S(t)\}$ is another C_0-semigroup having A as infinitesimal generator. Fix $x \in D(A)$, $t > 0$ and consider the function $u : [0,t] \to X$, defined by $u(s) := T(t-s)S(s)x$ for $s \in [0,t]$. We can see that u is differentiable in $[0,t]$ with identically vanishing derivative. So, this implies that $u(t) = u(0)$, i.e., $S(t)x = T(t)x$. Since $D(A)$ is dense in X and t is arbitrarily fixed in $(0,\infty)$, we conclude that $T(t) = S(t)$ for every $t > 0$. □

We now introduce the concepts of spectrum/resolvent set of a linear operator and of resolvent operator.

Definition 2.1.8. Let $A : D(A) \subset X \to X$ be a closed linear operator. The set $\sigma(A) := \{\lambda \in \mathbb{C} : \lambda I - A : D(A) \to X \text{ is not bijective}\}$ is called the spectrum of A. Its complement in $\mathbb{C}$ is called the resolvent set of A and will be denoted by $\rho(A)$.

For each $\lambda \in \rho(A)$, the operator $R(\lambda, A) := (\lambda I - A)^{-1}$ is called the resolvent of A at the point λ.

Remark 2.1.9. (i) The closed graph theorem shows that, if $\rho(A) \neq \varnothing$, then $R(\lambda, A)$ is a bounded operator in X for every $\lambda \in \rho(A)$.

(ii) We stress that the resolvent set of a closed operator may be empty. See Exercise 2.3.5.

Proposition 2.1.10. *The resolvent set of a closed operator $A : D(A) \subset X \to X$ is an open subset of $\mathbb{C}$. Moreover, the function $R(\lambda, A) : \rho(A) \to L(X)$ is holomorphic. Finally, the family of operators $\{R(\lambda, A) : \lambda \in \rho(A)\}$ satisfies the so-called resolvent identity, i.e.,*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A), \qquad \lambda, \mu \in \rho(A).$$

In particular, $R(\lambda, A)$ commutes with $R(\mu, A)$ for every $\lambda, \mu \in \rho(A)$.

In the proof of Proposition 2.1.10 we will use the following lemma, whose proof is left as an exercise.

Lemma 2.1.11 (Perturbation of the identity)**.** *Let X be a Banach space and $T \in L(X)$ with $\|T\|_{L(X)} < 1$. Then, the operator $I - T$ is invertible and its inverse is given by the Von Neumann series, i.e.,*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n. \tag{2.7}$$

Proof of Proposition 2.1.10. Fix $\lambda_0 \in \rho(A)$, $\lambda \in \mathbb{C}$ and observe that $\lambda I - A = (I - (\lambda_0 - \lambda)R(\lambda_0, A))(\lambda_0 I - A)$. Thus, $\lambda \in \rho(A)$ if and only if the operator $I - (\lambda_0 - \lambda)R(\lambda_0, A)$ is invertible and, by Lemma 2.1.11, this is the case if $\lambda \in B(\lambda_0, r)$, the open ball centered at λ_0 with radius $r = \|R(\lambda_0, A)\|_{L(X)}^{-1}$. Hence, $\rho(A)$ is open. Moreover, for each $\lambda \in B(\lambda_0, r)$,

$$R(\lambda, A) = R(\lambda_0, A)(I - (\lambda_0 - \lambda)R(\lambda_0, A))^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n (R(\lambda_0, A))^{n+1}, \tag{2.8}$$

and this shows that the function $R(\cdot, A)$ is holomorphic in $\rho(A)$ with values in $L(X)$.

The resolvent identity follows from the following chain of equalities:

$$R(\lambda, A) - R(\mu, A) = R(\mu, A)((\mu I - A) - (\lambda I - A))R(\lambda, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A)$$

for every $\lambda, \mu \in \rho(A)$. The last assertion is straightforward to prove. □

Now, we can go back to the infinitesimal generator of C_0-semigroups. In the following result, we show that the resolvent set of the infinitesimal generator A is not empty and, for each $\lambda \in \rho(A)$, the operator $R(\lambda, A)$ can be written in terms of the semigroup.

Proposition 2.1.12. *Let $\{T(t)\}$ be a C_0-semigroup with infinitesimal generator A and let $M \geq 1$ and $\omega \in \mathbb{R}$ be such that $\|T(t)\|_{L(X)} \leq M e^{\omega t}$ for every $t \geq 0$. Then, $\rho(A) \supset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \omega\}$ and*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \tag{2.9}$$

for every $x \in X$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$. Moreover,

$$\|(R(\lambda, A))^n\|_{L(X)} \leq \frac{M}{(\mathrm{Re}\,\lambda - \omega)^n}. \tag{2.10}$$

Proof. Fix $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$. Then, the operator defined by the right-hand side of (2.9) is well defined, linear and continuous in X, since $\|e^{-\lambda t}T(t)x\| \leq M e^{-(\mathrm{Re}\,\lambda - \omega)t}\|x\|$ for $t \geq 0$. In particular,

$$\left\| \int_0^\infty \mathrm{e}^{-\lambda t} T(t)x dt \right\| \leq M\|x\| \int_0^\infty e^{-(\mathrm{Re}\,\lambda - \omega)t} dt = \frac{M}{\mathrm{Re}\,\lambda - \omega}\|x\|, \qquad x \in X. \tag{2.11}$$

To prove that this operator, which we denote by R_λ, is the inverse of $\lambda I - A$, we observe that, since A is a closed operator and the function $t \mapsto e^{-\operatorname{Re}\lambda t}\|T(t)Ax\|$ is integrable in $[0,\infty)$ for every $x \in D(A)$, $R_\lambda x \in D(A)$ for any $x \in D(A)$ and, integrating by parts,

$$\begin{aligned}R_\lambda Ax &= \int_0^\infty e^{-\lambda t}T(t)Ax\,dt = \lim_{n\to\infty}\int_0^n e^{-\lambda t}D_tT(t)x\,dt\\ &= \lim_{n\to\infty}\left(e^{-\lambda n}T(n)x - x + \lambda\int_0^n e^{-\lambda t}T(t)x\,dt\right) = -x + \lambda R_\lambda x.\end{aligned}$$

Hence, $R_\lambda(\lambda I - A)x = x$ for every $x \in D(A)$. This shows that the operator $\lambda I - A$ is injective. To prove that it is also surjective, we fix $x \in X$ and prove that $R_\lambda x \in D(A)$ and $AR_\lambda x = \lambda R_\lambda - x$. For this purpose, we fix $h > 0$ and observe that

$$\begin{aligned}\frac{T(h)R_\lambda x - R_\lambda x}{h} =& \frac{1}{h}\bigg(\int_0^\infty e^{-\lambda t}T(t+h)x\,dt - \int_0^\infty e^{-\lambda t}T(t)x\,dt\bigg)\\ =& \frac{e^{\lambda h}}{h}\int_h^\infty e^{-\lambda s}T(s)\,ds - \frac{1}{h}\int_0^\infty e^{-\lambda t}T(t)x\,dt\\ =& \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda s}T(s)\,ds - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda s}T(s)x\,ds.\end{aligned}$$

Letting h tend to 0^+, we conclude that

$$\lim_{h\to 0^+}\frac{T(h)R_\lambda x - R_\lambda x}{h} = \lambda R_\lambda x - x,$$

as it has been claimed.

To complete the proof, let us prove (2.10) with $n > 1$, since the case $n = 1$ follows from (2.11). Fix $x \in X$. By Proposition 2.1.10, we know that the function $\lambda \mapsto R(\lambda, A)x$ is holomorphic in $\rho(A)$ and, applying the dominated convergence theorem to the formula (2.9), it can be easily checked that

$$\frac{d^n}{d\lambda^n}R(\lambda, A)x = \int_0^\infty (-1)^n t^n \mathrm{e}^{-\lambda t}T(t)x\,dt$$

for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$. Let us compute the derivatives of the function $R(\cdot, A)x$ in a different way, using formula (2.8). This formula shows that

$$\frac{d^n}{d\lambda^n}R(\lambda, A) = (-1)^n n!(R(\lambda, A))^{n+1}, \qquad \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > \omega.$$

From these last two formulas it follows that

$$(R(\lambda, A))^n x = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}T(t)x\,dt.$$

Hence,

$$\|(R(\lambda, A))^n x\| \le \frac{M}{(n-1)!}\|x\|\int_0^\infty t^{n-1}e^{-\operatorname{Re}\lambda t}e^{\omega t}dt = \frac{M}{(\operatorname{Re}\lambda - \omega)^n}\|x\|,$$

which completes the proof. □

Example 2.1.13. If $\{T(t)\}$ is given by (2.1), then it is clearly strongly continuous since the series defining the semigroup converges locally uniformly in $[0,\infty)$. Its infinitesimal generator is the operator A. Indeed,

$$\left\|\frac{T(t)-I}{t}-A\right\|_{L(X)}=\left\|\sum_{n=2}^{\infty}\frac{t^{n-1}}{n!}A^n\right\|_{L(X)}\le\sum_{n=2}^{\infty}\frac{t^{n-1}}{n!}\|A\|_{L(X)}^n\le t\sum_{n=0}^{\infty}\frac{1}{n!}\|A\|_{L(X)}^n=te^{\|A\|_{L(X)}}$$

for every $t\in(0,1]$. It follows that $\lim_{t\to 0^+}\|t^{-1}(T(t)-I)-A\|_{L(X)}=0$. As a byproduct, $t^{-1}(T(t)x-x)$ converges to Ax as $t\to 0^+$ for each $x\in X$.

Example 2.1.14. Let us go back to the semigroup of left translations and show that its infinitesimal generator A is the first order derivative with $BUC^1(\mathbb{R})$ as domain. Let $f\in D(A)$. Then, $t^{-1}(f(\cdot+t)-f)$ converges to Af in $BUC(\mathbb{R})$ as t tends to 0^+. In particular, for each $x\in\mathbb{R}$, the ratio $t^{-1}(f(x+t)-f(x))$ converges to $(Af)(x)$. It thus follows that f is differentiable in $\mathbb{R}$ and $f'=Af\in BUC(\mathbb{R})$. Hence, $f\in BUC^1(\mathbb{R})$.

Conversely, suppose that $f\in BUC^1(\mathbb{R})$. Then, by the fundamental theorem of calculus, we can write

$$\frac{f(x+t)-f(x)}{t}=\frac{1}{t}\int_0^t f'(x+s)ds,\qquad x\in\mathbb{R},\ t>0.$$

Fix $\varepsilon>0$ and let $\delta>0$ be such that $|f'(x_2)-f'(x_1)|\le\varepsilon$ for every $x_1,x_2\in\mathbb{R}$ such that $|x_2-x_1|\le\delta$. Then,

$$\left|\frac{f(x+t)-f(x)}{t}-f'(x)\right|\le\frac{1}{t}\int_0^t|f'(x+s)-f'(x)|ds\le\varepsilon$$

for every $x\in\mathbb{R}^d$ if $t\in(0,\delta]$. Hence, $t^{-1}(T(t)f-f)$ converges to f' uniformly in $\mathbb{R}^d$ as t tends to 0^+. This shows that $BUC^1(\mathbb{R})\subset D(A)$.

2.1.3 The Hille-Yosida theorem

By the results of the previous two subsections we know that to any C_0-semigroup $\{T(t)\}$ on X, we can associate an operator $A:D(A)\subset X\to X$, the infinitesimal generator, which satisfies the following properties:

(i) A is closed and densely defined;

(ii) $\rho(A)$ contains the right-halfplane $\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda>\omega_0\}$ for some $\omega_0\in\mathbb{R}$;

(iii) for each $\omega>\omega_0$ there exists a positive constant M such that $\|(R(\lambda,A))^n\|_{L(X)}\le M(\mathrm{Re}\lambda-\omega)^{-n}$ for every $n\in\mathbb{N}$ and $\lambda\in\mathbb{C}$ with $\mathrm{Re}\lambda>\omega$.

A natural question arises: is any linear operator $A:D(A)\subset X\to X$, which satisfies the above three properties, the infinitesimal generator of a C_0-semigroup on X? The answer is positive and given by the famous Hille-Yosida theorem

Theorem 2.1.15 (Hille-Yosida)**.** *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. Then the following properties are equivalent.*

(a) *A generates a C_0-semigroup $\{T(t)\}$ on X.*

(b) *A satisfies the above properties (i)-(iii).*

Proof. In view of the remarks at the very beginning of this subsection, we just need to show that (b)⇒(a). Without loss of generality we can assume that $\omega = 0$. Indeed, the operator $\widetilde{A} = A - \omega I$, satisfies properties (i)-(iii), with $\omega = 0$. If we prove that $\widetilde{A}$ generates a C_0-semigroup $\{S(t)\}$, then the operator A generates the C_0-semigroup $\{e^{\omega t}S(t)\}$.

Being rather long, we split the proof into steps.

Step 1. Here, we define the Yosida approximation of the operator A, i.e., the operators $A_n : X \to X$ defined by $A_n = nAR(n,A)$ for $n \in \mathbb{N}$. Since, $AR(n,A) = (A - nI)R(n,A) + nR(n,A) = nR(n,A) - I$, each operator A_n is bounded in X. We claim that $\lim_{n\to\infty} A_n x = Ax$ for each $x \in D(A)$. To prove the claim, it suffices to show that $nR(n,A)y$ tends to y for each $y \in X$. Indeed, $A_n x = nR(n,A)Ax$ for $x \in D(A)$. First, we suppose that $y \in D(A)$. Then, $nR(n,A)y = R(n,A)(ny - Ay + Ay) = y + R(n,A)Ay$ for every $n \in \mathbb{N}$ and $\|R(n,A)Ay\| \le Mn^{-1}\|Ay\|$ vanishes as $n \to \infty$. Hence, $nR(n,A)y$ tends to y as $n \to \infty$. If $y \in X$, then we consider a sequence $(y_m) \subset D(A)$, converging to y as $m \to \infty$. Since $\|nR(n,A)\|_{L(X)} \le M$ for every $n \in \mathbb{N}$, we can estimate

$$\begin{aligned}\|nR(n,A)y - y\| \le& \|nR(n,A)(y - y_m)\| + \|nR(n,A)y_m - y_m\| + \|y_m - y\|\\ \le& (M+1)\|y_m - y\| + \|nR(n,A)y_m - y_m\|\end{aligned}$$

for $m, n \in \mathbb{N}$. Hence,

$$\begin{aligned}\limsup_{n\to\infty}\|nR(n,A)y - y\| \le& (M+1)\|y - y_m\| + \lim_{n\to\infty}\|nR(n,A)y_m - y_m\|\\ =& (M+1)\|y_m - y\|\end{aligned}$$

for every $m \in \mathbb{N}$. Letting m tend to ∞, we conclude that $\limsup_{n\to\infty}\|nR(n,A)y - y\| = 0$. Hence, $nR(n,A)y$ converges to y as $n \to \infty$.

Step 2. For any $n \in \mathbb{N}$, we introduce the uniform continuous semigroup $\{T_n(t)\}$ on X, defined by

$$T_n(t) = e^{tA_n} := \sum_{k=0}^{\infty}\frac{t^k}{k!}A_n^k, \qquad t \ge 0,$$

and prove that, for each $x \in X$, $(T_n(\cdot)x)$ is a Cauchy sequence in $C([0,T];X)$ for every $T > 0$.

To begin with, we observe that, since $A_n = n^2R(n,A) - nI$, $T_n(t) = e^{-nt}e^{tn^2R(n,A)}$ for each $t \ge 0$. Hence, recalling that $\|(nR(n,A))^k\|_{L(X)} \le M$ for $k \in \mathbb{N}$, we deduce that

$$\|T_n(t)\|_{L(X)} \le e^{-nt}\bigg\|\sum_{k=0}^{\infty}\frac{t^k}{k!}(n^2R(n,A))^k\bigg\|_{L(X)} \le e^{-nt}\sum_{k=0}^{\infty}\frac{(tn)^k}{k!}\|(nR(n,A))^k\|_{L(X)}$$

$$\leq M e^{-nt} e^{nt} = M \tag{2.12}$$

for every $t \geq 0$ and $n \in \mathbb{N}$.

Next, we show that, for each $x \in X$, $(T_n(\cdot)x)$ is a Cauchy sequence in $C([0,T];X)$ for every $T > 0$. For this purpose, we fix $T > 0$, $t \in (0,T]$, $x \in D(A)$ and introduce the function $u_{m,n} : [0,t] \to X$, defined by $u_{m,n}(s) = T_m(t-s)T_n(s)x$ for $s \in [0,t]$. As it is easily checked $u_{m,n}(t) = T_n(t)x$ and $u_{m,n}(0) = T_m(t)x$. Moreover, $u_{m,n}$ is differentiable in $[0,t]$ and $u'_{m,n}(s) = -T_m(t-s)A_mT_n(s)x + T_m(t-s)T_n(s)A_nx$ for $s \in [0,t]$. Note that A_m commutes with A_n since $R(n,A)$ commutes with $R(m,A)$ (see Proposition 2.1.10). As a byproduct, A_m commutes with the semigroup $\{T_n(t)\}$ and, consequently, $u'_{m,n}(s) = T_m(t-s)T_n(s)(A_mx - A_nx)$ for every $s \in [0,t]$. Thus,

$$\begin{aligned}\|T_n(t)x - T_m(t)x\| \leq& \int_0^t \|T_m(t-s)T_n(s)(A_mx - A_nx)\|\,ds\\ \leq& M^2t\|A_mx - A_nx\| \leq M^2T\|A_mx - A_nx\|.\end{aligned}$$

Since T has been arbitrarily fixed and $x \in D(A)$, by Step 1 $(T_n(\cdot)x)$ is a Cauchy sequence in $C([0,T];X)$ for every $T > 0$.

If $x \in X$, then there exists a sequence $(x_k) \subset D(A)$ which converges to x as $k \to \infty$. Then, taking (2.12) into account, for each $k, m, n \in \mathbb{N}$ and $T > 0$, we can estimate

$$\begin{aligned}\|T_n(\cdot)x - T_m(\cdot)x\|_{C([0,T];X)} \leq& \|T_n(\cdot)(x - x_k)\|_{C([0,T];X)} + \|T_n(\cdot)x_k - T_m(\cdot)x_k\|_{C([0,T];X)}\\ &+ \|T_m(\cdot)(x - x_k)\|_{C([0,T];X)}\\ \leq& 2M\|x - x_k\| + M^2T\|A_mx_k - A_nx_k\|.\end{aligned}$$

Hence, for every fixed $\varepsilon > 0$, we can fix $k_0, n_0 \in \mathbb{N}$ such that $2M\|x - x_{k_0}\| \leq \varepsilon/2$ and $M^2T\|A_mx_{k_0} - A_nx_{k_0}\| \leq \varepsilon/2$ for each $m, n \geq n_0$. It thus follows that $\|T_n(\cdot)x - T_m(\cdot)x\|_{C([0,T];X)} \leq \varepsilon$ for each $m, n \geq n_0$. Hence, $(T_n(\cdot)x)$ is a Cauchy sequence in $C([0,T];X)$ for every $x \in X$.

Step 3. Here, we define a C_0-semigroup $\{T(t)\}$. Since each operator $T_n(t)$ is linear and $(T_n(\cdot)x)$ is a Cauchy sequence in $[0,T]$ for each $T > 0$ and $x \in X$, there exists a family of linear operators $\{T(t)\}$ such that $\lim_{n\to\infty} T_n(t)x = T(t)x$ for every $x \in X$ and the convergence is uniform on each interval $[0,T]$ $(T > 0)$. So, the function $t \mapsto T(t)x$ is continuous on $[0,\infty)$ for each $x \in X$. As is immediately seen, $T(0) = I$, since $T_n(0) = I$ for every $n \in \mathbb{N}$. Moreover, from the estimate $\|T_n(t)x\| \leq M\|x\|$, which holds true for every $t \geq 0$, $x \in X$ and $n \in \mathbb{N}$, it follows that $T(t)$ is a bounded linear operator and $\|T(t)\|_{L(X)} \leq M$ for any $t \geq 0$. To conclude that $\{T(t)\}$ is a C_0-semigroup we need to check the semigroup property. This follows from $T_n(t+s)x = T_n(t)T_n(s)x$ for each $t, s > 0$, $x \in X$ and $n \in \mathbb{N}$.

Step 4. Here, we complete the proof. Since $\{T(t)\}$ is a C_0-semigroup on X which satisfies the estimate $\|T(t)\|_{L(X)} \leq M$ for every $t \geq 0$, by Propositions 2.1.7 and 2.1.12 it admits an infinitesimal generator B whose resolvent set contains the line $(0,\infty)$.

To show that B coincides with the operator A, we begin by showing that $D(A) \subset D(B)$ and $Bx = Ax$ for every $x \in D(A)$. For this purpose, we fix $x \in D(A)$, $n \in \mathbb{N}$ and observe that the function $T_n(\cdot)x$ is differentiable in $[0,\infty)$ with $D_tT_n(t)x = T_n(t)A_nx$ for each $t \geq 0$. We claim that $D_tT_n(\cdot)x$ converges to the function $T(\cdot)Ax$ locally uniformly in $[0,\infty)$. To check the claim, we observe that

$$\|D_tT_n(\cdot)x - T(\cdot)Ax\|_{C([0,T];X)} \leq M\|A_nx - Ax\| + \|T_n(\cdot)Ax - T(\cdot)Ax\|_{C([0,T];X)}$$

for every $T > 0$ and the right-hand side of the previous inequality vanishes as $n \to \infty$ by Steps 1 and 3. The claim is proved.

Since $T_n(\cdot)x$ converges to $T(\cdot)x$, locally uniformly in $[0,\infty)$, the function $T(\cdot)x$ is differentiable in $[0,\infty)$ and $D_tT(t)x = T(t)Ax$ for every $t \geq 0$. This, in particular, shows that $x \in D(B)$ and $Bx = Ax$.

To show that $D(B) = D(A)$, we observe that $1 \in \rho(A) \cap \rho(B)$ and

$$X = (I - A)(D(A)) = (I - B)(D(A)) \subset (I - B)(D(B)) = X.$$

Hence, $(I - B)(D(A)) = (I - B)(D(B))$. Since the operator $I - B$ is injective, we obtain that $D(A) = D(B)$ and we are done. $\square$

Remark 2.1.16. Given a closed operator A, such that $\pi = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subseteq \rho(A)$ for some $\omega \in \mathbb{R}$, we need to check the infinitely many conditions $\|(R(\lambda, A))^n\| \leq M(\operatorname{Re}\lambda - \omega)^{-n}$ for every $\lambda \in \pi$ to establish whether A generates a C_0-semigroup or not. In the particular case when $M = 1$, things are easier since once the previous condition is proved with $n = 1$, it can be easily extended to every $n > 1$ just observing that $\|(R(\lambda, A))^n\|_{L(X)} \leq \|R(\lambda, A)\|_{L(X)}^n$ for each $n \in \mathbb{N}$.

Remark 2.1.17. At the very beginning of this lecture, we have introduced uniform continuous semigroups, which are actually groups since the operator e^{tA} is defined for every real value of t and the semigroup property is satisfied for every $s, t \in \mathbb{R}$. On the other hand, C_0-semigroups are in general defined only on the line $[0,\infty)$. Indeed, suppose that the C_0-semigroup is actually a group. Then, $\{T(t)\}$ and $\{T(-t)\}$ are C_0-semigroups. Clearly, if A is the infinitesimal generator of $\{T(t)\}$, then, $-A$ generates the semigroup $\{T(-t)\}$. Hence, the Hille-Yosida theorem implies that the resolvent set of A should contain the halfplanes $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}$ and $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\omega\}$ for some $\omega \geq 0$. Moreover, $\|(R(\lambda, A))^n\|_{L(X)} \leq M(\operatorname{Re}\lambda - \omega)^{-n}$ for every $\lambda \in \mathbb{C}$, such that $|\operatorname{Re}\lambda| > \omega$, every $n \in \mathbb{N}$ and some positive constant M. These conditions are also sufficient for an operator A to be the generator of a C_0-group.

Remark 2.1.18. An useful criterium to determine whether a closed operator generates a C_0-semigroup or not is the well celebrated Lumer-Phillips theorem. To state it, we introduce the concept of dissipative operator. The operator $A : D(A) \subset X \to X$ is called dissipative if $\|\lambda x - Ax\| \geq \lambda\|x\|$ for each $\lambda > 0$ and $x \in X$. (Note that, if A generates a C_0-semigroups of contractions, then A is dissipative.) The Lumer-Phillips theorem states that, if A is dissipative, with dense domain, and $\rho(A) \cap (0, +\infty) \neq \varnothing$, then A generates a C_0-semigroup of contractions on X. As a corollary of the Lumer-Phillips theorem one can infer that, if A is densely defined and dissipative and A^* is dissipative as well, then A generates a C_0-semigroup of contractions. For more details we refer to [1, Chapter II].

2.2 Notes and Remarks

Operator semigroups has been widely studied during the last decades and there are many monographs dealing with them. We mention here the excellent graduate texts by Engel and Nagel [1, 2]. The first milestone in the theory was the opus of Hille and Phillips [4]. An important later reference are the books by Goldstein [3] and Pazy [5].

2.3 Exercises

1. Prove Lemma 2.1.11.

2. On $X = \ell^p$ ($p \in [1,\infty)$) consider the operator A defined by $A(x_n) = (a_n x_n)$ for every $(x_n) \in D(A) = \{(x_n) \in \ell^p : (a_n x_n) \in \ell^p\}$, where $(a_n) \subset \mathbb{C}$ is a given sequence. Show that:

   (i) $A \in L(X)$ if and only if $(a_n) \in \ell^\infty$.

   (ii) A is a closed operator with dense domain.

   (iii) A generates a C_0-semigroup $\{T(t)\}$ if and only if there exists $\omega \in \mathbb{R}$ such that $\operatorname{Re} a_n \le \omega$ for all $n \in \mathbb{N}$. In this case, $T(t)(x_n) = (\mathrm{e}^{ta_n} x_n)$ for each $(x_n) \in X$.

   (iv) Prove that, if $a_n = -n^2$, then $\{T(t)\}$ is continuous in the operator norm on $(0,\infty)$, but not right continuous at $t = 0$.

3. Let $X = C_0(\mathbb{R})$ and $q \in C(\mathbb{R})$. Consider the operator $(Af)(s) := q(s)f(s)$ with $D(A) := \{f \in X : qf \in X\}$ and make analogous statements as in the previous exercise. Prove these statements.

4. On the Banach space $X_0 = \{f \in C([0,1]) : f(1) = 0\}$, endowed with the sup-norm, consider the family $\{T(t)\}$ of operators, defined by

$$(T(t)f)(x) = \begin{cases} f(x+t), & \text{if } x+t \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

   for every $x \in [0,1]$ and $t \ge 0$. Prove that $\{T(t)\}$ is a C_0-semigroup, called the semigroup of left translations on $C([0,1])$, and show that its growth bound ω_0 is $-\infty$.

5. Let $X = C([0,1])$. Prove that

   (i) the spectrum of the operator $A_1 : \{f \in C^1([0,1]) : f(0) = 0\} \subset X \to X$, defined by $A_1 f = f'$ for $f \in D(A_1)$, is empty.

   (ii) the resolvent set of the operator $A_2 : \{f \in C^1([0,1]) : f(0) = f(1) = 0\} \subset X \to X$, defined by $A_2 f = f'$ for $f \in D(A_2)$, is empty.

6. Determine which of the following operators generate a C_0-semigroup on $X = C([0,1])$:

(i) $A_1 f = f'$ for $f \in D(A_1) = \{f \in C^1([0,1]) : f(0) = 0\}$;

(ii) $A_2 f = f''$ for $f \in D(A_2) = C^2([0,1])$;

(iii) $A_3 f = f''$ for $f \in D(A_3) = \{f \in C^2([0,1]) : f(0) = f(1) = 0\}$

(iv) $A_4 f = f''$ for $f \in D(A_4) = \{f \in C^2([0,1]) : f''(0) = 0\}$.

7. On $X = C_0(\mathbb{R})$ consider the one-parameter family $\{T(t)\}$ defined by

$$(T(t)f)(x) = \exp\left(\int_{x-t}^{t} q(s)ds\right) f(x-t), \qquad x \in \mathbb{R},\ t \geq 0,$$

where $q : \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function. Prove that $\{T(t)\}$ is a C_0-semigroup and identify its infinitesimal generator.

8. Prove Remark 2.1.17.

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Lecture 3

Semigroups of bounded operators. Part II. Analytic semigroups

In this lecture, we keep on the study of semigroups of bounded operators introducing analytic semigroups and studying their main properties. By X we still denote a complex Banach space with norm $\|\cdot\|$. Moreover, given a curve $\gamma : I \to \mathbb{C}$ ($I \subset \mathbb{R}$ being an interval) and a function f defined at least on the support of f, we set $\int_{-\gamma} f(\lambda)d\lambda := -\int_{\gamma} f(\lambda)d\lambda$.

3.1 Prelude

In the previous lecture, we have seen that, to every bounded operator on X, we can associate a uniformly continuous semigroup by setting

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \qquad t \in \mathbb{R}. \tag{3.1}$$

This formula can not be extended to the case when A is not defined in the whole X since the domain of the powers A^n becomes smaller and smaller. On the other hand, when $\rho(A)$ is not empty, the resolvent operator is defined and bounded in X. Hence, the idea is to look for a formula for e^{tA} which involves the operators $R(\lambda, A)$.

Lemma 3.1.1. *For $A \in L(X)$ the following assertion holds:*

$$\sigma(A) \subseteq \overline{B(0, \|A\|_{L(X)})}.$$

Proof. Fix $\lambda \in \mathbb{C}$ and observe that $\lambda I - A = \lambda(I - \lambda^{-1}A)$. Hence, from Lemma 2.1.11 it follows that, if $|\lambda^{-1}|\|A\|_{L(X)} < 1$, then $\lambda I - A$ is invertible in X, and we are done. □

Now, we can prove the following integral representation formula for uniformly continuous semigroups.

Proposition 3.1.2. *Let $A \in L(X)$ and let γ_r ($r > \|A\|_{L(X)}$) be the curve, defined by $\gamma_r(t) = re^{it}$ for $t \in [0, 2\pi]$. Then,*

$$e^{tA} = \frac{1}{2\pi i}\int_{\gamma_r} e^{t\lambda}R(\lambda, A)d\lambda, \qquad t \in \mathbb{R}. \tag{3.2}$$

Proof. From (3.1) and Lemma 2.1.11 we can write

$$R(\lambda, A) = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}$$

for every $|\lambda| > \|A\|_{L(X)}$. Hence,

$$e^{\lambda t}R(\lambda, A) = \sum_{n=0}^{\infty} \frac{t^n\lambda^n}{n!} \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}.$$

Integrating both sides of the previous formula along the curve γ_r and observing that the series and the integral commute, we get

$$\frac{1}{2\pi i}\int_{\gamma_r} e^{t\lambda}R(\lambda, A)d\lambda = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\infty}A^k\int_{\gamma_r}\lambda^{n-k-1}d\lambda = e^{tA}, \tag{3.3}$$

since

$$\int_{\gamma_r}\lambda^{n-k-1}d\lambda = \begin{cases} 2\pi i, & n = k, \\ 0, & \text{otherwise.} \end{cases}$$

□

Note that the right-hand side of formula (3.2) makes sense for each closed operator $A : D(A) \subset X \to X$ whose spectrum is bounded, and is independent of $r > 0$. Moreover, if we set $T(0) = I$, then the family $\{T(t)\}$ defines a semigroup on X. Indeed, take $s, t > 0$ and let $r > 0$ be such that $\sigma(A) \subset B(0, r)$. Then,

$$\begin{aligned}
T(t)T(s) =& -\frac{1}{4\pi^2}\int_{\gamma_{2r}} e^{t\lambda}R(\lambda, A)d\lambda\int_{\gamma_r} e^{s\mu}R(\mu, A)d\mu \\
=& -\frac{1}{4\pi^2}\int_{\gamma_{2r}} e^{t\lambda}d\lambda\int_{\gamma_r} e^{s\mu}R(\lambda, A)R(\mu, A)d\mu \\
=& -\frac{1}{4\pi^2}\int_{\gamma_{2r}} e^{t\lambda}d\lambda\int_{\gamma_r} \frac{e^{s\mu}}{\mu-\lambda}(R(\lambda, A) - R(\mu, A))d\mu \\
=& -\frac{1}{4\pi^2}\int_{\gamma_{2r}} e^{t\lambda}R(\lambda, A)d\lambda\int_{\gamma_r}\frac{e^{s\mu}}{\mu-\lambda}d\mu - \frac{1}{4\pi^2}\int_{\gamma_r} e^{s\mu}R(\mu, A)d\mu\int_{\gamma_{2r}}\frac{e^{t\lambda}}{\lambda-\mu}d\lambda,
\end{aligned}$$

where in the last integral term we have changed the order of integration. Since $\lambda \notin \overline{B(0, r)}$, the function $\mu \mapsto (\mu - \lambda)^{-1}e^{s\mu}$ is holomorphic in $\overline{B(0, r)}$ and, consequently, by the Cauchy integral theorem, $\int_{\gamma_r}(\mu-\lambda)^{-1}e^{s\mu}d\mu = 0$. On the contrary, the function $\lambda \mapsto (\lambda-\mu)^{-1}e^{t\lambda}$ has a simple pole at $\lambda = \mu \in B(0,2r)$. Hence, by the residue theorem, $\int_{\gamma_{2r}}(\lambda-\mu)^{-1}e^{t\lambda}d\lambda = (2\pi i)e^{t\mu}$. It thus follows that

$$T(t)T(s) = \frac{1}{2\pi i}\int_{\gamma_{2r}} e^{(s+t)\mu}R(\mu,A)d\mu = T(t+s).$$

In general, the spectrum of a closed operator is not bounded. Hence, formula (3.2) can not used to define a semigroup associated to an unbounded operator A. But, as we will show in the next section, we can overcome this difficulty changing the path of integration, provided that the operator A satisfies nice spectral properties.

3.2 Sectorial operators and analytical semigroups

As announced, here we introduce an important class of closed operators, the so-called, *sectorial operators*, to which we can associate a semigroup through a variant of formula (3.2).

Definition 3.2.1. A linear operator $A : D(A) \subset X \to X$ is called *sectorial* in X if there exist $\omega \in \mathbb{R}$, $\theta_0 \in (\pi/2,\pi)$ and $M > 0$ such that the spectrum of A contains the sector $\Sigma_{\omega,\theta_0} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda-\omega)| < \theta_0\}$ and $\|R(\lambda,A)\|_{L(X)} \le M|\lambda-\omega|^{-1}$ for every $\lambda \in \Sigma_{\omega,\theta_0}$.

Throughout this lecture, we denote by $S(\omega,\theta_0,M)$ the set of all the sectorial operators A which satisfy the condition in Definition 3.2.1.

Given $A \in S(\omega,\theta_0,M)$, we can define an operator $T(t)$, for each $t > 0$, by using formula (3.2), where we replace the curve γ_r by the "union" of three curves $-\gamma_{1,r,\eta,\omega}$, $\gamma_{2,r,\eta,\omega}$ and $\gamma_{3,r,\eta,\omega}$, where $\gamma_{2k+1,r,\eta,\omega} : [r,\infty) \to \mathbb{C}$ $(k = 0,1)$ is defined by $\gamma_{2k+1,r,\eta,\omega}(\rho) = \omega + \rho e^{(-1)^{k+1}i\eta}$, for each $\rho \ge r$, and $\gamma_{2,r,\eta,\omega} : [-\eta,\eta] \to \mathbb{C}$ is defined by $\gamma_{2,r,\eta,\omega}(\theta) = \omega + re^{i\theta}$ for each $\theta \in [-\eta,\eta]$. Here, $r > 0$ and $\eta \in (\pi/2,\theta_0)$ are arbitrarily fixed.

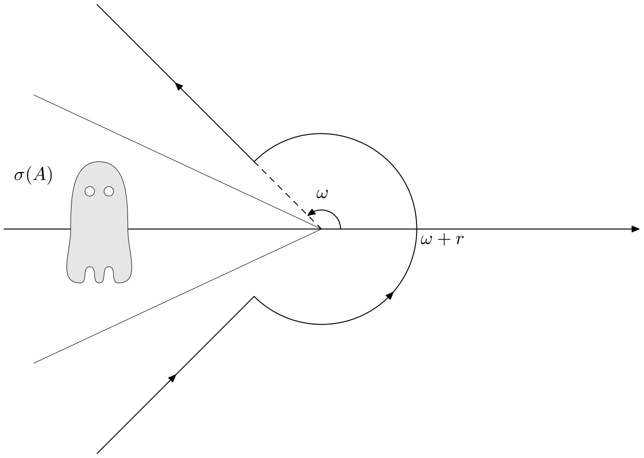

Figure 3.1: the support of the union of the three curves $\gamma_{1,r,\eta,\omega}$, $\gamma_{2,r,\eta,\omega}$ and $\gamma_{3,r,\eta,\omega}$.

More precisely, we set

$$\begin{aligned}
T(t) =& -\frac{1}{2\pi i}\int_{\gamma_{1,r,\eta,\omega}} e^{\lambda t}R(\lambda, A)d\lambda + \frac{1}{2\pi i}\int_{\gamma_{2,r,\eta,\omega}} e^{\lambda t}R(\lambda, A)d\lambda \\
&+ \frac{1}{2\pi i}\int_{\gamma_{3,r,\eta,\omega}} e^{\lambda t}R(\lambda, A)d\lambda \\
=& \frac{e^{\omega t}}{2\pi i}\bigg(\int_r^\infty e^{\rho\cos(\eta)t}[e^{i(\eta+\rho\sin(\eta)t)}R(\omega+\rho e^{i\eta}, A) - e^{-i(\eta+\rho\sin(\eta)t)}R(\omega+\rho e^{-i\eta}, A)]d\rho \\
&+ \int_{-\eta}^{\eta} e^{(r\cos(\theta)+ir\sin(\theta))t}R(\omega+re^{i\theta}, A)ire^{i\theta}d\theta\bigg).
\end{aligned} \tag{3.4}$$

To ease the notation, we denote the above integral by $\int_{\gamma_{r,\eta,\omega}} e^{t\lambda}R(\lambda, A)d\lambda$.

Note that $T(t)$ is well defined since the definition is independent of r and η. Indeed, by Proposition 2.1.10, the function $\lambda \mapsto v(\lambda) = e^{t\lambda}R(\lambda, A)$ is an holomorphic function in the sector Σ_{ω,θ_0}, with values in $L(X)$. This in particular shows that the integral of v over $\gamma_{2,r,\eta,\omega}$ is well defined. Similarly, since

$$\|e^{t\rho\cos(\eta)\pm it\rho\sin(\eta)}R(\omega+\rho\cos(\eta)\pm i\rho\sin(\eta), A)\|_{L(X)} \le Mr^{-1}e^{\rho\cos(\eta)} \tag{3.5}$$

for every $\rho \ge r$ and $\cos(\eta) < 0$, it follows immediately that also the integrals of v over the curves $\gamma_{1,r,\eta,\omega}$ and $\gamma_{3,r,\eta,\omega}$ are well defined.

Now, we fix $r' > 0$ and $\eta' \in (\pi/2, \theta_0)$ and denote by D be the region lying between the supports of the curves $\gamma_{j,r,\eta,\omega}$ and $\gamma_{j,r',\eta',\omega}$ $(j = 1, 2, 3)$ and by D_n $(n \in \mathbb{N})$ the intersection of D with the closed ball centered at zero with radius n.

Denote by γ a curve which parameterizes D_n, obtained from the curves $\gamma_{j,r,\eta,\omega}$, $\gamma_{j,r',\eta',\omega}$ and the canonical parametrization of the arc of $\partial B(0, n)$. The Cauchy integral theorem implies that

$$\int_\gamma e^{t\lambda}R(\lambda, A)d\lambda = 0.$$

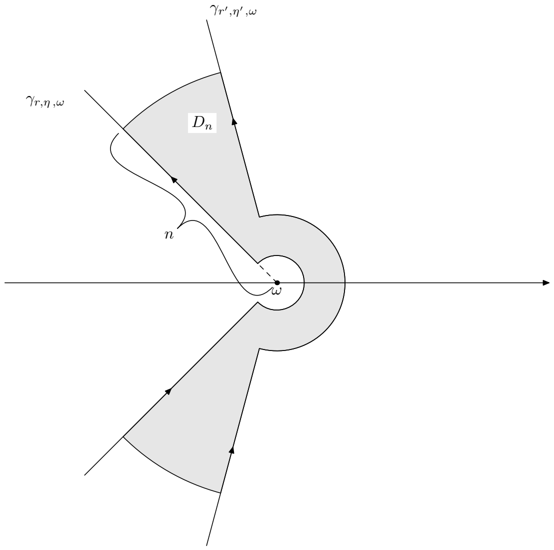

Figure 3.2: the region D_n.

By estimate (3.5) the integrals on the two arcs contained in $\partial B(0,n)$ vanish as n tends to ∞. From these remarks, we deduce that

$$\int_{\gamma_{r,\eta,\omega}} e^{t\lambda}R(\lambda,A)d\lambda = \int_{\gamma_{r',\eta',\omega}} e^{t\lambda}R(\lambda,A)d\lambda$$

as claimed.

We can now study the main properties of the operators $T(t)$, $t \geq 0$.

Theorem 3.2.2. *Let $A \in S(\omega,\theta_0,M)$ and let $T(t)$ be given by (3.4) for $t > 0$. Then, the following properties hold true.*

(i) *For each $x \in X$, $k \in \mathbb{N}$ and $t > 0$, $T(t)x \in D(A^k)$. Further, if $x \in D(A^k)$, then $A^kT(t)x = T(t)A^kx$ for every $t \geq 0$.*

(ii) *If we set $T(0) = I$, then the family $\{T(t)\}$ is a semigroup of bounded linear operators.*

(iii) *There exist positive constants M_k ($k \in \mathbb{N} \cup \{0\}$) such that*

$$\|t^k(A-\omega I)^kT(t)\|_{L(X)} \leq M_k, \qquad t>0,\ k \in \mathbb{N}\cup\{0\}. \tag{3.6}$$

(iv) *The function $t \mapsto T(t)$ belongs to $C^\infty((0,\infty);L(X))$ and $D_t^kT(t) = A^kT(t)$ for every $t > 0$. Moreover, the function $t \mapsto T(t)$ admits an analytic extension to the sector $\Sigma_{0,\theta_0-\pi/2}$, given by*

$$T(z) = \frac{1}{2\pi i}\int_{\gamma_{r,\theta'_z,\omega}} e^{\lambda z}R(\lambda,A)\,d\lambda, \qquad z \in \Sigma_{0,\theta_0-\pi/2},$$

where θ'_z is arbitrarily fixed in $(\pi/2, \theta - \arg(z))$.

Proof. As in the proof of Hille-Yosida theorem, we replace A with the operator $A-\omega I$, which is sectorial in $\Sigma_{0,\theta}$.

(i) The core of the proof is the case $k=1$. Fix $t>0$ and $x\in X$. Since $e^{t\lambda}AR(\lambda,A)=-e^{\lambda t}I+\lambda e^{\lambda t}R(\lambda,A)$,

$$\|e^{\lambda t}I-\lambda e^{\lambda t}R(\lambda,A)\|_{L(X)}\le e^{\mathrm{Re}\,\lambda t}(1+\|\lambda R(\lambda,A)\|_{L(X)})\le e^{\mathrm{Re}\,\lambda t}(1+M)$$

and $\cos(\eta)<0$, the function $\lambda\mapsto e^{t\lambda}AR(\lambda,A)x$ is integrable along the curves $\gamma_{1,r,\eta,0}$ and $\gamma_{3,r,\eta,0}$ for each $x\in X$. Of course, being a continuous function it is also integrable along the curve $\gamma_{2,r,\eta,0}$. Therefore, we can invoke Proposition A.2.3, which guarantees that $T(t)x\in D(A)$ and

$$\begin{aligned} AT(t)x =&\frac{1}{2\pi i}\int_{\gamma_{r,\eta,0}} e^{t\lambda}AR(\lambda,A)x\,d\lambda=-\frac{x}{2\pi i}\int_{\gamma_{r,\eta,0}} e^{t\lambda}\,d\lambda+\frac{1}{2\pi i}\int_{\gamma_{r,\eta,0}}\lambda e^{t\lambda}R(\lambda,A)x\,d\lambda\\ =&\frac{1}{2\pi i}\int_{\gamma_{r,\eta,0}}\lambda e^{t\lambda}R(\lambda,A)\,\lambda, \end{aligned}\tag{3.7}$$

since

$$\int_{\gamma_{r,\eta,0}} e^{t\lambda}\,d\lambda=0.\tag{3.8}$$

If $x\in D(A)$, then $AR(\lambda,A)x=R(\lambda,A)Ax$ and, hence, $AT(t)x=T(t)Ax$. Iterating this argument, we can show that $T(t)x\in D(A^k)$ for every $k\in\mathbb{N}$ and

$$A^kT(t)x=\frac{1}{2\pi i}\int_{\gamma_{r,\eta,0}}\lambda^k e^{\lambda t}R(\lambda,A)x\,d\lambda\tag{3.9}$$

and, if $x\in D(A^k)$, then $A^kT(t)x=T(t)A^kx$.

(ii) The semigroup property can be proved arguing as in the last part of Section 3.1. Fix $s,t>0$. Writing

$$T(t)=\frac{1}{2\pi}\int_{\gamma_{r,\eta,0}} e^{\lambda t}R(\lambda,A)\,d\lambda,\qquad T(s)=\frac{1}{2\pi}\int_{\gamma_{2r,\eta',0}} e^{\lambda s}R(\lambda,A)\,d\lambda$$

for each $r>0$, $\pi/2<\eta'<\eta<\theta$ and using the resolvent identity it follows that

$$\begin{aligned} T(t)T(s)=&-\frac{1}{4\pi^2}\int_{\gamma_{r,\eta,0}} e^{\lambda t}R(\lambda,A)\,d\lambda\int_{\gamma_{2r,\eta',0}}\frac{e^{\mu s}}{\mu-\lambda}\,d\mu\\ &+\frac{1}{4\pi^2}\int_{\gamma_{2r,\eta',0}} e^{\mu s}R(\mu,A)\,d\mu\int_{\gamma_{r,\eta,\omega}}\frac{e^{\lambda t}}{\mu-\lambda}\,d\lambda. \end{aligned}$$

Since

$$\int_{\gamma_{2r,\eta',0}} e^{\mu s}\frac{d\mu}{\mu-\lambda}=2\pi i e^{s\lambda},\quad \text{for }\ \lambda\in\gamma_{r,\eta,0}\ \text{ and }\ \int_{\gamma_{r,\eta,0}} e^{\lambda t}\frac{d\lambda}{\mu-\lambda}=0\ \text{ for }\ \mu\in\gamma_{2r,\eta',0},\tag{3.10}$$

we conclude that $T(t)T(s) = T(t+s)$.

(iii) We fix $t > 0$ and take the norm of the three integrals which define $T(t)$ (see (3.4)). Since we are assuming that $\omega = 0$, from the estimate in Definition 3.2.1 we get

$$\|T(t)\|_{L(X)} \le \frac{M}{\pi}\int_r^\infty e^{t\rho\cos(\eta)}\rho^{-1}d\rho + \frac{M}{2\pi}\int_{-\eta}^{\eta} e^{tr\cos(\theta)}d\theta \le \frac{M}{\pi}\left(\frac{e^{tr\cos(\eta)}}{tr|\cos(\eta)|} + e^{tr}\right)$$

for each $r > 0$ and $\eta \in (\pi/2, \theta_0)$. Note that, if we take the same r for every $t > 0$, we end up with an estimate for $\|T(t)\|_{L(X)}$ which is singular as t tends to 0. To overcome this difficulty, it suffices to replace r by $1/t$ and we get (3.6) with $M_0 = M\pi^{-1}(|\cos(\eta)|^{-1}e^{\cos(\eta)} + e)$.

Estimating the norm of $AT(t)$ is easier. We do not need to take a radius which depends on t. It is enough to use formula (3.7) observing that $\|\lambda R(\lambda, A)\|_{L(X)} \le M$. We thus get

$$\|AT(t)\|_{L(X)} \le \frac{M}{\pi}\left(\frac{e^{tr\cos(\eta)}}{t|\cos(\eta)|} + \frac{Mr}{2\pi}\int_{-\eta}^{\eta} e^{tr\cos\theta}d\theta\right)$$

for each $r > 0$. Letting r tend to 0^+, by dominated convergence we get

$$\|AT(t)\|_{L(X)} \le \frac{M}{\pi t|\cos(\eta)|}, \qquad t > 0. \tag{3.11}$$

To estimate the operator norm of $A^kT(t)$ for $k > 1$ it suffices to use (3.11), property (i) and the semigroup property to write $A^kT(t) = (AT(t/k))^k$ and, hence, estimate $\|A^kT(t)\|_{L(X)} \le \|AT(t)\|_{L(X)}^k$.

(iv) By applying repeatedly the dominated convergence theorem, it can be easily shown that the function $t \mapsto T(t)$ belongs to $C^\infty((0,\infty); L(X))$ and

$$D_t^kT(t) = \frac{1}{2\pi i}\int_{\gamma_{r,\eta,\omega}} \lambda^k e^{\lambda t}R(\lambda, A)\,d\lambda, \qquad t > 0.$$

From this formula and (3.9) it follows that $D_t^kT(t) = A^kT(t)$ for every $t > 0$ and $k \in \mathbb{N}$.

To complete the proof, we fix $\alpha \in (0, \theta_0 - \pi/2)$. Then, the function

$$z \mapsto T(z) = \frac{1}{2\pi i}\int_{\gamma_{r,\theta_0-\alpha,\omega}} e^{z\lambda}R(\lambda, A)\,d\lambda,$$

is well defined and holomorphic in the sector $\Sigma_{0,\theta_0-\pi/2-\alpha}$. Indeed, if $\lambda = \rho e^{i(\theta_0-\alpha)}$ and $z = |z|e^{i\varphi}$ belongs to $\Sigma_{0,\theta-\pi/2-\alpha}$, then $\mathrm{Re}(z\lambda) = \rho|z|\cos(\theta_0 - \alpha + \varphi)$ and $\cos(\theta_0 - \alpha + \varphi)$ is negative since $\theta_0 - \alpha + \varphi \in (\pi/2, 3\pi/2)$. Hence, we can differentiate under the integral sign, taking the dominated convergence theorem into account, and conclude that the map $z \mapsto T(z)$ is holomorphic in $\Sigma_{0,\theta_0-\pi/2-\alpha}$. Since $\Sigma_{0,\theta_0-\pi/2} = \bigcup_{\alpha\in(0,\theta_0-\pi/2)} \Sigma_{0,\theta_0-\pi/2-\alpha}$, the conclusion follows. □

Remark 3.2.3. From property (iii) in Theorem 3.2.2 it follows that there exists a positive constant C_k such that $\|t^kA^kT(t)\|_{L(X)} \le C_ke^{\omega t}$ for $t > 0$. Indeed,

$$A^k = (A - \omega I + \omega I)^k = \sum_{n=0}^{k}\binom{k}{n}\omega^{k-n}(A - \omega I)^n$$

and, therefore,

$$\|A^kT(t)\|_{L(X)} \le \sum_{n=0}^{k}\binom{k}{n}\omega^{k-n}\|(A-\omega I)^nT(t)\|_{L(X)} \le \sum_{n=0}^{k}\binom{k}{n}\omega^{k-n}M_nt^{-n}, \qquad t>0.$$

If $t \le 1$, then from the previous estimate we immediately get

$$\|A^kT(t)\|_{L(X)} \le t^{-k}\sum_{n=0}^{k}\binom{k}{n}\omega^{k-n}M_n.$$

On the other, if $t > 1$, then there exists a positive constant C_ω such that $t^{-n}\omega^{k-n} \le C_\omega t^{-k}e^{\omega t}$ for every $n < k$. We thus conclude that

$$\|A^kT(t)\|_{L(X)} \le C_\omega t^{-k}e^{\omega t}\sum_{n=0}^{k}\binom{k}{n}M_n$$

and we are done.

In view of Theorem 3.2.2 we can now give the following definition.

Definition 3.2.4. Let A be a sectorial operator. The family of operators $\{T(t)\}$, defined by (3.4) for $t > 0$ and such that $T(0) = I$, is called *analytic semigroup generated by* A (in X).

Is each analytic semigroup strongly continuous? The answer is in general negative since $D(A)$ may be not dense in X. In any case, for each $x \in X$, $T(t)x$ converges to x as $t \to 0^+$ in a “weak” sense as the following proposition shows.

Proposition 3.2.5. *Let $\{T(t)\}$ be an analytic semigroup associated with an operator $A \in S(\omega,\theta_0,M)$. Then, $\lim_{t\to 0^+} T(t)x = x$ if and only if $x \in \overline{D(A)}$. As a byproduct, for each $\lambda \in \rho(A)$ and $x \in X$, $\lim_{t\to 0^+} R(\lambda,A)T(t)x = R(\lambda,A)x$.*

Proof. By replacing A with $A - \alpha I$ for some constant α, we can assume without loss of generality that $\omega = 0$.
Using the Cauchy integral theorem, it follows that

$$\begin{aligned} T(t)x - x &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta,\omega}} e^{\lambda t}\left[R(\lambda,A)x - \frac{1}{\lambda}x\right]d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta,\omega}} \frac{e^{\lambda t}}{\lambda}R(\lambda,A)Ax\,d\lambda \end{aligned}$$

for $x \in D(A)$ and $t > 0$. Since $\|\lambda^{-1}R(\lambda,A)Ax\| \le C\|Ax\||\lambda|^{-2}$, it follows from the dominated convergence theorem and Cauchy's theorem that

$$\lim_{t\to 0^+} T(t)x - x = \frac{1}{2\pi i}\int_{\gamma_{r,\eta,\omega}} \frac{1}{\lambda}R(\lambda,A)Ax\,d\lambda = 0$$

for $x \in D(A)$ and hence for $x \in \overline{D(A)}$, since $\|T(t)\| \le M$ for every $t \in [0,1]$. The other implication is obtained from the inclusion $T(t)X \subset D(A)$ for every $t > 0$, see Theorem 3.2.2.
Now, the last statement follows from the fact that $R(\lambda, A)$ commute with $T(t)$ for every $t > 0$ and $\lambda \in \rho(A)$. □

Remark 3.2.6. By Theorem 3.2.2, $T(t)$ maps X into $D(A)$ for every $t > 0$. Hence, it leaves $\overline{D(A)}$ invariant. Moreover, by Proposition 3.2.5, $T(t)x$ converges to x as $t \to 0^+$ for every $x \in \overline{D(A)}$. It follows that the restriction of $\{T(t)\}$ to $\overline{D(A)}$ is a C_0-semigroup. Note that $\overline{D(A)}$ is the largest subspace of X, where the restriction of $\{T(t)\}$ is a C_0-semigroup. Indeed, as already remarked, $T(t)x \in D(A)$ for each $x \in X$ and $t > 0$. Hence, if $T(t)x$ converges to x as $t \to 0^+$, then, necessarily, $x \in \overline{D(A)}$.

In the following proposition, we show other interesting properties of analytic semigroups. In particular, we show that the infinitesimal generator of the restriction of $\{T(t)\}$ to $\overline{D(A)}$ is the part of A in $\overline{D(A)}$, i.e., the operator defined in $D := \{x \in D(A) : Ax \in \overline{D(A)}\}$ by $A_{|\overline{D(A)}}x = Ax$ for each $x \in D$.

Proposition 3.2.7. *The following properties hold true.*

(i) *For each $x \in X$ and $t > 0$, $\int_0^t T(s)x\,ds \in D(A)$ and*

$$A\int_0^t T(s)x\,ds = T(t)x - x. \tag{3.12}$$

If, in addition, $x \in D(A)$, then

$$T(t)x - x = \int_0^t T(s)Ax\,ds, \qquad t \ge 0. \tag{3.13}$$

(ii) *If $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, then*

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}T(t)\,dt. \tag{3.14}$$

(iii) *If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then $\lim_{t\to 0^+}(T(t)x - x)/t = Ax$. Conversely, if $z := \lim_{t\to 0^+}(T(t)x - x)/t$ exists, then $x \in D(A)$ and $Ax = z \in \overline{D(A)}$.*

Proof. (i) Fix $t > 0$ and $x \in X$. Since the function $T(\cdot)x$ is differentiable in $(0,\infty)$ and $D_tT(\cdot)x = AT(\cdot)x$ (see Theorem 3.2.2) then

$$\begin{aligned}\int_\varepsilon^t T(s)x\,ds =& \int_\varepsilon^t ((\omega+1)I - A)R(\omega+1, A)T(s)x\,ds\\ =&(\omega+1)\int_\varepsilon^t R(\omega+1,A)T(s)x\,ds - \int_\varepsilon^t \frac{d}{ds}(R(\omega+1,A)T(s)x)\,ds\end{aligned}$$

$$=(\omega+1)R(\omega+1,A)\int_\varepsilon^t T(s)x\,ds - T(t)R(\omega+1,A)x + T(\varepsilon)R(\omega+1,A)x$$

for each $\varepsilon \in (0,t)$. Recalling that the function $T(\cdot)x$ is bounded in $[0,t]$ and continuous in $(0,+\infty)$ and taking Proposition 3.2.5 into account, we can let ε tend to 0 and obtain

$$\int_0^t T(s)x\,ds = (\omega+1)R(\omega+1,A)\int_0^t T(s)x\,ds - R(\omega+1,A)(T(t)x-x). \tag{3.15}$$

Thus, $\int_0^t T(s)x\,ds \in D(A)$ and applying the operator $((\omega+1)I-A)$ to both the sides of (3.15), formula (3.12) follows.

If $x \in D(A)$, then the function $AT(\cdot)x$ is continuous in $(0,t]$ and bounded in $[0,t]$. Hence,

$$A\int_\varepsilon^t T(s)x\,ds = \int_\varepsilon^t AT(s)x\,ds$$

and, letting ε tend to 0^+, we conclude that

$$\lim_{\varepsilon\to 0^+} A\int_\varepsilon^t T(s)x\,ds = \int_0^t AT(s)x\,ds$$

and formula (3.13) follows.

(ii) Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and consider the operator $A-\lambda I$. This operator is sectorial and the associated semigroup $\{S(t)\}$ is defined by $S(t) = e^{-\lambda t}T(t)$ for $t \geq 0$. From formula (3.12) we know that

$$(A-\lambda)\int_0^t S(r)x\,dr = S(t)x - x, \qquad t>0,\ \ x\in X.$$

Since $\|T(t)\|_{L(X)} \leq Me^{\omega t}$ for $t \geq 0$ (see Theorem 3.2.2), the function $S(t)x$ vanishes as t tends to ∞. Hence, $\lim_{t\to\infty}(A-\lambda)\int_0^t S(r)x\,dr = -x$ for every $x \in X$. Since $\int_0^t S(r)x\,dr$ converges to $\int_0^\infty S(r)x\,dr$, by the closedness of A, $\int_0^t S(r)x\,dr$ belongs to $D(A)$ and

$$(\lambda - A)\int_0^\infty e^{-\lambda t}T(t)x\,dt = x, \qquad x \in X.$$

Since we know that $\lambda \in \rho(A)$, formula (3.14) follows at once.

(iii) Fix $x \in D(A)$ such that $Ax \in \overline{D(A)}$. Then, by property (i), we can write

$$\frac{T(t)x-x}{t} = \frac{1}{t}A\int_0^t T(s)x\,ds = \frac{1}{t}\int_0^t T(s)Ax\,ds. \tag{3.16}$$

Since $Ax \in \overline{D(A)}$, by Proposition 3.2.5 the function $T(\cdot)Ax$ is continuous in $[0,\infty)$. Hence, letting t tend to 0^+ in (3.16), we conclude that $\lim_{t\to 0^+}(T(t)x-x)/t = Ax$.

Vice versa, suppose that the limit $z := \lim_{t\to 0^+}(T(t)x - x)/t$ exists. Then, clearly, $T(t)x$ converges to x as $t \to 0^+$, so that x belong to $\overline{D(A)}$. Since $T(t)x \in D(A)$ for every $t > 0$, also z belongs to $\overline{D(A)}$. Moreover, using property (i) and recalling that $R(\omega + 1, A)$ commutes with $T(s)$, we get

$$\begin{aligned} R(\omega+1, A)z =& \lim_{t\to 0} R(\omega+1, A)\frac{T(t)x - x}{t} = \lim_{t\to 0} t^{-1} R(\omega+1, A)A\int_0^t T(s)x\, ds \\ =& -\lim_{t\to 0}\frac{1}{t}\int_0^t T(s)x\, ds + (\omega+1)\frac{1}{t}\int_0^t R(\omega+1, A)T(s)x\, ds. \end{aligned} \tag{3.17}$$

Since $x \in \overline{D(A)}$, the function $T(\cdot)x$ is continuous in $[0, \infty)$. Hence $\lim_{t\to 0^+} t^{-1}\int_0^t T(s)x\, ds = x$. Similarly, $R(\omega + 1, A)T(t)x$ converges to $R(\omega + 1, A)x$ as $t \to 0^+$, by Proposition 3.2.5. Hence, from (3.17) it follows that $x = R(\omega + 1, A)((\omega + 1)x - z) \in D(A)$ and $Ax = z$. □

We now provide a sufficient condition for a closed operator to be sectorial. This criterium is particularly used in the applications.

Proposition 3.2.8. *Let* $A : D(A) \subset X \mapsto X$ *be a closed operator such that* $\Pi = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ *for some* $\omega \in \mathbb{R}$, *and*

$$\|\lambda R(\lambda, A)\|_{L(X)} \le M, \qquad \lambda \in \Pi, \tag{3.18}$$

for some constant $M \ge 1$. *Then,* A *is a sectorial operator.*

Proof. The same argument as in the proof of Proposition 2.1.10 (which uses Lemma 2.1.11) shows that the open ball $B(\omega \pm ir, |\omega + ir|/M)$ is contained in $\rho(A)$ for each $r > 0$. Since $|\omega+ir| \ge r$, the union of such balls and the halfplane Π properly contains the sector $\Sigma_{0,\theta_{2M}}$, where $\theta_{2M} = \pi - \arctan(2M)$. Observe that each $\lambda \in \Sigma_{\omega,\theta_{2M}}$ with $\operatorname{Re}\lambda \le \omega$ can be written in the form $\lambda = \omega \pm ir - (2M)^{-1}(\theta r)$ for some $\theta \in [0, 1)$. Moreover, since

$$R(\lambda, A) = R(\omega \pm ir, A)(I - (2M)^{-1}\theta r R(\omega \pm ir, A))^{-1}$$

and $\|(I - (2M)^{-1}\theta r R(\omega \pm ir, A))^{-1}\|_{L(X)} \le 2$, it follows that

$$\|R(\lambda, A)\|_{L(X)} \le \frac{2M}{|\omega \pm ir|} \le \frac{2M}{r} \le \frac{\sqrt{4M^2+1}}{|\lambda - \omega|}.$$

On the other hand, if $\lambda \in \Sigma_{0,\theta_{2M}}$ with $\operatorname{Re}\lambda > \omega$, then from (3.18) it follows easily that $\|R(\lambda, A)\|_{L(X)} \le M|\lambda - \omega|^{-1}$ and this completes the proof. □

We conclude this lecture with two important remarks.

Remark 3.2.9. (i) In this and in the previous lecture we have dealt with *complex* Banach spaces. In particular, the sectorial operators have been defined through an integral on an unbounded curve in the complex plane.

As a matter of fact, in many applications one has to deal with closed operators A on *real* Banach spaces X. This (apparent) difficult can be overcome by complexifying both the Banach space X and the operator A. The complexification $X_{\mathbb{C}}$ of X is the set $X_{\mathbb{C}} = \{x+iy : x,y \in X\}$, which is a Banach space when endowed with the norm $\|x+iy\|_{\widetilde{X}} = \sup_{-\pi\le\theta\le\pi}\|x\cos\theta + y\sin\theta\|$ for every $x+iy \in X_{\mathbb{C}}$ and the operations $(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2)$ and $(\lambda_1+i\lambda_2)(x_1+iy_1) = \lambda_1x_1-\lambda_2y_1+i(\lambda_1y_1+\lambda_2x_1)$ for every $x_1+iy_1, x_2+iy_2 \in X_{\mathbb{C}}$ and every $\lambda_1+i\lambda_2 \in \mathbb{C}$. (Note that in general the map $(x+iy)\mapsto\sqrt{\|x\|^2+\|y\|^2}$ is not a norm, in general, see Exercise 3.4.5). The complexification of the operator A is the operator $A_{\mathbb{C}} : D(A_{\mathbb{C}}) \subset X_{\mathbb{C}} \to X_{\mathbb{C}}$, defined as follows; $D(A_{\mathbb{C}}) = \{x+iy : x,y \in D(A)\}$ and $A_{\mathbb{C}}(x+iy) = Ax+iAy$ for every $x+iy \in D(A_{\mathbb{C}})$. Clearly, if we identify X with the set $\{x+i0 : x \in X\}$, then the restriction of the operator $A_{\mathbb{C}}$ to X is the operator A.

Suppose that the operator $A_{\mathbb{C}}$ is sectorial in $X_{\mathbb{C}}$ and denote by $\{T_{\mathbb{C}}(t)\}$ the associated analytic semigroup. We claim that it leaves X invariant. To prove the claim, it turns out useful to replace the three curves used to define each operator $T(t)$ by the union of the two curves γ_- and γ_+, where $\gamma_\pm : [0,\infty) \to \mathbb{C}$ are defined by $\gamma_\pm(\rho) = \omega+1+\rho e^{\pm i\theta}$ for every $\rho \ge 0$. For each $t>0$ and $x \in X$ it holds that

$$\begin{aligned}T_{\mathbb{C}}(t)x = \frac{1}{2\pi i}\int_0^\infty & e^{(\omega+1)t}[e^{\rho te^{i\theta}}R(\omega+1+\rho e^{i\theta}, A_{\mathbb{C}})x \\ & - \overline{e^{\rho te^{i\theta}}}R(\omega+1+\rho e^{-i\theta}, A_{\mathbb{C}})x]d\rho.\end{aligned}$$

Note that $e^{\rho te^{i\theta}}R(\omega+1+\rho e^{i\theta}, A_{\mathbb{C}})x - \overline{e^{\rho te^{i\theta}}}R(\omega+1+\rho e^{-i\theta}, A_{\mathbb{C}})x$ belongs to iX, so that $T_{\mathbb{C}}(t)x$ belongs to X. Indeed, it is easy to check that, if we set $\overline{z+iw} = z-iw$ for every $z+iw \in X_{\mathbb{C}}$, then $R(\overline{\lambda}, A_{\mathbb{C}})x = \overline{R(\lambda, A_{\mathbb{C}})x}$ for every $\lambda \in \rho(A_{\mathbb{C}})$ and $x \in X$. Thus,

$$\begin{aligned}&e^{\rho te^{i\theta}}R(\omega+1+\rho e^{i\theta}, A_{\mathbb{C}})x - \overline{e^{\rho te^{i\theta}}}R(\omega+1+\rho e^{-i\theta}, A_{\mathbb{C}})x \\ =&e^{\rho te^{i\theta}}R(\omega+1+\rho e^{i\theta}, A_{\mathbb{C}})x - \overline{e^{\rho te^{i\theta}}R(\omega+1+\rho e^{i\theta}, A_{\mathbb{C}})x} \\ =&2i\mathrm{Im}(e^{\rho te^{i\theta}}R(\omega+1+\rho e^{i\theta}, A_{\mathbb{C}})x),\end{aligned}$$

where $\mathrm{Im}(x+iy) = y$ for every $x+iy \in X_{\mathbb{C}}$.

If we set $T(t) = T_{\mathbb{C}}(t)_{|X}$ for $t \ge 0$, then we define a semigroup of bounded operators in X, which satisfies the properties that we have established so far.

(ii) From this and the previous lecture, a very crucial difference from strongly continuous semigroups and analytic semigroup arises: the analytic semigroups take a datum x in the space X and make it smoother (indeed, $T(t)x$ belongs to $\bigcap_{k\in\mathbb{N}} D(A^k)$ for every $t>0$). Strongly continuous semigroups do not enjoy this property. Think for instance to the semigroup of left translations on $X = BUC(\mathbb{R}^d)$. Since $T(t)f = f(\cdot+t)$ for every $t>0$, $T(t)$ has no smoothing effects on f. For this reason, as we will see in the next lectures, analytic semigroups are naturally associated to solutions to parabolic equations.

3.3 Notes

As we will see later on, most parabolic problems are based on the theory of analytic semigroups. We mention here that the characterization of the sectoriality of a given operator A involves only a single resolvent estimate, see Proposition 3.2.8. This should be compared with the Hille-Yosida theorem, which characterizes the infinitesimal generators of C_0-semigroups. We also mention that the single resolvent estimate, which a sectorial operator A should satisfy, yields regularity properties for the solution to the associated Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t \in (0, \infty), \\ u(0) = x \end{cases}$$

which is $u = T(\cdot)x$, see Theorem 3.2.2.(i).
For more details on sectorial operators and analytic semigroups, we refer the reader, e.g., to [1–5].

3.4 Exercises

1. Prove formulas (3.8) and (3.10).

2. Let $A : D(A) \subset X \to X$ be sectorial, let $\alpha \in \mathbb{C}$, and set $B : D(B) := D(A) \to X$, $Bx = Ax - \alpha x$, $C : D(C) = D(A) \to X$, $Cx = \alpha Ax$. Prove that the operator B is sectorial, and that the associated semigroup $\{S(t)\}$ is defined by $S(t) = e^{-\alpha t}T(t)$ for every $t \geq 0$. For which α the operator C is sectorial?

3. Let $A : D(A) \subset X \to X$ be sectorial and let $x \in D(A)$ be an eigenvector of A with eigenvalue λ.
   (i) Prove that $R(\mu, A)x = (\mu - \lambda)^{-1}x$ for every $\mu \in \rho(A)$.
   (ii) Prove that $T(t)x = e^{\lambda t}x$ for every $t > 0$.
   (iii) Prove that if A and $-A$ are sectorial operators in X, then A is bounded.

4. Let X_k, $k = 1, \ldots, n$ be Banach spaces, and let $A_k : D(A_k) \to X_k$ be sectorial operators. Set
$$X = \prod_{k=1}^{n} X_k, \qquad D(A) = \prod_{k=1}^{n} D(A_k),$$
and $A(x_1, \ldots, x_n) = (A_1x_1, \ldots, A_nx_n)$, and show that A is a sectorial operator in X, endowed with the product norm $\|(x_1, \ldots, x_n)\| = \left(\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2}$.

5. Let X be a real Banach space. Prove that the function $f : X \times X \to \mathbb{R}$ defined by $f(x, y) = \sqrt{\|x\|^2 + \|y\|^2}$ for every $x, y \in X$, does not satisfy, in general, the homogeneity property.

6. (i) Prove that if A with domain $D(A)$ generates a C_0-group on a Banach space X, then A^2 generates an analytic semigroup on X.

   (ii) Prove that the operator $Af = f'$ for $f \in D(A) = W^{1,p}(\mathbb{R})$ generates a C_0-group on $L^p(\mathbb{R})$, $1 < p < \infty$.

   (iii) Deduce that the operator $Bf = f^{''}$ for $f \in D(B) = W^{2,p}(\mathbb{R})$ generates an analytic semigroup on $L^p(\mathbb{R})$, $1 < p < \infty$.

7. (a) Prove that the operator $Af = f''$ for any $f \in D(A) = C_b^2(\mathbb{R})$ is sectorial in $C_b(\mathbb{R})$ and, hence, generates an analytic semigroups.

   (b) Prove that the operator $Af = f''$ for any $f \in D(A) = \{u \in C_b^2([0,1]) : u(0) = u(1) = 0\}$ generates an analytic semigroup in $C([0,1])$. Is this semigroup strongly continuous?

   (c) Prove that the operator $Af = f''$ for any $f \in D(A) = \{u \in C_b^2([0,1]) : u'(0) = u'(1) = 0\}$ generates an analytic semigroup in $C([0,1])$. Is this semigroup strongly continuous?

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Lecture 4

The heat equation and the Gauss-Weierstrass semigroup in $C_b(\mathbb{R}^d)$. Part 1

This lecture is the prelude to the analysis of parabolic equations in the whole space and in bounded domains. We will deal with the homogeneous Cauchy problem

$$\begin{cases} D_t u(t,x) = \Delta u(t,x), & t>0, \quad x \in \mathbb{R}^d, \\ u(0,x) = f(x), & \qquad\quad\; x \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

where $f \in C_b(\mathbb{R}^d)$ (the space of all the bounded and continuous functions $f : \mathbb{R}^d \to \mathbb{R}$).

The Laplacian is the prototype of a uniformly elliptic operator with bounded coefficients in $\mathbb{R}^d$ and it has the peculiarity that, for every $f \in C_b(\mathbb{R}^d)$, there exists an explicit formula for the classical solution (see the forthcoming Definition 4.1.1). This greatly simplifies the analysis of the nonhomogeneous Cauchy problem

$$\begin{cases} D_t u(t,x) = \Delta u(t,x) + g(t,x), & t>0, \quad x \in \mathbb{R}^d, \\ u(0,x) = f(x), & \qquad\quad\; x \in \mathbb{R}^d, \end{cases} \tag{4.2}$$

as we will see.

In this lecture we will use also the space $C^\alpha_b(\mathbb{R}^d)$ ($\alpha > 0$) of all the functions $f \in C^\alpha(\mathbb{R}^d)$ which are bounded and admit bounded derivatives up to the order $[\alpha]$ and their derivatives of order $[\alpha]$ are $(\alpha - [\alpha])$-Hölder continuous in $\mathbb{R}^d$ (if $\alpha \notin \mathbb{N}$). This is a Banach space when endowed with the norm

$$\|f\|_{C^\alpha_b(\mathbb{R}^d)} = \sum_{|\beta| \le [\alpha]} \left\| \frac{\partial^\beta f}{\partial x^\alpha} \right\|_\infty + + \sum_{|\beta| = [\alpha]} \left[\frac{\partial^\beta f}{\partial x^\alpha} \right]_{C^{\alpha - [\alpha]}(\mathbb{R}^d)}$$

where $[g]_{C^{\alpha-[\alpha]}(\mathbb{R}^d)} = \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|^{\alpha-[\alpha]}} : x, y \in \mathbb{R}^d,\ x \neq y \right\}$.

4.1 The heat equation in $\mathbb{R}^d$. Classical solutions: existence and uniqueness

Definition 4.1.1. A function $u : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ is a classical solution to problem (4.1) if (i) $u \in C([0,\infty)\times\mathbb{R}^d)$, (ii) it is continuously differentiable in $(0,\infty)\times\mathbb{R}^d$, once with respect to the time variable and twice with respect to the spatial variables, (iii) it satisfies the differential equation and the initial condition in (4.1).

We now introduce the so-called *fundamental solution* to the differential equation in (4.1), i.e., the function $K : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ defined by

$$K(t,x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}, \qquad t>0,\ x\in\mathbb{R}^d, \tag{4.3}$$

and study its main properties.

Lemma 4.1.2. *The following properties are satisfied.*

- (i) $K \in C^\infty((0,\infty)\times\mathbb{R}^d)$;
- (ii) $\displaystyle\int_{\mathbb{R}^d} K(t,x)dx = 1$ *for all* $t>0$;
- (iii) $D_jK(t,x) = -\dfrac{x_j}{2t}K(t,x)$ *for all* $(t,x)\in(0,\infty)\times\mathbb{R}^d$;
- (iv) $D_{ij}K(t,x) = \left(\dfrac{x_ix_j}{4t^2} - \dfrac{\delta_{ij}}{2t}\right)K(t,x)$ *for all* $(t,x)\in(0,\infty)\times\mathbb{R}^d$;
- (v) $D_tK(t,x) = \Delta K(t,x)$ *for all* $(t,x)\in(0,\infty)\times\mathbb{R}^d$.

Proof. We limit ourselves to proving property (v), since the remaining ones are straightforward to prove.

From property (iv) it follows that

$$\Delta K(t,x) = \left(\frac{|x|^2}{4t^2} - \frac{d}{2t}\right)K(t,x), \qquad (t,x)\in(0,\infty)\times\mathbb{R}^d.$$

On the other hand,

$$D_tK(t,x) = -d(4\pi t)^{-\frac{d}{2}-1}2\pi e^{-\frac{|x|^2}{4t}} + (4\pi t)^{-\frac{d}{2}}\frac{|x|^2}{4t^2}e^{-\frac{|x|^2}{4t}} = \left(\frac{|x|^2}{4t^2} - \frac{d}{2t}\right)K(t,x)$$

for all $(t,x)\in(0,\infty)\times\mathbb{R}^d$. Property (v) follows at once. □

Now, we can prove the existence and uniqueness of a classical solution to the Cauchy problem

Theorem 4.1.3. *For each $f \in C_b(\mathbb{R}^d)$, the Cauchy problem* (4.1) *has a unique classical solution u given by the following formula:*

$$u(t,x) = \begin{cases} f(x), & t=0, \quad x \in \mathbb{R}^d, \\ \displaystyle\int_{\mathbb{R}^d} K(t,x-y)f(y)dy, & t>0, \quad x \in \mathbb{R}^d. \end{cases} \tag{4.4}$$

Moreover, $\|u(t,\cdot)\|_\infty \le \|f\|_\infty$ for all $t \ge 0$.

Proof. To begin with, we observe that, if u is given by (4.4), then

$$\begin{aligned} |u(t,x)| \le & \left| \int_{\mathbb{R}^d} K(t,x-y)f(y)dy \right| \\ \le & \|f\|_\infty \int_{\mathbb{R}^d} K(t,x-y)dy = \|f\|_\infty \int_{\mathbb{R}^d} K(t,y)dy = \|f\|_\infty \end{aligned}$$

for every $(t,x) \in (0,\infty) \times \mathbb{R}^d$, by property (ii) in Lemma 4.1.2. It thus follows that $\|u(t,\cdot)\|_\infty \le \|f\|_\infty$ for all $t \ge 0$ as claimed.

Next, we observe that, since $K \in C^\infty((0,\infty)\times\mathbb{R}^d)$ (see Lemma 4.1.2(i)), by the dominated convergence theorem we conclude that $u \in C^\infty((0,\infty)\times\mathbb{R}^d)$, see Exercise 4.2.3, and

$$D_t u(t,x) - \Delta u(t,x) = \int_{\mathbb{R}^d} (D_t K(t,x-y) - \Delta K(t,x-y)) f(y)dy = 0$$

for every $(t,x) \in (0,\infty)\times\mathbb{R}^d$, thanks to Lemma 4.1.2(v).

To prove that u is a classical solution to problem (4.1) we need to show that u is continuous on $\{0\}\times\mathbb{R}^d$, where it equals the function f. For this purpose, we fix $t>0$ and, using the fact that $\int_{\mathbb{R}^d} K(t,x)\,dx = 1$ for all $t>0$, we split

$$\begin{aligned} |u(t,x) - f(x_0)| = & \left| \int_{\mathbb{R}^d} K(t,y)f(x-y)\,dy - \int_{\mathbb{R}^d} K(t,y)f(x_0)\,dy \right| \\ \le & \int_{\mathbb{R}^d} K(t,y)|f(x-y)-f(x_0)|\,dy \\ \le & \int_{\mathbb{R}^d} K(t,y)|f(x-y)-f(x)|\,dy + |f(x)-f(x_0)| \end{aligned}$$

for $x, x_0 \in \mathbb{R}^d$. By the change of variable $z = y/\sqrt{t}$ we can rewrite

$$\begin{aligned} \int_{\mathbb{R}^d} K(t,y)|f(x-y)-f(x)|dy = & (4\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{4}} |f(x-\sqrt{t}z)-f(x)|dz \\ = & (4\pi)^{-\frac{d}{2}} \int_{B(0,r)} e^{-\frac{|z|^2}{4}} |f(x-\sqrt{t}z)-f(x)|dz \\ & + (4\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d\setminus B(0,r)} e^{-\frac{|z|^2}{4}} |f(x-\sqrt{t}z)-f(x)|dz \end{aligned}$$

$$\begin{aligned}&\leq(4\pi)^{-\frac{d}{2}}\int_{B(0,r)}e^{-\frac{|z|^2}{4}}|f(x-\sqrt{t}z)-f(x)|dz\\ &\quad+2(4\pi)^{-\frac{d}{2}}\|f\|_\infty\int_{\mathbb{R}^d\setminus B(0,r)}e^{-\frac{|z|^2}{4}}dz \end{aligned}\tag{4.5}$$

for each $r>0$. Fix $\varepsilon>0$. By the absolutely continuity of the measure $e^{-|z|^2/4}dz$, we can find out $r_0>0$ such that

$$2(4\pi)^{-\frac{d}{2}}\|f\|_\infty\int_{\mathbb{R}^d\setminus B(0,r_0)}e^{-\frac{|z|^2}{4}}dy\leq\frac{\varepsilon}{2}.$$

As far as the first term in the last side of (4.5) is concerned, we notice that since f is continuous in $\mathbb{R}^d$, it is uniformly continuous in each compact set K of $\mathbb{R}^d$. Hence, if $x\in B(x_0,1)$, $z\in B(0,r_0)$ and $t<1$, then both $x-\sqrt{t}z$ and x belong to $K=\overline{B(0,M)}$ where $M=|x_0|+1+r_0$. Let $\delta>0$ be such that $|f(z_2)-f(z_1)|\leq\varepsilon/2$ if $z_1,z_2\in K$ satisfy $|z_2-z_1|\leq\delta$. If $t\leq t_0:=\delta^2r_0^{-2}$, then $|f(x-\sqrt{t}z)-f(x)|\leq\varepsilon/2$ for all $x\in B(x_0,1)$, so that

$$\begin{aligned}(4\pi)^{-\frac{d}{2}}\int_{B(0,r_0)}e^{-\frac{|z|^2}{4}}|f(x-\sqrt{t}z)-f(x)|\,dz\leq&\frac{\varepsilon}{2}(4\pi)^{-\frac{d}{2}}\int_{B(0,r)}e^{-\frac{|z|^2}{4}}dz\\ \leq&\frac{\varepsilon}{2}(4\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\frac{|z|^2}{4}}dz=\frac{\varepsilon}{2}\end{aligned}$$

and we conclude that $|u(t,x)-f(x_0)|\leq\varepsilon+|f(x)-f(x_0)|$ for every $(t,x)\in(0,t_0]\times B(x_0,1)$. It thus follows that

$$\limsup_{(t,x)\to(0,x_0)}|u(t,x)-f(x_0)|\leq\varepsilon$$

and the arbitrariness of $\varepsilon>0$ implies that $u(t,x)$ tends to $f(x_0)$ as $(t,x)\to(0,x_0)$.

To complete the proof, we observe that the uniqueness of the classical solution to the Cauchy problem (4.1) is implied by the maximum principle in Exercise 1.4.4. □

Remark 4.1.4. Up to now we have just verified that formula (4.4) defines the (unique) classical solution to the Cauchy problem (4.1). A natural question is how formula (4.4) can be derived. The (formal) answer is based on the use of the Fourier transform. For each $f\in L^1(\mathbb{R}^d)\cup L^\infty(\mathbb{R}^d)$, the Fourier transform of f is the function $\mathcal{F}(f)$ defined by

$$\mathcal{F}(f)(\xi):=(2\pi)^{-\frac{d}{2}}\int_{-\infty}^{\infty}\mathrm{e}^{-i\langle\xi,x\rangle}f(x)dx,\qquad\xi\in\mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product of $\mathbb{R}^d$. We recall that

$$\mathcal{F}(D_jf)(\xi)=i\xi_j\mathcal{F}(f)(\xi),\qquad\xi\in\mathbb{R}^d,\ \ j=1,\dots,d,$$

for every smooth enough function f. If we take the Fourier transform of both the sides of the equation $D_tu=\Delta u$ with respect to x and interchange the actions of $\mathcal{F}$ and the time derivative, then we deduce that the function $\widehat{u}$, defined by $\widehat{u}(t,\xi) = (\mathcal{F}(w(t,\cdot)))(\xi)$ for every $t \geq 0$ and $\xi \in \mathbb{R}^d$, solves the Cauchy problem

$$\begin{cases} D_t\widehat{u}(t,\xi) = -|\xi|^2\widehat{u}(t,\xi), & (t,\xi) \in (0,\infty) \times \mathbb{R}^d, \\ \widehat{u}(0,\xi) = \widehat{f}(\xi), & \xi \in \mathbb{R}^d. \end{cases}$$

This is a Cauchy problem for an ordinary differential equation, since ξ plays the role of a parameter, and it is easy to see that

$$\widehat{u}(t,\xi) = \mathrm{e}^{-t|\xi|^2}\widehat{f}(\xi), \qquad t \geq 0,\ \ \xi \in \mathbb{R}^d. \tag{4.6}$$

To get back to u, we take the inverse Fourier transform of the right-hand side of (4.6), recalling that the inverse Fourier transform of the product of two functions is the convolution of the inverse Fourier transforms of the two factors, and the inverse Fourier transform of the function $\xi \mapsto \mathrm{e}^{-t|\xi|^2}$ is the function $x \mapsto (4\pi t)^{-d/2}\mathrm{e}^{-|x|^2/(4t)}$ for $t > 0$.

The results in Theorem 4.1.3 can be rephrased in the semigroup language. More precisely, if for each $f \in C_b(\mathbb{R}^d)$ and $t \geq 0$ we set $T(t)f = u(t,\cdot)$, where u is the unique classical solution to problem (4.1), we define a semigroup of bounded linear operators in $C_b(\mathbb{R}^d)$, usually called the *Gauss-Weierstrass* semigroup. Indeed, the uniqueness of the classical solution to problem (4.1) for each $f \in C_b(\mathbb{R}^d)$, implied by the maximum principle in Exercise 1.4.4, shows that each operator $T(t)$ is linear on $C_b(\mathbb{R}^d)$. It also implies the semigroup rule, i.e.

$$T(t+s)f = T(t)T(s)f, \qquad f \in C_b(\mathbb{R}^d),\ \ t,s \geq 0.$$

In other terms the value at time t of the classical solution of problem (4.1) with initial condition $T(s)f$ at time zero coincides with the value at time $t+s$ of the classical solution to problem (4.1) with f as initial condition at time zero. Moreover, for $t > 0$ and $x \in \mathbb{R}^d$,

$$(T(t)f)(x) = (4\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d} K(t,x-y)f(y)dy, \qquad t > 0,\ \ x \in \mathbb{R}^d. \tag{4.7}$$

The Gauss-Weierstrass semigroup is not strongly continuous. It turns out that $T(t)f$ converges to f in $C_b(\mathbb{R}^d)$ as $t \to 0^+$ if and only if $f \in BUC(\mathbb{R}^d)$. Indeed, adapting the arguments in the last part of the proof of Theorem 4.1.3, we can easily show that, if f is uniformly continuous in $\mathbb{R}^d$, then $T(t)f$ converges uniformly in $\mathbb{R}^d$ to f as t tends to 0^+. Conversely, the forthcoming Proposition 4.1.9 shows that $T(t)f \in C^1_b(\mathbb{R}^d)$ for each $t > 0$. Hence, if $T(t)f$ converges to f in $C_b(\mathbb{R}^d)$ as $t \to 0^+$, then f belongs to the closure of $C^1_b(\mathbb{R}^d)$ in $C_b(\mathbb{R}^d)$, which is $BUC(\mathbb{R}^d)$.

On the other hand, as the following theorem shows $\{T(t)\}$ is an analytic semigroup.

Theorem 4.1.5. *The Gauss-Weierstrass semigroup is analytic in $C_b(\mathbb{R}^d)$.*

To prove the theorem, we need the following result.

Lemma 4.1.6. *Let $I \subset \mathbb{R}$ be an interval and let $\psi : I \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that $\|\psi(t,\cdot)\|_\infty \le g(t)$ for each $t \in I$ and some function $g \in L^1(I)$. Then, the function $\Psi : \mathbb{R}^d \to \mathbb{R}$, defined by*

$$\Psi(x) = \int_I \psi(t,x)dt, \qquad x \in \mathbb{R}^d,$$

is bounded and continuous and

$$(T(t)\Psi)(x) = \int_I (T(t)\psi(s,\cdot))(x)ds, \qquad t>0,\ \ x \in \mathbb{R}^d. \tag{4.8}$$

Proof. The dominated convergence theorem shows that the function Ψ is well defined and belongs to $C_b(\mathbb{R}^d)$ and formula (4.8) follows from Fubini theorem. Indeed, the function $(s,y) \mapsto e^{-|x-y|^2/(4t)}\psi(s,y)$ belongs to $L^1(I \times \mathbb{R}^d)$ for each $x \in \mathbb{R}^d$. Hence,

$$\begin{aligned}(T(t)\Psi)(x) =&(4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} \bigg(\int_I \psi(s,y)\,ds\bigg)dy\\ =&(4\pi t)^{-\frac{d}{2}} \int_{I\times\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}}\psi(s,y)\,ds\,dy = (4\pi t)^{-\frac{d}{2}}\int_I ds \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}}\psi(s,y)\,dy\\ =&\int_I (T(t)\psi(s,\cdot))(x)\,ds\end{aligned}$$

for $t>0$ and $x \in \mathbb{R}^d$, and we are done ☐

Proof. We denote by Π the right-halfplane, i.e. the set of all $\lambda \in \mathbb{C}$ with positive real part. For each $\lambda \in \Pi$, we introduce the operator R_λ defined by

$$(R_\lambda f)(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)dt, \qquad x \in \mathbb{R}^d,\ \ f \in C_b(\mathbb{R}^d;\mathbb{C}),$$

Observe that, by the dominated convergence theorem, $R_\lambda f$ is a bounded and continuous function in $\mathbb{R}^d$ for every $f \in C_b(\mathbb{R}^d,\mathbb{C})$ and $\lambda \in \Pi$. Moreover, R_λ is a bounded operator for every λ as above. Finally, if f is real valued and $\lambda > 0$, then R_λ is real valued too.

It is easily seen that $\{R_\lambda : \lambda \in \Pi\}$ is a resolvent family: indeed, taking Lemma 4.1.6 into account, we can show that, for every $\lambda, \mu \in \Pi$, it holds that

$$\begin{aligned}(R_\lambda R_\mu f)(x) =& \int_0^\infty e^{-\lambda t}\bigg(T(t)\int_0^\infty e^{-\mu s}(T(s)f)(\cdot)\,ds\bigg)(x)\,dt\\ =&\int_0^\infty dt \int_0^\infty e^{-\lambda t-\mu s}(T(t+s)f)(x)\,ds\\ =&\int_0^\infty e^{-\mu\sigma}(T(\sigma)f)(x)\,d\sigma \int_0^\sigma e^{(\mu-\lambda)t}dt = \int_0^\infty e^{-\mu\sigma}(T(\sigma)f)(x)\frac{e^{(\mu-\lambda)\sigma}-1}{\mu-\lambda}d\sigma\\ =&\frac{1}{\mu-\lambda}[(R_\lambda f)(x) - (R_\mu f)(x)]\end{aligned}$$

for all $x \in \mathbb{R}^d$. Let us prove that R_λ is injective for each $\lambda \in \Pi$. For this purpose, we fix $\lambda_0 \in \Pi$ and $f \in C_b(\mathbb{R}^d)$ such that $R_{\lambda_0} f \equiv 0$. The resolvent identity proved above shows that $R_\lambda f \equiv 0$ for each $\lambda \in \Pi$. Since, for every $x \in \mathbb{R}^d$, the function $\lambda \mapsto (R_\lambda f)(x)$ is the Laplace transform of the bounded and continuous function $t \mapsto (T(t)f)(x)$, by the uniqueness of the Laplace transform, we conclude that $(T(t)f)(x) = 0$ for all $t \geq 0$. Taking $t = 0$ we conclude that $f(x) = 0$. The arbitrariness of $x \in \mathbb{R}^d$ implies that $f \equiv 0$. Hence, R_λ is injective.

By Proposition A.3.1, it follows that there exists a closed operator $A : D(A) \subset C_b(\mathbb{R}^d; \mathbb{C}) \to C_b(\mathbb{R}^d; \mathbb{C})$, whose resolvent set contains the right-halfplane, and such that $R(\lambda, A) = R_\lambda$ for each $\lambda \in \Pi$.

We claim that A is a sectorial operator. To check the claim, we will show that there exists a positive constant C such that

$$\|R(\lambda, A)\|_{L(C_b(\mathbb{R}^d,\mathbb{C}))} \leq C|\lambda|^{-1}, \qquad \lambda \in \mathbb{C},\ \mathrm{Re}\lambda \geq 1. \tag{4.9}$$

Indeed, from (4.9) and Proposition 3.2.8, the sectoriality of A follows at once.

We begin by observing that, for each $f \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the function $t \mapsto (T(t)f)(x)$ can be extended to the right-halfplane with a holomorphic function. Indeed, for $x \in \mathbb{R}^d$, the function $K(\cdot, x) : \Pi \to \mathbb{C}$, defined by $K(z, x) = (4\pi z)^{-d/2} e^{-|x|^2/(4z)}$ is holomorphic in Π. Moreover, $K(z, \cdot) \in L^1(\mathbb{R}^d, \mathbb{C})$ for every $z \in \Pi$ and

$$\int_{\mathbb{R}^d} |K(z, x)| dx = \left(\frac{\mathrm{Re} z}{|z|}\right)^{-\frac{d}{2}}, \qquad z \in \Pi.$$

Hence, the function

$$x \mapsto (T(z)f)(x) = \int_{\mathbb{R}^d} K(z, x - y) f(y)\, dz \tag{4.10}$$

is well defined, bounded and continuous in $\mathbb{R}^d$, for every $z \in \Pi$, as it can be easily seen again applying the dominated convergence theorem. Moreover,

$$\|T(z)f\|_\infty \leq \left(\frac{\mathrm{Re} z}{|z|}\right)^{-\frac{d}{2}} \|f\|_\infty, \qquad z \in \Pi.$$

In particular, if for each $\vartheta_0 \in (0, \pi/2)$, we denote by Σ_{ϑ_0} the sector of all $\lambda \in \mathbb{C} \setminus \{0\}$ such that $|\arg(\lambda)| \leq \vartheta_0$, then, from the previous estimate it follows immediately that

$$\|T(z)\|_{C_b(\mathbb{R}^d,\mathbb{C})} \leq [1 + (\tan(\vartheta_0))^2]^{\frac{d}{4}} \|f\|_\infty, \qquad z \in \Sigma_{\vartheta_0}.$$

Now, we can prove (4.9). If $\lambda = a + iy$ with $a \geq 1$ and $y \geq 0$, then by Cauchy integral theorem we have

$$(R(\lambda, A)f)(x) = \int_0^\infty e^{-\lambda t} (T(t)f)(x)\, dt = \int_\gamma e^{-\lambda z} (T(z)f)(x)\, dz, \qquad x \in \mathbb{R}^d,$$

where $\gamma(s) = s - is$ for every $s \geq 0$. Therefore

$$\|R(\lambda, A)f\|_\infty \leq 2^{\frac{d}{4}}\|f\|_\infty \int_0^\infty e^{-(a+y)s} ds \leq 2^{\frac{d}{4}}|\lambda|^{-1}\|f\|_\infty.$$

If $y \leq 0$ then one gets the same estimate replacing the curve γ with the curve $\widetilde{\gamma}$, defined by $\widetilde{\gamma}(s) = s + is$ for every $s \geq 0$. Estimate (4.9) follows.

Denote by $\{S(t)\}$ the analytic semigroup generated in $C_b(\mathbb{R}^d, \mathbb{C})$ by A. By Proposition 2.1.12, if $\lambda \in \mathbb{C}$ has sufficiently large real part, then

$$\int_0^\infty e^{-\lambda t}(S(t)f)(x)\, dt = (R(\lambda, A)f)(x) = (R_\lambda f)(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)\, dt$$

for $x \in \mathbb{R}^d$, i.e.

$$\int_0^\infty e^{-\lambda t}[(S(t)f)(x) - (T(t)f)(x)]\, dt = 0, \qquad x \in \mathbb{R}^d.$$

Again, the uniqueness of the Laplace transform implies that $S(t)f \equiv T(t)f$ in $(0, \infty)$. We have so proved that the Gauss-Weierstrass semigroup is analytic in $C_b(\mathbb{R}^d)$. □

Remark 4.1.7. Without much effort, we can show that the sectorial operator A associated with the Gauss-Weierstrass semigroup is an extension of the operator $(\Delta, C_b^2(\mathbb{R}^d))$. Indeed, fix $f \in C_b^2(\mathbb{R}^d)$. Recalling that each operator $T(t)$ commutes with the Laplacian and integrating by parts, we get

$$\begin{aligned}(R(1,A)(f-\Delta f))(x) =& \int_0^\infty e^{-t}(T(t)(f-\Delta f))(x)\, dt \\ =& \int_0^\infty e^{-t}((T(t)f)(x) - (\Delta T(t)f)(x))\, dt \\ =& \int_0^\infty e^{-t}(T(t)f)(x)\, dt - \int_0^\infty e^{-t}(D_t T(t)f)(x)\, dt \\ =& (R(1,A)f)(x) + f(x) - (R(1,A)f)(x) = f(x)\end{aligned}$$

for $x \in \mathbb{R}^d$. Hence, $R(1, A)(f - \Delta f) = f$. Applying the operator $I - A$ to both the sides of the previous equality we obtain $f - \Delta f = f - Af$, i.e., $Af = \Delta f$. Therefore, A is an extension of the operator $(\Delta, C_b^2(\mathbb{R}^d))$.

If $N = 1$, then A actually coincides with the second order derivative with $C_b^2(\mathbb{R})$ as domain. On the other hand, if $N > 2$, then A is a proper extension of $(\Delta, C_b^2(\mathbb{R}^d))$. In fact, $D(A) = \{u \in C_b(\mathbb{R}^d) \cap W^{2,p}_{\rm loc}(\mathbb{R}^d)$, for all $p \in [1, \infty)$, and $\Delta u \in C_b(\mathbb{R}^d)\}$ and $Au = \Delta u$ for each $u \in D(A)$. The proof of the above characterization of $(A, D(A))$ is beyond the purposes of this lecture. We refer the interested reader to [1, Proposition 4.1.10]

4.1.1 Estimates of the spatial derivatives of $T(t)f$

To begin with, let us consider the following lemma.

Lemma 4.1.8. *Fix $t > 0$, $a \in \mathbb{R}^d$, $k \in \mathbb{N}$. Then,*

$$\int_{\mathbb{R}^d} (\langle x, a\rangle)^k K(t,x)dx = \begin{cases} 0, & k \text{ odd}, \\ \dfrac{(2m)!}{m!} t^m |a|^{2m}, & k = 2m, \ m \in \mathbb{N}. \end{cases} \tag{4.11}$$

Proof. Fix $\lambda > 0$, $t > 0$ and observe that

$$\int_{\mathbb{R}^d} e^{\lambda\langle x,a\rangle} K(t,x)dx = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\lambda\langle x,a\rangle} e^{-\frac{|x|^2}{4t}} dx = \frac{1}{(4\pi t)^{\frac{d}{2}}} \prod_{i=1}^{d} \int_{\mathbb{R}} e^{\lambda x_i a_i - \frac{x_i^2}{4t}} dx_i. \tag{4.12}$$

Note that

$$\int_{\mathbb{R}} e^{\lambda x_i a_i - \frac{x_i^2}{4t}} dx = \int_{\mathbb{R}} e^{-\left(\frac{x_i}{2\sqrt{t}} - \lambda a_i\sqrt{t}\right)^2} e^{\lambda^2 a_i^2 t} dx_i \underset{(s=\frac{x_i}{2\sqrt{t}} - \lambda a_i\sqrt{t})}{=} e^{\lambda^2 a_i^2 t} \int_{\mathbb{R}} e^{-s^2} 2\sqrt{t} ds = \sqrt{4\pi t} e^{\lambda^2 a_i^2 t}$$

for $i = 1, \dots, d$. Replacing this formula in (4.12), we obtain

$$\int_{\mathbb{R}^d} e^{\lambda\langle x,a\rangle} K(t,x)dx = e^{t\lambda^2|a|^2}.$$

Differentiating k-times with respect to λ both sides of the previous formula and then computing the so obtained derivative at $\lambda = 0$ gives

$$\left(\frac{\partial^k}{\partial\lambda^k} e^{t\lambda^2|a|^2}\right)_{|\lambda=0} = \int_{\mathbb{R}^d} (\langle x,a\rangle)^k K(t,x)dx.$$

Now, an easy computation yields

$$\left(\frac{\partial^k}{\partial\lambda^k} e^{t\lambda^2|a|^2}\right)_{|\lambda=0} = \begin{cases} 0, & k \text{ odd}, \\ \dfrac{(2m)!}{m!} t^m |a|^{2m}, & k = 2m, \ m \in \mathbb{N}. \end{cases}$$

□

Proposition 4.1.9. *The following estimates hold true.*

(i) $\|\nabla_x T(t)f\|_\infty \le \dfrac{1}{\sqrt{2t}} \|f\|_\infty$ *for $t > 0$ and $f \in C_b(\mathbb{R}^d)$;*

(ii) $\|D_{ij} T(t)f\|_\infty \le \dfrac{\sqrt{1+\delta_{ij}}}{2t} \|f\|_\infty$ *for $t > 0$, $f \in C_b(\mathbb{R}^d)$ and $i, j = 1, \dots, d$;*

(iii) $\|D_{ij} T(t)f\|_\infty \le \dfrac{1}{\sqrt{2t}} \|\nabla_x f\|_\infty$ *for $t > 0$, $f \in C_b^1(\mathbb{R}^d)$ and $i, j = 1, \dots, d$;*

(iv) $\|D_{ijk}T(t)f\|_\infty \le \dfrac{1}{\sqrt{8}t\sqrt{t}}[1+\delta_{ij}+\delta_{jk}+\delta_{ik}+2\delta_{ij}\delta_{jk}]^{\frac{1}{2}}\|f\|_\infty$ *for* $f\in C_b(\mathbb{R}^d)$ *and* $i,j,k=1,\dots,d$*;*

(v) $\|D_{ijk}T(t)f\|_\infty \le \dfrac{\sqrt{1+\delta_{ij}\delta_{jk}}}{2t}\|\nabla_x f\|_\infty$ *for* $t>0$*,* $f\in C^1_b(\mathbb{R}^d)$ *and* $i,j,k=1,\dots,d$*;*

(vi) $\|D_{ijk}T(t)f\|_\infty \le \dfrac{1}{\sqrt{2t}}\|D^2 f\|_\infty$ *for* $t>0$*,* $f\in C^2_b(\mathbb{R}^d)$ *and* $i,j,k=1,\dots,d$*.*

Proof. (i) Fix $f\in C_b(\mathbb{R}^d)$. By differentiating under the integral sign and taking Lemma 4.1.2 into account, we get

$$\begin{aligned}(\nabla_x T(t)f)(x) =&\left(\nabla_x\int_{\mathbb{R}^d}K(t,\cdot-y)f(y)dy\right)(x)=\int_{\mathbb{R}^d}\nabla_x K(t,x-y)f(y)dy\\ =&-\int_{\mathbb{R}^d}\frac{x-y}{2t}K(t,x-y)f(y)dy\end{aligned}$$

for all $t>0$ and $x\in\mathbb{R}^d$. It thus follows that

$$\langle(\nabla_x T(t)f)(x),a\rangle=-\frac{1}{2t}\int_{\mathbb{R}^d}\langle x-y,a\rangle K(t,x-y)f(y)\,dy,\qquad t>0,\ \ a,x\in\mathbb{R}^d,$$

and, consequently, using Cauchy-Schwarz inequality and (4.11), we conclude that

$$\begin{aligned}|\langle(\nabla_x T(t)f)(x),a\rangle|^2=&\frac{1}{4t^2}\left(\int_{\mathbb{R}^d}[\langle x-y,a\rangle\sqrt{K(t,x-y)}]\sqrt{K(t,x-y)}f(y)\,dy\right)^2\\ \le&\frac{1}{4t^2}\int_{\mathbb{R}^d}(\langle x-y,a\rangle)^2K(t,x-y)dy\int_{\mathbb{R}^d}K(t,x-y)(f(y))^2dy\\ \le&\frac{1}{2t}|a|^2\int_{\mathbb{R}^d}K(t,x-y)(f(y))^2dy\\ \le&\frac{1}{2t}|a|^2\|f\|_\infty^2\end{aligned}$$

for all $t>0$, $a,x\in\mathbb{R}^d$. Taking $a=(\nabla_x T(t)f)(x)$, yields

$$|(\nabla_x T(t)f)(x)|^4\le\frac{1}{2t}|(\nabla_x T(t)f)(x)|^2\|f\|_\infty^2.$$

So, property (i) follows.

(ii) Fix $f\in C_b(\mathbb{R}^d)$. Applying Lemma 4.1.2(iv), one obtains, using again Hölder's inequality,

$$\begin{aligned}|(D_{ij}T(t)f)(x)|^2=&\left|\int_{\mathbb{R}^d}\left(\frac{y_iy_j}{4t^2}-\frac{\delta_{ij}}{2t}\right)K(t,y)f(x-y)\,dy\right|^2\\ \le&\int_{\mathbb{R}^d}K(t,y)(f(x-y))^2dy\int_{\mathbb{R}^d}\left(\frac{y_iy_j}{4t^2}-\frac{\delta_{ij}}{2t}\right)^2K(t,y)\,dy\end{aligned}$$

$$
\begin{aligned}
\leq&\|f\|_\infty^2\int_{\mathbb{R}^d}\left(\frac{y_iy_j}{4t^2}-\frac{\delta_{ij}}{2t}\right)^2K(t,y)\,dy\\
=&\|f\|_\infty^2\left(\frac{1}{16t^4}\int_{\mathbb{R}^d}y_i^2y_j^2K(t,y)\,dy+\frac{\delta_{ij}}{4t^2}-\frac{\delta_{ij}}{4t^3}\int_{\mathbb{R}^d}y_iy_jK(t,y)\,dy\right).
\end{aligned}
$$

Now, if $i=j$, then by (4.11) with $k=2,4$ and a being the i-th element of the Euclidean basis of $\mathbb{R}^d$, we get

$$
(a)\ \int_{\mathbb{R}^d}y_i^2K(t,y)dy=2t,\qquad (b)\ \int_{\mathbb{R}^d}y_i^4K(t,y)dy=12t^2. \tag{4.13}
$$

On the other hand, if $i\neq j$, then, using the one-dimensional version of formula (4.11), we get

$$
\begin{aligned}
\int_{\mathbb{R}^d}y_i^2y_j^2K(t,y)\,dy=&\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}y_i^2e^{-\frac{y_i^2}{4t}}dy_i\right)\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}y_j^2e^{-\frac{y_j^2}{4t}}dy_j\right)\prod_{h\neq i,j}\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}e^{-\frac{y_h^2}{4t}}dy_h\right)\\
=&4t^2. \qquad (4.14)
\end{aligned}
$$

Hence,

$$
\frac{1}{16t^4}\int_{\mathbb{R}^d}y_i^2y_j^2K(t,y)\,dy+\frac{\delta_{ij}}{4t^2}-\frac{\delta_{ij}}{4t^3}\int_{\mathbb{R}^d}y_iy_jK(t,y)\,dy=\begin{cases}\dfrac{1}{4t^2}, & i\neq j,\\[2mm] \dfrac{1}{2t^2}, & i=j.\end{cases}
$$

Property (ii) follows.

(iii) As it is immediately seen, if $f\in C_b^1(\mathbb{R}^d)$, then $(D_iT(t)f)(x)=(T(t)D_if)(x)$ for $(t,x)\in[0,\infty)\times\mathbb{R}^d$, $i=1,\dots,d$ and, consequently, thanks to property (i), we can estimate

$$
\|D_{ij}T(t)\|_\infty=\|D_iT(t)D_jf\|_\infty\leq\|\nabla_xT(t)D_jf\|_\infty\leq\frac{1}{\sqrt{2t}}\|D_jf\|_\infty\leq\frac{1}{\sqrt{2t}}\|\nabla f\|_\infty
$$

and we are done.

The proofs of (iv), (v) and (vi) are left to the reader as an exercise, see Exercise 4.2.5. □

Remark 4.1.10. From Proposition 4.1.9(ii), it follows that $\Delta T(t)$ is a bounded operator in $C_b(\mathbb{R}^d)$ for every $t>0$ and

$$
\|\Delta T(t)\|_{L(C_b(\mathbb{R}^d))}\leq\frac{d}{t\sqrt{2}},\qquad t>0. \tag{4.15}
$$

Now, we want to prove estimates similar to those in Proposition 4.1.9, when $C_b(\mathbb{R}^d)$, $C_b^1(\mathbb{R}^d)$ and $C_b^2(\mathbb{R}^d)$ are replaced by some subspaces of Hölder continuous functions. For this purpose, we need the following preliminary result.

Lemma 4.1.11. *For every $\theta \in (0,1)$ there exists a positive constant $C = C(\theta)$ such that*

$$\|f\|_{C^\theta_b(\mathbb{R}^d)} \le C\|f\|^\theta_{C^1_b(\mathbb{R}^d)}\|f\|^{1-\theta}_\infty, \qquad f \in C^1_b(\mathbb{R}^d). \tag{4.16}$$

Proof. Fix $x, y \in \mathbb{R}^d$. Using the mean value theorem, we can estimate $|f(x) - f(y)| \le \|\nabla f\|_\infty |x-y|$. Moreover, $|f(x)-f(y)| \le |f(x)| + |f(y)| \le 2\|f\|_\infty$. Hence, writing $|f(x) - f(y)| = |f(x) - f(y)|^\theta |f(x) - f(y)|^{1-\theta}$ and using the previous estimates, we conclude that $|f(x)-f(y)| \le 2^{1-\theta}\|\nabla f\|^\theta_\infty \|f\|^{1-\theta}_\infty |x-y|^\theta$, which shows that $[f]_{C^\theta_b(\mathbb{R}^d)} \le 2^{1-\theta}\|\nabla f\|^\theta_\infty\|f\|^{1-\theta}_\infty$. Now, estimate (4.16) follows at once. □

Using the previous lemma and Proposition 4.1.9, we can now prove the following result.

Theorem 4.1.12. *For every $0 \le \alpha \le \theta \le 3$ and $T > 0$ there exists a positive constant $C_{\alpha,\theta,T}$ such that*

$$\|T(t)f\|_{C^\theta_b(\mathbb{R}^d)} \le C_{\alpha,\theta,T}t^{-\frac{\theta-\alpha}{2}}\|f\|_{C^\alpha_b(\mathbb{R}^d)}, \qquad t \in (0,T],\ \ f \in C^\alpha_b(\mathbb{R}^d). \tag{4.17}$$

Proof. Fix $T > 0$. In view of Proposition 4.1.9, we can confine ourselves to the case when at least one between α and θ is not an integer. The crucial point is the proof of (4.17) with $\alpha \in (0,1)$ and $\theta = 1$. For this purpose, we observe that

$$(D_iT(t)f)(x) = \frac{1}{2t(4\pi t)^{d/2}}\int_{\mathbb{R}^d}(y_i - x_i)e^{-\frac{|y-x|^2}{4t}}f(y)dy, \qquad t > 0,\ \ x \in \mathbb{R}^d,\ \ i = 1,\dots,d.$$

As it is immediately seen,

$$\int_{\mathbb{R}}(y_i - x_i)e^{-\frac{(y_i-x_i)^2}{4t}}dy_i = 0, \qquad i = 1,\dots,d,$$

whence

$$\int_{\mathbb{R}^d}(y_i - x_i)e^{-\frac{|y-x|^2}{4t}}dy = \int_{\mathbb{R}}(y_i - x_i)e^{-\frac{(y_i-x_i)^2}{4t}}dy_i\prod_{j\ne i}\int_{\mathbb{R}}e^{-\frac{(y_j-x_j)^2}{4t}}dy_j = 0.$$

It thus follows that

$$\begin{aligned}
|(D_iT(t)f)(x)| =&\frac{1}{2t(4\pi t)^{d/2}}\left|\int_{\mathbb{R}^d}(y_i - x_i)e^{-\frac{|y-x|^2}{4t}}(f(y) - f(x))dy\right|\\
\le&\frac{1}{2t(4\pi t)^{d/2}}\int_{\mathbb{R}^d}|y_i - x_i|e^{-\frac{|y-x|^2}{4t}}|f(y) - f(x)|dy\\
\le&\frac{1}{2t(4\pi t)^{d/2}}[f]_{C^\alpha_b(\mathbb{R}^d)}\int_{\mathbb{R}^d}|y-x|^{1+\alpha}e^{-\frac{|y-x|^2}{4t}}dy\\
=&\frac{1}{2(4\pi)^{d/2}}t^{\frac{\alpha-1}{2}}[f]_{C^\alpha_b(\mathbb{R}^d)}\int_{\mathbb{R}^d}|z|^{1+\alpha}e^{-\frac{|z|^2}{4}}dz
\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $t > 0$. We thus conclude that $\|\nabla_xT(t)f\|_\infty \le c_\alpha t^{(\alpha-1)/2}\|f\|_{C^\alpha_b(\mathbb{R}^d)}$ for all $t > 0$. Since $\|T(t)f\|_\infty \le \|f\|_\infty$ for $t > 0$, estimate (4.17) follows at once in this case.

Using the semigroup rule and Proposition 4.1.9(i) we can prove (4.17) with $\alpha = 2$ and $\theta = 3$. Indeed, we can estimate

$$\begin{aligned}\|D_{ij}T(t)f\|_\infty =&\|D_{ij}T(t/2)T(t/2)f\|_\infty = \|D_iT(t/2)D_jT(t/2)f\|_\infty\\ \le&\|D_iT(t/2)\|_{L(C_b(\mathbb{R}^d))}\|D_jT(t/2)f\|_\infty \le c'_\alpha t^{\frac{\alpha}{2}-1}[f]_{C^\alpha_b(\mathbb{R}^d)}\end{aligned}$$

for all $t > 0$. Similarly, using Proposition 4.1.9(ii) we get

$$\begin{aligned}\|D_{ijk}T(t)f\|_\infty =&\|D_{ij}T(t/2)D_kT(t/2)f\|_\infty \le \|D_{ij}T(t/2)\|_{L(C_b(\mathbb{R}^d))}\|D_kT(t/2)f\|_\infty\\ \le& c''_\alpha t^{\frac{\alpha-3}{2}}[f]_{C^\alpha_b(\mathbb{R}^d)}\end{aligned}$$

for all $t > 0$, $i, j, k = 1, \dots, d$.

Now, suppose that $f \in C^\alpha_b(\mathbb{R}^d)$ for some $\alpha \in [1, 2)$. Since $D_iT(t)f = T(t)D_if$ for all $t > 0$ and $i = 1, \dots, d$, and $D_if \in C^{\alpha-1}_b(\mathbb{R}^d)$, we can estimate

$$\|D_{ij}T(t)f\|_\infty = \|D_jT(t)D_if\|_\infty \le c_{\alpha-1}t^{-1+\frac{\alpha}{2}}[D_if]_{C^{\alpha-1}_b(\mathbb{R}^d)}$$

for all $t > 0$ and $i, j = 1, \dots, d$. Since

$$\|T(t)f\|_{C^1_b(\mathbb{R}^d)} = \|T(t)f\|_\infty + \sum_{j=1}^d \|D_jT(t)f\|_\infty = \|T(t)f\|_\infty + \sum_{j=1}^d \|T(t)D_jf\|_\infty \le \|f\|_{C^1_b(\mathbb{R}^d)}$$

for all $t > 0$, from the previous estimate, (4.17) with $\theta = 2$ follows. Similarly,

$$\|D_{ijk}T(t)f\|_\infty = \|D_{jk}T(t)D_if\|_\infty \le c'_{\alpha-1}t^{-\frac{3-\alpha}{2}}[f]_{C^{\alpha-1}_b(\mathbb{R}^d)}$$

for all $t > 0$ and $i, j, k = 1, \dots, d$ and (4.17), with $\theta = 3$ follows.

The proof of (4.17) for $\alpha \in [2, 3)$ and $\theta = 3$ is completely similar and, hence, left to the reader.

To conclude the proof, we should check (4.17) for $\theta \notin \mathbb{N}$. For this purpose, it suffices to apply Lemma 4.1.11. To fix the ideas, we prove it for $\theta \in (2, 3)$ and $\alpha \in (0, 1)$, but all the other cases can be analyzed just in the same way.

By the above results, we know that

$$\|T(t)f\|_{C^2_b(\mathbb{R}^d)} \le C_{\alpha,2}t^{\frac{\alpha-2}{2}}\|f\|_{C^\alpha_b(\mathbb{R}^d)}, \qquad t > 0$$

and

$$\|T(t)f\|_{C^3_b(\mathbb{R}^d)} \le C_{\alpha,3}t^{\frac{\alpha-3}{2}}\|f\|_{C^\alpha_b(\mathbb{R}^d)}, \qquad t > 0.$$

Moreover, from Exercise 4.2.7 it follows that

$$\|T(t)f\|_{C^\theta_b(\mathbb{R}^d)} \le C_\theta\|T(t)f\|^{\theta-2}_{C^3_b(\mathbb{R}^d)}\|T(t)f\|^{3-\theta}_{C^2_b(\mathbb{R}^d)}$$

for each $f \in C^\alpha_b(\mathbb{R}^d)$, $t > 0$ and some positive constant C_θ. Hence,

$$\begin{aligned}\|T(t)f\|_{C^\theta_b(\mathbb{R}^d)} \le& C_\theta\|T(t)f\|^{\theta-2}_{C^3_b(\mathbb{R}^d)}\|T(t)f\|^{3-\theta}_{C^2_b(\mathbb{R}^d)}\\ \le& C_\theta\big[C_{\alpha,3}t^{\frac{\alpha-3}{2}}\|f\|_{C^\alpha_b(\mathbb{R}^d)}\big]^{\theta-2}\big[C_{\alpha,2}t^{\frac{\alpha-2}{2}}\|f\|_{C^\alpha_b(\mathbb{R}^d)}\big]^{3-\theta}\\ =& C_\theta C^{\theta-2}_{\alpha,3}C^{3-\theta}_\alpha t^{\frac{\alpha-\theta}{2}}\|f\|_{C^\alpha_b(\mathbb{R}^d)}\end{aligned}$$

for all $t > 0$. This completes the proof. □

4.2 Exercises

1. Prove that $BUC(\mathbb{R}^d)$ is the closure of $C^1_b(\mathbb{R}^d)$ in $C_b(\mathbb{R}^d)$.
2. Prove that the function $z \mapsto T(z)f$, defined in (4.10) is holomorphic in the sector Σ_ϑ for every $\vartheta \in (0, \pi/2)$.
3. Prove that the function
$$u(t,x) = \int_{\mathbb{R}^d} K(t, x-y) f(y)\, dy, \quad t > 0,\, x \in \mathbb{R}^d,$$
belongs to $C^\infty((0,+\infty) \times \mathbb{R}^d)$ for each $f \in C_b(\mathbb{R}^d)$.
4. Prove that the sectorial operator associated with the one-dimensional Gauss-Weierstrass semigroup is the second-order derivative with $C^2_b(\mathbb{R})$ as domain.
5. Prove properties (iv), (v) and (vi) in Proposition 4.1.9.
6. Use (4.9) and Proposition 3.2.8 to prove that the operator
$$A + b(\cdot)\nabla \ \text{ with domain } \ D(A)$$
generates an analytic semigroup in $C_b(\mathbb{R}^d)$, where $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ and A the generator of the Gauss-Weierstrass semigroup $\{T(t)\}$ in $C_b(\mathbb{R}^d)$.
7. Prove that for each $\theta \in (2,3)$ there exists a positive constant C_θ such that
$$\|f\|_{C^\theta_b(\mathbb{R}^d)} \le C_\theta \|f\|^{\theta-2}_{C^3_b(\mathbb{R}^d)} \|f\|^{3-\theta}_{C^2_b(\mathbb{R}^d)}$$
for every $f \in C^\theta_b(\mathbb{R}^d)$.

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Lecture 5

The heat equation and the Gauss-Weierstrass semigroup in $C_b(\mathbb{R}^d)$. Part 2

In this lecture, we begin by proving that the Hölder spaces can be characterized by means of the Gauss-Weierstrass semigroup and, then, we study the nonhomogeneous heat equation.

5.1 Two equivalent characterizations of the Hölder spaces

Let us introduce the space $D_\Delta(\theta,\infty)$.

Definition 5.1.1. For each $\theta \in (0,1)$, we denote by $D_\Delta(\theta,\infty)$ the set of all functions $f \in C_b(\mathbb{R}^d)$ such that

$$[f]_{D_\Delta(\theta,\infty)} := \sup_{t\in(0,\infty)} t^{1-\theta}\|\Delta T(t)f\|_\infty < \infty.$$

Remark 5.1.2. For each $\theta \in (0,1)$, $D_\Delta(\theta,\infty)$ is a Banach space when endowed with the norm

$$\|f\|_{D_\Delta(\theta,\infty)} = \|f\|_\infty + [f]_{D_\Delta(\theta,\infty)}, \qquad f \in D_\Delta(\theta,\infty).$$

To begin with, we prove the following (abstract) equivalent characterization of the space $D_\Delta(\theta,\infty)$.

Proposition 5.1.3. *For each $\theta \in (0,1)$, $D_\Delta(\theta,\infty)$ is the space of all functions $f \in C_b(\mathbb{R}^d)$ such that*

$$[[f]]_\theta = \sup_{t>0} t^{-\theta}\|T(t)f - f\|_\infty < \infty.$$

Moreover, the norm of $D_\Delta(\theta,\infty)$ is equivalent to the norm $\|\cdot\|_\infty + [[\cdot]]$.

Remark 5.1.4. Since $\|T(t)f\|_\infty \le \|f\|_\infty$ for all $t \ge 0$, it is immediate to see that $[[f]]_\theta < \infty$ if and only if

$$[[[f]]]_\theta = \sup_{t\in(0,1)} t^{-\theta}\|T(t)f - f\|_\infty < \infty.$$

Moreover, the norms $\|\cdot\|_\infty + [[\cdot]]$ and $\|\cdot\|_\infty + [[[\cdot]]]$ are equivalent.

Proof of Proposition 5.1.3. Suppose that $f \in D_\Delta(\theta,\infty)$. Then, in view of the fundamental theorem of calculus we can write

$$(T(t)f)(x) - (T(\varepsilon)f)(x) = \int_\varepsilon^t (D_tT(s)f)(x)ds = \int_\varepsilon^t (\Delta T(s)f)(x)ds$$

for $0 < \varepsilon < t$ and $x \in \mathbb{R}^d$. Since, by assumptions $\|\Delta T(t)f\|_\infty \le t^{\theta-1}[f]_{D_\Delta(\theta,\infty)}$ for $t > 0$, from the previous formula we can infer that

$$|(T(t)f)(x) - (T(\varepsilon)f)(x)| \le [f]_{D_\Delta(\theta,\infty)} \int_\varepsilon^t s^{\theta-1}ds = [f]_{D_\Delta(\theta,\infty)}\theta^{-1}(t^\theta - \varepsilon^\theta).$$

Letting ε tend to 0^+ shows that

$$[[f]]_\theta \le \theta^{-1}[f]_{D_\Delta(\theta,\infty)}. \tag{5.1}$$

Vice versa, let us suppose that $[[f]]_\theta < \infty$ and let us prove that $f \in D_\Delta(\theta,\infty)$. For this purpose, we split

$$f(x) = \frac{1}{t-\varepsilon}\int_\varepsilon^t (f(x) - (T(s)f)(x))ds + \frac{1}{t-\varepsilon}\int_\varepsilon^t (T(s)f)(x)ds. \tag{5.2}$$

for $0 < \varepsilon < t$ and $x \in \mathbb{R}^d$. Applying $\Delta T(t)$ to both the sides of the previous formula, by dominated convergence we obtain

$$(\Delta T(t)f)(x) = \frac{1}{t-\varepsilon}\bigg(\Delta T(t)\int_\varepsilon^t (f - T(s)f)ds\bigg)(x) + \frac{1}{t-\varepsilon}\bigg(\Delta T(t)\int_\varepsilon^t T(s)f ds\bigg)(x). \tag{5.3}$$

Taking (4.15) into account we can estimate

$$\begin{aligned}\bigg\|\frac{1}{t-\varepsilon}\Delta T(t)\int_\varepsilon^t (f(\cdot) - (T(s)f)(\cdot))ds\bigg\|_\infty \le& \frac{d}{t\sqrt{2}(t-\varepsilon)}[[f]]_\theta\int_\varepsilon^t s^\theta ds\\ \le& \frac{d}{\sqrt{2}(1+\theta)}\frac{t^\theta}{t-\varepsilon}[[f]]_\theta.\end{aligned} \tag{5.4}$$

As far as the second integral term in (5.2) is concerned, we observe that Proposition 4.1.9 and the dominated convergence theorem show that the function $x \mapsto \int_\varepsilon^t (T(s)f)(x)ds$

belongs to $C_b^2(\mathbb{R}^d)$. Since $T(t)$ commutes with Δ on $C_b^2(\mathbb{R}^d)$ and $\|T(t)\|_{L(C_b(\mathbb{R}^d))} = 1$, we conclude that

$$\begin{aligned}\left\|\frac{1}{t-\varepsilon}\Delta T(t)\int_\varepsilon^t (T(s)f(\cdot))ds\right\|_\infty =&\frac{1}{t-\varepsilon}\left\|T(t)\int_\varepsilon^t (\Delta T(s)f)(\cdot)ds\right\|_\infty\\ =&\frac{1}{t-\varepsilon}\left\|T(t)\int_\varepsilon^t (D_sT(s)f)(\cdot)ds\right\|_\infty\\ =&\frac{1}{t-\varepsilon}\|T(t)(T(t)f-T(\varepsilon)f)\|_\infty\\ \le&\frac{1}{t-\varepsilon}\|T(t)f-T(\varepsilon)f\|_\infty\\ \le&\frac{1}{t-\varepsilon}(\|(T(t)f-f\|_\infty+\|T(\varepsilon)f-f\|_\infty)\\ \le&\frac{1}{t-\varepsilon}[[f]]_\theta(t^\theta+\varepsilon^\theta). \end{aligned}\tag{5.5}$$

From (5.3), (5.4) and (5.5), we conclude that

$$\|\Delta T(t)f\|_\infty \le [[f]]_\theta\left(\frac{d}{\sqrt{2}(1+\theta)}\frac{t^\theta}{t-\varepsilon}+\frac{1}{t-\varepsilon}(t^\theta+\varepsilon^\theta)\right)$$

for every $t > 0$ and $\varepsilon \in (0,t)$. Letting ε tend to 0^+, from the previous estimate we can infer that

$$\|\Delta T(t)f\|_\infty \le t^{\theta-1}[[f]]_\theta\left(\frac{d}{\sqrt{2}(1+\theta)}+1\right)$$

so that

$$[f]_{D_\Delta(\theta,\infty)} \le \left(\frac{d}{\sqrt{2}(1+\theta)}+1\right)[[f]]_\theta. \tag{5.6}$$

We have so proved that $f \in D_\Delta(\theta,\infty)$.

To conclude the proof it suffices to observe that (5.1) and (5.6) show that the norms $\|\cdot\|_\infty + [\,\cdot\,]_{D_\Delta(\theta,\infty)}$ and $\|\cdot\|_\infty + [[\,\cdot\,]]_{D_\Delta(\theta,\infty)}$ are equivalent. □

Remark 5.1.5. In view of Proposition 5.1.3, Remark 5.1.4 and the semigroup property, it follows that $D_\Delta(\theta,\infty)$ can be characterized as the set of all $f \in C_b(\mathbb{R}^d)$ such that the function $t \mapsto T(t)f$ belongs to $C_b^\theta([0,\infty);C_b(\mathbb{R}^d))$ for every $T > 0$, where, as usually, b means bounded. Indeed, if $0 \le s < t$, and $f \in D_\Delta(\theta,\infty)$, then

$$\|T(t)f-T(s)f\|_\infty = \|T(s)[T(t-s)f-f]\|_\infty \le \|T(t-s)f-f\|_\infty \le [[f]]_\theta(t-s)^\theta.$$

We can now give an explicit characterization of the space $D_\Delta(\theta,\infty)$.

Theorem 5.1.6. *For each $\theta \in (0,1)\setminus\{1/2\}$ it holds that $D_\Delta(\theta,\infty) = C_b^{2\theta}(\mathbb{R}^d)$, with equivalence of the respective norms.*

Proof. We begin by observing that the embedding $C_b^{2\theta}(\mathbb{R}^d) \hookrightarrow D_\Delta(\theta,\infty)$ follows immediately from Theorem 4.1.12.

Conversely, assume that $f \in D_\Delta(\theta,\infty)$. We first consider the case $\theta \in (0,\frac{1}{2})$ and observe that for $t>0$ and $x,y \in \mathbb{R}^d$ we can estimate

$$\begin{aligned}|f(x)-f(y)| \leq &|(T(t)f)(x)-f(x)|+|(T(t)f)(x)-(T(t)f)(y)|+|(T(t)f)(y)-f(y)|\\ \leq &2[[f]]_{D_\Delta(\theta,\infty)}t^\theta+\|\nabla_x T(t)f\|_\infty|x-y|. \end{aligned} \tag{5.7}$$

We would like to take $t=|x-y|^2$ in the previous formula. Unfortunately, the estimate in Proposition 4.1.9(i) is not enough sharp for our purposes. Indeed, using that estimate we would get $|f(x)-f(y)| \leq 2[[f]]_{D_\Delta(\theta,\infty)}|x-y|^{2\theta}+2^{-1/2}$. To overcome this difficulty, we differentiate the formula

$$(T(n)f)(x)-(T(t)f)(x)=\int_t^n (D_tT(s)f)(x)ds=\int_t^n (\Delta T(s)f)(x)ds,$$

to get

$$(D_iT(n)f)(x)-(D_iT(t)f)(x)=\int_t^n (D_i\Delta T(s)f)(x)ds, \qquad t\in(0,n),\ \ x\in\mathbb{R}^d, \tag{5.8}$$

for every $i=1,\dots,d$. Since $f \in D_\Delta(\theta,\infty)$, we can estimate $\|\Delta T(t)f\|_\infty \leq t^{\theta-1}[f]_\theta$ for all $t>0$. Hence, taking Proposition 4.1.9(i) into account, using the semigroup property and the fact that $T(t)$ and Δ commute on $C_b^2(\mathbb{R}^d)$, we get

$$\begin{aligned}\|D_i\Delta T(s)f\|_\infty =&\|D_iT(s/2)\Delta T(s/2)f\|_\infty \leq \|D_iT(s/2)\|_{L(C_b(\mathbb{R}^d))}\|\Delta T(s/2)f\|_\infty\\ \leq &Cs^{\theta-\frac{3}{2}}[f]_{D_\Delta(\theta,\infty)},\end{aligned} \tag{5.9}$$

so that we may let n tend to ∞ in (5.8) to get

$$(D_iT(t)f)(x)=-\int_t^\infty (D_i\Delta T(s)f)(x)ds, \qquad t>0,\ \ x\in\mathbb{R}^d,$$

and

$$\|D_iT(t)f\|_\infty \leq C\|f\|_{D_\Delta(\theta,\infty)}\int_t^\infty s^{\theta-\frac{3}{2}}ds = C_\theta t^{\theta-\frac{1}{2}}\|f\|_{D_\Delta(\theta,\infty)} \tag{5.10}$$

for any $i=1,\dots,d$. This estimate is what we need to prove that f is 2θ-Hölder continuous in $\mathbb{R}^d$. Indeed, replacing (5.10) in (5.7) and taking $t=|x-y|^2$, we obtain $|f(x)-f(y)| \leq \widetilde{C}_\theta\|f\|_{D_\Delta(\theta,\infty)}|x-y|^{2\theta}$, so that $f \in C_b^{2\theta}(\mathbb{R}^d)$ and $[f]_{C_b^{2\theta}(\mathbb{R}^d)} \leq \widetilde{C}_\theta\|f\|_{D_\Delta(\theta,\infty)}$. We have so proved that $D_\Delta(\theta,\infty) \hookrightarrow C_b^{2\theta}(\mathbb{R}^d)$. This completes the proof in the case $\theta<1/2$.

Let us now suppose that $\theta \in (1/2,1)$. As a first step, we prove that $f \in C_b^1(\mathbb{R}^d)$. For this purpose, we fix $0<s<t$ and observe that

$$(D_iT(t)f)(x)-(D_iT(s)f)(x)=\int_s^t (D_i\Delta T(r)f)(x)dr, \qquad x\in\mathbb{R}^d,\ \ i=1,\dots,d.$$

Using estimate (5.9) we deduce that

$$|(D_iT(t)f)(x)-(D_iT(s)f)(x)|\le C\frac{2}{2\theta-1}[f]_{D_\Delta(\theta,\infty)}(t^{\theta-\frac12}-s^{\theta-\frac12}),\qquad x\in\mathbb{R}^d. \tag{5.11}$$

This estimate shows that $(\nabla_xT(1/n)f)$ is a Cauchy sequence in $(C_b(\mathbb{R}^d))^d$. On the other hand, applying (5.7) with $t=|x-y|$ and Proposition 4.1.9.(i), we deduce that $D_\Delta(\theta,\infty)\subset BUC(\mathbb{R}^d)$. So, $T(1/n)f$ converges to f uniformly in $\mathbb{R}^d$ as n tends to ∞. As a byproduct, we conclude that f is continuously differentiable in $\mathbb{R}^d$. Moreover, from (5.11) it follows that

$$\|D_iT(t)f-D_if\|_\infty\le\widetilde{C}_\theta[f]_{D_\Delta(\theta,\infty)}t^{\theta-\frac12},\qquad t>0,\ \ i=1,\dots,d. \tag{5.12}$$

Hence, taking Proposition 4.1.9(i) into account, we deduce that

$$\|D_if\|_\infty\le\|D_iT(1)f-D_if\|_\infty+\|D_iT(1)f\|_\infty\le\widetilde{C}_\theta[f]_{D_\Delta(\theta,\infty)}+\frac{1}{\sqrt2}\|f\|_\infty \tag{5.13}$$

for every $i=1,\dots,d$. So, $f\in C^1_b(\mathbb{R}^d)$.

Fix $i\in\{1,\dots,d\}$. Since f is differentiable in $\mathbb{R}^d$, taking (5.12) into account we can estimate

$$\begin{aligned}|D_if(x)-D_if(y)|\le&|(D_iT(t)f)(x)-D_if(x)|+|(D_iT(t)f)(x)-(D_iT(t)f)(y)|\\&+|(D_iT(t)f)(y)-D_if(y)|\\\le&2\widetilde{C}_\theta[f]_{D_\Delta(\theta,\infty)}t^{\theta-\frac12}+\|\nabla_xD_iT(t)f\|_\infty|x-y|.\end{aligned} \tag{5.14}$$

With the same arguments used to prove (5.9), we can show that

$$\|D_{ij}\Delta T(s)f\|_\infty\le\widetilde{C}s^{\theta-2}[f]_{D_\Delta(\theta,\infty)},\qquad s>0. \tag{5.15}$$

Therefore, since

$$(D_{ij}T(t)f)(x)=-\int_t^\infty(D_{ij}\Delta T(s)f)(x)ds,\qquad t>0,\ \ x\in\mathbb{R}^d,$$

using (5.15) we obtain that $\|\nabla_xD_iT(t)f\|_\infty\le\widehat{C}_\theta t^{\theta-1}\|f\|_{D_\Delta(\theta,\infty)}$. Replacing this estimate in (5.14), we get

$$|D_if(x)-D_if(y)|\le2\widetilde{C}_\theta[f]_{D_\Delta(\theta,\infty)}t^{\theta-\frac12}+\widehat{C}_\theta t^{\theta-1}\|f\|_{D_\Delta(\theta,\infty)}|x-y|$$

for $x,y\in\mathbb{R}^d$ and $t>0$. Taking $t=|x-y|^2$ we can infer that

$$[D_if]_{C_b^{2\theta-1}(\mathbb{R}^d)}\le(2C_\theta+\widehat{C}_\theta)\|f\|_{D_\Delta(\theta,\infty)} \tag{5.16}$$

From (5.13) and (5.16) we conclude that $D_if\in C_b^{2\theta-1}(\mathbb{R}^d)$ and $\|D_if\|_{C_b^{2\theta-1}(\mathbb{R}^d)}\le K_\theta\|f\|_{D_\Delta(\theta,\infty)}$ for every $i=1,\dots,d$. The inclusion $D_\Delta(\theta,\infty)\hookrightarrow C_b^{2\theta}(\mathbb{R}^d)$ follows at once. □

Remark 5.1.7. The previous theorem does not cover the case $\theta = 1/2$. One may think that $D_\Delta(1/2,\infty) = C_b^1(\mathbb{R}^d)$ or $D_\Delta(1/2,\infty) = \mathrm{Lip}_b(\mathbb{R}^d)$ (the space of bounded Lipschitz continuous functions $f : \mathbb{R} \to \mathbb{R}$). But this is not the case. Indeed,

$$D_A(1/2,\infty) = \left\{ f \in C_b(\mathbb{R}^d) : \sup_{x\neq y} \frac{|f(x) + f(y) - 2f((x+y)/2)|}{|x-y|} < \infty \right\},$$

and $\mathrm{Lip}_b(\mathbb{R}^d)$ is one of its proper subspaces (see e.g., [4]).

We are going to prove that the semigroup $T(t)$ leaves $C_b^\alpha(\mathbb{R}^d)$ invariant for any $t > 0$ and $\alpha \in (0,1)$.

Proposition 5.1.8. *For each $f \in C_b^\alpha(\mathbb{R}^d)$ and $t > 0$, the function $T(t)f$ belongs to $C_b^\alpha(\mathbb{R}^d)$. More precisely,*

$$\|T(t)f\|_{C_b^\alpha(\mathbb{R}^d)} \le C\|f\|_{C_b^\alpha(\mathbb{R}^d)}, \qquad t > 0,$$

for some positive constant C, independent of f.

Proof. To prove the assertion, we observe that since $T(s)$ and Δ commute on $C_b^2(\mathbb{R}^d)$ and each operator $T(t)$ is a contraction in $C_b(\mathbb{R}^d)$, we can estimate

$$\|\Delta T(s+t)f\|_\infty = \|T(s)\Delta T(t)f\|_\infty \le \|\Delta T(t)f\|_\infty \le t^{\frac{\alpha}{2}-1}[f]_{D_\Delta(\alpha/2,\infty)}$$

for all $t, s > 0$. Hence, $[T(s)f]_{D_\Delta(\alpha/2,\infty)} \le [f]_{D_\Delta(\alpha/2,\infty)}$ for any $s > 0$. Theorem 5.1.6 allows us to complete the proof. □

In terms of the bounded classical solution to problem (4.1), the previous result says that if $f \in C_b^\alpha(\mathbb{R}^d)$ for some $\alpha \in (0,1)$, then $u(t,\cdot) \in C_b^\alpha(\mathbb{R}^d)$ for any $t > 0$ and

$$\sup_{t\ge 0} \|u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \le C\|f\|_{C_b^\alpha(\mathbb{R}^d)}$$

for some constant C, independent of f.

5.2 Optimal Schauder estimates for solutions to the heat equation in $\mathbb{R}^d$: preliminary results

The following lemma plays a crucial role in the proof of the forthcoming Theorem 6.1.2.

Lemma 5.2.1. *For each $\alpha \in (0,1)$, there exists a positive constant $C = C(\alpha)$ such that*

$$\|\nabla f\|_\infty \le C\|f\|_{C_b^\alpha(\mathbb{R}^d)}^{\frac{\alpha+1}{2}} \|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)}^{\frac{1-\alpha}{2}}, \qquad f \in C_b^{2+\alpha}(\mathbb{R}^d) \tag{5.17}$$

and

$$\|f\|_{C_b^2(\mathbb{R}^d)} \le C\|f\|_{C_b^\alpha(\mathbb{R}^d)}^{\frac{\alpha}{2}} \|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)}^{1-\frac{\alpha}{2}}, \qquad f \in C_b^{2+\alpha}(\mathbb{R}^d). \tag{5.18}$$

Proof. To begin with, we observe that it suffices to prove (5.17) and (5.18) when $d = 1$. Indeed, if $d > 1$, then it suffices to apply the one-dimensional result to the functions $x_i \mapsto f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d)$ $(i = 1, \ldots, d)$.

Since the proof is rather long, we split it into three steps. Throughout the proof, c denotes a positive constant, depending on α but being independent of the functions that we consider, which may vary from line to line.

Step 1. Here, we prove that

$$(5.19) \qquad (i)\ \|g'\|_\infty \le c\|g\|^{\alpha}_{C^\alpha_b(\mathbb{R})}[g']^{1-\alpha}_{C^\alpha_b(\mathbb{R})}, \qquad (ii)\ \|g'\|_\infty \le c\|g\|_\infty^{\frac{\alpha}{\alpha+1}}[g']^{\frac{1}{1+\alpha}}_{C^\alpha_b(\mathbb{R})}$$

for every $g \in C^{1+\alpha}_b(\mathbb{R})$. For this purpose, we fix $g \in C^{1+\alpha}_b(\mathbb{R})$ and use the fundamental theorem of calculus to write

$$(5.20) \qquad g(y) = g(0) + \int_0^y g'(t)dt = g(0) + g'(0)y + \int_0^y (g'(t) - g'(0))dt, \qquad y > 0,$$

and then estimate

$$|g'(0)| \le \left|\frac{g(y) - g(0)}{y}\right| + \frac{1}{y}\int_0^y |g'(t) - g'(0)|dt \le [g]_{C^\alpha_b(\mathbb{R})}y^{\alpha-1} + \frac{y^\alpha}{1+\alpha}[g']_{C^\alpha_b(\mathbb{R})}.$$

Minimizing with respect to $y > 0$, we get

$$|g'(0)| \le c[g]^{\alpha}_{C^\alpha_b(\mathbb{R})}[g']^{1-\alpha}_{C^\alpha_b(\mathbb{R})}.$$

Next, replacing g by the function $g_x = g(\cdot + x)$ and observing that $[g_x]_{C^\alpha_b(\mathbb{R})} = [g]_{C^\alpha_b(\mathbb{R})}$ and $[g_x']_{C^\alpha_b(\mathbb{R})} = [g']_{C^\alpha_b(\mathbb{R})}$, we immediately conclude that

$$|g'(x)| \le c[g]^{\alpha}_{C^\alpha_b(\mathbb{R})}[g']^{1-\alpha}_{C^\alpha_b(\mathbb{R})}, \qquad x \in \mathbb{R}.$$

Finally, taking the supremum with respect to $x \in \mathbb{R}$, estimate (5.19)(i) follows at once.

The proof of estimate (5.19)(ii) is very similar. Starting from (5.20), we can infer that

$$|g'(0)| \le \frac{2}{y}\|g\|_\infty + \frac{1}{1+\alpha}y^\alpha[g']_{C^\alpha_b(\mathbb{R})}$$

for all $y > 0$. Again, minimizing with respect to y and then replacing g with the function g_x as above, estimate (5.19)(ii) follows.

Step 2. Here, we prove that

$$(5.21) \qquad \|g'\|_\infty \le c\|g\|^{\frac{1}{2-\alpha}}_{C^\alpha_b(\mathbb{R})}\|g''\|_\infty^{\frac{1-\alpha}{2-\alpha}}, \qquad g \in C^2_b(\mathbb{R}).$$

For this purpose, we fix $g \in C^2_b(\mathbb{R})$ and apply estimates (4.17), with $\theta = \alpha$ and (5.19)(i) to get

$$\|g'\|_\infty \le c\|g\|^{\alpha}_{C^\alpha_b(\mathbb{R})}[g']^{1-\alpha}_{C^\alpha_b(\mathbb{R})} \le c\|g\|^{\alpha}_{C^\alpha_b(\mathbb{R})}\big(\|g'\|_\infty^{1-\alpha}\|g''\|_\infty^{\alpha}\big)^{1-\alpha}$$

$$=c\|g\|_{C^\alpha_b(\mathbb{R})}^{\alpha}\|g'\|_\infty^{(1-\alpha)^2}\|g''\|_\infty^{\alpha(1-\alpha)}$$

or, equivalently,

$$\|g'\|_\infty^{2\alpha-\alpha^2}\le c\|g\|_{C^\alpha_b(\mathbb{R})}^{\alpha}\|g''\|_\infty^{\alpha(1-\alpha)},$$

from which (5.21) follows immediately.

Step 3. Here, we complete the proof of (5.17). We fix $f\in C^{2+\alpha}_b(\mathbb{R})$ and use (5.21) together with (5.19)(ii) to estimate

$$\|f'\|_\infty\le c\|f\|_{C^\alpha_b(\mathbb{R})}^{\frac{1}{2-\alpha}}\|f''\|_\infty^{\frac{1-\alpha}{2-\alpha}}\le c\|f\|_{C^\alpha_b(\mathbb{R})}^{\frac{1}{2-\alpha}}\big(\|f'\|_\infty^{\frac{\alpha}{\alpha+1}}[f'']_{C^\alpha_b(\mathbb{R})}^{\frac{1}{1+\alpha}}\big)^{\frac{1-\alpha}{2-\alpha}},$$

from which we deduce that

$$\|f'\|_\infty^{\frac{2}{(\alpha+1)(2-\alpha)}}\le c\|f\|_{C^\alpha_b(\mathbb{R})}^{\frac{1}{2-\alpha}}[f'']_{C^\alpha_b(\mathbb{R})}^{\frac{1-\alpha}{(1+\alpha)(2-\alpha)}}$$

and (5.17) follows.

Step 4. Finally, here we check estimate (5.18) with $d=1$. We use again Taylor formula to get

$$g(y)=g(0)+g'(0)y+\frac{1}{2}g''(0)y^2+\int_0^y(g''(t)-g''(0))(y-t)dt,\qquad y>0,$$

for any $g\in C^{2+\alpha}_b(\mathbb{R})$. Hence,

$$\begin{aligned}|g''(0)|\le&\frac{2|g(y)-g(0)|}{y^2}+\frac{2|g'(0)|}{y}+\frac{2}{y^2}\int_0^y|g''(t)-g''(0)|(y-t)dt\\ \le&2y^{\alpha-2}[g]_{C^\alpha_b(\mathbb{R})}+2y^{-1}\|g'\|_\infty+2y^{-2}[g'']_{C^\alpha_b(\mathbb{R})}\int_0^y t^\alpha(y-t)dt\\ \le&2y^{\alpha-2}[g]_{C^\alpha_b(\mathbb{R})}+2y^{-1}\|g'\|_\infty+cy^\alpha[g'']_{C^\alpha_b(\mathbb{R})}.\end{aligned}$$

Now, using (5.17) we can estimate

$$\begin{aligned}y^{-1}\|g'\|_\infty\le&cy^{-1}\|g\|_{C^\alpha_b(\mathbb{R})}^{\frac{\alpha+1}{2}}\|g\|_{C^{2+\alpha}_b(\mathbb{R})}^{\frac{1-\alpha}{2}}\\ =&c(y^{\alpha-2}\|g\|_{C^\alpha_b(\mathbb{R})})^{\frac{\alpha+1}{2}}(y^\alpha\|g\|_{C^{2+\alpha}_b(\mathbb{R})})^{\frac{1-\alpha}{2}}\\ \le&c(y^{\alpha-2}\|g\|_{C^\alpha_b(\mathbb{R})}+y^\alpha\|g\|_{C^{2+\alpha}_b(\mathbb{R})}),\end{aligned}$$

where we have taken advantage of the Young inequality $a^\beta b^{1-\beta}\le\beta^{-1}a+(1-\beta)^{-1}b$, which holds true for any $a,b\ge 0$ and $\beta\in(0,1)$, see Exercise 5.4.1.
From the previous two inequalities we can infer that

$$|g''(0)|\le2y^{\alpha-2}[g]_{C^\alpha_b(\mathbb{R})}+2y^{-1}\|g'\|_\infty+cy^\alpha[g'']_{C^\alpha_b(\mathbb{R})}\le\quad cy^{\alpha-2}\|g\|_{C^\alpha_b(\mathbb{R})}+cy^\alpha\|g\|_{C^{2+\alpha}_b(\mathbb{R})}.$$

Minimizing with respect to $y>0$ we get

$$|g''(0)|\le c\|g\|_{C^\alpha_b(\mathbb{R})}^{\frac{\alpha}{2}}\|g\|_{C^{2+\alpha}_b(\mathbb{R})}^{1-\frac{\alpha}{2}}.$$

Replacing g with g_x as in the proof of (5.17) we can complete the proof. □

Remark 5.2.2. We stress that estimate (5.17) holds true also with $\alpha = 0$, as it can be easily seen adapting the proof of the above lemma. The only difference is in the estimate of the term $y^{-1}|g(y) - g(0)|$, which now is bounded from above by $2\|g\|_\infty y^{-1}$.

5.3 Notes

The notation $D_\Delta(\theta, \infty)$, used to define the space of functions f such that

$$\sup_{t>0} t^{1-\theta}\|\Delta T(t) f\|_\infty < +\infty,$$

may sound a bit strange, but it is the commonly used notation to define the so-called interpolation space of order θ between X and the domain $D(A)$ of the generator of an analytic semigroup. More precisely, if A is a sectorial operator, then $D_A(\theta, \infty) = \{x \in X : [x]_{D_A(\theta,\infty)} = \sup_{t\in(0,1)} t^{1-\theta}\|AT(t)x\| < +\infty\}$. This is a Banach space when endowed with the norm $\|\cdot\|_{D_A(\theta,\infty)} = \|\cdot\|_X + [\,\cdot\,]_{D_A(\theta,\infty)}$. It can be proved that $D_A(\theta, \infty)$ coincides with the real interpolation space $(X, D(A))_{\theta,\infty}$ between X and $D(A)$. See also Exercises 5.4.5 and 5.4.6. For further details, we refer the reader to [3].
Finally, the interested reader may find in [2] a very exhaustive and nice exposition of the heat equation in the one dimensional case. For the multidimensional case we refer the reader to [1].

5.4 Exercises

1. Prove the Young inequality

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$$

for any $a, b \ge 0$ and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$.

2. Prove that, for any $\alpha \in (0, 1)$ and $T > 0$ there exists positive constants C_α and $C_{\alpha,T}$ such that the interpolation estimates

$$\|\zeta\|_{C^\alpha_b(\mathbb{R}^d)} \le C_\alpha \|\zeta\|_\infty^{\frac{2}{2+\alpha}} \|\zeta\|_{C^{2+\alpha}_b(\mathbb{R}^d)}^{\frac{\alpha}{2+\alpha}}, \qquad \zeta \in C^{2+\alpha}_b(\mathbb{R}^d), \tag{5.22}$$

$$\|\varphi\|_{C^\alpha([0,T])} \le C_{\alpha,T} \|\varphi\|_\infty^{\frac{1}{1+\alpha}} \|\varphi\|_{C^{1+\alpha}_b([0,T])}^{\frac{\alpha}{1+\alpha}}, \qquad \varphi \in C^{1+\alpha}_b([0, T]), \tag{5.23}$$

$$\|\varphi\|_{C^{\alpha/2}([0,T])} \le C_{\alpha,T} \|\varphi\|_\infty^{\frac{1}{1+\alpha}} \|\varphi\|_{C^{(1+\alpha)/2}([0,T])}^{\frac{\alpha}{1+\alpha}}, \qquad \varphi \in C^{(1+\alpha)/2}([0, T]) \tag{5.24}$$

hold true.

3. Prove the interpolation estimates

$$\|f\|_{C^k_b(\mathbb{R}^d)} \le C_\alpha \|f\|_\infty^{\frac{2+\alpha-k}{2+\alpha}} \|f\|_{C^{2+\alpha}_b(\mathbb{R}^d)}^{\frac{k}{2+\alpha}}, \qquad f \in C^{2+\alpha}_b(\mathbb{R}^d),\ k = 1, 2, \tag{5.25}$$

$$\|f\|_{C^1_b(\mathbb{R}^d)} \le C_\alpha \|f\|_\infty^{\frac{\alpha}{1+\alpha}} \|f\|_{C^{1+\alpha}_b(\mathbb{R}^d)}^{\frac{1}{1+\alpha}}, \qquad f \in C^{1+\alpha}_b(\mathbb{R}^d), \tag{5.26}$$

for any $\alpha \in (0,1)$ and some positive constant C_α.

4. Let A be the infinitesimal generator of the Gauss-Weierstrass semigroup and $\theta \in (0,1)\setminus\{1\}$. Prove that its part in $C^{2\theta}_b(\mathbb{R}^d)$ (i.e., the operator $A_{|C^{2\theta}_b(\mathbb{R}^d)} : D_\Delta(\theta+1,\infty) := \{f \in D(A) : Af \in C^{2\theta}_b(\mathbb{R}^d)\} \to C^{2\theta}_b(\mathbb{R}^d)$, defined by $A_{|C^{2\theta}_b(\mathbb{R}^d)}f = Af$ for any $f \in D_\Delta(\theta+1,\infty)$) is still sectorial and the analytic semigroup that it generates is the restriction of the Gauss-Weierstrass semigroup to $C^{2\theta}_b(\mathbb{R}^d)$. In the next lecture it will become clear what $D_\Delta(\theta+1,\infty)$ is.

5. For each $z \in X+Y$ and $t > 0$ set

$$K(t,z) := \inf_{z=x+y:x\in X,\, y\in Y} (\|x\|_X + t\|y\|_Y),$$

where X and Y are two Banach spaces endowed respectively with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. For $\theta \in (0,1)$ define the set

$$(X,Y)_{\theta,\infty} := \left\{ z \in X+Y : \sup_{t>0} t^{-\theta} K(t,z) < +\infty \right\}.$$

   a) Prove that

   $$\|z\|_{\theta,\infty} := \sup_{t>0} t^{-\theta} K(t,z), \quad z \in (X,Y)_{\theta,\infty},$$

   defines a norm in $(X,Y)_{\theta,\infty}$.

   b) Prove that $(X,Y)_{\theta,\infty}$ endowed with $\|\cdot\|_{\theta,\infty}$ is a Banach space.

6. Let A be the generator of the Gauss-Weierstrass semigroup $\{T(t)\}$ in $C_b(\mathbb{R}^d)$. Here $D(A)$ is endowed with its graph norm. With the notations of the above exercise prove that

   a) $(C_b(\mathbb{R}^d), D(A))_{\theta,\infty} = D_\Delta(\theta,\infty)$ with equivalent norms.
   Hints:

      (i) using the characterization of $D_\Delta(\theta,\infty)$ in Proposition 5.1.3, prove that $D_\Delta(\theta,\infty)$ is continuously embedded into $(X,D(A))_{\theta,\infty}$;

      (ii) prove that, if $f \in (X,D(A))_{\theta,\infty}$, then $[f]_{D_\Delta(\theta,\infty)} \le K[f]_{(X,D(A))_{\theta,\infty}}$ for some positive constant K, independent of f;

   b)

   $$(C_b(\mathbb{R}^d), D(A))_{\theta,\infty} = \left\{ f \in C_b(\mathbb{R}^d) : \sup_{\lambda>0} \lambda^\theta \|AR(\lambda,A)f\|_\infty < \infty \right\} =: \Lambda$$

and the norms $\|\cdot\|_{\theta,\infty}$ and

$$\|f\|^*_{\theta,\infty} := \|f\|_\infty + \sup_{\lambda>0} \lambda^\theta \|AR(\lambda, A)f\|_\infty, \quad f \in \Lambda$$

are equivalent.
Hints: Adapt the idea of the proof of a).

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Lecture 6

The heat equation and the Gauss-Weierstrass semigroup in $C_b(\mathbb{R}^d)$. Part 3

6.1 Optimal Schauder estimates

In this section we prove the fundamental optimal Schauder estimates for solutions to the Cauchy problem

$$\begin{cases} D_t u(t,x) = \Delta u(t,x) + g(t,x), & t \in (0,T], \quad x \in \mathbb{R}^d, \\ u(0,x) = f(x), & \qquad\qquad\quad x \in \mathbb{R}^d, \end{cases} \tag{6.1}$$

proving also that, under suitable assumptions on f and g, it admits a unique bounded classical solution.

Definition 6.1.1. A function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is called "a bounded classical solution" to problem (6.1) if (i) $u \in C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$, (ii) $u(0,\cdot) = f$ in $\mathbb{R}^d$ and (iii) u satisfies the differential equation in (6.1).

To begin with, we introduce the following functional spaces that we use in this lecture.

Definition 6.1.2. For any $\alpha \in (0,1)$,

(a) $C_b^{0,\alpha}([0,T] \times \mathbb{R}^d)$ denotes the space of all the bounded functions $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that the function $f(t,\cdot)$ is α-Hölder continuous in $\mathbb{R}^d$ for any $t \in [0,T]$. It is a Banach space with the norm

$$\|f\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)} = \sup_{t\in[0,T]} \|f(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}$$

for each $f \in C_b^{0,\alpha}([0,T] \times \mathbb{R}^d)$.

(b) $C_b^{\alpha,0}([0,T]\times\mathbb{R}^d)$ denotes the space of all the bounded functions $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ such that the function $f(\cdot,x)$ is α-Hölder continuous in $[0,T]$ for any $x\in\mathbb{R}^d$. It is a Banach space with the norm

$$\|f\|_{C_b^{\alpha,0}([0,T]\times\mathbb{R}^d)}=\sup_{x\in\mathbb{R}^d}\|f(\cdot,x)\|_{C^\alpha([0,T])}$$

for each $f\in C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)$.

(c) $C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)=C_b^{\alpha/2,0}([0,T]\times\mathbb{R}^d)\cap C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)$. It is a Banach space with the norm

$$\|f\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}=\|f\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}+\sup_{x\in\mathbb{R}^d}[f(\cdot,x)]_{C^{\alpha/2}([0,T])}$$

for each $f\in C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)$.

(d) $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$ denotes the space of all the bounded functions $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ which are once continuously differentiable with respect to the time variable and twice continuously differentiable with respect to the spatial variables in $[0,T]\times\mathbb{R}^d$ with D_tf and $D_{ij}u$ in $C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)$ for any $i,j=1,\dots,d$. It is a Banach space with the norm

$$\begin{aligned}\|f\|_{C_b^{1+\alpha/2,2\alpha}([0,T]\times\mathbb{R}^d)}=&\|f\|_\infty+\sum_{j=1}^d\|D_jf\|_\infty+\sum_{i,j=1}^d\|D^2_{ij}f\|_{C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\\&+\|D_tf\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\end{aligned}$$

Remark 6.1.3. $C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)$ can be also characterized as the space of all bounded functions $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ which are α-Hölder continuous with respect to the parabolic distance $d((t_1,x_1),(t_2,x_2))=\sqrt{|t_2-t_1|+|x_2-x_1|^2}$ for every $t_1,t_2\in[0,T]$ and $x_1,x_2\in\mathbb{R}^d$.

The aim of this lecture is to prove the following Schauder estimate for the solution to (6.1).

Theorem 6.1.4. *For each $f\in C_b^{2+\alpha}(\mathbb{R}^d)$ and $g\in C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)$, $\alpha\in(0,1)$, the Cauchy problem* (6.1) *admits a unique bounded classical solution u. Further, u belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$ and there exists a positive constant C, independent of u and f, such that*

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\le C(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)}+\|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}).\tag{6.2}$$

Remark 6.1.5. Estimate (6.2) is usually referred to as an *optimal Schauder estimate.* The optimality comes from the following facts:

(i) the first-order time derivative and all the second-order derivatives of the solution u have the same degree of smoothness of the datum g;

(ii) for any $t > 0$, the function $u(t, \cdot)$ preserves the same regularity it has at $t = 0$;

(iii) the assumptions $f \in C_b^{2+\alpha}(\mathbb{R}^d)$ and $g \in C_b^{\alpha/2,\alpha}([0,T] \times \mathbb{R}^d)$ are necessary for u to belong to $C_b^{1+\alpha/2,2+\alpha}([0,T] \times \mathbb{R}^d)$.

Proof of Theorem 6.1.4. To prove the uniqueness of the classical solution to problem (6.1), let us consider two bounded classical solutions u and v to (6.1). Then $u - v$ is a bounded classical solution to

$$\begin{cases} D_t w(t,x) = \Delta w(t,x), & t \in [0,T], \quad x \in \mathbb{R}^d, \\ w(0,x) = 0, & \qquad\qquad\quad x \in \mathbb{R}^d. \end{cases}$$

So, by Exercise 1.4.4, one obtains $u = v$.

As far as the existence part is concerned, we will show that the function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, defined by the variation of constants formula

(6.3)
$$u(t,x) = (T(t)f)(x) + \int_0^t (T(t-s)g(s,\cdot))(x)ds =: u_0(t,x) + u_1(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R}^d,$$

solves problem (6.1) and satisfies estimate (6.2). Throughout the proof, we denote by c a positive constant, independent of u and f, which may vary from line to line. Clearly, $u(0,\cdot) = f$. Being rather long, we spit the rest of the proof into several steps.

Step 1. Here, we show that it suffices to prove that the function u given by formula (6.3) solves the Cauchy problem (6.1), belongs to $C^{1,2}([0,T] \times \mathbb{R}^d) \cap C^{0,2+\alpha}([0,T] \times \mathbb{R}^d)$ and

$$\|u\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} \le c(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}). \tag{6.4}$$

Indeed, if the previous properties are satisfied, then, for any $t_1, t_2 \in [0,T]$, with $t_1 \le t_2$ and $x \in \mathbb{R}^d$, we can write

$$u(t_2,x) - u(t_1,x) = \int_{t_1}^{t_2} D_t u(s,x)ds = \int_{t_1}^{t_2} (\Delta u(s,x) + g(s,x))ds.$$

Using estimate (6.4) we conclude that

$$\|u(t_2,\cdot) - u(t_1,\cdot)\|_\infty \le c(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)})|t_2 - t_1|.$$

Similarly,

$$\begin{aligned} &|u(t_2,x) - u(t_1,x) - u(t_2,y) + u(t_1,y)| \\ =&\left| \int_{t_1}^{t_2} (\Delta u(s,x) + g(s,x) - \Delta u(s,y) - g(s,y))ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_1}^{t_2} (|\Delta u(s,x) - \Delta u(s,y)| + |g(s,x) - g(s,y)|)ds \\
&\leq c \sup_{t\in[0,T]} ([\Delta u(t,\cdot)]_{C_b^\alpha(\mathbb{R}^d)} + [g(t,\cdot)]_{C_b^\alpha(\mathbb{R}^d)})|x-y|^\alpha |t_2 - t_1| \\
&\leq c(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)})|x-y|^\alpha |t_2 - t_1|
\end{aligned}$$

for all $x, y \in \mathbb{R}^d$. Summing up, we have proved that

$$\|u(t_2,\cdot) - u(t_1,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \leq c(\|f\|_{C_b^\alpha(\mathbb{R}^d)} + \|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)})|t_2 - t_1|, \qquad 0 \leq t_1 \leq t_2 \leq T.$$

Using the interpolation estimate (5.18) together with (6.4) we deduce that

$$\begin{aligned}
\|u(t_2,\cdot) - u(t_1,\cdot)\|_{C_b^2(\mathbb{R}^d)} \leq & c\|u(t_2,\cdot) - u(t_1,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}^{\frac{\alpha}{2}} \|u(t_2,\cdot) - u(t_1,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}^{1-\frac{\alpha}{2}} \\
\leq & c(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)})|t_2 - t_1|^{\frac{\alpha}{2}}
\end{aligned} \tag{6.5}$$

for all $t_1, t_2 \in [0,T]$. Hence, for every $x \in \mathbb{R}^d$ the functions $u(\cdot,x)$, $D_i u(\cdot,x)$ and $D_{ij}u(\cdot,x)$ ($i,j = 1,\ldots,d$) are $\alpha/2$-Hölder continuous uniformly with respect to $x \in \mathbb{R}^d$. We thus conclude that $u \in C_b^{\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$ and, from (6.4) and (6.5), it follows that

$$\|u\|_{C_b^{\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \leq c(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}).$$

Since u is supposed to be a classical solution to the Cauchy problem (6.1), $D_t u = \Delta u + g$ and, therefore, $D_t u \in C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)$ and

$$\begin{aligned}
\|D_t u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} \leq & c(\|\Delta u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} + \|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}) \\
\leq & c(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}).
\end{aligned}$$

From the previous two estimates, (6.2) follows at once.

By Theorems 4.1.3 and 4.1.12, u_0 is the classical solution to problem (6.1) with $g \equiv 0$ and

$$\sup_{t\in[0,T]} \|u_0(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} \leq c\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)}.$$

Hence, to complete the proof we have to consider the function u_1 in (6.3), show that it belongs to $C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)$, is continuously differentiable with respect to t, solves the Cauchy problem

$$\begin{cases} D_t u_1(t,x) = \Delta u_1(t,x) + g(t,x), & t\in[0,T], \quad x \in \mathbb{R}^d, \\ u_1(0,x) = 0, & \qquad\qquad\qquad x \in \mathbb{R}^d \end{cases}$$

and satisfies the estimate

$$\|u_1\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} \leq c\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}. \tag{6.6}$$

This is done in Steps 2 to 4.

Step 2. Here, we prove that u_1 is continuous in $[0,T]\times\mathbb{R}^d$. To begin with, we observe that, for any $t>0$ and $x\in\mathbb{R}^d$, the function $s\mapsto (T(t-s)g(s,\cdot))(x)$ is continuous in $[0,t]$ Hence it can be integrated over $[0,t]$. This can be easily checked using the representation formula (4.7) and the dominated convergence theorem.

To prove that the function u_1 is continuous in $[0,T]\times\mathbb{R}^d$, by a straightforward change of variable we rewrite it in the form

$$u_1(t,x)=\int_0^t (T(s)g(t-s,\cdot))(x)ds, \qquad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Fix $(t_0,x_0)\in[0,T]\times\mathbb{R}^d$. To ease the notation, we write $(t,x)\to(t_0^+,x_0)$ (resp. $(t,x)\mapsto(t_0^-,x_0)$) to denote that (t,x) tends to (t_0,x_0) with $t>t_0$ (resp. $t<t_0$).

As a first step, we prove that $u_1(t,x)$ converges to $u_1(t_0,x_0)$ as $(t,x)\to(t_0^+,x_0)$. For this purpose, for $t>t_0$ and $x\in\mathbb{R}^d$, we split

$$\begin{aligned}u_1(t,x)-u_1(t_0,x_0)=&\int_{t_0}^t (T(s)g(t-s,\cdot))(x)ds\\&+\int_0^{t_0}[(T(s)g(t-s,\cdot))(x)-(T(s)g(t_0-s,\cdot))(x_0)]ds\\=:&\ I_1^+(t,x)+I_2^+(t,x).\end{aligned}$$

Since $\|T(s)g(t-s,\cdot)\|_\infty\le\|g(t-s,\cdot)\|_\infty\le\|g\|_{C_b([0,T]\times\mathbb{R}^d)}$ for any $0\le s\le t$, we immediately conclude that $I_1^+(t,x)$ vanishes as (t,x) tends to (t_0^+,x_0). As far as $I_2^+(t,x)$ is concerned, using (4.7) we can write

$$\begin{aligned}&(T(s)g(t-s,\cdot))(x)-(T(s)g(t_0-s,\cdot))(x_0)\\=&(4\pi s)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\big(e^{-\frac{|x-y|^2}{4s}}g(t-s,y)-e^{-\frac{|x_0-y|^2}{4s}}g(t_0-s,y)\big)dy.\end{aligned}$$

This formula and the dominated convergence theorem imply that $(T(s)g(t-s,\cdot))(x)$ converges to $(T(s)g(t_0-s,\cdot))(x_0)$ as (t,x) tends to (t_0,x_0), for every $s\in[0,t_0]$. Indeed, $e^{-\frac{|x-y|^2}{4s}}g(t-s,y)-e^{-\frac{|x_0-y|^2}{4s}}g(t_0-s,y)$ vanishes as $(t,x)\to(t_0^+,x_0)$ for every $s\in[0,t_0]$ and $y\in\mathbb{R}^d$. Moreover, if $x\in B(x_0,1)$, then $|x-y|\ge|y|-|x|\ge|y|-|x_0|-1$. Therefore, if $|y|\ge 2(|x_0|+1)$, then we can estimate

$$|e^{-\frac{|x-y|^2}{4s}}g(t-s,y)-e^{-\frac{|x_0-y|^2}{4s}}g(t_0-s,y)|\le 2e^{-\frac{|y|^2}{16s}}\|g\|_{C_b([0,T]\times\mathbb{R}^d)}$$

for every $s\in[0,t_0]$. On the other hand, if $|y|\le 2(|x_0|+1)$, then we get

$$|e^{-\frac{|x-y|^2}{4s}}g(t-s,y)-e^{-\frac{|x_0-y|^2}{4s}}g(t_0-s,y)|\le 2\|g\|_{C_b([0,T]\times\mathbb{R}^d)}.$$

Hence, $|e^{-\frac{|x-y|^2}{4s}}g(t-s,y)-e^{-\frac{|x_0-y|^2}{4s}}g(t_0-s,y)|\le\psi(y)$ for any $y\in\mathbb{R}^d$, where $\psi(y)=2(e^{-\frac{|y|^2}{16s}}\chi_{\mathbb{R}^d\setminus B(0,2(|x_0|+1))}+\chi_{B(0,2(|x_0|+1))})\|g\|_{C_b([0,T]\times\mathbb{R}^d)}$, and the function ψ belongs to $L^1(\mathbb{R}^d)$.

Since $|(T(s)g(t-s,\cdot))(x)-(T(s)g(t_0-s,\cdot))(x_0)|\le 2\|g\|_{C_b([0,T]\times\mathbb{R}^d)}$ for every $0\le s\le t_0\le t$ and $x,x_0\in\mathbb{R}^d$, we can apply the dominated convergence theorem once more and deduce that $I_2^+(t,x)$ tends to 0 as (t,x) tends to (t_0^+,x_0). Hence, $u_1(t,x)$ converges to $u_1(t_0,x_0)$ as $(t,x)\to(t_0^+,x_0)$. If t tends to t_0 from the left, then the proof is similar. We split

$$\begin{aligned}u_1(t,x)-u_1(t_0,x_0)=&-\int_t^{t_0}(T(s)g(t_0-s,\cdot))(x)ds\\&+\int_0^{t_0}[(T(s)g(t-s,\cdot))(x)-(T(s)g(t_0-s,\cdot))(x_0)]\chi_{[0,t]}(s)\,ds\\=:&\,I_1^-(t,x)+I_2^-(t,x).\end{aligned}$$

Again $I_1^-(t,x)$ tends to 0 and $[(T(s)g(t-s,\cdot))(x)-(T(s)g(t_0-s,\cdot))(x_0)]\chi_{[0,t]}(s)$ vanishes as $(t,x)\to(t_0^-,x_0)$, for every $s\in[0,t_0]$. The dominated convergence theorem shows that also $I_2^-(t,x)$ vanishes as $(t,x)\to(t_0^-,x_0)$, and the continuity of u_1 at (t_0,x_0) is proved. Moreover, since each operator $T(t)$ is a contraction, we deduce that $\|u_1\|_{C_b([0,T]\times\mathbb{R}^d)}\le T\|g\|_{C_b([0,T]\times\mathbb{R}^d)}$.

Step 3. Here, we prove that the function u_1 is twice continuously differentiable with respect to the spatial variables, $u_1(t,\cdot)\in C_b^{2+\alpha}(\mathbb{R}^d)$ for any $t\in[0,T]$ and estimate (6.6) is satisfied.

By Theorem 4.1.12, for any $0\le s<t\le T$, the function $T(t-s)g(s,\cdot)$ is differentiable in $\mathbb{R}^d$ and $\|\nabla_xT(t-s)g(s,\cdot)\|_\infty\le c(t-s)^{\frac{\alpha-1}{2}}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}$. Hence, the dominated convergence theorem shows that $u_1(t,\cdot)$ is differentiable in $\mathbb{R}^d$ for any $t\in[0,T]$ and

$$\nabla_xu_1(t,x)=\int_0^t(\nabla_xT(t-s)g(s,\cdot))(x)ds,\qquad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Similarly, since $\|D_x^2T(t-s)g(s,\cdot)\|_\infty\le c(t-s)^{\frac{\alpha}{2}-1}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}$, we deduce that $u_1(t,\cdot)$ is twice differentiable in $\mathbb{R}^d$ and

$$D_x^2u_1(t,x)=\int_0^t(D_x^2T(t-s)g(s,\cdot))(x)ds,\qquad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Moreover,

$$\begin{aligned}|D_x^ju_1(t,x)|\le&\int_0^t(D_x^jT(t-s)g(s,\cdot))(x)ds\le\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\int_0^t(t-s)^{\frac{\alpha-j}{2}}ds\\=&\frac{2}{\alpha-j+2}T^{\frac{\alpha-j+2}{2}}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\end{aligned}$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$ and $j=1,2$. Moreover, for every $\theta\in(0,\alpha)$, $t\in[0,T]$ and $x_1,x_2\in\mathbb{R}^d$ we can estimate

$$|D_x^2u_1(t,x_2)-D_x^2u_1(t,x_1)|\le\int_0^t|(D_x^2T(t-s)g(s,\cdot))(x_2)-(D_x^2T(t-s)g(s,\cdot))(x_1)|ds$$

$$
\begin{aligned}
&\le C\|g\|_{C^{0,\alpha}_b([0,T]\times\mathbb{R}^d)}|x_2-x_1|^\theta\int_0^t (t-s)^{-1+\frac{\alpha-\theta}{2}}ds\\
&\le \frac{2C}{\alpha-\theta}T^{\frac{\alpha-\theta}{2}}|x_2-x_1|^\theta\|g\|_{C^{0,\alpha}_b([0,T]\times\mathbb{R}^d)},
\end{aligned}
$$

which shows that, for any $t\in[0,T]$, $u_1(t,\cdot)$ belongs to $C^{2+\theta}_b(\mathbb{R}^d)$ and $[D^2_x u_1(t,\cdot)]_{C^\theta_b(\mathbb{R}^d)}\le 2C(\alpha-\theta)^{-1}T^{(\alpha-\theta)/2}\|g\|_{C^{0,\alpha}_b([0,T]\times\mathbb{R}^d)}$. Note that, as θ tends to α from the left, the right-hand side of the previous inequality blows up. This prevents us from concluding that $u_1(t,\cdot)\in C^{2+\alpha}_b(\mathbb{R}^d)$ for any $t\in[0,T]$. To overcome this difficulty, we perform a different strategy based on Theorem 5.1.6: to prove that $D_{ij}u_1(t,\cdot)\in C^\alpha_b(\mathbb{R}^d)$ for every $t\in[0,T]$ and $i,j=1,\dots,d$, we will show that

$$
\sup_{\tau>0}\tau^{1-\frac{\alpha}{2}}\|\Delta T(\tau)D_{ij}u_1(t,\cdot)\|_\infty<\infty,\qquad t\in[0,T].
$$

For this purpose, using Lemma 4.1.6 we can easily show that

$$
\begin{aligned}
(T(\tau)D_{ij}u_1(t,\cdot))(x)&=\int_0^t (T(\tau)D_{ij}T(t-s)g(s,\cdot))(x)\,ds\\
&=\int_0^t (D_{ij}T(t+\tau-s)g(s,\cdot))(x)\,ds,\qquad (t,x)\in[0,T]\times\mathbb{R}^d,\ \ \tau>0.
\end{aligned}
\tag{6.7}
$$

Thanks to (6.7) we can write

$$
(\Delta T(\tau)D_{ij}u_1(t,\cdot))(x)=\int_0^t(\Delta D_{ij}T(t+\tau-s)g(s,\cdot))(x)ds,\qquad (t,x)\in[0,T]\times\mathbb{R}^d,\ \ \tau>0.
$$

Hence, by Proposition 4.1.9 and Theorem 4.1.12, we get

$$
\begin{aligned}
\|\Delta D_{ij}T(t+\tau-s)g(s,\cdot)\|_\infty&=\left\|\Delta T\left(\frac{t+\tau-s}{2}\right)D_{ij}T\left(\frac{t+\tau-s}{2}\right)g(s,\cdot)\right\|_\infty\\
&\le\left\|\Delta T\left(\frac{t+\tau-s}{2}\right)\right\|_{L(C_b(\mathbb{R}^d))}\left\|D_{ij}T\left(\frac{t+\tau-s}{2}\right)g(s,\cdot)\right\|_\infty\\
&\le c(t+\tau-s)^{\frac{\alpha}{2}-2}\|g\|_{C^{0,\alpha}_b([0,T]\times\mathbb{R}^d)}
\end{aligned}
$$

so that

$$
\tau^{1-\frac{\alpha}{2}}\|\Delta T(\tau)D_{ij}u_1(t,\cdot)\|_\infty\le c\tau^{1-\frac{\alpha}{2}}\|g\|_{C^{0,\alpha}_b([0,T]\times\mathbb{R}^d)}\int_0^t(t+\tau-s)^{\frac{\alpha}{2}-2}ds
\tag{6.8}
$$

for all $t\in[0,T]$, $\tau>0$ and $i,j=1,\dots,d$. Performing the change of variable $t-s=\tau\sigma$, we can estimate

$$
\int_0^t(t+\tau-s)^{\frac{\alpha}{2}-2}ds=\tau^{\frac{\alpha}{2}-1}\int_0^{\frac{t}{\tau}}(1+\sigma)^{\frac{\alpha}{2}-2}d\sigma\le\tau^{\frac{\alpha}{2}-1}\int_0^\infty(1+\sigma)^{\frac{\alpha}{2}-2}d\sigma=\frac{2}{2-\alpha}\tau^{\frac{\alpha}{2}-1}
\tag{6.9}
$$

From (6.8) and (6.9) we deduce that

$$\tau^{1-\frac{\alpha}{2}}\|\Delta T(\tau)D_{ij}u_1(t,\cdot)\|_\infty \le c\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}, \qquad t\in[0,T],\ \ \tau>0,$$

i.e., each second-order derivative of $u_1(t,\cdot)$ belongs to $D_\Delta(\alpha/2,\infty)=C_b^\alpha(\mathbb{R}^d)$ and

$$\|D_{ij}u_1(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \le c\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}, \qquad t\in[0,T],\ \ i,j=1,\dots,d.$$

Summing up, we have proved that u_1 is bounded in $[0,T]$ with values in $C_b^{2+\alpha}(\mathbb{R}^d)$ and satisfies (6.6).

Step 4. Here, we prove that u_1 is continuously differentiable with respect to t in $[0,T]\times\mathbb{R}^d$. For notational convenience, we denote by D_t^+ (resp. D_t^-) the right (resp. left) time derivative.

For each $\varepsilon\in(0,1)$ we introduce the function $u_{1,\varepsilon}:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ defined by

$$u_{1,\varepsilon}(t,\cdot)=\int_0^{\varepsilon t}(T(t-s)g(s,\cdot))(x)ds, \qquad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Note that $u_{1,\varepsilon}$ converges to u_1 as $\varepsilon\to 1^-$, uniformly in $[0,T]\times\mathbb{R}^d$. Indeed,

$$|u_1(t,x)-u_{1,\varepsilon}(t,x)|=\left|\int_{\varepsilon t}^{t}(T(t-s)g(s,\cdot))(x)ds\right|\le\|g\|_{C_b([0,T]\times\mathbb{R}^d)}(1-\varepsilon)T$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$ and all $\varepsilon\in(0,1)$.

Let us prove that $u_{1,\varepsilon}$ is differentiable in $[0,T]\times\mathbb{R}^d$ with respect to t. Fix $t\in[0,T)$ and $h\in(0,T-t)$. Then,

$$\begin{aligned}\frac{u_{1,\varepsilon}(t+h,\cdot)-u_{1,\varepsilon}(t,x)}{h}=&\frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)}(T(t+h-s)g(s,\cdot))(x)ds\\&+\int_0^{\varepsilon t}\frac{(T(t+h-s)g(s,\cdot))(x)-(T(t-s)g(s,\cdot))(x)}{h}ds\\=:&\,J_1^+(h)+J_2^+(h).\end{aligned}$$

We claim that $J_1^+(h)$ converges to $\varepsilon(T((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)$ as $h\to 0^+$. Indeed,

$$\begin{aligned}&J_1^+(h)-\varepsilon(T((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)\\=&\frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)}(T(t+h-s)g(s,\cdot))(x)ds-\varepsilon(T((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)\\=&\frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)}[(T(t+h-s)g(s,\cdot))(x)-(T((1-\varepsilon)t)g(s,\cdot))(x)]ds\\&+\int_{\varepsilon t}^{\varepsilon(t+h)}\frac{1}{h}(T((1-\varepsilon)t)(g(s,\cdot)-g(\varepsilon t,\cdot)))(x)\,ds.\end{aligned}$$

It is clear that the second integral term in the last side of the previous chain of equalities converges to 0 as $h \to 0^+$. As far as the first integral term is concerned, we observe that an easy computation (based on the representation formula for the Gauss-Weierstrass semigroup) reveals that the function $(r,s) \mapsto (T(r)g(s,\cdot))(x)$ is continuous on $[0,T]\times[0,T]$. Therefore, it is therein uniformly continuous. Hence, for each $\rho > 0$ there exists $\delta > 0$ such that $|(T(r_2)g(s,\cdot))(x) - (T(r_1)g(s,\cdot))(x)| \le \rho$ if $|r_2 - r_1| \le \delta$ and $s \in [0,T]$. Consequently, if $|h| \le \delta$, then $|(T(t+h-s)g(s,\cdot))(x) - (T((1-\varepsilon)t)g(s,\cdot))(x)| \le \rho$ for $s \in [\varepsilon t, \varepsilon(t+h)]$ and[1]

$$\frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)} [(T(t+h-s)g(s,\cdot))(x) - (T((1-\varepsilon)t)g(s,\cdot))(x)]ds \le \varepsilon\rho$$

Therefore,

$$\lim_{h\to 0}\frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)} [(T(t+h-s)g(s,\cdot))(x) - (T((1-\varepsilon)t)g(s,\cdot))(x)]ds = 0.$$

As far as $J_2^+(h)$ is concerned, we observe that the function under the integral sign converges to $(D_tT(t-s)g(s,\cdot))(x) = (\Delta T(t-s)g(s,\cdot))(x)$ as $h \to 0^+$. Moreover, using the mean value theorem, we estimate

$$\begin{aligned}\left|\frac{(T(t+h-s)g(s,\cdot))(x) - (T(t-s)g(s,\cdot))(x)}{h}\right| =&|(\Delta T(t+\xi-s)g(s,\cdot))(x)|\\ \le&\frac{c}{t-s}\|g\|_{C_b([0,T]\times\mathbb{R}^d)}\end{aligned}$$

for all $s \in [0,\varepsilon t]$, where ξ is a suitable point on the line joining 0 to h. Hence, we can apply the dominated convergence theorem and conclude that

$$\lim_{h\to 0^+} J_2^+(h) = \int_0^{\varepsilon t} (\Delta T(t-s)g(s,\cdot))(x)ds.$$

We have so proved that $u_{1,\varepsilon}$ is differentiable from the right in $[0,T) \times \mathbb{R}^d$, with respect to t, and

$$\begin{aligned}D_t^+ u_{1,\varepsilon}(t,x) =&\varepsilon(T((1-\varepsilon)t)g(\varepsilon t,\cdot))(x) + \int_0^{t\varepsilon} (\Delta T(t-s)g(s,\cdot))(x)ds\\ =&\varepsilon(T((1-\varepsilon)t)g(\varepsilon t,\cdot))(x) + \Delta u_{1,\varepsilon}(t,x).\end{aligned}$$

We can argue similarly to prove that $u_{1,\varepsilon}$ is differentiable from the left in $(0,T] \times \mathbb{R}^d$, with respect to t and

$$D_t^- u_{1,\varepsilon}(t,x) =\varepsilon(T((1-\varepsilon)t)g(\varepsilon t,\cdot))(x) + \int_0^{t\varepsilon} (\Delta T(t-s)g(s,\cdot))(x)ds$$

---

[1] Note that $|t+h-s-(1-\varepsilon)t| = |h-s+\varepsilon t| \le |h|$ if $s \in [\varepsilon t, \varepsilon(t+h)]$.

$$=\varepsilon(T((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)+\Delta u_{1,\varepsilon}(t,x).$$

Summing up, $u_{1,\varepsilon}$ is differentiable with respect to t in $[0,T]\times\mathbb{R}^d$ and

$$\begin{aligned}D_t u_{1,\varepsilon}(t,x)=&\varepsilon(T((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)+\int_0^{t\varepsilon}(\Delta T(t-s)g(s,\cdot))(x)ds\\=&\varepsilon(T((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)+\Delta u_{1,\varepsilon}(t,x)\end{aligned}$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$. Now, we observe that

$$\lim_{\varepsilon\to 1^-}D_t u_{1,\varepsilon}(t,x)=g(t,x)+\int_0^t(\Delta T(t-s)g(s,\cdot))(x)ds=g(t,x)+\Delta u_1(t,x)$$

for any $(t,x)\in[0,T]\times\mathbb{R}^d$. Since $u_{1,\varepsilon}$ converges to u_1 uniformly in $[0,T]\times\mathbb{R}^d$, it follows that u_1 is differentiable with respect to t in $[0,T]\times\mathbb{R}^d$ and therein $D_t u_1=\Delta u_1+g$. The proof is now complete. □

Remark 6.1.6. As the proof of Theorem 6.1.4 shows, if we assume that $f\in C_b^{2+\alpha}(\mathbb{R}^d)$ and $g\in C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)$, then, problem (6.1) admits a unique solution $u\in C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)$ such that $D_t u\in C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)$. Moreover,

$$\|u\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)}\le c(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)}+\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)})$$

for some positive constant c, independent of f and g.

Remark 6.1.7. Under the same assumptions of Theorem 6.1.4, the first-order spatial derivatives of the solution u to the Cauchy problem (6.1) belong to $C^{(1+\alpha)/2,0}([0,T]\times\mathbb{R}^d)$. Indeed, Step 1 of the proof of the quoted theorem shows that

$$\|u(t_2,\cdot)-u(t_1,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\le c_1\|D_t u\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}|t_2-t_1|$$

for all $t_1,t_2\in[0,T]$. Using (5.17), we deduce that

$$\begin{aligned}\|\nabla_x u(t_2,\dot{)}-\nabla_x u(t_1,\cdot)\|_\infty\le&c_2\|u(t_2,\cdot)-u(t_1,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}^{\frac{1+\alpha}{2}}\|u(t_2,\cdot)-u(t_1,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}^{\frac{1-\alpha}{2}}\\\le&c_3\|u\|_{C_b^{1,2+\alpha}(\mathbb{R}^d)}|t_2-t_1|^{\frac{1+\alpha}{2}}.\end{aligned}$$

The proof of the following result can be obtained adapting the arguments in the proof of Theorem 6.1.4. Hence, it is left as an exercise.

Theorem 6.1.8. *For each* $f\in C_b^{\beta}(\mathbb{R}^d)$, $(\beta\in[0,2+\alpha))$ *and each* $g\in C((0,T]\times\mathbb{R}^d)$ *such that* $\sup_{t\in(0,T]}t^\theta\|g(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}<\infty$ $(\alpha,\theta\in(0,1))$, *the Cauchy problem* (6.1) *admits a unique classical solution* $u\in C^{1,2}((0,T]\times\mathbb{R}^d)$ *such that*

$$\sup_{t\in(0,T]}t^{\max\{\theta,(2+\alpha-\beta)/2\}}\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}\le C\bigg(\|f\|_{C_b^{\beta}(\mathbb{R}^d)}+\sup_{t\in(0,T]}t^\theta\|g(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg)$$

for some positive constant C, *independent of* f *and* g. *Moreover,* u *belongs to* $C_b^{0,\gamma}([0,T]\times\mathbb{R}^d)$, *where* $\gamma=\min\{\beta,2+\alpha-2\theta\}$.

6.2 Notes

As we start to see in Exercise 6.3. optimal Schauder estimates plays an important role in the study of nonlinear equations. We will see other examples in the next lectures when parabolic equations will be considered in a bounded (and smooth enough) domain. In these situations, the relevance of the optimal Schauder estimates will be made much more clear.

For a different proof of the optimal Schauder estimates and for further details, we refer the interested reader to [1]

6.3 Exercises

1. Prove Remark 6.1.3.

2. (i) Prove that there exists a positive constant C, independent of t and τ such that

$$\int_0^t s^{-\theta}(t+\tau-s)^{\frac{\alpha}{2}-2}ds \le C\tau^{\frac{\alpha}{2}}t^{-\theta}, \qquad t,\tau>0,\ \ \alpha,\theta\in(0,1),$$

   and observe that one can take

$$C = 2^{\theta}\frac{4+\alpha\theta-\alpha-2\theta}{2-\alpha}.$$

   (ii) Prove that

$$\sup_{t\ge 0}\int_0^t s^{-\theta}(t-s)^{1-\theta}ds < \infty.$$

   Use this result to prove that, for each continuous function $g : (0,T]\times\mathbb{R}^d \to \mathbb{R}$ such that $\sup_{t\in(0,T]} t^{\theta}\|g(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)} < \infty$, the function

$$(t,x)\mapsto \int_0^t (T(t-s)g(s,\cdot))(x)ds$$

   belongs to $C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)$.

   (ii) Prove Theorem 6.1.8.

3. On a complete metric space (Y,d) we consider a contractive map $F : Y\to Y$, i.e.

$$d(F(x),F(y)) \le \gamma d(x,y), \qquad x,y\in Y$$

   for some $\gamma\in(0,1)$. Given $y\in Y$, define $F^n(y)$ inductively by $F^0(y)=y$ and $F^{n+1}(y) = F(F^n(y))$, $y\in Y$.
   Prove that there is a unique $u\in Y$ such that $F(u)=u$ and $\lim_{n\to+\infty}F^n(y)=u$ for each $y\in Y$. This is the well-known **Banach fixed point theorem**.

4. Let us consider the nonlinear convection-diffusion equation

$$\begin{cases} D_t u(t,x) = \Delta u(t,x) - u(t,x)^3, & (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d. \end{cases} \tag{6.10}$$

Assume that $f \in C_b^\alpha(\mathbb{R}^d)$ for some $\alpha \in (0,1)$ and define the map

$$F(v) := T(t)f - \int_0^t T(t-s)(v(s,\cdot))^3 \, ds, \qquad v \in C([0,T], C_b^\alpha(\mathbb{R}^d)).$$

a. Prove that F maps $C([0,T], C_b^\alpha(\mathbb{R}^d))$ into $C([0,T], C_b^\alpha(\mathbb{R}^d))$.

b. Prove that for each $R > 0$ there exists $T > 0$, depending only on R, such that (6.10) has a mild solution $u \in C([0,T], C_b^\alpha(\mathbb{R}^d))$, i.e,

$$u(t,x) = (T(t)f)(x) - \int_0^t (T(t-s)(u(s,\cdot))^3)(x) \, ds, \qquad t \in [0,T], \; x \in \mathbb{R}^d,$$

for every initial data $f \in C_b^\alpha(\mathbb{R}^d)$ such that $\|f\|_{C_b^\alpha(\mathbb{R}^d)} < R$.
Hints: Use the above Banach fixed point theorem.

c. Prove that

$$\|u\|_{C_b([0,T]\times\mathbb{R}^d)} \leq \|f\|_\infty. \tag{6.11}$$

d. Prove that the above obtained mild solution u can be extended to all time and satisfies

$$\|u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \leq \|f\|_{C_b^\alpha(\mathbb{R}^d)} e^{Ct\|f\|_\infty^2}, \quad t \geq 0,$$

for some constant $C > 0$.
Hints: Use Proposition 5.1.8,

$$\|\varphi^3\|_{C_b^\alpha(\mathbb{R}^d)} \leq C\|\varphi\|_\infty^2 \|\varphi\|_{C_b^\alpha(\mathbb{R}^d)}, \quad \forall \varphi \in C_b^\alpha(\mathbb{R}^d), \tag{6.12}$$

(6.11) and Gronwall's lemma.

e. Prove that $u \in C^{1,2+\alpha}((0,T] \times \mathbb{R}^d)$ and u is a classical solution to (6.10).
Hints: Use Theorem 4.1.12 with $\theta = \alpha + 2$, (6.11), (6.12) and the ideas of the proof of Theorem 6.1.4.

Bibliography

[1] N.V. Krylov, *Lectures on Elliptic and Parabolic Equations in Hölder spaces.*, Graduate Studies in Mathematics, vol. 12, AMS, 1996.

Lecture 7

Parabolic equations in $\mathbb{R}^d$. Part I

In this lecture, we start to study the nonhomogeneous Cauchy problem

$$
\left\{\begin{array}{lll} D_t u(t,x) = \mathcal{A}u(t,x) + g(t,x), & t \in (0,T], & x \in \mathbb{R}^d, \\ u(0,x) = f(x), & & x \in \mathbb{R}^d, \end{array}\right. \tag{7.1}
$$

where $\mathcal{A}$ is the operator defined on smooth functions $\psi : \mathbb{R}^d \to \mathbb{R}$ by

$$
\begin{aligned}
\mathcal{A}\psi(x) =& \sum_{i,j=1}^d q_{ij} D_{ij}\psi(x) + \sum_{j=1}^d b_j(x) D_j \psi(x) + c(x)\psi(x) \\
=& \mathrm{Tr}(Q(x)D^2\psi(x)) + \langle b(x), \nabla_x \psi(x)\rangle + c(x)\psi(x)
\end{aligned}
$$

with $Q(x) = (q_{ij}(x))_{1\le i,j\le d}$ for $x \in \mathbb{R}^d$. Throughout the lecture, we assume the following conditions on the coefficients of the operator $\mathcal{A}$.

Hypotheses 7.0.1. (i) The coefficients $q_{ij} = q_{ji}$, b_j $(i,j = 1,\ldots,d)$ and c are bounded and α-Hölder continuous in $\mathbb{R}^d$ for some $\alpha \in (0,1)$;

(ii) there exists a positive constant μ such that $\langle Q(x)\xi,\xi\rangle \ge \mu|\xi|^2$ for all $x, \xi \in \mathbb{R}^d$.

Remark 7.0.2. The condition (ii) in Hypothesis 7.0.1 is usually rephrased saying that the operator $\mathcal{A}$ is uniformly elliptic in $\mathbb{R}^d$.

Our aim consists in extending the results in Lecture 6 to the more general Cauchy problem (7.1). Two are the main tools which we use to prove the counterpart of Theorem 6.1.4 for the solution to problem (7.1):

(i) some *apriori* estimates for solutions to the Cauchy problem (7.1);

(ii) the continuity method.

The continuity method states that, if we have a "segment" of bounded linear operators $L_\sigma = (1-\sigma)L_0 + \sigma L_1$ $(\sigma \in [0,1])$ mapping a Banach space X into a Banach space

Y, L_0 is invertible and there exists a positive constant C, independent of σ, such that $\|L_\sigma x\|_Y \geq C\|x\|_X$ for any $x \in X$ and $\sigma \in [0,1]$, then each operator L_σ is invertible. We will apply the continuity method with the following choices:

$$\begin{aligned} &X = C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d), \qquad && Y = redC_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)\times C_b^{2+\alpha}(\mathbb{R}^d),\\ &L_0u = (D_tu-\Delta u, u(0,\cdot)), && L_1u = (D_tu-\mathcal{A}u, u(0,\cdot)). \end{aligned}$$

Since, by Theorem 6.1.4, the operator L_0 is invertible, to make the continuity method work we need to prove the *apriori* estimate $\|L_\sigma u\|_Y \geq c\|u\|_X$, for each $u \in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$, $\sigma \in [0,1]$ and some positive constant c, independent of σ. Such an estimate is proved using the classical method of *freezing* the coefficients.

7.1 The continuity method

Here, we restate and prove the continuity method.

Theorem 7.1.1 (Continuity method)**.** *Let X, Y be two Banach spaces, L_0, L_1 be two bounded linear operators mapping X into Y. For each $\sigma \in [0,1]$, set $L_\sigma = (1-\sigma)L_0 + \sigma L_1$. Suppose that the operator L_0 is invertible and there exists a positive constant C such that*

$$\|L_\sigma x\|_Y \geq C\|x\|_X, \qquad x \in X,\ \sigma\in[0,1]. \tag{7.2}$$

Then, each operator L_σ is invertible and $\|L_\sigma^{-1}\|_{L(Y,X)} \leq C^{-1}$.

Proof. To begin with, we observe that estimate (7.2) implies that each operator L_σ is injective. To prove that L_σ is also surjective for all $\sigma \in [0,1]$, we prove that if L_{σ_0} is invertible for some $\sigma_0 \in [0,1)$, then there exists $\delta > 0$, independent of σ_0, such that L_σ is invertible for all $\sigma \in [\sigma_0, \sigma_0 + \min\{\delta, 1-\sigma_0\}]$. This is enough for our purposes. Indeed, since L_0 is invertible, starting from $\sigma_0 = 0$ and moving to the right by steps of length δ we reach $\sigma = 1$ in a finite number of steps. And this shows that all the operators L_σ are invertible.

So, let us suppose that L_{σ_0} is invertible and fix $\sigma \in (0,1]$. Observe that

$$\begin{aligned} L_\sigma =& L_{\sigma_0} + L_\sigma - L_{\sigma_0} = L_{\sigma_0}(I + L_{\sigma_0}^{-1}(L_\sigma - L_{\sigma_0}))\\ =& L_{\sigma_0}(I + (\sigma-\sigma_0)L_{\sigma_0}^{-1}(L_1 - L_0)). \end{aligned}$$

Since L_{σ_0} is invertible, the operator L_σ is invertible if and only if the bounded operator $I + (\sigma-\sigma_0)L_{\sigma_0}^{-1}(L_1-L_0)$, mapping X into itself, is invertible. Applying Lemma 2.1.11 and (7.2) with $\sigma = \sigma_0$, we conclude that, if $|\sigma-\sigma_0|\|L_1 - L_0\|_{L(X,Y)} < C$, then the operator $I + (\sigma-\sigma_0)L_{\sigma_0}^{-1}(L_1-L_0)$ is invertible. Hence, if $\delta = 2^{-1}C\|L_1-L_0\|_{L(X,Y)}^{-1}$, then, for each $\sigma \in [\sigma_0, \sigma_0 + \min\{\delta, 1-\sigma_0\}]$ the operator L_σ is invertible. Since δ is independent of σ_0, we are done. □

7.2 A priori estimates

This section is devoted to prove the following result.

Theorem 7.2.1. *Let the coefficients of the operator $\mathcal{A}$ satisfy Hypotheses 7.0.1. Then, for each $u \in C^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$ there exists a positive constant K, which depends only on d, T, μ (the ellipticity constant of the operator $\mathcal{A}$) and the C^α-norm of the coefficients of q_{ij}, b_j $(i,j=1,\dots,d)$ and c, such that*

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \le K(\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_tu-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}). \tag{7.3}$$

In view of Theorem 6.1.4, we know that estimate (7.3) is satisfied when $\mathcal{A}$ is the Laplacian. By a straightforward change of variables, we can show that (7.3) holds true when $\mathcal{A} = \mathrm{Tr}(QD^2)$ and Q is a positive definite and constant matrix.

Lemma 7.2.2. *Suppose that $\mathcal{A} = \mathrm{Tr}(QD^2)$ for some constant symmetric and positive definite matrix Q. Then, estimate (7.3) holds true.*

Proof. We split the proof into two steps.

Step 1. Here, we suppose that $Q = \mathrm{diag}(\lambda_1,\dots,\lambda_d)$ and denote, by $\lambda_{\max}$ the maximum eigenvalue of Q. Note that the best possible choice of μ is the minimum eigenvalue of Q. Given $u \in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$, we introduce the function $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ defined by $v(t,x) = u(t,\sqrt{\lambda_1}x,\dots,\sqrt{\lambda_d}x_d)$ for $(t,x)\in[0,T]\times\mathbb{R}^d$. Clearly, v belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$. Hence, applying (6.2), we get

$$\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \le K_1(\|v(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_tv-\Delta v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}), \tag{7.4}$$

where the constant K_1 depends only on d, T and α. Now, we observe that

$$D_tv(t,x) - \Delta v(t,x) = D_tu(t,\sqrt{\lambda_1}x_1,\dots,\sqrt{\lambda_d}x_d) - \mathrm{Tr}(QD^2u(t,\sqrt{\lambda_1}x_1,\dots,\sqrt{\lambda_d}x_d)).$$

Therefore,

$$\begin{aligned}\|D_tv-\Delta v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} \le& \|D_tu-\mathcal{A}u\|_{C_b^{\alpha/2,0}([0,T]\times\mathbb{R}^d)} + \lambda_{\max}^{\alpha/2}[D_tu-\mathcal{A}u]_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\\ \le& \max\{1,\lambda_{\max}^{\alpha/2}\}\|D_tu-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \ge& \|u\|_{C_b([0,T]\times\mathbb{R}^d)} + \sqrt{\mu}\sum_{j=1}^d \|D_ju\|_{C_b([0,T]\times\mathbb{R}^d)} + \|D_tu\|_{C_b^{\alpha/2,0}([0,T]\times\mathbb{R}^d)}\\ &+ \mu\sum_{i,j=1}^d \|D^2_{ij}u\|_{C_b^{\alpha/2,0}([0,T]\times\mathbb{R}^d)} + \mu^{\frac{\alpha}{2}}[D_tu]_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\\ &+ \mu^{1+\frac{\alpha}{2}}\sum_{i,j=1}^d [D^2_{ij}u]_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\end{aligned}$$

$$\geq \min\{\mu^{1+\frac{\alpha}{2}}, 1\}\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}$$

and

$$\|v(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} \leq \max\{1, \lambda_{\max}^{1+\frac{\alpha}{2}}\}\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}.$$

Replacing these three estimates in (7.4), we get

$$\begin{aligned}\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \leq &\max\{\mu^{-1-\frac{\alpha}{2}}, 1\}\max\{1, \lambda_{\max}^{1+\alpha/2}\}\\ &\times K_1(\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u - \mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)})\end{aligned}$$

and (7.3) follows with $K = \max\{\mu^{-1-\frac{\alpha}{2}}, 1\}\max\{1, \lambda_{\max}^{1+\alpha/2}\}K_1$.

Step 2. Now, we consider the general case and fix a matrix P such that $PP^* = P^*P = Id$, the $(d\times d)$-identity matrix, and $PQP^* = D_\lambda := \mathrm{diag}(\lambda_1,\ldots,\lambda_d)$. Then, the function v, defined by $v(t,x) = u(t,P^*x)$ for $(t,x)\in[0,T]\times\mathbb{R}^d$, belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$. By Step 1, we know that

$$\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \leq K(\|v(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t v - \mathrm{Tr}(D_\lambda D_x^2 v)\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}). \tag{7.5}$$

Since $\nabla_x v(t,x) = P\nabla_x u(t,P^*x)$, $D_x^2 v(t,x) = PD_x^2 u(t,P^*x)P^*$ for $(t,x)\in[0,T]\times\mathbb{R}^d$ and P is an isometry with Euclidean norm $|||P|||$ equal to 1, it is immediate to check that

$$\|D_j u\|_\infty \leq \|\nabla_x u\|_\infty = \|\nabla_x v\|_\infty \leq \sum_{j=1}^d \|D_j v\|_\infty,$$

$$\begin{aligned}\|D_{ij}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} \leq &\|D_x^2 u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} = \|P^* D_x^2 v(t,P\cdot)P\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\\ \leq &|||P^*|||\|D_x^2 v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}|||P||| = \|D_x^2 v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\\ \leq &\sum_{h,j=1}^d \|D_{hk}v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\end{aligned}$$

for every $i,j = 1,\ldots,d$. Hence.

$$\begin{aligned}\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \leq &\|v\|_\infty + \sum_{j=1}^d \|D_j v\|_\infty + \sum_{i,j=1}^d \|D_{ij}v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\\ &+ \|D_t v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\\ \leq &\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}.\end{aligned} \tag{7.6}$$

Moreover,

$$D_t v(t,x) - \mathrm{Tr}(D_\lambda D_x^2 v(t,x)) = D_t u(t,P^*x) - \mathrm{Tr}(D_\lambda P D_x^2 u(t,P^*x)P^*)$$

$$
\begin{aligned}
&= D_t u(t, P^* x) - \mathrm{Tr}(P^* D_\lambda P D_x^2 u(t, P^* x)) \\
&= D_t u(t, P^* x) - \mathrm{Tr}(Q D_x^2 u(t, P^* x)).
\end{aligned}
$$

Hence,

$$
\|D_t v - \mathrm{Tr}(D_\lambda D_x^2 v)\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} = \|D_t u - \mathrm{Tr}(Q D_x^2 u)\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}. \tag{7.7}
$$

Replacing (7.6) and (7.7) in (7.5) the assertion follows also in this case. □

Lemma 7.2.3. *For any $\alpha \in (0,1)$, a function f belongs to $C_b^\alpha(\mathbb{R}^d)$ if and only if f is bounded and, for some (and, hence, all) $r > 0$, $|f(x) - f(y)| \le C_r |x-y|^\alpha$ for every $x, y \in \mathbb{R}^d$, such that $|x - y| \le r$, and some positive constant C_r. Moreover, for every $r > 0$, the norm*

$$
\|f\|_\infty + \sup_{x,y\in\mathbb{R}^d:\, 0<|x-y|\le r} \frac{|f(x) - f(y)|}{|x-y|^\alpha}, \qquad u \in C_b^\alpha(\mathbb{R}^d), \tag{7.8}
$$

is equivalent to the classical norm of $C_b^\alpha(\mathbb{R}^d)$.

Proof. Clearly, if $f \in C_b^\alpha(\mathbb{R}^d)$, then we can take $C_r = [f]_{C_b^\alpha(\mathbb{R}^d)}$. On the other hand, if f is bounded and $|f(x) - f(y)| \le C_r|x-y|^\alpha$ for all $x, y \in \mathbb{R}^d$ such that $|x-y| \le r$, for some $r > 0$, then for $|x - y| > r$ we can estimate $|f(x) - f(y)| \le 2\|f\|_\infty \le 2r^{-\alpha}\|f\|_\infty |x-y|^\alpha$ Hence, $f \in C_b^\alpha(\mathbb{R}^d)$ and $[f]_{C_b^\alpha(\mathbb{R}^d)} \le \max\{C_r, 2r^{-\alpha}\|f\|_\infty\}$. This estimate also shows that the classical norm of $C_b^\alpha(\mathbb{R}^d)$ and the norm in (7.8) are equivalent. □

Lemma 7.2.4. *$C_b^{1,2}([0,T]\times\mathbb{R}^d)$ is continuously embedded into*[1] *$C_b^{\alpha/2,1+\alpha}([0,T]\times\mathbb{R}^d)$ for each $\alpha \in (0,1)$. Moreover, for every $\varepsilon > 0$ there exists a positive constant $C_{\alpha,\varepsilon}$ such that*

$$
\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)} \le \varepsilon \|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} + C_{\alpha,\varepsilon}(\|D_t u - \mathcal{A}u\|_{C_b([0,T]\times\mathbb{R}^d)} + \|u(0,\cdot)\|_\infty) \tag{7.9}
$$

for all $u \in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$.

Proof. We begin by proving that $C_b^{1,2}([0,T]\times\mathbb{R}^d) \hookrightarrow C_b^{\alpha/2,1+\alpha}([0,T]\times\mathbb{R}^d)$. For this purpose, we just need to estimate the $C^{\alpha/2,\alpha}$-seminorm of $u \in C_b^{1,2}([0,T]\times\mathbb{R}^d)$. Applying the mean value theorem, we can easily show that

$$
|u(t,x) - u(s,x)| \le \|D_t u\|_{C_b([0,T]\times\mathbb{R}^d)}|t-s| \le \|D_t u\|_{C_b([0,T]\times\mathbb{R}^d)} T^{1-\frac{\alpha}{2}}|t-s|^{\alpha/2}
$$

for all $x \in \mathbb{R}^d$ and $s, t \in [0,T]$, and

$$
|u(t,x) - u(t,y)| \le \|\nabla_x u\|_{C_b([0,T]\times\mathbb{R}^d)}|x-y| \le \|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}|x-y|^\alpha,
$$

---

[1] $C_b^{\alpha/2,1+\alpha}([0,T]\times\mathbb{R}^d)$ is the space of all the functions $f \in C_b^{\alpha/2,0}([0,T]\times\mathbb{R}^d)$ such that $D_j f \in C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)$ for any $j = 1,\dots,d$. It is a Banach space with the norm $\|f\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} = \|f\|_{C_b^{\alpha/2,0}([0,T]\times\mathbb{R}^d)} + \sum_{j=1}^d \|D_j f\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}$.

$$|D_ju(t,x)-D_ju(t,y)|\le \|\nabla_x D_ju\|_{C_b([0,T]\times\mathbb{R}^d)}|x-y|\le \|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}|x-y|^\alpha$$

for $t\in[0,T]$, $x,y\in\mathbb{R}^d$, with $|x-y|\le 1$, and $j=1,\ldots,d$. Taking Lemma 7.2.3 into account, we conclude that

$$[u]_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+[D_ju]_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\le C_1\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}$$

for every $j=1,\ldots,d$ and some positive constant C_1, independent of u. Finally, we recall (see Remark 5.2.2) that $\|\nabla_x\psi\|_\infty\le C_2\|\psi\|_\infty^{1/2}\|\psi\|_{C_b^2(\mathbb{R}^d)}^{1/2}$ for every $\psi\in C_b^2(\mathbb{R}^d)$ and some positive constant C_2, independent of ψ. Hence,

$$\begin{aligned}|D_ju(t,x)-D_ju(s,x)|\le&C_2\|u(t,\cdot)-u(s,\cdot)\|_\infty^{\frac12}\|u(t,\cdot)-u(s,\cdot)\|_{C_b^2(\mathbb{R}^d)}^{\frac12}\\ \le&C_2\|D_tu\|_{C_b([0,T]\times\mathbb{R}^d)}^{\frac12}(2\|u\|_{C_b^{0,2}([0,T]\times\mathbb{R}^d)})^{\frac12}|t-s|^{\frac12}\\ \le&\widetilde{C}_2T^{\frac{1-\alpha}{2}}\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}|t-s|^{\frac{\alpha}{2}}\end{aligned}$$

for all $s,t\in[0,T]$, $x\in\mathbb{R}^d$ and $j=1,\ldots,d$. We thus conclude that

$$[D_ju]_{C_b^{\alpha/2,0}([0,T]\times\mathbb{R}^d)}\le \widetilde{C}_2T^{\frac{1-\alpha}{2}}\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}$$

for every $j=1,\ldots,d$, The embedding $C_b^{1,2}([0,T]\times\mathbb{R}^d)\hookrightarrow C_b^{\alpha/2,1+\alpha}([0,T]\times\mathbb{R}^d)$ follows.

To complete the proof, let us check estimate (7.9). For this purpose, we fix $u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$ and begin by observing that, since u is a classical solution to the Cauchy problem

$$\begin{cases}D_tu(t,x)=\mathcal{A}u(t,x)+g(t,x), & t>0,\quad x\in\mathbb{R}^d,\\ u(0,x)=f(x), & \qquad\quad\; x\in\mathbb{R}^d,\end{cases}$$

with $f=u(0,\cdot)$ and $g=D_tu-\mathcal{A}u$, Exercise 7.3.3 implies that

$$\|u\|_{C_b([0,T]\times\mathbb{R}^d)}\le C_3(\|D_tu-\mathcal{A}u\|_{C_b([0,T]\times\mathbb{R}^d)}+\|u(0,\cdot)\|_\infty)\tag{7.10}$$

for some positive constant C_3, independent of u. Now, applying the interpolation estimate (5.25) with $k=2$ to the function $u(t,\cdot)$ and Young's inequality, we get

$$\begin{aligned}\|u\|_{C_b^{0,2}([0,T]\times\mathbb{R}^d)}\le&C\|u\|_{C_b([0,T]\times\mathbb{R}^d)}^{\frac{\alpha}{2+\alpha}}\|u\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)}^{\frac{2}{2+\alpha}}\\ \le&\frac{\varepsilon}{2}\|u\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)}+\widehat{C}_{\varepsilon,1}\|u\|_{C_b([0,T]\times\mathbb{R}^d)}\end{aligned}$$

for all $\varepsilon>0$ and some positive constant $\widehat{C}_{\varepsilon,1}$.

Similarly, estimate (5.19)(ii) (with $\mathbb{R}$ and α being replaced, respectively, by $[0,T]$ and $\alpha/2$), and again Young's inequality yield

$$\|u\|_{C_b^{1,0}([0,T]\times\mathbb{R}^d)}\le C\|u\|_{C_b([0,T]\times\mathbb{R}^d)}^{\frac{\alpha}{2+\alpha}}\|u\|_{C_b^{1+\alpha/2,0}([0,T]\times\mathbb{R}^d)}^{\frac{2}{2+\alpha}}$$

$$\leq \frac{\varepsilon}{2}\|u\|_{C_b^{1+\alpha/2,0}([0,T]\times\mathbb{R}^d)} + \widehat{C}_{\varepsilon,2}\|u\|_{C_b([0,T]\times\mathbb{R}^d)}$$

for all $\varepsilon > 0$ and some positive constant $\widehat{C}_{\varepsilon,2}$. Hence,

$$\begin{aligned}
\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)} \leq & \|u\|_{C_b^{0,2}([0,T]\times\mathbb{R}^d)} + \|u\|_{C_b^{1,0}([0,T]\times\mathbb{R}^d)} \\
\leq & \frac{\varepsilon}{2}(\|u\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} + \|u\|_{C_b^{1+\alpha/2,0}([0,T]\times\mathbb{R}^d)}) + (\widehat{C}_{\varepsilon,1} + \widehat{C}_{\varepsilon,2})\|u\|_{C_b([0,T]\times\mathbb{R}^d)} \\
\leq & \varepsilon\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} + (\widehat{C}_{\varepsilon,1} + \widehat{C}_{\varepsilon,2})\|u\|_{C_b([0,T]\times\mathbb{R}^d)}.
\end{aligned}$$

Replacing (7.10) in this estimate, (7.9) follows at once. □

Now, we are ready to prove Theorem 7.2.1 in its full generality.

Proof of Theorem 7.2.1. Throughout the proof, we denote by C a positive constant, which may vary from line to line and depends only on T, μ and the C^α-norm of the coefficients.

For each $x_0 \in \mathbb{R}^d$ and $r > 0$, we introduce a cut-off function $\vartheta_{x_0,r} \in C_c^\infty(\mathbb{R}^d)$, such that

$$\begin{aligned}
&\chi_{B(x_0,r/2)} \leq \vartheta_{x_0,r} \leq \chi_{B(x_0,r)} \text{ and} \\
&r\|\nabla\vartheta_{x_0,r}\|_\infty + r^2\|D^2\vartheta_{x_0,r}\|_\infty + r^3\|D^3\vartheta_{x_0,r}\|_\infty \leq M_1
\end{aligned}\tag{7.11}$$

for some positive constant M_1, independent of x_0 and r. As a byproduct,

$$r^\alpha\|\vartheta_{x_0,r}\|_{C_b^\alpha(\mathbb{R}^d)} + r^{1+\alpha}\|\vartheta_{x_0,r}\|_{C_b^{1+\alpha}(\mathbb{R}^d)} + r^{2+\alpha}\|\vartheta_{x_0,r}\|_{C_b^{2+\alpha}(\mathbb{R}^d)} \leq M_2.\tag{7.12}$$

We fix $u \in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$, $r \in (0,1)$ and apply Lemma 7.2.2 to the function $\vartheta_{x_0,r}u$ and the operator $\mathcal{A}_{x_0} = \mathrm{Tr}(Q(x_0)D^2)$, obtaining

$$\begin{aligned}
&\|\vartheta_{x_0,r}u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \\
\leq & \widetilde{K}(\|\vartheta_{x_0,r}D_tu - \mathcal{A}_{x_0}(\vartheta_{x_0,r}u)\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} + \|\vartheta_{x_0,r}u(0,\cdot)\|_{C_b^{2+\alpha}([0,T]\times\mathbb{R}^d)}).
\end{aligned}\tag{7.13}$$

Let us estimate the right-hand side of (7.13). To begin with, we observe that $\vartheta_{x_0,r}D_tu - \mathcal{A}_{x_0}(\vartheta_{x_0,r}u) = \vartheta_{x_0,r}(D_tu - \mathcal{A}_{x_0}u) - u\mathcal{A}_{x_0}\vartheta_{x_0,r} - 2\langle Q(x_0)\nabla\vartheta_{x_0,r}, \nabla_x u\rangle$. Hence, taking Lemma 7.2.4 and (7.12) into account, we can estimate

$$\begin{aligned}
&\|\vartheta_{x_0,r}D_tu - \mathcal{A}_{x_0}(\vartheta_{x_0,r}u)\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} \\
\leq & \|\vartheta_{x_0,r}\|_{C_b^\alpha(\mathbb{R}^d)}\|D_tu - \mathcal{A}_{x_0}u\|_{C([0,T]\times\overline{B(x_0,r)})} + \|\vartheta_{x_0,r}\|_\infty\|D_tu - \mathcal{A}_{x_0}u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})} \\
& + \|\mathcal{A}\vartheta_{x_0,r}\|_{C_b^\alpha(\mathbb{R}^d)}\|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})} + 2|||Q(x_0)|||\|\nabla\vartheta_{x_0,r}\|_{C_b^\alpha(\mathbb{R}^d)}\|\nabla_x u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})} \\
\leq & \|D_tu - \mathcal{A}_{x_0}u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})} + M_2r^{-\alpha}\|D_tu - \mathcal{A}_{x_0}u\|_{C_b([0,T]\times\mathbb{R}^d)} \\
& + Cr^{-2-\alpha}\|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} + Cr^{-1-\alpha}\|\nabla_x u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} \\
\leq & \|D_tu - \mathcal{A}_{x_0}u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})} + Cr^{-\alpha}\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)} + Cr^{-2-\alpha}\|u\|_{C_b^{\alpha/2,1+\alpha}([0,T]\times\mathbb{R}^d)} \\
\leq & \|D_tu - \mathcal{A}_{x_0}u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})} + Cr^{-2-\alpha}\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}.
\end{aligned}$$

(7.14)

Analogously, since $r \in (0,1)$, $\|\vartheta_{x_0,r}u(0,\cdot)\|_{C^{2+\alpha}_b([0,T]\times\mathbb{R}^d)} \le Cr^{-2-\alpha}\|u(0,\cdot)\|_{C^{2+\alpha}_b(\mathbb{R}^d)}$. From this estimate, (7.13), (7.14) and observing that

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{B(x_0,r/2)})} \le \|\vartheta_{x_0,r}u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)},$$

we deduce that

$$\begin{aligned}\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{B(x_0,r/2)})} \le {}& C(\|D_tu-\mathcal{A}_{x_0}u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})}\\ &+ r^{-2-\alpha}\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)} + r^{-2-\alpha}\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}).\end{aligned}\tag{7.15}$$

Next, we replace the operator $\mathcal{A}_{x_0}$ with the operator $\mathcal{A}$ in the right-hand side of (7.15). For this purpose, we observe that

$$\begin{aligned}&\|D_tu-\mathcal{A}_{x_0}u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})}\\ \le&\|D_tu-\mathcal{A}u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})} + \|\mathcal{A}u-\mathcal{A}_{x_0}u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})}\\ \le&\|D_tu-\mathcal{A}u\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})} + \|\mathrm{Tr}((Q-Q(x_0))D^2_xu)\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})}\\ &+\sum_{j=1}^d\|b_j\|_{C^\alpha_b(\mathbb{R}^d)}\|D_ju\|_{C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} + \|c\|_{C^\alpha_b(\mathbb{R}^d)}\|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}.\end{aligned}\tag{7.16}$$

Since the diffusion coefficients of the operator $\mathcal{A}$ are α-Hölder continuous in $\mathbb{R}^d$, we can estimate

$$\|\mathrm{Tr}((Q-Q(x_0))D^2_xu(t,\cdot))\|_{C(\overline{B(x_0,r)})} \le Cr^\alpha\|D^2_xu\|_{C([0,T]\times\overline{B(x_0,r)})},$$

$$\begin{aligned}&[\mathrm{Tr}((Q-Q(x_0))D^2_xu(t,\cdot))]_{C^\alpha(\overline{B(x_0,r)})}\\ \le&\|\mathrm{Tr}((Q-Q(x_0)))\|_{C^\alpha(\overline{B(x_0,r)})}\|D^2_xu\|_{C([0,T]\times\overline{B(x_0,r)})}\\ &+\|\mathrm{Tr}(Q-Q(x_0))\|_{C(\overline{B(x_0,r)})}[D^2_xu]_{C^{0,\alpha}([0,T]\times\overline{B(x_0,r)})}\\ \le&\sum_{i,j=1}^d\|q_{ij}\|_{C^\alpha_b(\mathbb{R}^d)}\|D^2_xu\|_{C([0,T]\times\overline{B(x_0,r)})} + r^\alpha\sum_{i,j=1}^d\|q_{ij}\|_{C^\alpha_b(\mathbb{R}^d)}\|D^2_xu\|_{C^{0,\alpha}([0,T]\times\overline{B(x_0,r)})}\end{aligned}$$

for all $t \in [0,T]$ and

$$[\mathrm{Tr}((Q(x)-Q(x_0))D^2_xu(\cdot,x))]_{C^{\alpha/2}([0,T])} \le r^\alpha\sum_{i,j=1}^d\|q_{ij}\|_{C^\alpha_b(\mathbb{R}^d)}[D^2_xu]_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})}$$

for all $x \in \overline{B(x_0,r)}$. Hence, we conclude that

$$\|\mathrm{Tr}((Q-Q(x_0))D^2_xu)\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})} \le C\|D^2_xu\|_{C_b([0,T]\times\mathbb{R}^d)} + Cr^\alpha\|D^2_xu\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)},$$

which, replaced in (7.16), gives

$$\begin{aligned}\|D_t u-\mathcal{A}_{x_0}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,r)})}\leq&\|D_t u-\mathcal{A}u\|_{C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+C\|u\|_{C_b^{0,2}([0,T]\times\mathbb{R}^d)}\\&+C\|u\|_{C_b^{\alpha/2,1+\alpha}([0,T]\times\mathbb{R}^d)}+Cr^\alpha\|D_x^2u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}.\end{aligned}\tag{7.17}$$

Recalling once more that $r\in(0,1)$, from (7.15), (7.17) and Lemma 7.2.4 we obtain

$$\begin{aligned}\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{B(x_0,r/2)})}\leq C(&\|D_t u-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+r^\alpha\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\\&+r^{-2-\alpha}\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}+r^{-2-\alpha}\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}).\end{aligned}\tag{7.18}$$

Since the constant C in the previous estimate is independent of $x_0\in\mathbb{R}^d$, we immediately deduce that

$$\begin{aligned}\|u\|_{C_b^{1+\alpha/2,2}([0,T]\times\mathbb{R}^d)}\leq C(&\|D_t u-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+r^\alpha\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\\&+r^{-2-\alpha}\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}+r^{-2-\alpha}\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}).\end{aligned}\tag{7.19}$$

Moreover, for every $(t,x),(t,y)\in[0,T]\times\mathbb{R}^d$, with $|x-y|\leq\frac{r}{2}$, and $i,j=1,\dots,d$, we can estimate

$$\begin{aligned}&|D_{ij}u(t,x)-D_{ij}u(t,y)|+|D_tu(t,x)-D_tu(t,y)|\\\leq&([D_{ij}u(t,\cdot)]_{C^\alpha(\overline{B(x,r/2)})}+[D_tu(t,\cdot)]_{C^\alpha(\overline{B(x,r/2)})})|x-y|^\alpha\\\leq&\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{B(x,r/2)})}|x-y|^\alpha.\end{aligned}$$

By (7.18) and Lemma 7.2.3, we conclude that

$$\begin{aligned}&\sup_{t\in[0,T]}[D_{ij}(t,\cdot)]_{C_b^\alpha(\mathbb{R}^d)}+\sup_{t\in[0,T]}[D_t(t,\cdot)]_{C_b^\alpha(\mathbb{R}^d)}\\\leq&C(\|D_t u-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+r^\alpha\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\\&\quad+r^{-2-\alpha}\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}+r^{-2-\alpha}\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)})\end{aligned}\tag{7.20}$$

and, as byproduct, by (7.19) and (7.20),

$$\begin{aligned}\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\leq C(&\|D_t u-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+r^\alpha\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\\&+r^{-2-\alpha}\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}+r^{-2-\alpha}\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}).\end{aligned}$$

Now, we fix $r_0\in(0,1)$ such that $Cr_0^\alpha\leq 1/2$ to move the term $Cr^\alpha\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}$ to the left-hand side of the previous estimate and thus conclude that

$$\begin{aligned}\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\leq 2Cr_0^{-2-\alpha}(&\|D_t u-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+\|u\|_{C_b^{1,2}([0,T]\times\mathbb{R}^d)}\\&+\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)})\end{aligned}\tag{7.21}$$

Using Lemma 7.2.4 we estimate the $C^{1,2}$-norm of u and from (7.21) we get

$$\begin{aligned}&\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\\ \le&2Cr_0^{-2-\alpha}[\|D_tu-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+\varepsilon\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\\ &\qquad+C'_{\alpha,\varepsilon}(\|D_tu-\mathcal{A}u\|_{C_b([0,T]\times\mathbb{R}^d)}+\|u(0,\cdot)\|_\infty)+\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}]\end{aligned}$$

for all $\varepsilon>0$. Choosing $\varepsilon_0>0$ sufficiently small, we conclude that

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\le C(\|D_tu-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)})$$

and the proof is complete. □

Remark 7.2.5. We stress that the constant K in (7.3) can be split as

$$K=\max\{\mu^{-1-\frac{\alpha}{2}},1\}C\left(d,T,\max_{i,j=1,\dots,d}\|q_{ij}\|_{C_b^\alpha(\mathbb{R}^d)},\max_{j=1,\dots,d}\|b_j\|_{C_b^\alpha(\mathbb{R}^d)},\|c\|_{C_b^\alpha(\mathbb{R}^d)}\right),$$

where $C(d,T,\cdot,\cdot,\cdot)$ is a locally bounded function. Indeed, the ellipticity constant μ appears only in the constant $\widetilde{K}$ of the estimate (7.13) and the proof of Lemma 7.2.2 shows that this constant is given by $\max\{1,\mu^{-1-\alpha/2}\}\widetilde{C}(d,T,\alpha)\max\left\{1,\sum_{i,j=1}^d\|q_{ij}\|_\infty^\alpha\right\}$.

On the other hand the constants C in estimates (7.14) and (7.17) are locally bounded functions of the $C_b^\alpha(\mathbb{R}^d)$-norm of the coefficients of the operator $\mathcal{A}$.

Adapting the proof of Theorem 7.2.1 and taking Remark 6.1.6 into account the following result can be proved.

Theorem 7.2.6. *For each $T>0$ there exists a positive constant C such that*

$$\|u\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)}\le C(\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}+\|D_tu-\mathcal{A}u\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)})\tag{7.22}$$

for all $u\in C_b^{1,2+\alpha}([0,T]\times\mathbb{R}^d)$.

7.3 Exercises

1. Prove that, for any pair of compact sets K_1 and K_2 with $K_1\Subset K_2\subset\mathbb{R}^d$, there exists a function $\varphi\in C_c^\infty(\mathbb{R}^d)$ such that $\varphi\equiv1$ in K_1 and $\varphi\equiv0$ outside K_2.

2. Given $x_0\in\mathbb{R}^d$ and $r>0$, construct a function $\vartheta_{x_0,r}$ satisfying (7.11) and prove estimate (7.12).

3. Assume that $u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$ is a classical solution to the following Cauchy problem

$$\begin{cases}D_tu(t,x)=\mathcal{A}u(t,x)+g(t,x), & t>0, \quad x\in\mathbb{R}^d,\\ u(0,x)=f(x), & \qquad\quad\; x\in\mathbb{R}^d,\end{cases}$$

where $f \in C_b(\mathbb{R}^d)$ and $g \in C_b([0,T] \times \mathbb{R}^d)$. Prove that

$$\|u\|_{C_b([0,T]\times\mathbb{R}^d)} \leq C(\|g\|_{C_b([0,T]\times\mathbb{R}^d)} + \|f\|_\infty)$$

for some constant $C > 0$.
Hint: Use the weak maximum principle for the function $(t,x) \mapsto \theta_n(x)u(t,x)$, where θ_n is a cut-off function.

4. Prove Theorem 7.2.6.

Lecture 8

Parabolic equations in $\mathbb{R}^d$. Part II

8.1 Solving problem (7.1)

Now, we are in a position to prove the main result of this lecture.

Theorem 8.1.1. *Let Hypotheses* 7.0.1 *be satisfied. Then, for every* $f \in C_b^{2+\alpha}(\mathbb{R}^d)$ *and* $g \in C_b^{\alpha/2,\alpha}([0,T] \times \mathbb{R}^d)$ *there exists a unique classical solution* u *to problem* (7.1)*. In addition,* u *belongs to* $C_b^{1+\alpha/2,2+\alpha}([0,T] \times \mathbb{R}^d)$ *and there exists a positive constant* C*, independent of* f *and* g*, such that*

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \le C(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}). \tag{8.1}$$

Proof. The uniqueness of the classical solution to problem (7.1) follows from the maximum principle in Exercise 8.4.2. The existence of a solution to problem (7.1) will be proved, as already claimed, using the continuity method. For this purpose, for every $\sigma \in [0,1]$, we introduce the operator $\mathcal{A}_\sigma$ defined on smooth functions $\psi : \mathbb{R}^d \to \mathbb{R}$ by

$$\mathcal{A}_\sigma\psi = \sum_{i,j=1}^d q_{ij}(\cdot,\sigma)D_{ij}\psi + \sum_{j=1}^d b_j(\cdot,\sigma)D_j\psi + c(\cdot,\sigma)\psi,$$

where $q_{ij}(\cdot,\sigma) = (1-\sigma)\delta_{ij} + \sigma q_{ij}$, $b_j(\cdot,\sigma) = \sigma b_j$ and $c(\cdot,\sigma) = \sigma c$ for every $i,j = 1,\ldots,d$. As it is immediately seen, $\mathcal{A}_0 = \Delta$ and $\mathcal{A}_1 = \mathcal{A}$. Each operator $\mathcal{A}_\sigma$ is elliptic since

$$\sum_{i,j=1}^d q_{ij}(x,\sigma)\xi_i\xi_j = (1-\sigma)|\xi|^2 + \sigma\sum_{i,j=1}^d q_{ij}(x)\xi_i\xi_j \ge (1-\sigma)|\xi|^2 + \sigma\mu|\xi|^2 \ge \min\{1,\mu\}|\xi|^2 \tag{8.2}$$

for every $x, \xi \in \mathbb{R}^d$ and $\sigma \in [0,1]$. Moreover,

$$\|q_{ij}(\cdot,\sigma)\|_{C_b^\alpha(\mathbb{R}^d)} \le \max\{\|q_{ij}\|_{C_b^\alpha(\mathbb{R}^d)}, 1\}, \tag{8.3}$$

$$\|b_j(\cdot,\sigma)\|_{C_b^\alpha(\mathbb{R}^d)} \le \|b_j\|_{C_b^\alpha(\mathbb{R}^d)}, \qquad \|c(\cdot,\sigma)\|_{C_b^\alpha(\mathbb{R}^d)} \le \|c\|_{C_b^\alpha(\mathbb{R}^d)} \tag{8.4}$$

for each $i,j=1,\dots,d$ and $\sigma\in[0,1]$. Consider the operators $L_0,L_1 : C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)\to C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)\times C_b^{2+\alpha}(\mathbb{R}^d)$ defined by

$$L_0u=(D_tu-\Delta u,u(0,\cdot)),\qquad L_1u=(D_tu-\mathcal{A}u,u(0,\cdot)),\qquad u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d),$$

Note that the operator $L_\sigma=(1-\sigma)L_0+\sigma L_1$ is associated with the operator $\mathcal{A}_\sigma$ in the sense that $L_\sigma u=(D_tu-\mathcal{A}_\sigma u,u(0,\cdot))$ for each $u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$.

By estimate (7.3) we known that

$$\begin{aligned}\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\leq&C(\|D_tu-\mathcal{A}_\sigma u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+\|u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)})\\ =&C\|L_\sigma u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)\times C_b^{2+\alpha}(\mathbb{R}^d)},\end{aligned}$$

where the constant C is independent of σ in view of Remark 7.2.5 and the estimates (8.2)-(8.4).

To conclude the proof, we observe that Theorem 6.1.4 implies that the operator L_0 is invertible from $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$ into $C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)\times C_b^{2+\alpha}(\mathbb{R}^d)$. Hence, we can apply Theorem 7.1.1 which shows that the operator L_1 is invertible, i.e., for each $f\in C_b^{2+\alpha}(\mathbb{R}^d)$ and $g\in C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)$, the Cauchy problem (7.1) admits a unique solution $u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$, which satisfies estimate (8.1). □

8.2 More on the Cauchy problem (7.1)

In the previous section, we have seen that, if $f\in C_b^{2+\alpha}(\mathbb{R}^d)$ and $g\in C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)$ for some $\alpha\in(0,1)$, then the Cauchy problem (7.1) admits a unique classical solution, which in addition belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$. Thinking at the ordinary differential equations, one might think that, to get a classical solution to problem (7.1), the continuity of f and g are enough. This is not the case in general. To justify this claim we show that there exists a function $u\in C^2(\mathbb{R}^2\setminus\{(0,0)\})\cap C_b^1(\mathbb{R}^2)$ such that $D_{xx}u$ and $D_{yy}u$ are bounded and continuous in $\mathbb{R}^2$ and $D_{xy}u(x,y)$ diverges to ∞ as (x,y) tends to $(0,0)$. Clearly, u solves the Cauchy problem (7.1) with $g=-\Delta u\in C_b(\mathbb{R}^2)$. This Cauchy problem admits no classical solutions since every classical solution should coincide with u on $[0,T]\times(\mathbb{R}^d\setminus\{0\})$, which does not belong to $C^2(\mathbb{R}^2)$, see Exercise 8.4.4.

Example 8.2.1. Let $u:\mathbb{R}^2\to\mathbb{R}$ be the function defined by

$$u(x,y)=\sum_{n=1}^{\infty}n^{-1}xy\vartheta(2^nx,2^ny),\qquad (x,y)\in\mathbb{R}^2,$$

where $\vartheta\in C_c^\infty(\mathbb{R}^2)$ is such that $\chi_{B(0,1)}\leq\vartheta\leq\chi_{B(0,2)}$. We observe that the series defining u converges uniformly in $\mathbb{R}^2$. Indeed,

$$n^{-1}|xy\vartheta(2^nx,2^ny)|\leq 4^{-n}n^{-1}\sup_{(z,w)\in\mathbb{R}^2}|zw\vartheta(z,w)|=:4^{-n}n^{-1}C,\qquad (x,y)\in\mathbb{R}^2,\ n\in\mathbb{N}.$$

Similarly, the series $\sum_{n=1}^{\infty} n^{-1}D_x(xy\vartheta(2^nx,2^ny))$ and $\sum_{n=1}^{\infty} n^{-1}D_y(xy\vartheta(2^nx,2^ny))$ converge uniformly in $\mathbb{R}^2$ since

$$|D_x(n^{-1}xy\vartheta(2^nx,2^ny))| \le 2^{-n}n^{-1}\bigg(\sup_{(z,w)\in\mathbb{R}^2}|w\vartheta(z,w)| + \sup_{(z,w)\in\mathbb{R}^2}|zwD_x\vartheta(z,w)|\bigg)$$

for every $(x,y)\in\mathbb{R}^2$, and $D_y(n^{-1}xy\vartheta(2^nx,2^ny))$ satisfies a similar estimate. Hence, $u\in C_b^1(\mathbb{R}^2)$. The same arguments show that the classical derivatives $D_{xx}u$ and $D_{yy}u$ exist in $\mathbb{R}^2$ and are therein continuous. In particular, Δu is a continuous function in $\mathbb{R}^2$. As far as the mixed derivative is concerned, we observe that if $(x,y)\in\mathbb{R}^2\setminus B(0,r)$ for some $r>0$, then $|(2^nx,2^ny)|\ge 2^nr\ge 2$ for each $n\ge n_0$, where n_0 is a suitable integer independent of (x,y). Hence, on $\mathbb{R}^2\setminus B(0,r)$ the series defining u reduces to a finite sum and thus it belongs to $C^\infty(\mathbb{R}^2\setminus B(0,r))$. The arbitrariness of $r>0$ implies that $u\in C^\infty(\mathbb{R}^d\setminus\{0\})$. On the other hand, if $2^{-2k-4}\le x^2+y^2\le 2^{-2k}$ for some $k\in\mathbb{N}$, then $\vartheta(2^nx,2^ny)=0$ if $n\ge k+3$. Hence,

$$\begin{aligned}D_{xy}u(x,y) =& D_{xy}\sum_{n=1}^{k+2} n^{-1}xy\vartheta(2^nx,2^ny)\\ =& \sum_{n=1}^{k+2} n^{-1}\vartheta(2^nx,2^ny) + \sum_{n=1}^{k+2} 4^nn^{-1}xyD_{xy}\vartheta(2^nx,2^ny)\\ &+\sum_{n=1}^{k+2} 2^nn^{-1}[xD_x\vartheta(2^nx,2^ny)+yD_y\vartheta(2^nx,2^ny)].\end{aligned}$$

Note that, if $n\le k$, then $|2^n(x,y)|\le 1$ so that $\vartheta(2^nx,2^ny)=1$. Therefore,

$$\begin{aligned}D_{xy}u(x,y) =& \sum_{n=1}^{k}\frac{1}{n} + \frac{1}{k+1}\vartheta(2^{k+1}x,2^{k+1}y) + \frac{1}{k+2}\vartheta(2^{k+2}x,2^{k+2}y)\\ &+\frac{4^{k+1}}{k+1}xyD_{xy}\vartheta(2^{k+1}x,2^{k+1}y) + \frac{4^{k+2}}{k+2}xyD_{xy}\vartheta(2^{k+2}x,2^{k+2}y)\\ &+\frac{2^{k+1}}{k+1}[xD_x\vartheta(2^{k+1}x,2^{k+1}y)+yD_y\vartheta(2^{k+1}x,2^{k+1}y)]\\ &+\frac{2^{k+2}}{k+2}[xD_x\vartheta(2^{k+2}x,2^{k+2}y)+yD_y\vartheta(2^{k+2}x,2^{k+2}y)].\end{aligned}$$

Hence,

$$\begin{aligned}D_{xy}u(x,y) \ge& \sum_{n=1}^{k}\frac{1}{n} - 2\sup_{(z,w)\in\mathbb{R}^2}|zwD_{xy}\vartheta(z,w)|\\ &-2\bigg(\sup_{(z,w)\in\mathbb{R}^2}|zD_x\vartheta(z,w)| + \sup_{(z,w)\in\mathbb{R}^2}|wD_y\vartheta(z,w)|\bigg).\end{aligned}$$

The above estimate shows that $D_{xy}u(x,y)$ diverges to ∞ as $(x,y)\to(0,0)$.

Nevertheless, the assumptions on f and g may be weakened as the following theorem shows.

Theorem 8.2.2. *Let Hypotheses* 7.0.1 *be satisfied. Suppose that* $f \in C_b^{2+\alpha-2\theta}(\mathbb{R}^d)$ *and* $g \in C((0,T]\times\mathbb{R}^d)$ *is such that* $\sup_{t\in(0,T]} t^\theta \|g(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} < \infty$ *for some* $\theta \in (0,1)$*. Then, the Cauchy problem* (7.1) *admits a unique classical solution* u*. In addition,* $u(t,\cdot) \in C_b^{2+\alpha}(\mathbb{R}^d)$ *and*

$$\begin{aligned}&\sup_{t\in(0,T]} t^\theta(\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}) + \|u\|_{C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)}\\ \le & C_0\bigg(\|f\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + \sup_{t\in(0,T]} t^\theta \|g(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg),\end{aligned}$$

for some positive constant C_0*, independent of* u*,* f *and* g*.*

Proof. We just sketch the proof, which is similar to that of Theorem 8.1.1. Throughout the proof, by C, we denote a positive constant, which is independent of u, f and g.

To begin with, we observe that, by Theorem 6.1.8 (with $\beta = 2+\alpha-2\theta$), for every $f \in C_b^{2+\alpha-2\theta}(\mathbb{R}^d)$ and $g \in C((0,T]\times\mathbb{R}^d)$ such that $\sup_{t\in(0,T]} t^\theta\|g(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} < \infty$, the Cauchy problem

$$\begin{cases} D_t u(t,x) = \Delta u(t,x) + g(t,x), & t\in(0,T], \quad x\in\mathbb{R}^d,\\ u(0,x) = f(x), & \qquad\qquad\quad\; x\in\mathbb{R}^d,\end{cases} \tag{8.5}$$

admits a unique solution $u \in C_b([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T]\times\mathbb{R}^d)$ such that

$$\sup_{t\in(0,T]} t^\theta(\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}) \le C\bigg(\|f\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + \sup_{t\in(0,T]} t^\theta\|g(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg).$$

Using the same arguments as in the proof of Lemma 7.2.2, it can be easily checked that the Cauchy problem (8.5) with the Laplacian being replaced with the operator $\mathcal{A}_{x_0} = \mathrm{Tr}(Q(x_0)D_x^2)$, admits a (unique) solution $u \in C_b([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T]\times\mathbb{R}^d)$ for every $x_0 \in \mathbb{R}^d$. In particular, if $f \in C_b^{2+\alpha-2\theta}(\mathbb{R}^d)$, then u is bounded in $[0,T]$ with values in $C_b^{2+\alpha-2\theta}(\mathbb{R}^d)$ and

$$\begin{aligned}&\sup_{t\in(0,T]} t^\theta(\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}) + \|u\|_{C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)}\\ \le & C\bigg(\|f\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + \sup_{t\in(0,T]} t^\theta\|g(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg)\end{aligned} \tag{8.6}$$

with the constant C being independent of $x_0 \in \mathbb{R}^d$.

Take $u \in C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T]\times\mathbb{R}^d)$ such that $\sup_{t\in(0,T]}(t^\theta\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + t^\theta\|D_t u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}) < \infty$ for some $\theta\in(0,1)$. Since u solves the Cauchy problem (8.5) with

Δ being replaced by the operator $\mathcal{A}_{x_0}$, $f = u(0,\cdot) \in C_b(\mathbb{R}^d)$ and $g = D_t u - \mathcal{A}_{x_0}u$, from (8.6) we conclude that

$$
\begin{aligned}
&\sup_{t\in(0,T]} t^\theta(\|u(t,\cdot)\|_{C^{2+\alpha}_b(\mathbb{R}^d)} + \|D_t u(t,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)}) + \|u\|_{C^{0,2+\alpha-2\theta}_b([0,T]\times\mathbb{R}^d)}\\
\leq & C\bigg(\|u(0,\cdot)\|_{C^{2+\alpha-2\theta}_b(\mathbb{R}^d)} + \sup_{t\in(0,T]} t^\theta\|D_t u(t,\cdot) - \mathcal{A}_{x_0}u(t,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)}\bigg).
\end{aligned}
\tag{8.7}
$$

Fix u as above and apply (8.7) to the function $v = \vartheta_{x_0,r}u$, where the function $\vartheta_{x_0,r} \in C^\infty_c(\mathbb{R}^d)$ satisfies (7.11) and $r \in (0,1)$, obtaining

$$
\begin{aligned}
&\sup_{t\in(0,T]} t^\theta(\|u(t,\cdot)\|_{C^{2+\alpha}(\overline{B(x_0,r/2)})} + \|D_t u(t,\cdot)\|_{C^\alpha(\overline{B(x_0,r/2)})}) + \|u\|_{C^{0,2+\alpha-2\theta}_b([0,T]\times\overline{B(x_0,r/2)})}\\
\leq & C\bigg(\|\vartheta_{x_0,r}u(0,\cdot)\|_{C^{2+\alpha-2\theta}_b(\mathbb{R}^d)} + \sup_{t\in(0,T]} t^\theta\|\vartheta_{x_0,r}D_t u(t,\cdot) - \mathcal{A}_{x_0}(\vartheta_{x_0,r}u(t,\cdot))\|_{C^\alpha_b(\mathbb{R}^d)}\bigg).
\end{aligned}
\tag{8.8}
$$

We note that (see (7.16) and (7.17))

$$
\begin{aligned}
&t^\theta\|\vartheta_{x_0,r}D_t u(t,\cdot) - \mathcal{A}_{x_0}(\vartheta_{x_0,r}u(t,\cdot))\|_{C^\alpha_b(\mathbb{R}^d)}\\
\leq & t^\theta\|D_t u(t,\cdot) - \mathcal{A}_{x_0}u(t,\cdot)\|_{C^\alpha(\overline{B(x_0,r)})} + Mr^{-\alpha}t^\theta\|D_t u(t,\cdot) - \mathcal{A}_{x_0}u(t,\cdot)\|_\infty\\
&+ C_1 r^{-2-\alpha}t^\theta\|u(t,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)} + C_1 r^{-1-\alpha}t^\theta\sum_{j=1}^d\|D_j u(t,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)}\\
\leq & (1+Mr^{-\alpha})t^\theta\|D_t u(t,\cdot) - \mathcal{A}u(t,\cdot)\|_{C^\alpha(\overline{B(x_0,r)})} + C_1 t^\theta r^{-2-\alpha}\|u(t,\cdot)\|_{C^2_b(\mathbb{R}^d)}\\
&+ C_1 t^\theta r^\alpha\|u(t,\cdot)\|_{C^{2+\alpha}_b(\mathbb{R}^d)}
\end{aligned}
$$

for some positive constant C_1, independent of t and r. Next, applying (5.25), we can estimate

$$
\begin{aligned}
C_1 t^\theta\|u(t,\cdot)\|_{C^2_b(\mathbb{R}^d)} \leq & C_1 t^\theta(\varepsilon C_1^{-1}\|u(t,\cdot)\|_{C^{2+\alpha}_b(\mathbb{R}^d)} + C_\varepsilon\|u\|_\infty)\\
\leq & \varepsilon t^\theta\|u(t,\cdot)\|_{C^{2+\alpha}_b(\mathbb{R}^d)} + C'_\varepsilon T^\theta\|u(t,\cdot)\|_\infty
\end{aligned}
$$

for each $\varepsilon > 0$ and some positive constant C'_ε Replacing these estimates in (8.8), we conclude that

$$
\begin{aligned}
&\sup_{t\in(0,T]} t^\theta(\|u(t,\cdot)\|_{C^{2+\alpha}(\overline{B(x_0,r/2)})} + \|D_t u(t,\cdot)\|_{C^\alpha(\overline{B(x_0,r/2)})}) + \|u\|_{C^{0,2+\alpha-2\theta}([0,T]\times\overline{B(x_0,r/2)})}\\
\leq & C_2\bigg(r^{-2-\alpha}\|u(0,\cdot)\|_{C^{2+\alpha-2\theta}_b(\mathbb{R}^d)} + (1+Mr^{-\alpha})\sup_{t\in(0,T]} t^\theta\|D_t u(t,\cdot) - \mathcal{A}u(t,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)}\\
&+ (\varepsilon r^{-2-\alpha} + C_1 r^\alpha)\sup_{t\in(0,T]} t^\theta\|u(t,\cdot)\|_{C^{2+\alpha}_b(\mathbb{R}^d)} + C'_\varepsilon T^\theta\|u(t,\cdot)\|_\infty\bigg)
\end{aligned}
$$

for each $\varepsilon > 0$ and some positive constant C_2, independent of r, f and g. Now, arguing as in the proof of Theorem 7.2.1, we can show that

$$\begin{aligned}&\sup_{t\in(0,T]} t^\theta(\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_tu(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}) + \|u\|_{C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)}\\ \le& C_3\bigg(\|u(0,\cdot)\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + \sup_{t\in(0,T]} t^\theta\|D_tu(t,\cdot)-\mathcal{A}u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} + C'_\varepsilon T^\theta\|u(t,\cdot)\|_\infty\bigg).\end{aligned}$$

with C_3, independent of u. In particular, C_3 does not blow up as $T \to 0^+$. Hence, if $T \le T_0$, where T_0 is any positive constant such that $C_3C_\varepsilon T_0^\theta \le 1/2$, then,

$$\begin{aligned}&\sup_{t\in(0,T]} t^\theta(\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_tu(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}) + \|u\|_{C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)}\\ \le& 2C_3\bigg(\|u(0,\cdot)\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + \sup_{t\in(0,T]} t^\theta\|D_tu(t,\cdot)-\mathcal{A}u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg).\end{aligned}\tag{8.9}$$

Now, suppose that $T > T_0$. Then, (8.9) holds true with T being replaced by T_0, i.e.,

$$\begin{aligned}&\sup_{t\in(0,T_0]} t^\theta(\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_tu(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}) + \|u\|_{C_b^{0,2+\alpha-2\theta}([0,T_0]\times\mathbb{R}^d)}\\ \le& 2C_3\bigg(\|u(0,\cdot)\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + \sup_{t\in(0,T_0]} t^\theta\|D_tu(t,\cdot)-\mathcal{A}u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg).\end{aligned}\tag{8.10}$$

Moreover, since u belongs to $C_b^{1,2+\alpha}([T_0,T]\times\mathbb{R}^d)$, we can apply Theorem 7.2.6, which, together with 8.10 yields

$$\begin{aligned}&\|u\|_{C_b^{0,2+\alpha}([T_0,T]\times\mathbb{R}^d)}\\ \le& C(\|u(T_0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_tu-\mathcal{A}u\|_{C_b^{0,\alpha}([T_0,T]\times\mathbb{R}^d)})\\ \le& CT_0^{-\theta}\bigg[2C_3\bigg(\|u(0,\cdot)\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + \sup_{t\in(0,T_0]} t^\theta\|D_tu(t,\cdot)-\mathcal{A}u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg)\\ &\qquad + \sup_{t\in[T_0,T]} t^\theta\|D_tu(t,\cdot)-\mathcal{A}u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg]\\ \le& CT_0^{-\theta}\bigg[2C_3\|u(0,\cdot)\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + (2C_3+1)\sup_{t\in(0,T]} t^\theta\|D_tu(t,\cdot)-\mathcal{A}u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg)\bigg].\end{aligned}$$

Hence,

$$\begin{aligned}&\sup_{t\in(T_0,T]} t^\theta\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}\\ \le& CT^\theta T_0^{-\theta}\bigg[2C_3\|u(0,\cdot)\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + (2C_3+1)\sup_{t\in(0,T]} t^\theta\|D_tu(t,\cdot)-\mathcal{A}u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg)\bigg]\end{aligned}\tag{8.11}$$

.

From (8.10) and (8.11), we get (8.9) also in this situation.

To complete the proof, we introduce the sets $X = \{u \in C^{1,2}((0,T]\times\mathbb{R}^d)\cap C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d) : \sup_{t\in(0,T]} t^{\theta}(\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}) < \infty\}$ and $Y = C_b^{2+\alpha-2\theta}(\mathbb{R}^d) \times \{g \in C((0,T]\times\mathbb{R}^d) : \sup_{t\in(0,T]} t^{\theta}\|g(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)} < \infty\}$, endowed with the norms

$$\begin{aligned}
&\|u\|_X = \|u\|_{C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)} + \sup_{t\in(0,T]} t^{\theta}(\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}),\\
&\|(f,g)\|_Y = \|f\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + \sup_{t\in(0,T]} t^{\theta}\|g(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)},
\end{aligned}$$

and the same operator L_σ ($\sigma \in (0,1)$) as in the proof of Theorem 7.1.1. Since $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, see Exercise 8.4.1, using the above results and the continuity method, we can complete the proof. ☐

To conclude this section, we consider the following result which will be used in the next lecture.

Theorem 8.2.3. *Let $u \in C^{1,2+\alpha}([0,T]\times\mathbb{R}^d)$ be a classical solution to the differential equation $D_t u = \mathcal{A}u$, Then, for every $T > 0$, there exists a positive constant C_T such that*

$$\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} \le C_T t^{-\frac{2+\alpha}{2}}\|u(0,\cdot)\|_\infty, \qquad t \in (0,T]. \tag{8.12}$$

Proof. We fix $s_0 \in (0,T]$ and consider the sequence (t_n) as in the proof of Theorem B.0.4, with $\tau_0 = s_0/2$ and $\tau = s_0$. We also introduce the same sequence (φ_n) of functions defined in the proof of the quoted theorem. Observe that $\|\varphi_n'\|_\infty \le 2^n C s_0^{-1}$ for every $n \in \mathbb{N}$, C being independent of n.

As it is easily seen, for every $n \in \mathbb{N}$ the function $v_n = \varphi_n u$ belongs to $C^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$, vanishes in $[0,s_0/2]\times\mathbb{R}^d$ and solves the equation $D_t v_n = \mathcal{A}v_n + \varphi_n' u$ in $[0,T]\times\mathbb{R}^d$. By (7.22) it follows that

$$\begin{aligned}
\|v_n\|_{C^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} \le& C\|\varphi_n'\|_\infty\|u\|_{C_b^{0,\alpha}([t_{n+1},T]\times\mathbb{R}^d)} \le 2^n C s_0^{-1}\|u\|_{C_b^{0,\alpha}([t_{n+1},T]\times\mathbb{R}^d)}\\
\le& 2^n C s_0^{-1}\|v_{n+1}\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\\
\le& 2^n C s_0^{-1}(\varepsilon\|v_{n+1}\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} + \varepsilon^{-\frac{\alpha}{2}}\|v_{n+1}\|_{C_b([0,T]\times\mathbb{R}^d)})
\end{aligned}$$

for each $\varepsilon > 0$ and $n \in \mathbb{N}$, since $v_{n+1} = u$ on $[t_{n+1},T]\times\mathbb{R}^d$, (5.22) and Young's inequality.

Now, we fix $\eta \in (0, 2^{-1-\alpha/2})$ and choose $\varepsilon = \varepsilon_n = 2^{-n}C^{-1}s_0\eta$; from the previous estimate we obtain

$$\|v_n\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} \le \eta\|v_{n+1}\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} + 2^{n\left(1+\frac{\alpha}{2}\right)}C^{1+\frac{\alpha}{2}}\eta^{-\frac{\alpha}{2}}s_0^{-\frac{2+\alpha}{2}}\|u\|_{C_b([0,T]\times\mathbb{R}^d)}.$$

Multiplying both sides of this estimate by η^n and summing from 1 to $m \in \mathbb{N}$, we get

$$\|v_1\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} - \eta^{m+1}\|v_{m+1}\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} \le C^{1+\frac{\alpha}{2}}\eta^{-\frac{\alpha}{2}}s_0^{-\frac{2+\alpha}{2}}\sum_{k=1}^{m}(\eta 2^{1+\frac{\alpha}{2}})^k\|u\|_{C_b([0,T]\times\mathbb{R}^d)}$$

$$\leq C_1 s_0^{-\frac{2+\alpha}{2}} \|u\|_{C_b([0,T]\times\mathbb{R}^d)} \tag{8.13}$$

for some constant C_1, independent of s_0, due to the choice of η. Letting m tend to ∞ in (8.13) we conclude that $\|v_1\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)} \leq C_1 s_0^{-\frac{2+\alpha}{2}} \|u\|_{C_b([0,T]\times\mathbb{R}^d)}$ and, hence,

$$\|u\|_{C_b^{0,2+\alpha}([s_0,T]\times\mathbb{R}^d)} \leq C_1 s_0^{-\frac{2+\alpha}{2}} \|u\|_{C_b([0,T]\times\mathbb{R}^d)}.$$

Estimate (8.12) follows, from the maximum principle in Exercise 8.4.3, which shows that $\|u\|_{C_b([0,T]\times\mathbb{R}^d)} \leq e^{c_0 T}\|u(0,\cdot)\|_\infty$, where c_0 denotes the supremum over $\mathbb{R}^d$ of the function c. □

8.3 Notes

For further details on parabolic equations in the whole $\mathbb{R}^d$, besides Appendix B we refer the interested reader to e.g., [**?**, 1, 2].

8.4 Exercises

1. Prove that the sets $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, defined in the proof of Theorem 8.2.2 are Banach spaces.
2. Let $u \in C_b([0,T]\times\mathbb{R}^d) \cap C^{1,2}((0,T]\times\mathbb{R}^d)$ be such that $D_t u - \mathcal{A}u \leq 0$ on $(0,T]\times\mathbb{R}^d$ and $u(0,\cdot) \leq 0$ on $\mathbb{R}^d$, where $\mathcal{A}$ is the elliptic operator introduce at the beginning of Lecture 7.
    (a) Prove that there exists $\lambda > 0$ such that the function $\varphi : \mathbb{R}^d \to \mathbb{R}$, defined by $\varphi(x) = 1 + |x|^2$ for every $x \in \mathbb{R}^d$, satisfies the inequality $\mathcal{A}\varphi \leq \lambda\varphi$ on $[0,T]\times\mathbb{R}^d$.
    (b) Prove that the function $v_n : [0,T]\times\mathbb{R}^d \to \mathbb{R}$, defined by $v_n(t,x) = e^{-(\lambda+1)t}(u(t,x) - n^{-1}(1+|x|^2))$ for every $(t,x) \in [0,T]\times\mathbb{R}^d$ belongs to $C_b([0,T]\times\mathbb{R}^d) \cap C^{1,2}((0,T]\times\mathbb{R}^d)$, achieves its maximum on $[0,T]\times\mathbb{R}^d$ and satisfies the differential inequality $D_t v_n - \mathcal{A}v_n < 0$ on $(0,T]\times\mathbb{R}^d$ for every $n \in \mathbb{N}$.
    (c) Use the above result and the arguments in the proof of Theorem 1.1.1 to prove, first that $v_n \leq 0$ on $[0,T]\times\mathbb{R}^d$ and, then, that $u \leq 0$ on $[0,T]\times\mathbb{R}^d$.
    (d) Conclude that, for every $f \in C_b(\mathbb{R}^d)$ and $g \in C_b((0,T]\times\mathbb{R}^d)$, the Cauchy problem (7.1) admits at most a unique solution $u \in C_b([0,T]\times\mathbb{R}^d) \cap C^{1,2}((0,T]\times\mathbb{R}^d)$.
3. Let $\mathcal{A}$ be the elliptic operator in Lecture 7 and denote by c_0 the supremum over $[0,T]\times\mathbb{R}^d$ of the function c. Assume that $u \in C_b([0,T]\times\mathbb{R}^d) \cap C^{1,2}((0,T]\times\mathbb{R}^d)$ satisfies the inequality $D_t u \leq \mathcal{A}u$ on $(0,T]\times\mathbb{R}^d$.
    (a) Prove that the function $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}$, defined by $v(t,x) = e^{-c_0 t}u(t,x) - \|u(0,\cdot)\|_\infty$ for every $(t,x) \in [0,T]\times\mathbb{R}^d$, satisfies the inequalities $D_t v - \mathcal{A}_0 v \leq 0$ on $(0,T]\times\mathbb{R}^d$, where $\mathcal{A}_0 = \mathrm{Tr}(QD^2) + \langle b, \nabla u\rangle$.

   (b) Using Exercise 8.4.2 conclude that $u(t,x) \le e^{c_0 t}\|u(0,\cdot)\|_\infty$ for every $(t,x) \in [0,T] \times \mathbb{R}^d$.

   (c) Prove that, if $D_t u = \mathcal{A}u$ on $(0,T] \times \mathbb{R}^d$, then $\|u(t,\cdot)\|_\infty \le e^{c_0 t}\|u(0,\cdot)\|_\infty$ for every $t \in [0,T]$.

4. Taking advantage of the previous exercises prove that, for each $f \in C_b(\mathbb{R}^2)$ and $g \in C_b([0,T]\times\mathbb{R}^2)$, there exists at most a solution $u \in C_b([0,T]\times\mathbb{R}^2)$ to the Cauchy problem $D_t u = \Delta u$ on $(0,T] \times \mathbb{R}^2$, $u(0,\cdot) = f$ on $\mathbb{R}^2$, such that $D_t u, D_x u, D_y u, D_{xx} u, D_{yy} u$ are bounded and continuous on $(0,T]\times\mathbb{R}^2$ and $D_{xy}u$ is continuous on $(0,T]\times(\mathbb{R}^2\setminus\{(0,0)\})$.

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Lecture 9

Parabolic equations in $\mathbb{R}^d$. Part III

In this chapter, we are interested in proving that a semigroup $\{T(t)\}$ of bounded linear operators in $C_b(\mathbb{R}^d)$ can be associated with the operator $\mathcal{A}$, which satisfies Hypotheses 7.0.1.

As a byproduct of Theorems 8.2.2 and 8.2.3, we can prove the following result.

Theorem 9.0.1. *For each $f \in C_b(\mathbb{R}^d)$, the Cauchy problem*

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x), & t>0, \quad x \in \mathbb{R}^d, \\ u(0,x) = f(x), & \qquad\quad\; x \in \mathbb{R}^d, \end{cases} \tag{9.1}$$

admits a unique solution $u \in C^{1,2}((0,\infty)\times\mathbb{R}^d) \cap C([0,\infty)\times\mathbb{R}^d)$, which is bounded in $[0,T]\times\mathbb{R}^d$ for every $T>0$, and, in addition, belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,\infty)\times\mathbb{R}^d)$. As a byproduct, we can associate a semigroup of bounded operators $\{T(t)\}$ in $C_b(\mathbb{R}^d)$ with the operator $\mathcal{A}$, defined as follows: $T(t)f = u(t,\cdot)$. Moreover, for each $0 \le \beta \le \gamma \le 2+\alpha$, each operator $T(t)$ is bounded from $C^\beta_b(\mathbb{R}^d)$ into $C^\gamma_b(\mathbb{R}^d)$ and there exist two positive constants $\omega_{\beta,\gamma}$ and $C_{\beta,\gamma}$ such that

$$\|T(t)f\|_{C^\gamma_b(\mathbb{R}^d)} \le C_{\beta,\gamma} t^{-\frac{\gamma-\beta}{2}} e^{\omega_{\beta,\gamma}t}\|f\|_{C^\beta_b(\mathbb{R}^d)}, \qquad t>0. \tag{9.2}$$

Proof. Being rather long, we split the proof into several step.

Step 1. Here, we prove that, for each $f \in BUC(\mathbb{R}^d)$, problem (9.1) admits a unique solution $u \in C^{1,2}((0,\infty)\times\mathbb{R}^d) \cap C([0,\infty)\times\mathbb{R}^d)$, which is bounded in $[0,T]\times\mathbb{R}^d$ for every $T>0$. The uniqueness follows from the maximum principle in Exercise 8.4.3. Hence, we just need to prove the existence part.

Fix $f \in BUC(\mathbb{R}^d)$. Observe that there exists $(f_n) \subset C^{2+\alpha}_b(\mathbb{R}^d)$ which converges to f uniformly in $\mathbb{R}^d$. (For instance, one can take $f_n = S(1/n)f$, where $\{S(1/n)\}$ denotes the Gauss-Weierstrass semigroup in $C_b(\mathbb{R}^d)$.) By Theorem 8.1.1, for every $n \in \mathbb{N}$ the Cauchy problem (9.1), with f being replaced by f_n, admits a unique classical solution $u_n : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ which belongs to $C^{1+\alpha/2,2+\alpha}_b([0,T]\times\mathbb{R}^d)$ for every $T>0$. By the maximum principle in Exercise 8.4.3,

$$\|u_n(t,\cdot) - u_m(t,\cdot)\|_\infty \le e^{c_0 t}\|f_n - f_m\|_\infty, \qquad t>0, \;\; m,n \in \mathbb{N},$$

where $c_0 = \sup_{x\in\mathbb{R}^d} c(x)$. Hence, u_n converges uniformly in $[0,T]\times\mathbb{R}^d$, for every $T>0$, to a function $u\in BUC(\mathbb{R}^d)$ which, of course, satisfies the condition $u(0,\cdot)=f$. From the interior Schauder estimates in Theorem B.0.4, we deduce that

$$\|u_n-u_m\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times K)} \le C\|u_n-u_m\|_{C_b([0,T]\times\mathbb{R}^d)}, \qquad m,n\in\mathbb{N},$$

for every $0<\varepsilon<T$, every compact set $K\subset\mathbb{R}^d$ and some constant C, independent of $m,n\in\mathbb{N}$, which blows up as ε tend to 0. Hence, u_n converges to u in $C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times K)$ for every ε, T and K as above. Thus, u belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,\infty)\times\mathbb{R}^d)$. Finally, since $D_t u_n=\mathcal{A}u_n$ in $(0,\infty)\times\mathbb{R}^d$, letting n tend to ∞, we conclude that u solves the differential equation in (9.1). We have so proved that the Cauchy problem (9.1), corresponding to $f\in BUC(\mathbb{R}^d)$ admits a classical solution $u\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,\infty)\times\mathbb{R}^d)\cap C([0,\infty)\times\mathbb{R}^d)$, which satisfies the estimate $\|u(t,\cdot)\|_\infty\le e^{c_0t}$ for every $t>0$. We can thus define, for each $t>0$, the bounded linear operator $T(t)$ on $BUC(\mathbb{R}^d)$ as described in the statement. The linearity and the boundedness follow from the maximum principle.

Step 2. Here, we prove that problem (9.1) admits a unique solution $u\in C([0,\infty)\times\mathbb{R}^d)\cap C^{1,2}((0,\infty)\times\mathbb{R}^d)$ for each $f\in C_b(\mathbb{R}^d)$. For this purpose, we introduce a bounded sequence $(f_n)\subset BUC(\mathbb{R}^d)$, converging to f locally uniformly in $\mathbb{R}^d$ and set $u_n=T(\cdot)f_n$ ($n\in\mathbb{N}$). Arguing as in Step 1, we conclude that the sequence (u_n) is bounded in $C^{1+\alpha/2,2+\alpha}([1/k,k]\times\overline{B(0,k)})$ for each $k\in\mathbb{N}$. Applying the Arzelà-Ascoli theorem to u_n and its first-order time derivative and first- and second-order spatial derivatives we conclude that, for each $k\in\mathbb{N}$, there exist a function $u_k\in C^{1+\alpha/2,2+\alpha}([1/k,k]\times\overline{B(0,k)})$ and a subsequence $(u_{n_h^k})$ such that $u_{n_h^k}$ converges to u_k in $C^{1,2}([1/k,k]\times\overline{B(0,k)})$ as $h\to\infty$. Without loss of generality, we can assume that (n_h^{k+1}) is a subsequence of (n_h^k). It thus follows that $u_k\equiv u_{k+1}$ in $[1/k,k]\times\overline{B(0,k)}$ for each $k\in\mathbb{N}$. Let $u:(0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ be the function defined as follows: for each $(t,x)\in(0,\infty)\times\mathbb{R}^d$, $u(t,x)=u_k(t,x)$, where k is any integer such that $(t,x)\in[1/k,k]\times\mathbb{R}^d$. By the above results u is well defined, it belongs to $C^{1,2}((0,\infty)\times\mathbb{R}^d)$ and, if we consider the diagonal sequence (n_h^h), then we immediately realize that $u_{n_h^h}$ converges to u in $C^{1,2}([a,b]\times\overline{B(0,M)})$ for each $0<a<b$ and $M>0$. In particular, $D_tu=\mathcal{A}u$ in $(0,\infty)\times\mathbb{R}^d$. To prove that u is the solution to problem (9.1) we are looking for, we need to show that u can be extended by continuity at $t=0$ with $u(0,\cdot)=f$. For this purpose, we use an argument from [1]. Given $M>0$, we consider a function $\vartheta_M\in C^\infty_c(\mathbb{R}^d)$ such that $\chi_{B(0,M)}\le\vartheta\le\chi_{B(0,2M)}$. By the linearity of each operator $T(t)$, it follows that

$$u_{n_h^h}(t,\cdot)=T(t)f_{n_h^h}=T(t)(\vartheta_M f_{n_h^h})+T(t)((1-\vartheta_M)f_{n_h^h}), \qquad t>0. \tag{9.3}$$

We claim that $|T(t)((1-\vartheta_M)f_{n_h^h})|\le\sup_{h\in\mathbb{N}}\|f_{n_h^h}\|_\infty(e^{c_0t}-T(t)\vartheta_M)$ in $\mathbb{R}^d$, for every $t>0$. Indeed, assume first that $c\le 0$ on $\mathbb{R}^d$. Then the function $v=T(\cdot)((1-\vartheta_M)f_{n_h^h})-\sup_{h\in\mathbb{N}}\|f_{n_h^h}\|_\infty(1-T(\cdot)\vartheta_M)$ satisfies $D_tv\le\mathcal{A}v$ in $(0,\infty)\times\mathbb{R}^d$. Moreover, $v(0,\cdot)=(1-\vartheta_M)f_{n_h^h}-\sup_{h\in\mathbb{N}}\|f_{n_h^h}\|_\infty(1-\vartheta_M)\le 0$. Hence, by Exercise 8.4.2, v is nonpositive in $[0,\infty)\times\mathbb{R}^d$, i.e., $T(t)((1-\vartheta_M)f_{n_h^h})\le\sup_{h\in\mathbb{N}}\|f_{n_h^h}\|_\infty(1-T(t)\vartheta_M)$ in $\mathbb{R}^d$, for all $t\ge 0$. The same argument, applied to the function $w = -T(\cdot)((1-\vartheta_M)f_{n_h^h}) - \sup_{h\in\mathbb{N}}\|f_{n_h^h}\|_\infty(1-T(\cdot)\vartheta_M)$, reveals that $w \le 0$, i.e., $T(t)((1-\vartheta_M)f_{n_h^h}) \ge -\sup_{h\in\mathbb{N}}\|f_{n_h^h}\|_\infty(1-T(t)\vartheta_M)$ in $\mathbb{R}^d$, for all $t \ge 0$, and the claim follows in the general case by considering the semigroup $\{e^{-c_0 t}T(t)\}$ instead of $\{T(t)\}$. From the claim and (9.3) we conclude that

$$|u_{n_h^h}(t,\cdot) - f| \le |T(t)(\vartheta_M f_{n_h^h}) - f| + \sup_{h\in\mathbb{N}}\|f_{n_h^h}\|_\infty(e^{c_0 t} - T(t)\vartheta_M), \qquad t > 0. \tag{9.4}$$

Since $f_{n_h^h}$ converges to f, locally uniformly as $h \to \infty$, the function $\vartheta_M f_{n_h^h}$ converges to $\vartheta_M f \in BUC(\mathbb{R}^d)$ uniformly in $\mathbb{R}^d$. Hence, letting h tend to ∞ in both the sides of (9.4), we conclude that

$$|u(t,x) - f(x)| \le |(T(t)(\vartheta_M f))(x) - f(x)| + \sup_{h\in\mathbb{N}}\|f_{n_h^h}\|_\infty[e^{c_0 t} - (T(t)\vartheta_M)(x)] \tag{9.5}$$

for every $t > 0$ and $x \in \mathbb{R}^d$. Fix $x_0 \in B(0,M)$. Since $\vartheta_M \equiv 1$ in $B(0,M)$, as (t,x) tends to $(0,x_0)$ the right-hand side of (9.5) vanishes. It thus follows that $u(t,x)$ tends to $f(x_0)$ as $(t,x) \to (0,x_0)$ for every $x_0 \in B(0,M)$. By the arbitrariness of M, we thus conclude that u can be extended by continuity to $\{0\} \times \mathbb{R}^d$, by setting $u(0,x) = f(x)$ for $x \in \mathbb{R}^d$. We have so proved that the function u is the solution to problem (9.1) we were looking for.

Step 3. By the above results, we can extend each operator $T(t)$ to $C_b(\mathbb{R}^d)$. By the maximum principle in Exercise 8.4.3, $T(t)$ is bounded linear operator in $C_b(\mathbb{R}^d)$ and $\|T(t)\|_{L(C_b(\mathbb{R}^d))} \le e^{c_0 t}$ for every $t > 0$. Moreover, the semigroup property is satisfied, still thanks to the maximum principle. Hence, if we set $T(0) = I$, then the family $\{T(t) : t \ge 0\}$ is a semigroup of bounded operators in $C_b(\mathbb{R}^d)$.

Step 4. Here, we prove estimate (9.2) for every $\gamma \in (2, 2+\alpha]$ and $\beta < \gamma$ and for $\gamma = \beta \in [0, 2+\alpha]$.

Theorem 8.1.1 shows that each operator $T(t)$ is bounded from $C_b^{2+\alpha}(\mathbb{R}^d)$ in itself and

$$\|T(t)f\|_{L(C_b^{2+\alpha}(\mathbb{R}^d))} \le C_1\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)}, \qquad t \in [0,1],$$

for some positive constant C_1, independent of f, which, without loss of generality, we can assume to be larger than one. Using the semigroup property, we extend this estimate to each $t > 0$. Indeed, if $t > 1$, then $t = n + \sigma$, where $n \in \mathbb{N}$ and $\sigma \in [0,1)$. So, $T(t) = T(n)T(\tau) = (T(1))^n T(\tau)$. Hence,

$$\begin{aligned}\|T(t)\|_{L(C_b^{2+\alpha}(\mathbb{R}^d))} \le& \|T(1)\|^n_{L(C_b^{2+\alpha}(\mathbb{R}^d))}\|T(\sigma)\|_{L(C_b^{2+\alpha}(\mathbb{R}^d))}\\ \le& C_1^{n+1} = \exp((n+1)\log(C_1)) = C_1 e^{\omega_{2+\alpha,2+\alpha}t},\end{aligned}$$

where $\omega_{2+\alpha,2+\alpha} = \log C_1$. Similarly, taking $g \equiv 0$ and choosing θ properly in Theorem 8.2.2 we can show that $T(t)$ is bounded from $C_b^\gamma(\mathbb{R}^d)$ in itself for each $\gamma \in (0, 2+\alpha)$, and it satisfies (9.2). Still Theorem 8.2.2 shows that estimate (9.2) is satisfied if $\beta \le 2 < \gamma \le 2+\alpha$, such that $\beta > \gamma - 2$ To prove it with $\beta \le \gamma - 2$, we use again the semigroup property and the results so far obtained. More precisely, we have proved that

$$\|T(t)\|_{L(C_b^\beta(\mathbb{R}^d), C_b^{2+\beta/2}(\mathbb{R}^d))} \le C_{\beta,2+\beta/2}t^{-1+\frac{\beta}{4}}e^{\omega_{\beta,2+\beta/2}t}, \qquad t > 0,$$

and

$$\|T(t)\|_{L(C_b^{2+\beta/2}(\mathbb{R}^d),C_b^\gamma(\mathbb{R}^d))} \le C_{2+\beta/2,\gamma}t^{\frac{-2\gamma+4+\beta}{4}}e^{\omega_{2+\beta/2,\gamma}t}, \qquad t>0.$$

Hence,

$$\begin{aligned}\|T(t)\|_{L(C_b^\beta(\mathbb{R}^d),C_b^\gamma(\mathbb{R}^d))} \le&\|T(t/2)\|_{L(C_b^\beta(\mathbb{R}^d),C_b^{2+\beta/2}(\mathbb{R}^d))}\|T(t/2)\|_{L(C_b^{2+\beta/2}(\mathbb{R}^d),C_b^\gamma(\mathbb{R}^d))}\\ \le&2^{\frac{\gamma-\beta}{2}}t^{-\frac{\gamma-\beta}{2}}C_{\beta,2+\beta/2}C_{2+\beta/2,\gamma}e^{\frac{1}{2}(\omega_{\beta,2+\beta/2}+\omega_{2+\beta/2,\gamma})t}\end{aligned}$$

and we are done.

Step 5. Here, we prove estimate (9.2) for $0<\beta<\gamma\le 2$. Applying the estimate

$$\|g\|_{C_b^\gamma(\mathbb{R}^d)} \le C_2\|g\|_{C_b^\beta(\mathbb{R}^d)}^{\frac{4+\beta-2\gamma}{4-\beta}}\|g\|_{C_b^{2+\beta/2}(\mathbb{R}^d)}^{\frac{2(\gamma-\beta)}{4-\beta}}, \qquad g\in C_b^{2+\beta/2}(\mathbb{R}^d), \tag{9.6}$$

(see Exercise 9.1.3) with $g=T(t)f$, we deduce that

$$\begin{aligned}\|T(t)f\|_{C_b^\gamma(\mathbb{R}^d)} \le&C_2\|T(t)f\|_{C_b^\beta(\mathbb{R}^d)}^{\frac{4+\beta-2\gamma}{4-\beta}}\|T(t)f\|_{C_b^{2+\beta/2}(\mathbb{R}^d)}^{\frac{2(\gamma-\beta)}{4-\beta}}\\ \le&C_2(C_{\beta,\beta}e^{\omega_{\beta,\beta}t}\|f\|_{C_b^\beta(\mathbb{R}^d)})^{\frac{4+\beta-2\gamma}{4-\beta}}(C_{\beta,2+\beta/2}t^{-1+\frac{\beta}{4}}e^{\omega_{\beta,2+\beta/2}t}\|f\|_{C_b^\beta(\mathbb{R}^d)})^{\frac{2(\gamma-\beta)}{4-\beta}}\\ \le&C_2C_{\beta,\beta}^{\frac{4+\beta-2\gamma}{4-\beta}}C_{\beta,2+\beta/2}^{\frac{2(\gamma-\beta)}{4-\beta}}t^{-\frac{\gamma-\beta}{2}}e^{\omega_{\beta,\gamma}t}\|f\|_{C_b^\beta(\mathbb{R}^d)}\end{aligned}$$

for all $t>0$, where $\omega_{\beta,\gamma}=\omega_{\beta,\beta}\frac{4+\beta-2\gamma}{4-\beta}+\omega_{\beta,2+\beta/2}\frac{2(\gamma-\beta)}{4-\beta}$.

Step 6. Finally, we prove estimate (9.2) for $\beta=0$ and $\gamma\in(0,2+\alpha]$. First we suppose that $f\in BUC(\mathbb{R}^d)$ and observe that by Theorem 8.2.3 it follows that

$$\|u_n(t,\cdot)-u_m(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} \le Ct^{-\frac{2+\alpha}{2}}\|f_n-f_m\|_\infty, \qquad t\in(0,1],$$

for every $m,n\in\mathbb{N}$, where u_m and u_n are as in Step 1. Letting n tend to ∞, we conclude that $T(t)f\in C_b^{2+\alpha}(\mathbb{R}^d)$ for every $t\in(0,1]$, $f\in BUC(\mathbb{R}^d)$ and $\|T(t)f\|_{C_b^{2+\alpha}(\mathbb{R}^d)}\le Ct^{-\frac{2+\alpha}{2}}\|f\|_\infty$ for every $t\in(0,1]$. Using the semigroup property and arguing as above, we easily show that $T(t)f\in C_b^{2+\alpha}(\mathbb{R}^d)$ for every $t>0$ and estimate (9.2) follows with $\beta=0$. Observing that

$$\|g\|_{C_b^\gamma(\mathbb{R}^d)} \le \|g\|_\infty^{\frac{2+\alpha-\gamma}{2+\alpha}}\|g\|_{C_b^{2+\alpha}(\mathbb{R}^d)}^{\frac{\gamma}{2+\alpha}}, \tag{9.7}$$

for every $g\in C_b^{2+\alpha}(\mathbb{R}^d)$ and $\gamma\in(0,2+\alpha)$, (see Exercise 9.1.3) and arguing as above, estimate (9.2) follows with $\beta=0$ and $\gamma\in(0,2+\alpha)$.

Now, suppose that $f\in C_b(\mathbb{R}^d)$. Let $(f_n)\subset BUC(\mathbb{R}^d)$ be a sequence converging to f locally uniformly in $\mathbb{R}^d$ and such that $\|f_n\|_\infty\le\|f\|_\infty$ for all $n\in\mathbb{N}$. By Step 2, $T(t)f_n$ converges to $T(t)f$ in $C^2(\overline{B(0,M)})$ for every $M>0$ and $t>0$. Since

$$\|T(t)f_n\|_{C_b^\gamma(\mathbb{R}^d)} \le C_{0,\gamma}t^{-\frac{\gamma}{2}}e^{\omega_{0,\gamma}t}\|f_n\|_\infty \le C_{0,\gamma}t^{-\frac{\gamma}{2}}e^{\omega_{0,\gamma}t}\|f\|_\infty, \qquad t>0,$$

for all $n\in\mathbb{N}$ and $\gamma\in(0,2+\alpha]$, letting $n\to\infty$, we conclude that $\|T(t)f\|_{C_b^\gamma(\mathbb{R}^d)}\le C_{0,\gamma}t^{-\gamma/2}e^{\omega_{0,\gamma}t}\|f\|_\infty$ for all $t>0$, completing the proof. $\square$

Remark 9.0.2. Some important remarks are in order.

(i) $\{T(t)\}$ is not strongly continuous in $C_b(\mathbb{R}^d)$. This is clear if $\mathcal{A} = \Delta$, but it can be proved for each operator $\mathcal{A}$ satisfying Hypotheses 7.0.1. The proof of Theorem 9.0.1 shows that $T(t)f$ converges to f, as $t \to 0^+$, locally uniformly in $\mathbb{R}^d$.

(ii) On the other hand, the restriction of $\{T(t)\}$ to $BUC(\mathbb{R}^d)$ is strongly continuous. Indeed, by estimate (9.2), with $\beta = 0$ and $\gamma = 1$, it follows that $T(t)f \in C^1_b(\mathbb{R}^d) \hookrightarrow BUC(\mathbb{R}^d)$ for every $t > 0$. Moreover, let $(f_n) \subset C^{2+\alpha}_b(\mathbb{R}^d)$ be a sequence converging to $f \in BUC(\mathbb{R}^d)$, uniformly in $\mathbb{R}^d$. Note that

$$|(T(t)f)(x) - f(x)| \le \sup_{r\in[0,1]} \|T(r)f_n - T(r)f\|_\infty + |(T(t)f_n)(x) - f_n(x)| + \|f_n - f\|_\infty \tag{9.8}$$

for all $t \in [0,1]$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, and, by Step 1 in the proof of Theorem 9.0.1, $T(\cdot)f_n$ converges to $T(\cdot)f$ uniformly in $[0,T] \times \mathbb{R}^d$ for every $T > 0$. Moreover, using again (9.2) and recalling that $D_tT(\cdot)f_n = \mathcal{A}T(\cdot)f_n$ for every $n \in \mathbb{N}$, we get

$$|(T(t)f_n)(x) - f_n(x)| \le \int_0^t |(\mathcal{A}T(s)f_n)(x)|ds \le C\|f_n\|_{C^{2+\alpha}_b(\mathbb{R}^d)}t$$

for every $t \in [0,1]$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and some positive constant C, independent of $t \in [0,1]$, which, replaced in (9.8), gives

$$\|T(t)f - f\|_\infty \le \sup_{r\in[0,1]} \|T(r)f_n - T(r)f\|_\infty + C\|f_n\|_{C^{2+\alpha}_b(\mathbb{R}^d)}t + \|f_n - f\|_\infty,$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. Thus,

$$\limsup_{t\to 0^+} \|T(t)f - f\|_\infty \le \sup_{r\in[0,1]} \|T(r)f_n - T(r)f\|_\infty + \|f_n - f\|_\infty, \qquad n \in \mathbb{N},$$

and letting n tend to ∞, we conclude that $\limsup_{t\to 0^+} \|T(t)f - f\|_\infty = 0$, i.e., $T(t)f$ converges to f uniformly in $\mathbb{R}^d$ as $t \to 0^+$, showing that the restriction of $\{T(t)\}$ to $BUC(\mathbb{R}^d)$ is strongly continuous.

(iii) Actually, $\{T(t)\}$ is an analytic semigroup in $C_b(\mathbb{R}^d)$ whose associated sectorial operator A is defined as follows:

$$D(A) = \{u \in C_b(\mathbb{R}^d) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d), \text{ for each } p < \infty \text{ and } \mathcal{A}u \in C_b(\mathbb{R}^d)\}$$

and $Au = \mathcal{A}u$ for every $u \in D(A)$.

(iv) The arguments in the proof of Step 2 of the proof of Theorem 9.0.1 can be used to prove that, if $(f_n) \subset C_b(\mathbb{R}^d)$ is a bounded sequence, converging to $f \in C_b(\mathbb{R}^d)$, locally uniformly in $\mathbb{R}^d$, then $T(\cdot)f_n$ converges to $T(\cdot)f$ in $C^{1,2}([a,b] \times \overline{B(0,k)})$ for every $0 < a < b$ and $k \in \mathbb{N}$. Indeed, the quoted arguments show that a suitable subsequence of $(T(\cdot)f_n)$ converges in $C^{1,2}([a,b]\times\overline{B(0,K)})$ (for every $0<a<b$ and $K>0$) to $T(\cdot)f$. In particular, each subsequence of $(T(\cdot)f_n)$ admits a subsequence which converges in $C^{1,2}([a,b]\times\overline{B(0,K)})$ for every a,b,K as above, to $T(\cdot)f$. From Exercise 9.1(ii) it follows that the whole sequence $(T(\cdot)f_n)$ converges to $T(\cdot)f$ in $C^{1,2}([a,b]\times\overline{B(0,K)})$ for every $0<a<b$ and $K>0$.

(v) Actually, the result in point (iv) holds true also in the case when the bounded sequence (f_n) converges to $f\in C_b(\mathbb{R}^d)$ pointwise in $\mathbb{R}^d$. Indeed, the same arguments used in Step 1 of the proof of Theorem 9.0.1 show that, up to a subsequence, $(T(\cdot)f_n)$ converges in $C^{1,2}([a,b]\times\overline{B(0,K)})$, for every $0<a<b$ and $K>0$, to some function $u\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,\infty)\times\mathbb{R}^d)$. To identify u with $T(\cdot)f$, we observe that since, for all $t>0$ and $x\in\mathbb{R}^d$, the operator $g\mapsto (T(t)g)(x)$ belongs to the dual of $C_0(\mathbb{R}^d)$ (the space of continuous functions over $\mathbb{R}^d$ vanishing at infinity), there exists a positive Borel measure on $\mathbb{R}^d$, which we denote by $p(t,x,dy)$, such that

$$(T(t)g)(x)=\int_{\mathbb{R}^d} g(y)p(t,x,dy),\qquad g\in C_0(\mathbb{R}^d),\ \ t>0,\ \ x\in\mathbb{R}^d.$$

Since each function in $C_b(\mathbb{R}^d)$ is the local uniform limit of a bounded sequence of functions in $C_0(\mathbb{R}^d)$, by property (iv) the previous equality can be extended to each $g\in C_b(\mathbb{R}^d)$. Now, we are done. Indeed, by dominated convergence $(T(t)f_n)(x)$ converges to $(T(t)f)(x)$ for all $t>0$ and $x\in\mathbb{R}^d$. Hence, $u\equiv T(\cdot)f$ and we are done.

(vi) The difference between the case when f_n converges to f locally uniformly in $\mathbb{R}^d$ and the case when the convergence is only pointwise in $\mathbb{R}^d$, is that, in the first case, the convergence of $T(\cdot)f_n$ to $T(\cdot)f$ is uniform in $[0,T]\times\overline{B(0,M)}$ for every $M,T>0$. Indeed, suppose that f_n tends to f locally uniformly in $\mathbb{R}^d$, set $K=\sup_{n\in\mathbb{N}}\|f_n\|_\infty$ and fix $M>0$. Then, by (9.5) we get

$$\begin{aligned}\|T(t)f_n-f_n\|_{C(\overline{B(0,M)})}\le&\|T(t)(\vartheta_M f_n)-\vartheta_M f_n\|_{C(\overline{B(0,M)})}+K\|e^{c_0t}-T(t)\vartheta_M\|_{C(\overline{B(0,M)})}\\ \le&\|T(t)(\vartheta_M f_n-\vartheta_M f)\|_\infty+\|(T(t)(\vartheta_M f)-\vartheta_M f\|_\infty\\ &+\|f_n-f\|_{C(\overline{B(0,M)})}+K\|e^{c_0t}-T(t)\vartheta_M\|_{C(\overline{B(0,M)})}\\ \le&2\|f_n-f\|_{C(\overline{B(0,2M)})}+\|(T(t)(\vartheta_M f)-\vartheta_M f\|_\infty\\ &+K\|e^{c_0t}-T(t)\vartheta_M\|_{C(\overline{B(0,M)})}\end{aligned}$$

for all $t>0$. Fix $\varepsilon>0$ and let $n_0\in\mathbb{N}$ be such that $2\|f_n-f\|_{C(\overline{B(0,2M)})}\le\varepsilon/4$ for every $n\ge n_0$. With this choice of n_0 it follows that

$$\begin{aligned}\|T(t)f_n-f_n\|_{C(\overline{B(0,M)})}\le&\frac{\varepsilon}{4}+\|T(t)(\vartheta_M f)-\vartheta_M f\|_{C(\overline{B(0,M)})}\\ &+K\|e^{c_0t}-T(t)\vartheta_M\|_{C(\overline{B(0,M)})}\end{aligned}$$

for all $n\ge n_0$. Since $\|T(t)(\vartheta_M f)-\vartheta_M f\|_{C(\overline{B(0,M)})}$ and $\|e^{c_0t}-T(t)\vartheta_M\|_{C(\overline{B(0,M)})}$ vanish as t tends to 0^+, we can determine $t_0>0$ such that $\|T(t)f_n-f_n\|_{C(\overline{B(0,M)})}\le\varepsilon/2$ for all $t \in [0,t_0]$ and $n \geq n_0$. Now, we are almost done. Indeed, we can estimate

$$\begin{aligned}\|T(t)f_n - T(t)f\|_{C(\overline{B(0,M)})} \leq & \|T(t)f_n - f_n\|_{C(\overline{B(0,M)})} + \|f_n - f\|_{C(\overline{B(0,M)})} \\ & + \|T(t)f - f\|_{C(\overline{B(0,M)})}\end{aligned} \tag{9.9}$$

for $t > 0$ and $n \in \mathbb{N}$. Up to replacing t_0 with a smaller value and n_0 with a larger integer, if needed, we can assume that $\|T(t)f_n - f_n\|_{C(\overline{B(0,M)})} + \|f_n - f\|_{C(\overline{B(0,M)})} \leq \varepsilon/4$ for $t \in [0,t_0]$ and $n \geq n_0$. Hence, from (9.9) it follows that $\|T(t)f_n - T(t)f\|_{C(\overline{B(0,M)})} \leq \varepsilon$ for all $t \in [0,t_0]$ and $n \geq n_0$.

Fix $T > 0$. Without loss of generality, we can assume that $T > t_0$. Hence,

$$\begin{aligned}&\|T(\cdot)f_n - T(\cdot)f\|_{C([0,T]\times\overline{B(0,M)})} \\ =& \max\{\|T(\cdot)f_n - T(\cdot)f\|_{C([0,t_0]\times\overline{B(0,M)})}, \|T(\cdot)f_n - T(\cdot)f\|_{C([t_0,T]\times\overline{B(0,M)})}\} \\ \leq& \max\{\varepsilon, \|T(\cdot)f_n - T(\cdot)f\|_{C([t_0,T]\times\overline{B(0,M)})}\}\end{aligned}$$

for $n \geq n_0$. Since, $\|T(\cdot)f_n - T(\cdot)f\|_{C([t_0,T]\times\overline{B(0,M)})}$ vanishes as n tends to ∞, from the previous estimate, it follows that $\|T(\cdot)f_n - T(\cdot)f\|_{C([0,T]\times\overline{B(0,M)})}$ does not exceed ε if n is sufficiently large.

9.1 Exercises

1. (i) Prove that if a sequence $(g_n) \subset C_b^\gamma(\mathbb{R}^d)$ ($\gamma > 0$) satisfies the estimate $\|g_n\|_{C_b^\gamma(\mathbb{R}^d)} \leq K$, for some positive constant K and all $n \in \mathbb{N}$, and converges to some function $g : \mathbb{R}^d \to \mathbb{R}$, locally uniformly in $\mathbb{R}^d$, then $g \in C_b^\gamma(\mathbb{R}^d)$ and $\|g\|_{C_b^\gamma(\mathbb{R}^d)} \leq K$.

   (ii) Prove that, if $\gamma \in (0,1)$, then the same conclusion holds just assuming that g_n converges to g pointwise in $\mathbb{R}^d$.

2. Let (X,d) be a metric space and let (x_n) be a sequence with the following property: there exists a subsequence (x_{n_k}) with converges to some $x \in X$ and each subsequence of (x_n) admits a subsequence which converges to x. Prove that the sequence (x_n) converges.

3. Prove estimates (9.6) and (9.7).

4. Prove the first part of Remark 9.0.2.

Bibliography

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Lecture 10

Parabolic equations in $\mathbb{R}^d_+$ with homogeneous Dirichlet boundary conditions. Part I

In this lecture we study the Cauchy-Dirichlet problem

$$
\begin{cases}
D_t u(t,x) = \mathcal{A}u(t,x) + g(t,x), & t \in (0,T], \quad x \in \mathbb{R}^d_+,\\
u(t,x',0) = 0, & t \in (0,T], \quad x' \in \mathbb{R}^{d-1},\\
u(0,x) = f(x), & \qquad\qquad\quad x \in \mathbb{R}^d_+,
\end{cases}
\tag{10.1}
$$

where $\mathbb{R}^d_+ = \{(x_1,\ldots,x_d) \in \mathbb{R}^d : x_d > 0\}$ and we find it convenient to split $x = (x', x_d)$ for every $x \in \mathbb{R}^d$. Here, $\mathcal{A}$ is the elliptic operator defined on smooth functions $\psi : \mathbb{R}^d_+ \to \mathbb{R}$ by

$$
\begin{aligned}
\mathcal{A}\psi(x) =& \sum_{i,j=1}^d q_{ij}(x) D_{ij}\psi(x) + \sum_{j=1}^d b_j(x) D_j\psi(x) + c(x)\psi(x)\\
=& \mathrm{Tr}(Q(x)D^2\psi(x)) + \langle b(x), \nabla_x \psi(x)\rangle + c(x)\psi(x)
\end{aligned}
$$

with, as usually, $Q(x) = (q_{ij}(x))_{1\le i,j\le d}$ for $x \in \mathbb{R}^d$. Throughout the lecture, we assume the following conditions on the coefficients of the operator $\mathcal{A}$.

Hypotheses 10.0.1. (i) The coefficients $q_{ij} = q_{ji}$, b_j $(i,j = 1,\ldots,d)$ and c are bounded and α-Hölder continuous in $\overline{\mathbb{R}^d_+} := \mathbb{R}^{d-1} \times [0,\infty)$ for some $\alpha \in (0,1)$;

(ii) there exists a positive constant μ such that $\langle Q(x)\xi, \xi\rangle \ge \mu|\xi|^2$ for all $x \in \mathbb{R}^d_+$ and $\xi \in \mathbb{R}^d$.

The following result is the counterpart of Theorem 8.1.1.

Theorem 10.0.2. *Let Hypotheses* 10.0.1 *be satisfied. Then, for every* $f \in C^{2+\alpha}_b(\overline{\mathbb{R}^d_+})$ *and* $g \in C^{\alpha/2,\alpha}_b([0,T]\times\overline{\mathbb{R}^d_+})$ *such that* $\mathcal{A}f + g(t,\cdot) = 0$ *on* $\partial\mathbb{R}^d_+ := \mathbb{R}^{d-1}\times\{0\}$ *for each* $t \in [0,T]$*,* *there exists a unique classical solution u to problem* (10.1). *In addition, u belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$ and there exists a positive constant C, independent of f and g, such that*

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}\le C(\|f\|_{C_b^{2+\alpha}(\overline{\mathbb{R}^d_+})}+\|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}). \tag{10.2}$$

Remark 10.0.3. The conditions $f\equiv\mathcal{A}f+g(0,\cdot)\equiv 0$ on $\partial\mathbb{R}^d_+$ are *necessary* for problem (10.1) to have a solution $u\in C^{1,2}([0,T]\times\overline{\mathbb{R}^d_+})$. Indeed, since $u(t,\cdot)$ vanishes on $\partial\mathbb{R}^d_+$ for each $t\in[0,T]$, then $f=u(0,\cdot)$ and $D_tu(t,\cdot)$ vanish on $\partial\mathbb{R}^d_+$ for each $t\in[0,T]$. This latter condition implies that $\mathcal{A}u(t,x)+g(t,x)=0$ for each $t\in[0,T]$ and $x\in\partial\mathbb{R}^d_+$. Taking $t=0$ it follows that $\mathcal{A}f(x)+g(0,x)$ for each $x\in\partial\mathbb{R}^d_+$.

In this lecture, we prove Theorem 10.0.2 in the case when $\mathcal{A}=\Delta$ under stronger assumptions on the functions f and g. These stronger conditions make the proof easier since they easily allow to transform the Cauchy problem 10.1 into a Cauchy problem in the whole space.

Throughout the lecture we use the notation u_o to denote the odd extension with respect to the last variable of a function $u:\overline{\mathbb{R}^d_+}\to\mathbb{R}$, i.e., the function $u_o:\mathbb{R}^d\to\mathbb{R}$ defined by $u_o(x)=u(x)$ for each $x\in\overline{\mathbb{R}^d_+}$ and $u_o(x)=-u(x',-x_d)$ for each $x=(x',x_d)\in\mathbb{R}^d$ with $x_d<0$.

Now, we can prove 10.0.2 in the particular case when $\mathcal{A}=\Delta$.

Theorem 10.0.4. *For every $f\in C_b^{2+\alpha}(\overline{\mathbb{R}^d_+})$ and $g\in C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$ such that f, Δf and $g(t,\cdot)$ vanish on $\partial\mathbb{R}^d_+$ for every $t\in[0,T]$, there exists a unique classical solution u to problem*

$$\begin{cases} D_tu(t,x)=\Delta u(t,x)+g(t,x), & t\in(0,T], \quad x\in\mathbb{R}^d_+,\\ u(t,x',0)=0, & t\in(0,T], \quad x'\in\mathbb{R}^{d-1},\\ u(0,x)=f(x), & \qquad\qquad\quad x\in\mathbb{R}^d_+.\end{cases} \tag{10.3}$$

In addition, u belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$ and there exists a positive constant C, independent of f and g, such that

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}\le C(\|f\|_{C_b^{2+\alpha}(\overline{\mathbb{R}^d_+})}+\|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}). \tag{10.4}$$

Proof. Let us begin by proving the uniqueness part. For this purpose, we assume that $u\in C_b^{1,2}((0,T]\times\overline{\mathbb{R}^d_+})\cap C_b([0,T]\times\overline{\mathbb{R}^d_+})$ solves the Cauchy problem

$$\begin{cases} D_tu(t,x)=\Delta u(t,x), & t\in(0,T], \quad x\in\mathbb{R}^d_+,\\ u(t,x',0)=0, & t\in(0,T], \quad x'\in\mathbb{R}^{d-1},\\ u(0,x)=0, & \qquad\qquad\quad x\in\mathbb{R}^d_+,\end{cases}$$

and prove that u identically vanishes on $\mathbb{R}^d_+$. Since u vanishes on $[0,T]\times\partial\mathbb{R}^d_+$, function u_o is once continuously differentiable with respect to the time and spatial variables on $[0,T]\times\mathbb{R}^d$

and the first-order derivatives of u_o are bounded on $[0,T]\times\mathbb{R}^d$. Similarly, since $D_{ij}u=0$ on $[0,T]\times\partial\mathbb{R}^d_+$ $(i,j=1,\ldots,d-1)$, then $D_{ij}u_o$ exists and it is continuous on $[0,T]\times\mathbb{R}^d$. From the differential equation satisfied by u we conclude that $D_{dd}u$ vanishes on $[0,T]\times\partial\mathbb{R}^d_+$. Hence, the derivative $D_{dd}u_o$ exists and it is a continuous function in $[0,T]\times\mathbb{R}^d$. Moreover, u_o solves the equation $D_tu_o=\Delta u_o$ on $(0,T]\times\mathbb{R}^d$ and vanishes at $t=0$. Fix $t>0$ and let $v:[0,t]\times\mathbb{R}^d\to\mathbb{R}$ be the function defined by $v(s,x)=(T(t-s)u_o(s,\cdot))(x)$ where $\{T(t)\}$ denotes the Gauss-Weierstrass semigroup. A straightforward computation shows that

$$\begin{aligned}D_sv(s,\cdot)=&-\Delta T(t-s)u_o(s,\cdot)+T(t-s)D_su_o(s,\cdot)\\=&-\Delta T(t-s)u_o(s,\cdot)+T(t-s)\Delta u_o(s,\cdot)\\=&-\Delta T(t-s)u_o(s,\cdot)+\Delta T(t-s)u_o(s,\cdot)=0\end{aligned}$$

Hence, for any $x\in\mathbb{R}^d$ the function $v(\cdot,x)$ is constant in $[0,t]$ and, consequently $v(t,x)=v(0,x)$, i.e., $u_o(t,x)=0$. We have so proved that $u\equiv 0$ and the uniqueness of the solution to the Cauchy problem (10.3) follows.

Let us prove the existence part. As a first step, we observe that u solves the Cauchy problem 10.3 if and only if the function $v=u-f$ solves the Cauchy problem

$$\begin{cases}D_tv(t,x)=\Delta v(t,x)+g(t,x)+\Delta f(x), & t\in(0,T],\quad x\in\mathbb{R}^d_+,\\ v(t,x',0)=0, & t\in(0,T],\quad x'\in\mathbb{R}^{d-1},\\ v(0,x)=0, & \qquad\qquad\quad x\in\mathbb{R}^d_+.\end{cases}$$

To find a solution to this problem, we set $\psi=g+\Delta f$ and consider the function ψ_o. Due to the conditions $g=0$ on $[0,T]\times\partial\mathbb{R}^d_+$ and $\Delta f=0$ on $\partial\mathbb{R}^d_+$, function ψ_o belongs to $C^{\alpha/2,\alpha}_b([0,T]\times\mathbb{R}^d)$ and

$$\|\psi_o\|_{C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\le 2\|g+\Delta f\|_{C^{\alpha/2,\alpha}_b([0,T]\times\overline{\mathbb{R}^d_+})}\le C_1(\|f\|_{C^{2+\alpha}_b(\overline{\mathbb{R}^d_+})}+\|g\|_{C^{\alpha/2,\alpha}_b([0,T]\times\overline{\mathbb{R}^d_+})})$$

for some positive constant C_1, independent of f and g. Hence, we can apply Theorem 6.1.4 and conclude that the Cauchy problem

$$\begin{cases}D_tv(t,x)=\Delta v(t,x)+\psi_o(t,x), & t\in(0,T],\quad x\in\mathbb{R}^d,\\ v(0,x)=0, & \qquad\qquad\quad x\in\mathbb{R}^d,\end{cases}\tag{10.5}$$

admits a unique solution $v\in C^{1+\alpha/2,2+\alpha}_b([0,T]\times\mathbb{R}^d)$ which satisfies the estimate

$$\begin{aligned}\|v\|_{C^{1+\alpha/2,2+\alpha}_b([0,T]\times\mathbb{R}^d)}\le&C\|\psi_o\|_{C^{\alpha/2,\alpha}_b([0,T]\times\mathbb{R}^d)}\\ \le&C_2(\|f\|_{C^{2+\alpha}_b(\overline{\mathbb{R}^d_+})}+\|g\|_{C^{\alpha/2,\alpha}_b([0,T]\times\overline{\mathbb{R}^d_+})}).\end{aligned}$$

The restriction u of v to $[0,T]\times\mathbb{R}^d_+$ clearly belongs to $C^{1+\alpha/2,2+\alpha}_b([0,T]\times\overline{\mathbb{R}^d_+})$, solves the differential equation in (10.3), equals the function f on $\{0\}\times\overline{\mathbb{R}^d_+}$ and satisfies estimate (10.4). To complete the proof, we need to show that u vanishes on $[0,T]\times\partial\mathbb{R}^d_+$. For this purpose, it suffices to show that $v(t,x)=-v(t,x',-x_d)$ for each $(t,x)\in[0,T]\times\mathbb{R}^d$, but this follows from the uniqueness of the solution to the Cauchy problem (10.5), since the function $(t,x)\mapsto -v(t,x',-x_d)$ is a bounded classical solution to (10.5). □

Now, we show that, for each $f \in C_b(\mathbb{R}^d_+)$ the Cauchy problem (10.1) (with $g \equiv 0$) admits a classical solution u. Since f is not defined on $\partial\mathbb{R}^d_+$ (and even if it is defined on $\overline{\mathbb{R}^d_+}$ in general it does not identically vanish on $\partial\mathbb{R}^d_+$) we can not expect u to be continuous on $\{0\} \times \partial\mathbb{R}^d_+$. Hence, we need to slightly modify the definition of classical solution.

Definition 10.0.5. A classical solution to problem (10.3) with $g \equiv 0$ and $f \in C_b(\mathbb{R}^d_+)$ is a function $u \in C([0,\infty) \times \mathbb{R}^d_+) \cap C((0,\infty) \times \overline{\mathbb{R}^d_+}) \cap C^{1,2}((0,\infty) \times \mathbb{R}^d_+)$ which solves the equation $D_t u = \Delta u$ on $(0,\infty) \times \mathbb{R}^d_+$ and satisfies the boundary and initial condition in (10.3).

Theorem 10.0.6. *For each $f \in C_b(\mathbb{R}^d_+)$ there exists a unique bounded classical solution u of the Cauchy problem* (10.3)*, with $g \equiv 0$. Further,*

$$\|u\|_{C_b([0,\infty)\times\overline{\mathbb{R}^d_+})} \le \|f\|_{C_b(\overline{\mathbb{R}^d_+})}. \tag{10.6}$$

Finally,

$$u(t,x) = \int_{\mathbb{R}^d_+} k(t,x,y) f(y)dy, \qquad t \in (0,\infty),\ x \in \overline{\mathbb{R}^d}_+, \tag{10.7}$$

where $k(t,x,y) = (4\pi t)^{-d/2}(e^{-|x-y|^2/(4t)} - e^{-(|x'-y'|^2+|x_d+y_d|^2)/(4t)})$ for each $t > 0$ and $x, y \in \mathbb{R}^d_+$.

Proof. The uniqueness of the bounded classical solution to problem (10.3) follows from the proof of Theorem 10.0.4.

To prove the existence part, we introduce a bounded sequence (f_n) of functions in $C_b(\overline{\mathbb{R}^d_+})$ which vanish on $\partial\mathbb{R}^d_+$ and converge to f locally uniformly on $\mathbb{R}^d_+$. We denote by $u_n \in C_b([0,\infty) \times \overline{\mathbb{R}^d_+}) \cap C^{1,2}((0,\infty) \times \mathbb{R}^d_+)$ the solution of the Cauchy problem (10.1). By the interior Schauder estimates in Theorem B.0.4 for each $0 < \varepsilon < T$, each compact set $K \subset \mathbb{R}^d_+$ and each $\alpha \in (0,1)$, there exists a positive constant C such that

$$\|u_n\|_{C_b^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times K)} \le C\|u_n\|_\infty \le C\|f_n\|_\infty \le C\|f\|_\infty$$

for every $n \in \mathbb{N}$. This estimate allows us to apply Arzelà-Ascoli theorem to u_n and its time derivative and first-order and second-order spatial derivatives to infer that, up to a subsequence u_n converges in $C^{1,2}([\varepsilon,T] \times K)$ to a function $u \in C^{1+\alpha/2,2+\alpha}([\varepsilon,T] \times K)$. The arbitrariness of ε, T and K yields that $u \in C^{1+\alpha/2,2+\alpha}((0,\infty) \times \mathbb{R}^d_+)$ and it solves the differential equation in (10.1). Since

$$u_n(t,x) = \int_{\mathbb{R}^d_+} k(t,x,y) f_n(y)dy, \qquad t \in (0,\infty),\ x \in \mathbb{R}^d_+,$$

letting n tend to ∞, by dominated convergence we conclude that

$$u(t,x) = \int_{\mathbb{R}^d_+} k(t,x,y) f(y)dy, \qquad t \in (0,\infty),\ x \in \mathbb{R}^d_+, \tag{10.8}$$

or, equivalently,

$$u(t,x)=\int_{\mathbb{R}^d} e^{-\frac{|y|^2}{4}} f_o(x+\sqrt{t}y)dy, \qquad (t,x)\in(0,\infty)\times\mathbb{R}^d_+. \tag{10.9}$$

where f_o is continuous on $\mathbb{R}^d\setminus\partial\mathbb{R}^d_+$. Using this formula we can show that $u(t,x)$ converges to $f(x^0)$ as $(t,x)\to(0,x^0)$ for every $x^0\in\mathbb{R}^d_+$. Indeed, fix $\varepsilon>0$ and let $R>0$ be sufficiently large such that

$$\|f\|_\infty\int_{\mathbb{R}^d\setminus B(0,R)} e^{-\frac{|y|^2}{4}}dy\le\frac{\varepsilon}{2}.$$

Hence, if $\sqrt{t}R<x^0_d$ and $x\in\mathbb{R}^d_+$, then we can estimate

$$\begin{aligned}|u(t,x)-f(x^0)| =&\frac{\varepsilon}{2}+\left|\int_{B(0,R)} e^{-\frac{|y|^2}{4}} f(x+\sqrt{t}y)dy-f(x^0)\right|\\ \le&\frac{\varepsilon}{2}+\int_{B(0,R)} e^{-\frac{|y|^2}{4}}|f(x+\sqrt{t}y)-f(x^0)|dy+\|f\|_\infty\bigg(\int_{B(0,R)} e^{-\frac{|y|^2}{4}}dy-1\bigg)\\ \le&\varepsilon+\int_{B(0,R)} e^{-\frac{|y|^2}{4}}|f(x+\sqrt{t}y)-f(x^0)|dy.\end{aligned}$$

Letting (t,x) tend to $(0,x^0)$, we conclude that

$$\limsup_{(t,x)\to(0,x^0)}|u(t,x)-f(x^0)|\le\varepsilon.$$

The arbitrariness of $\varepsilon>0$ implies that $u(t,x)$ tends to $f(x_0)$ as $(t,x)\to(0,x^0)$. We have so proved that $u\in C([0,\infty)\times\mathbb{R}^d_+)$. On the other hand, using formula (10.8) together with the dominated convergence theorem it can be easily checked that, for each $t\in(0,\infty)$, the function $u(t,\cdot)$ can be extended by continuity to $\overline{\mathbb{R}^d_+}$, i.e., $u\in C((0,\infty)\times\overline{\mathbb{R}^d_+})$. Finally, since $u_n(t,x)=0$ for every $t>0$, $x\in\partial\mathbb{R}^d_+$, $n\in\mathbb{N}$ and $u_n(t,x)$ converges to $u(t,x)$ pointwise on $(0,\infty)\times\overline{\mathbb{R}^d_+}$, we conclude that $u(t,x)=0$ for every $t\in(0,\infty)$ and $x\in\partial\mathbb{R}^d_+$. Summing up, we have proved that u is a classical solution to the Cauchy problem 10.3 and this completes the proof. □

As in the previous lectures, we can associate a semigroup of bounded operators $\{S(t)\}$ with the Cauchy problem (10.3). For each $f\in C_b(\mathbb{R}^d_+)$ $S(\cdot)f$ is the unique solution to problem (10.3) with the regularity properties in the statement of Theorem 10.0.4. The semigroup property follows from the uniqueness part of the proof of Theorem 10.0.6. As a consequence of Theorem 5.1.6 and Theorem 4.1.12 we can prove the following interesting result.

Proposition 10.0.7. *Let* $C^\beta_0(\overline{\mathbb{R}^d_+})=\{f\in C^\beta_b(\overline{\mathbb{R}^d_+}):\ f\equiv 0 \text{ on } \partial\mathbb{R}^d_+\}$, *if* $\beta\le 2$, *and* $C^\beta_0(\overline{\mathbb{R}^d_+})=\{f\in C^\beta_b(\overline{\mathbb{R}^d_+}):\ f\equiv\Delta f\equiv 0 \text{ on } \partial\mathbb{R}^d_+\}$, *if* $\beta\in(2,3]$. *Then, each operator* $S(t)$

maps $C_0(\overline{\mathbb{R}^d_+})$ *into* $C^3_0(\overline{\mathbb{R}^d_+})$ *and, for each* $\alpha,\theta\in[0,3]$*, such that* $\alpha\le\theta$ *and* $T>0$*, there exists a positive constant* $\widetilde{C}_{\alpha,\theta,T}>0$ *such that*

$$\|S(t)\|_{L(C^\alpha_0(\mathbb{R}^d_+),C^\theta_0(\mathbb{R}^d_+))}\le \widetilde{C}_{\alpha,\theta,T}t^{-\frac{\theta-\alpha}{2}},\qquad t\in(0,T]. \tag{10.10}$$

Proof. To begin with, let us prove that, if $f\in C^\alpha_0(\overline{\mathbb{R}^d_+})$ ($\alpha\in[0,2)\cup(2,3)$), then the function $f_o\in C^\alpha_b(\mathbb{R}^d)$ and $\|f\|_{C^\alpha_b(\mathbb{R}^d_+)}\le 2\|f\|_{C^2_b(\overline{\mathbb{R}^d_+})}$. This is clear if $\alpha\in[0,2)$. Let us suppose that $\alpha\in(2,3)$. Then, the same argument used in the first part of the proof of Theorem 10.0.4 can be used to prove that function f_o has the following properties:

- it belongs to $C^1_b(\mathbb{R}^d)$;
- it admits the classical derivative $D_{jj}f_o$ ($j\in\{1,\dots,d\}$) which is a continuous function on $\mathbb{R}^d$.

Based on this remark, it is easy to check that $\Delta T(t)f_o=T(t)(\Delta f)_o$ for every $t>0$, where as usually in this lecture $\{T(t)\}$ denotes the Gauss-Weierstrass semigroup. Therefore, for every $i=1,\dots,d$ and $t>0$

$$\begin{aligned}\|\Delta T(t)D_if_o\|_\infty =&\|\Delta D_iT(t)f_o\|_\infty=\|D_i\Delta T(t)f_o\|_\infty=\|D_iT(t)(\Delta f)_o\|_\infty\\ \le& C_1t^{-\frac{1-\alpha}{2}}\|(\Delta f)_o\|_{C^\alpha_b(\mathbb{R}^d)}\le 2C_1t^{-\frac{1-\alpha}{2}}\|\Delta f\|_{C^\alpha_b(\overline{\mathbb{R}^d_+})}\end{aligned}$$

for some positive constant C_1, independent of t and f, so that, by Theorem 5.1.6, D_if_o belongs to $D_\Delta((1+\alpha)/2,\infty)=C^{1+\alpha}_b(\mathbb{R}^d)$. We have[1]so proved that f_o belongs to $C^{2+\alpha}_b(\mathbb{R}^d)$. Clearly, $\|f_o\|_{C^\alpha_b(\mathbb{R}^d)}\le 2\|f\|_{C^\alpha_b(\overline{\mathbb{R}^d_+})}$.
Now, Theorem 4.1.12 and formula (10.9), which shows that, for every $t>0$ and $f\in C_0(\overline{\mathbb{R}^d_+})$, $S(t)f$ coincides with the restriction $T(t)f_o$ to $\overline{\mathbb{R}^d_+}$, allow us to conclude that $S(t)f\in C^3_b(\overline{\mathbb{R}^d_+})$ for every $f\in C_0(\overline{\mathbb{R}^d_+})$, $t>0$ and

$$\|S(t)\|_{L(C^\alpha_0(\overline{\mathbb{R}^d_+}),C^\theta_b(\mathbb{R}^d_+))}\le 2C_{\alpha,\theta,T}t^{-\frac{\theta-\alpha}{2}},\qquad t\in(0,T] \tag{10.11}$$

for every $T>0$, $\alpha\in[0,2)\cup(2,3)$ and $\beta\in[\alpha,3]$, where $C_{\alpha,\theta,T}$ is the constant in (4.18). The case $\alpha=2\le\theta$ follows from observing that $\|f\|_{C^\alpha_b(\overline{\mathbb{R}^d_+})}\le C_2\|f\|_{C^2_b(\overline{\mathbb{R}^d_+})}$ for each $\alpha\in(1,2)$, $f\in C^2_0(\mathbb{R}^d_+)$ and some positive constant C_2, independent of f, so that from (10.11) we can infer that

$$\|S(t)f\|_{C^\theta_b(\overline{\mathbb{R}^d_+})}\le C_2C_{\alpha,\theta,T}t^{-\frac{\theta-\alpha}{2}}\|f\|_{C^2_b(\overline{\mathbb{R}^d_+})}, \tag{10.12}$$

and the function $\alpha\mapsto C_{\alpha,\theta,T}$ is bounded in a left neighborhood of 2, letting $\alpha\to 2^-$, (10.10) follows also in this case. A similar argument can be used in the case $\alpha=3$.

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[1] The previous argument is an adaption of an idea by Hendrik Voigt on the Discussion board: thanks Hendrik!

Fix $t > 0$ and $f \in C_0(\overline{\mathbb{R}^d_+})$. Since $S(t)f$ is the restriction to $\overline{\mathbb{R}^d}_+$ of the function $T(t)f_o$ and the function $T(\cdot)f_o$ belongs to $C^{1,2}((0,\infty) \times \mathbb{R}^d)$, it follows that $S(\cdot)f \in C^{1,2}((0,\infty) \times \overline{\mathbb{R}^d_+})$. Recalling that $S(t)f$ vanishes on $\partial\mathbb{R}^d_+$ for every $t > 0$ and $D_t S(t)f = \Delta S(t)f$ on $(0,\infty) \times \overline{\mathbb{R}^d_+}$ it is immediate to conclude that $\Delta S(t)f = 0$ on $\partial\mathbb{R}^d_+$ for every $t > 0$, so that $S(t)f \in C^3_0(\overline{\mathbb{R}^d_+})$ for every $t > 0$ and from (10.11), estimate (10.10) follows at once. □

10.1 Exercises

1. Adapting the arguments used in Lemma 7.2.2, prove Theorem 10.0.2 in the case when $\mathcal{A} = \mathrm{Tr}(QD^2)$ for some constant positive definite matrix Q. (Note that the case when Q is not diagonal demands some more efforts, since in general the transformation used in the proof of the quoted lemma does not preserve $\mathbb{R}^d_+$.)

2. For $f \in C_b(\overline{\mathbb{R}^d_+})$ prove that the solution u to the Cauchy problem (10.3) with $g \equiv 0$ is given by

$$u(t,x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d_+} \left(e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x'-y'|^2+|x_d+y_d|^2}{4t}} \right) f(y)\, dy, \qquad t > 0,\ x \in \mathbb{R}^d_+.$$

Lecture 11

Parabolic equations in $\mathbb{R}^d_+$ with homogeneous Dirichlet boundary conditions. Part II

In this lecture we keep on the study the Cauchy-Dirichlet problem (10.1), proving Theorem 10.0.2. The following proposition is the uniqueness part of Theorem 10.0.2. Its proof will be adapted to cover the case of elliptic operators with unbounded coefficients in one of the forthcoming lectures.

Proposition 11.0.1. *Let $u \in C_b([0,T] \times \overline{\mathbb{R}^d_+}) \cap C^{1,2}((0,T] \times \mathbb{R}^d_+)$ satisfy the inequalities $D_t u - \mathcal{A}u \le 0$ on $(0,T] \times \mathbb{R}^d_+$, $u \le 0$ on $(0,T] \times \partial\mathbb{R}^d_+$ and $u(0,\cdot) \le 0$ on $\overline{\mathbb{R}^d_+}$. Then, u is everywhere non positive. As a consequence, the Cauchy problem (10.1) admits at most a unique bounded classical solution $u \in C_b([0,T] \times \overline{\mathbb{R}^d_+}) \cap C^{1,2}((0,T] \times \mathbb{R}^d_+)$.*

Proof. Let u be as in the statement of the proposition and consider the function $\varphi : \mathbb{R}^d \to \mathbb{R}$, defined by $\varphi(x) = 1 + |x|^2$ for every $x \in \mathbb{R}^d$. Clearly, $\lim_{|x|\to\infty} \varphi(x) = +\infty$. Moreover, since $\mathcal{A}\varphi(x) = 2\mathrm{Tr}(Q(x)) + 2\langle b(x), x\rangle + c(x)(1 + |x|^2)$ for every $x \in \overline{\mathbb{R}^d_+}$, we can estimate

$$|\mathcal{A}\varphi(x)| \le 2\sum_{j=1}^d \|q_{jj}\|_\infty + \left(\sum_{j=1}^d \|b_j\|_\infty^2\right)^{\frac{1}{2}} |x| + \|c\|_\infty \varphi(x), \qquad x \in \overline{\mathbb{R}^d_+}, \tag{11.1}$$

and conclude that $\mathcal{A}\varphi < \lambda\varphi$ on $\overline{\mathbb{R}^d_+}$ for some positive constant λ.

For each $n \in \mathbb{N}$, let v_n be the function defined by $v_n(t,x) = e^{-\lambda t}u(t,x) - n^{-1}\varphi(x)$ for every $t \in [0,T]$ and $x \in \overline{\mathbb{R}^d_+}$. We claim that $v_n \le 0$ on $[0,T] \times \overline{\mathbb{R}^d_+}$ for all $n \in \mathbb{N}$. For this purpose we observe that each function v_n is as smooth as u is and $\lim_{|x|\to\infty} v_n(t,x) = -\infty$, uniformly with respect to $t \in [0,T]$. As a consequence, for each $n \in \mathbb{N}$ v_n has a maximum point $(t_n, x_n) \in [0,T] \times \overline{\mathbb{R}^d_+}$. If (t_n, x_n) belongs to the parabolic boundary of $(0,T) \times \mathbb{R}^d_+$, then v_n is clearly nonpositive on $[0,T] \times \overline{\mathbb{R}^d_+}$. Suppose that $(t_n, x_n) \in (0,T] \times \mathbb{R}^d_+$. Then, $v(t_n, x_n)$ should be nonpositive. Indeed, the function v_n satisfies the differential inequality $D_t v_n - \mathcal{A}v_n + \lambda v_n < 0$. The argument in the proof of Theorem 1.1.1 leads to a contradiction if $v(t_n,x_n)>0$. Thus, $v_n\le 0$ on $[0,T]\times\overline{\mathbb{R}^d_+}$. Letting n tend to ∞, we conclude that $u\le 0$ on $[0,T]\times\overline{\mathbb{R}^d_+}$.

To complete the proof, let us assume that $u\in C_b([0,T]\times\overline{\mathbb{R}^d_+})\cap C^{1,2}((0,T]\times\mathbb{R}^d_+)$ solves the Cauchy problem (10.1) with $f\equiv 0$ and $g\equiv 0$. Since both u and $-u$ satisfy the assumptions of the first part of the proof, they both are nonpositive on $[0,T]\times\overline{\mathbb{R}^d_+}$. As a byproduct, u identically vanishes on $[0,T]\times\overline{\mathbb{R}^d_+}$ and we are done. □

The existence part of the proof of Theorem 10.0.2 demands much more effort also in the simplest case when $\mathcal{A}$ is the Laplacian. We need some preliminary results which are the content of the next section.

11.1 Technical results

Lemma 11.1.1. *There exists a positive constant C such that*

(11.2)
$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}\le C\bigg(\|u\|_\infty+\|D_tu\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}+\sum_{i,j=1}^d\|D_{ij}u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}\bigg)$$

for each $u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$.

Proof. The proof follows the same lines of that of Lemma 5.2.1. We just need to show that $\|\psi'\|_\infty\le C\|\psi\|_\infty^{1/2}\|\psi''\|_\infty^{1/2}$ for every $\psi\in C^2_b([0,\infty))$. This follows from expanding ψ using Taylor's formula to write

$$\psi(y+t)=\psi(y)+\psi'(y)t+\int_0^t(t-s)\psi''(y+s)\,ds,\qquad y\ge 0,\ \ t>0,$$

and thus estimate

$$|\psi'(y)|\le\left|\frac{\psi(y+t)-\psi(y)}{t}\right|+\frac{1}{t}\int_0^t(t-s)|\psi''(y+s)|\,ds\le 2\|\psi\|_\infty t^{-1}+\frac{t}{2}\|\psi''\|_\infty.$$

Minimizing with respect to $t>0$, we get $|\psi'(y)|\le 2\|\psi\|_\infty^{1/2}\|\psi''\|_\infty^{1/2}$ and, taking the supremum with respect to $y\ge 0$, the wished estimate follows with $C=2$. □

To ease the notation, in the rest of the lecture, we set $\mathcal{C}_r(t_0,x_0)=(t_0-r^2,t_0)\times B(x_0,r)$ for every $r>0$ and $(t_0,x_0)\in\mathbb{R}^{d+1}$. We also set $[\zeta]_{\alpha,d}=\sup_{t\in I}[\zeta(t,\cdot)]_{C_b^\alpha(D)}+\sup_{x\in D}[\zeta(\cdot,x)]_{C_b^{\alpha/2}(I)}$ for every $\zeta\in C_b^{\alpha/2,\alpha}(I\times D)$, $I\times D$ being either the whole $\mathbb{R}^{d+1}$ or the set $(-\infty,T]\times\overline{\mathbb{R}^d_+}$.

Theorem 11.1.2. *There exists a positive constant C such that*

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})}\le C(\|D_tu-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}(\mathbb{R}^{d+1})}+\|u\|_\infty),\tag{11.3}$$

for every $u\in C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})$.

Proof. Being rather long, we split the proof into some steps. Throughout the proof, C denotes a positive constant, independent of u, which may vary from line to line.

Step 1. Here, we show that

$$[D_t u]_{\alpha,d} + \sum_{i,j=1}^{d} [D_{ij} u]_{\alpha,d} \le C \sup_{(t,x)\in\mathbb{R}^{d+1}} \sup_{r>0} r^{-2-\alpha} \inf_{p\in \mathrm{Pol}_{1,2}} \|u-p\|_{C(\overline{\mathcal{C}_r(t,x)})} =: C[u]_{1+\alpha,d},$$

for every $u \in C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})$, where $\mathrm{Pol}_{1,2}$ denotes the set of all the polynomials in (t,x) which are of at most first-order in t and at most second-order in x. To ease the notation a bit more, we set $[D_x^2 w]_{\alpha,d} = \sum_{i,j=1}^{d} [D_{ij} w]_{\alpha,d}$.

We fix $u \in C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})$ and begin by estimating $[D_{ij}u]_{\alpha,d}$ for arbitrarily fixed $i,j \in \{1,\ldots,d\}$. For every $h>0$ and $(t,x) \in \mathbb{R}^{d+1}$ we set

$$R_h u(t,x) = \frac{1}{h^2}[u(t,x+h(e_i+e_j)) - u(t,x+he_j) - u(t,x+he_i) + u(t,x)] =: \frac{1}{h^2}\zeta_{t,x}(h).$$

The function $\zeta_{t,x}$ belongs to $C_b^{2+\alpha}([0,\infty))$ for every $(t,x) \in \mathbb{R}^{d+1}$. Moreover, $\zeta_{t,x}(0) = \zeta'_{t,x}(0) = 0$ and

$$\begin{aligned}
&|\zeta''_{t,x}(h) - 2D_{ij}u(t,x)| \\
\le&|D_{ii}u(t,x+h(e_i+e_j)) - D_{ii}u(t,x+he_i)| + |D_{jj}u(t,x+h(e_i+e_j)) - D_{jj}u(t,x+he_j)| \\
&+ 2|D_{ij}u(t,x+h(e_i+e_j)) - D_{ij}u(t,x)| \\
\le&4h^\alpha [D_x^2 u]_{\alpha,d}.
\end{aligned}$$

Hence, expanding $\zeta_{t,x}$ by Taylor's formula centered at zero, we can estimate

$$\begin{aligned}
|R_h u(t,x) - D_{ij}u(t,x)| =& \left| h^{-2}\left(\zeta_{t,x}(0) + \zeta'_{t,x}(0)h + \frac{1}{2}\zeta''_{t,x}(h')h^2\right) - D_{ij}u(t,x)\right| \\
=&\frac{1}{2}|\zeta''_{t,x}(h')h^2 - D_{ij}u(t,x)| \le 2h^\alpha [D_x^2 u]_{\alpha,d}
\end{aligned} \tag{11.4}$$

for each $h>0$ and $(t,x) \in \mathbb{R}^{d+1}$, with some $h' \in (0,h)$.

Using (11.4) we can estimate $[D_x^2 u]_{\alpha,d}$. Fix (t,x_1) and (t,x_2) in $\mathbb{R}^{d+1}$ and set $r = |x_2 - x_1|$. As it is immediately seen, the points (t,x_k), $(t,x_k+h(e_i+e_j))$, (t,x_k+he_i) and (t,x_k+he_j) belong to $\overline{\mathcal{C}_{3r}(t,x_k)}$, for $k=1,2$, if $|h| \le r$. Noting that if $p \in \mathrm{Pol}_{1,2}$, then $R_h p$ is constant and coincides with the coefficients of the monomial $x_i x_j$, we can estimate

$$\begin{aligned}
&|D_{ij}u(t,x_2) - D_{ij}u(t,x_1)| \\
\le&\sum_{k=1}^{2} |D_{ij}u(t,x_k) - R_h u(t,x_k)| + |[R_h(u-p)](t_2,x_2) - [R_h(u-p)](t_1,x_1)| \\
\le&4h^\alpha [D_x^2 u]_{\alpha,d} + 4h^{-2}\|u-p\|_{C(\overline{\mathcal{C}_{3r}(t,x_1)})} + 4h^{-2}\|u-p\|_{C(\overline{\mathcal{C}_{3r}(t,x_2)})}
\end{aligned}$$

for each $p \in \mathrm{Pol}_{1,2}$. Minimizing with respect to p, gives $|D_{ij}u(t,x_2) - D_{ij}u(t,x_1)| \leq 4h^\alpha[D^2_x u]_{\alpha,d} + 8h^{-2}(3r)^{2+\alpha}[u]_{1+\alpha,d}$ for every $h > 0$. We now choose $h = \gamma r$, where $4\gamma^\alpha = 1/4$ to obtain

$$|D_{ij}u(t,x_2) - D_{ij}u(t,x_1)| \leq \left(\frac{1}{4}[D^2_x u]_{\alpha,d} + C[u]_{1+\alpha,d}\right)|x_2 - x_1|^\alpha. \tag{11.5}$$

Similarly, if we take (t_1,x), (t_2,x) in $\mathbb{R}^{d+1}$ with $t_1 < t_2$, set $r = \sqrt{t_2 - t_1}$ and argue as above, then we conclude that

$$|D_{ij}u(t_2,x) - D_{ij}u(t_2,x)| \leq \left(\frac{1}{4}[D^2_x u]_{\alpha,d} + C[u]_{1+\alpha,d}\right)|t_2 - t_1|^{\frac{\alpha}{2}}. \tag{11.6}$$

Estimates (11.5) and (11.6) yield $[D_{ij}u]_{\alpha,d} \leq 2^{-1}[D^2_x u]_{\alpha,d} + C[u]_{1+\alpha,d}$ and summing with respect to i,j we finally get $[D^2_x u]_{\alpha,d} \leq C[u]_{1+\alpha,d}$.

To estimate $[D_t u]_{\alpha,d}$ we replace the differential quotient $R_h u$ with the differential quotient $S_h u = h^{-2}[u(\cdot - h^2, \cdot) - u]$. Since $\|S_h u - D_t u\|_\infty \leq h^\alpha \|D_t u\|_{C^{\alpha/2,0}_b(\mathbb{R}^{d+1})}$ for every $h > 0$ and $S_h p$ coincides with the coefficients of the monomial t, we can repeat the same arguments used to estimate $D_{ij}u$ and conclude that $[D_t u]_{\alpha,d} \leq C[u]_{1+\alpha,d}$.

Step 2. Here, we prove that $[u]_{1+\alpha,d} \leq C[D_t u - \Delta u]_{\alpha,d}$ for every $u \in C^{1+\alpha/2,2+\alpha}_b(\mathbb{R}^{d+1})$ with compact support on $\mathbb{R}^{d+1}$. We fix u as above, $(t_0,x_0) \in \mathbb{R}^{d+1}$, $r > 0$ and consider the polynomial p_u defined by

$$\begin{aligned} p_u(t,x) =& u(t_0,x_0) + D_t u(t_0,x_0)(t - t_0) + \langle \nabla_x u(t_0,x_0), x - x_0\rangle \\ &+ \frac{1}{2}\langle D^2_x u(t_0,x_0)(x - x_0), (x - x_0)\rangle \end{aligned}$$

for every $(t,x) \in \mathbb{R}^{d+1}$. Clearly, $D_t p_u - \Delta p_u = D_t u(t_0,x_0) - \Delta u(t_0,x_0) =: g(t_0,x_0)$ on $\mathbb{R}^{d+1}$. Next, we fix a function $\zeta \in C^\infty_c(\mathbb{R}^{d+1})$, $\zeta = 1$ on $\mathcal{C}_{r_M}(t_0,x_0)$, where $r_M = (M+1)r$, with $M > 1$ to be properly chosen later on. The function $u - \zeta p_u$ is smooth, compactly supported on $\mathbb{R}^{d+1}$ and $D_t(u - \zeta p_u) - \Delta(u - \zeta p_u) = g - (D_t(\zeta p_u) - \Delta(\zeta p_u)) =: g - g_1$. Therefore, $u - \zeta p_u$ is the convolution in $\mathbb{R}^{d+1}$ of the functions $g - g_1$ and K, where $K(t,x) = e^{-\frac{|x|^2}{4t}}\chi_{(0,\infty)}(t)$ for any $(t,x) \in \mathbb{R}^{d+1}$, see Exercise 11.5.2. In particular, $u - p_u = K \star (g - g_1)$ on $\mathcal{C}_{r_K}(t_0,x_0)$. We split $K \star (g - g_1)$ into the sum of the integral over $\mathcal{C}_{r_M}(t_0,x_0)$ and over its complement on $\mathbb{R}^{d+1}$. The so obtained functions are denoted by z_1 and z_2 respectively. Since $g_1 = g(t_0,x_0)$ on $\mathcal{C}_{r_M}(t_0,x_0)$, we can estimate $|z_1| \leq r^\alpha_M [g]_{\alpha,d} K \star \chi_{\mathcal{C}_{r_M}(t_0,x_0)}$ on $\mathcal{C}_{r_M}(t_0,x_0)$. Performing the change of variables $(s',y') = (r^{-2}_M(s - t_0), r^{-1}_M(y - x_0))$ gives

$$\begin{aligned} |(K \star \chi_{\mathcal{C}_{r_M}(t_0,x_0)})(t,x)| =& r^{d+2}_M \left| \int_{R^{d+1}} K(t - t_0 - r^2_M s', x - x_0 - r_M y')\chi_{\mathcal{C}_1((0,0))}(s',y')dy'ds' \right| \\ =& r^2_M |(K \star \chi_{\mathcal{C}_1(0,0)})(r^{-2}_M(t - t_0), r^{-1}_M(x - x_0))| \leq r^2_M \end{aligned}$$

for every $(t,x) \in \mathbb{R}^{d+1}$. Hence, $\|z_1\|_{C(\overline{\mathcal{C}_{r_M}(t_0,x_0)})} \leq r^{2+\alpha}_M [g]_{\alpha,d}$.

As far as z_2 is concerned, we observe that it belongs to $C^\infty(\mathcal{C}_{r_M}(t_0,x_0))$ and $D_tz_2-\Delta z_2$ identically vanishes on $\mathcal{C}_{r_M}(t_0,x_0)$. Moreover, twice using Taylor's formula we can write

$$\begin{aligned}z_2(t,x) =&z_2(t_0,x)+D_tz_2(t_*,x)(t-t_0)\\ =&z_2(t_0,x_0)+\langle\nabla z_2(t_0,x_0),x-x_0\rangle+\frac{1}{2}\langle D_x^2z_2(t_0,x_*)(x-x_0),x-x_0\rangle\\ &+D_tz_2(t_*,x)(t-t_0)\end{aligned}$$

for every $(t,x)\in\mathcal{C}_{r_M}(t_0,x_0)$ and some suitable point $(t_*,x_*)\in\mathcal{C}_{r_M}(t_0,x_0)$, depending on (t,x). Hence, if we denote by p_{z_2} the second-order polynomial defined as p_u with u being replaced by z_2, then we can estimate

$$\begin{aligned}&|z_2(t,x)-p_{z_2}(t,x)|\\ \le&|D_tz_2(t_*,x)-D_tz_2(t_0,x_0)||t-t_0|+\frac{1}{2}\bigg|\sum_{i,j=1}^d(D_{ij}z_2(t_0,x_*)-D_{ij}z_2(t_0,x_0))(x-x_0)_i(x-x_0)_j\bigg|\\ \le&r^2|D_tz_2(t_*,x)-D_tz_2(t_0,x)|+r^2|D_tz_2(t_0,x)-D_tz_2(t_0,x_0)|+r^3\sum_{i,j,h=1}^d\|D_{ijh}z_2\|_{C(\overline{\mathcal{C}_r(t_0,x_0)})}\\ \le&r^4\|D_t^2z_2\|_{C(\overline{\mathcal{C}_r(t_0,x_0)})}+r^3\sum_{j=1}^d\|D_{jt}z_2\|_{C(\overline{\mathcal{C}_r(t_0,x_0)})}+r^3\sum_{i,j,h=1}^d\|D_{ijh}z_2\|_{C(\overline{\mathcal{C}_r(t_0,x_0)})}\end{aligned}\tag{11.7}$$

for every $(t,x)\in\mathcal{C}_r(t_0,x_0)$. Since $(t_1,x_1)+\overline{\mathcal{C}_{Mr}(0,0)}\subset\mathcal{C}_{r_M}(t_0,x_0)$ for every $(t_1,x_1)\in\mathcal{C}_r(t_0,x_0)$ we can apply Exercise 11.5.4 to the function $z_2(\cdot+t_1,\cdot+x_1)$ and conclude that

$$M^2r^2|D_{ij}z_2(t_1,x_1)|+M^3r^3|D_{ijh}z_2(t_1,x_1)|+M^4r^4|D_{ijhk}z_2(t_1,x_1)|\le\Lambda\|z_2\|_{C(\overline{\mathcal{C}_{r_M}(t_0,x_0)})}$$

for every $i,j,h,k=1,\dots,d$ and some positive constant Λ, independent of z_2 and (t_1,x_1). Since $D_{jt}z_2(t_1,x_1)=D_j\Delta z_2(t_1,x_1)$ and $D_t^2z_2(t_1,x_1)=\sum_{i,j=1}^dD_{iijj}z_2(t_1,x_1)$, from the previous estimate, we conclude that

$$M^3r^3|D_{jt}z_2(t_1,x_1)|\le d\Lambda\|z_2\|_{C(\overline{\mathcal{C}_{r_M}(t_0,x_0)})},\qquad M^4r^4|D_t^2z_2(t_1,x_1)|\le d^2\Lambda\|z_2\|_{C(\overline{\mathcal{C}_{r_M}(t_0,x_0)})}.$$

Thus, we can continue estimate (11.7) obtaining that $|z_2(t,x)-p_{z_2}(t,x)|\le C(M^{-3}+M^{-4})\|z_2\|_{C(\overline{\mathcal{C}_{r_M}(t_0,x_0)})}$ for every $(t,x)\in\mathcal{C}_r(t_0,x_0)$.

Let us estimate $\|z_2\|_{C(\overline{\mathcal{C}_{r_M}(t_0,x_0)})}=\|u-p_u-z_1\|_{C(\overline{\mathcal{C}_{r_M}(t_0,x_0)})}$. Since $\|u-p_u\|_{C(\overline{\mathcal{C}_{r_M}(t_0,x_0)})}\le Cr_M^{2+\alpha}([D_tu]_{\alpha,d}+[D_x^2u]_{\alpha,d})$, taking the above estimate of z_1 and Step 1 into account we get

$$\begin{aligned}\|z_2\|_{C(\overline{\mathcal{C}_{r_M}(t_0,x_0)})}\le&C(M+1)^{2+\alpha}r^{2+\alpha}([g]_{\alpha,d}+[D_tu]_{\alpha,d}+[D_x^2u]_{\alpha,d})\\ \le&CM^{2+\alpha}r^{2+\alpha}([g]_{\alpha,d}+[u]_{1+\alpha,d}).\end{aligned}$$

Summing up, we have proved that $\|z_2-p_{z_2}\|_{C(\overline{\mathcal{C}_r(t_0,x_0)})}\le CM^{\alpha-1}r^{2+\alpha}([g]_{\alpha,d}+[u]_{1+\alpha,d})$.

Now, we are done. Indeed,

$$\begin{aligned}[u]_{1+\alpha,d} \leq & \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}} \sup_{r>0} r^{-2-\alpha}\|u-p_u-p_{z_1}\|_{C(\overline{\mathcal{C}_r(t_0,x_0)})} \\ \leq & \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}} \sup_{r>0} r^{-2-\alpha}\|z_1\|_{C(\overline{\mathcal{C}_r(t_0,x_0)})} + \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}} \sup_{r>0} r^{-2-\alpha}\|z_2-p_{z_2}\|_{C(\overline{\mathcal{C}_r(t_0,x_0)})} \\ \leq & C(M^{2+\alpha}[g]_{\alpha,d}+K^{\alpha-1}[u]_{1+\alpha,d}).\end{aligned}$$

Taking M large enough, we conclude that $[u]_{1+\alpha,d} \leq C[g]_{\alpha,d}$.

Step 3. By Steps 1, 2, $[D_tu]_{\alpha,d}+[D^2_xu]_{\alpha,d} \leq C[D_tu-\mathcal{A}u]_{\alpha,d}$ for each $u \in C^{1+\alpha/2,2+\alpha}_b(\mathbb{R}^{d+1})$ with compact support. We now remove the condition on the support of u. For this purpose, we fix $u \in C^{1+\alpha/2,2+\alpha}_b(\mathbb{R}^{d+1})$ and consider a sequence $(\vartheta_n) \subset C^\infty_c(\mathbb{R}^{d+1})$ such that $\chi_{[-n,n]\times\overline{B(0,n)}} \leq \vartheta_n \leq 1$ on $\mathbb{R}^{d+1}$ and $[D_t\vartheta_n]_\alpha + [D^2_x\vartheta_n]_\alpha$ vanishes as n tend to ∞. The function $u_n = u\vartheta_n$ satisfies the estimate $[D_tu_n]_{\alpha,d}+[D^2_xu_n]_{\alpha,d} \leq C[D_tu_n-\mathcal{A}u_n]_{\alpha,d}$ for every $n \in \mathbb{N}$. Since $u_n = u$ on $[-n,n] \times \overline{B(0,n)}$, a straightforward computation shows that

$$\begin{aligned}&\sup_{x\in\overline{B(0,n)}}[D_tu(\cdot,x)]_{C^{\alpha/2}([-n,n])} + \sum_{i,j=1}^d \sup_{x\in\overline{B(0,n)}}[D_{ij}u(\cdot,x)]_{C^{\alpha/2}([-n,n])} \\ &+ \sup_{t\in[-n,n]}[D_tu(t,\cdot)]_{C^\alpha(\overline{B(0,n)})} + \sum_{i,j=1}^d \sup_{t\in[-n,n]}[D_{ij}u(t,\cdot)]_{C^\alpha(\overline{B(0,n)})} \leq C[D_tu-\mathcal{A}u]_{\alpha,d}+a_n\end{aligned}$$

for each $n \in \mathbb{N}$ and some sequence (a_n) converging to 0. Letting n tend to ∞, we conclude that $[D_tu]_{\alpha,d}+[D^2_xu]_{\alpha,d} \leq C[D_tu-\mathcal{A}u]_{\alpha,d}$. Note that this estimate can be straightforwardly extended to functions u taking values in $\mathbb{C}$: it is enough to apply it to the real and imaginary parts of u.

Step 4. Here, we prove estimate (11.3) when $\mathcal{A} = \Delta$. For this purpose, we use a trick commonly used in the analysis of elliptic equations on L^p spaces, which consists of adding a new variable. More precisely, for every $u \in C^{1+\alpha/2,2+\alpha}_b(\mathbb{R}^{d+1})$ we consider the function $v : \mathbb{R}^{d+2} \to \mathbb{C}$ defined by $v(t,x,x_{d+1}) = u(t,x)e^{ix_{d+1}}$. By Step 3, we know that $[D_tv]_{\alpha,d+1}+[D^2_xv]_{\alpha,d+1} \leq C[D_tv-\Delta v]_{\alpha,d+1}$, where Δ is now the Laplacian on $\mathbb{R}^{d+1}$. As it is immediately seen, $D_tv(\cdot,\cdot,x_{d+1})-\Delta v(\cdot,\cdot,x_{d+1}) = e^{ix_{d+1}}(D_tu-\Delta u+u)$ for every $x_{d+1} \in \mathbb{R}$. Hence, $[D_tv-\Delta v]_{\alpha,d+1} \leq C\|D_tu-\Delta u-u\|_{C^{\alpha/2,\alpha}_b(\mathbb{R}^{d+1})}$.

We now observe that

$$2|D_{ij}u(t,x)| = |D_{ij}v(t,x,\pi)-D_{ij}v(t,x,0)| \leq \pi^\alpha[D_{ij}v]_{\alpha,d+1}, \qquad (t,x) \in \mathbb{R}^{d+1}$$

for every $i,j = 1,\dots,d$, so that $\|D_{ij}u\|_\infty \leq 2^{-1}\pi^\alpha[D_{ij}v]_{\alpha,d+1}$. Similarly, $\|D_tu\|_\infty \leq 2^{-1}\pi^\alpha[D_tv]_{\alpha,d+1}$ and, consequently,

$$\|D_tu\|_\infty + \sum_{i,j=1}^d \|D_{ij}u\|_\infty \leq 2^{-1}\pi^\alpha([D_tv]_{\alpha,d+1}+[D^2_xv]_{\alpha,d+1}).$$

Finally, notice that $[D_{ij}v(\cdot,\cdot,0)]_{\alpha,d} = [D_{ij}u]_{\alpha,d}$ for every $i,j = 1,\dots,d$ and $[D_t v(\cdot,\cdot,0)]_{\alpha,d} = [D_t u]_{\alpha,d}$. From these estimates and Lemma 11.1.1 we deduce that

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})} \le C([D_t v]_{\alpha,d+1} + [D_x^2 v]_{\alpha,d+1}) \le C\|D_t u - \Delta u + u\|_{C_b^{\alpha/2,\alpha}(\mathbb{R}^{d+1})}. \tag{11.8}$$

Using (11.8) and the interpolation estimate $\|u\|_{C_b^{\alpha/2,2+\alpha}(\mathbb{R}^{d+1})} \le \varepsilon\|u\|_{C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})} + C_\varepsilon\|u\|_\infty$ we obtain

$$\begin{aligned}\|u\|_{C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})} \le & C(\|D_t u - \Delta u\|_{C_b^{\alpha/2,\alpha}(\mathbb{R}^{d+1})} + \|u\|_{C_b^{\alpha/2,\alpha}(\mathbb{R}^{d+1})}) \\ \le & C(\|D_t u - \Delta u\|_{C_b^{\alpha/2,\alpha}(\mathbb{R}^{d+1})} + \varepsilon\|u\|_{C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})} + C_\varepsilon\|u\|_\infty)\end{aligned}$$

for every $\varepsilon > 0$ and some positive constant C_ε, independent of u and blowing up as $\varepsilon \to 0^+$. Taking ε sufficiently small, we immediately get (11.3).

Step 5. To extend (11.3) to every operator $\mathcal{A}$, one can use the same arguments as in Lecture 7 first considering the case when $\mathcal{A} = \mathrm{Tr}(QD^2)$ for some constant and positive definite matrix Q and then freezing the coefficients to handle the more general case. Since there are no sensible differences with the proof of Theorem 7.2.1 we omit the details. □

To complete this section we prove the following property, which will be used in the proof of Theorem 11.2.1.

Lemma 11.1.3. *There exists a positive constant C such that*

$$[D_{id}u]_{\alpha,d} \le C\left([D_t u]_{\alpha,d} + \sum_{i,j=1}^{d-1}[D_{ij}u]_{\alpha,d} + [D_{dd}u]_{\alpha,d}\right)$$

for every $u \in C^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ and $i = 1,\dots,d$.

Proof. Throughout the proof, we denote by C a positive constant, which is independent of u and may vary from line to line. Moreover, we set $\|\cdot\|_\infty = \|\cdot\|_{C_b((-\infty,T]\times\overline{\mathbb{R}^d_+})}$. We split the proof into two steps.

Step 1. Let $\Delta_0^{(h)}$ and $\Delta_i^{(h)}$ be the operators defined by $(\Delta_0^{(h)}\zeta)(t,x) = \zeta(t,x) - \zeta(t-h^2,x)$ and $(\Delta_i^{(h)}\zeta)(t,x) = \zeta(t,x+he_i) - \zeta(t,x)$ for every $(t,x) \in (-\infty,T]\times\overline{\mathbb{R}^d_+}$, $h > 0$, and every function $\zeta : (-\infty,T]\times\overline{\mathbb{R}^d_+} \to \mathbb{R}$. Using the previous operators we can define some new operators that will be used in the proof. More precisely, for every $i,j,k \in \{1,\dots,d\}$ and $h \ge 0$ we set

$$\Delta_{i,j}^{(h)} = \Delta_i^{(h)} \circ \Delta_j^{(h)}, \qquad \Delta_{i,j,k}^{(h)} = \Delta_{i,j}^{(h)} \circ \Delta_k^{(h)}, \qquad \Delta_{0,j}^{(h)} = \Delta_0^{(h)} \circ \Delta_j^{(h)}.$$

As it is immediately seen, the operator $\Delta_{ijk}^{(h)}$ is invariant with respect to each perturbation of the triplet (i,j,k). Moreover,

$$\Delta_{i,j,k}^{(h)}u = \int_0^h dr\int_0^h [D_{ij}u(\cdot,\cdot+re_i+se_j+he_k) - D_{ij}u(\cdot,\cdot+re_i+se_j)]ds$$

for every $i,j,k=1,\dots,d$ and $h>0$ and $u\in C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$. Therefore,

$$\|\Delta^{(h)}_{i,j,k}u\|_\infty\le \sup_{t\le T}[D_{ij}u(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}h^{2+\alpha}\tag{11.9}$$

for every $h>0$, $i,j,k=1,\dots,d$. Arguing similarly, it can be easily shown that

$$\|\Delta^{(h)}_{0,j}u\|_\infty\le \sup_{t\le T}[D_tu(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}h^{2+\alpha},\tag{11.10}$$

$$\|D_{ij}u-h^{-2}\Delta^{(h)}_{i,j}u\|_\infty\le 2\sup_{t\le T}[D_{ij}u(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}h^{\alpha},\tag{11.11}$$

for every $h>0$, $i,j=1,\dots,d$ and $u\in C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$.

Step 2. Here, we set

$$[u]'_{1+\alpha}:=\sup_{h>0}h^{-2-\alpha}\bigg(\|\Delta^{(h)}_{0,i}u\|_\infty+\sum_{k=1}^d\|\Delta^{(h)}_{i,d,k}u\|_\infty\bigg)$$

for every $u\in C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ and prove that $[D_{id}u]_{\alpha,d}\le C[u]'_{1+\alpha}$ for some positive constant, independent of u (and of T) and every $i=1,\dots,d$. For this purpose, we fix $i,k=1,\dots,d$, $n\in\mathbb{N}$ and split

$$\begin{aligned}\Delta^{(h)}_kD_{id}u=&D_{id}u(\cdot,\cdot+he_k)-h^{-2}n^2\Delta^{(h/n)}_{i,d}u(\cdot,\cdot+he_k)+h^{-2}n^2\Delta^{(h/n)}_{i,d}u-D_{id}u\\&+h^{-2}n^2\sum_{r=1}^n(\Delta^{(h/n)}_{i,d}u(\cdot,\cdot+rhn^{-1}e_k)-\Delta^{(h/n)}_{i,d}u(\cdot,\cdot+(r-1)hn^{-1}e_k).\end{aligned}$$

Since $\Delta^{(h/n)}_{i,d}u(\cdot,\cdot+rhn^{-1}e_k)-\Delta^{(h/n)}_{i,d}u(\cdot,\cdot+(r-1)hn^{-1}e_k)=\Delta^{(h/n)}_{i,d,k}u(\cdot,\cdot+(r-1)hn^{-1}e_k)$, taking (11.9) and (11.11) into account, we can estimate

$$\begin{aligned}\|\Delta^{(h)}_kD_{id}u\|_\infty\le&4n^{-\alpha}h^\alpha\sup_{t\le T}[D_{id}u(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}+h^{-2}n^3\|\Delta^{(h/n)}_{i,d,k}u\|_\infty\\\le&4n^{-\alpha}h^\alpha\sup_{t\le T}[D_{id}u(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}+h^\alpha n^{1-\alpha}[u]'_{1+\alpha}\end{aligned}$$

for every $h>0$. This shows that, if $y,x\in\overline{\mathbb{R}^d_+}$ are such that $y-x=(y_k-x_k)e_k$ for some $k=1,\dots,d$, then

$$|D_{id}u(t,y)-D_{id}u(t,x)|\le\bigg(4n^{-\alpha}\sup_{t\le T}[D_{id}u(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}+n^{1-\alpha}[u]'_{1+\alpha}\bigg)|x-y|^\alpha,$$

for every $t\le T$. Now, for every $x,y\in\overline{\mathbb{R}^d_+}$ and $k=1,\dots,d$, we set $x^{(0)}=x$, $x^{(k)}=x^{(k-1)}+(y_k-x_k)e_k$ and split $D_{id}u(t,y)-D_{id}u(t,x)=\sum_{k=1}^d(D_{id}u(t,x^{(k)})-D_{id}u(t,x^{(k-1)}))$ and conclude that

$$|D_{id}u(t,y)-D_{id}u(t,x)|\le\bigg(4dn^{-\alpha}\sup_{t\le T}[D_{id}u(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}+dn^{1-\alpha}[u]'_{1+\alpha}\bigg)|x-y|^\alpha$$

so that, taking n sufficiently large, it follows that $\sup_{t\le T}[D_{id}u(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}\le C[u]'_{1+\alpha}$.

To estimate the $\alpha/2$-Hölder seminorms of $D_{id}u$, we fix $s<t\le T$ and set $h=\sqrt{t-s}$. Then, as above we get

$$\begin{aligned}|D_{id}u(t,\cdot)-D_{id}u(s,\cdot)|\le&|D_{id}u(t,\cdot)-h^{-2}\Delta^{(h)}_{i,d}u(t,\cdot)|+|D_{id}u(s,\cdot)-h^{-2}\Delta^{(h)}_{i,d}u(s,\cdot)|\\&+h^{-2}|(\Delta^{(h)}_0\circ\Delta^{(h)}_{i,d}u)(t,\cdot)|\\\le&2\sup_{t\le T}[D_{id}u(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}|t-s|^{\frac{\alpha}{2}}+2h^{-2}\|\Delta^{(h)}_{0,i}u\|_\infty\\\le&2\sup_{t\le T}[D_{id}u(t,\cdot)]_{C^\alpha_b(\overline{\mathbb{R}^d_+})}|t-s|^{\frac{\alpha}{2}}+2|t-s|^{\frac{\alpha}{2}}[u]'_{1+\alpha},\end{aligned}$$

where we have also used the estimate $\|\Delta^h_0\circ\Delta^{(h)}_{i,d}u\|_{C_b((-\infty,T]\times\overline{\mathbb{R}^d_+})}\le 2\|\Delta^h_{0,i}u\|_{C_b((-\infty,T]\times\overline{\mathbb{R}^d_+})}$. We have so proved that $\sup_{x\in\overline{\mathbb{R}^d_+}}[D_{id}u(\cdot,x)]_{C^{\alpha/2}((-\infty,T])}\le C[u]'_{1+\alpha}$. The estimate $[D_{id}u]_{\alpha,d}\le C[u]'_{1+\alpha}$ now follows for every $i=1,\dots,d$.

So, using (11.10) and (11.9), we deduce that

$$\begin{aligned}[D_{id}u]_{\alpha,d}&\le C\sup_{h>0}h^{-2-\alpha}\bigg(\|\Delta^{(h)}_{0,i}u\|_\infty+\sum_{k=1}^d\|\Delta^{(h)}_{i,d,k}u\|_\infty\bigg)\\&\le C\bigg([D_tu]_{\alpha,d}+\sum_{k=1}^d\|\Delta^{(h)}_{i,d,k}u\|_\infty\bigg)\\&=C\bigg([D_tu]_{\alpha,d}+\sum_{k=1}^{d-1}\|\Delta^{(h)}_{i,d,k}u\|_\infty+\|\Delta^{(h)}_{i,d,d}u\|_\infty\bigg)\\&\le C\bigg([D_tu]_{\alpha,d}+\sum_{j=1}^{d-1}[D_{ij}u]_{\alpha,d}+[D_{dd}u]_{\alpha,d}\bigg)\end{aligned}$$

and the assertion follows at once. □

11.2 An auxiliary boundary value problem

In this section we deal with the boundary value problem

$$\left\{\begin{array}{ll}D_tu(t,x)=\mathcal{A}u(t,x)+g(t,x), & t\in(-\infty,T],\quad x\in\mathbb{R}^d_+,\\ u(t,x',0)=\psi(t,x'), & t\in(-\infty,T],\quad x'\in\mathbb{R}^{d-1},\end{array}\right.\tag{11.12}$$

under Hypotheses 10.0.1, with $0>c_0=\sup_{x\in\mathbb{R}^d}c(x)$, and prove the following important result

Theorem 11.2.1. *For each $g\in C^{\alpha/2,\alpha}_b((-\infty,T]\times\overline{\mathbb{R}^d_+})$ and $\psi\in C^{1+\alpha/2,2+\alpha}_b((-\infty,T]\times\mathbb{R}^{d-1})$, problem (11.12) admits a unique solution $u\in C^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ which satisfies the estimate*

$$\|u\|_{C^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\le C_\star(\|g\|_{C^{\alpha/2,\alpha}_b((-\infty,T]\times\overline{\mathbb{R}^d_+})}+\|\psi\|_{C^{1+\alpha/2,2+\alpha}_b((-\infty,T]\times\mathbb{R}^{d-1})})\tag{11.13}$$

for some positive constant $C_\star$, independent of u and g.

The following proposition, yields the uniqueness of the solution to the Cauchy problem (11.12).

Proposition 11.2.2. *Suppose that $u \in C_b^{1+\alpha/2,2+\alpha}((-\infty,T] \times \overline{\mathbb{R}^d_+})$ solves the boundary value problem* (11.12)*. Then,*

$$\|u\|_{C_b((-\infty,T]\times\overline{\mathbb{R}^d_+})} \le |c_0|^{-1}\|g\|_{C_b((-\infty,T]\times\overline{\mathbb{R}^d_+})} + \|\psi\|_{C_b((-\infty,T]\times\mathbb{R}^{d-1})}. \tag{11.14}$$

Proof. We introduce the function $v = u + c_0^{-1}\|g\|_{C_b((-\infty,T]\times\overline{\mathbb{R}^d_+})} - \|\psi\|_{C_b(-\infty,T]\times\mathbb{R}^{d-1}}$, which is nonpositive on $(-\infty,T] \times \partial\mathbb{R}^d_+$ and satisfies the differential equation $D_tv - \mathcal{A}v \le 0$ on $(-\infty,T] \times \mathbb{R}^d_+$. As in (11.1), the function $\varphi_a : \mathbb{R}^d \to \mathbb{R}$, defined by $\varphi_a(x) = a + |x|^2$ for every $x \in \mathbb{R}^d$, satisfies the estimate $\mathcal{A}\varphi_a(x) \le d_1 + d_2|x| + c_0a + c_0|x|^2$ for every $x \in \mathbb{R}^d_+$ and some positive constants d_1 and d_2. Using Young's inequality, we can estimate $d_2|x| \le d_3 - c_0|x|^2$ for every $x \in \mathbb{R}^d_+$, so that we can choose a large enough such that $\mathcal{A}\varphi_a \le 0$ on $\mathbb{R}^d_+$. The function $v_n = v - n^{-1}\varphi_a$ is nonpositive on $(-\infty,T] \times \partial\mathbb{R}^d_+$ and $D_tv_n - \mathcal{A}v_n \le 0$ on $(-\infty,T] \times \overline{\mathbb{R}^d_+}$. Since $\lim_{|x|\to\infty} v_n(t,\cdot) = -\infty$, uniformly with respect to $t \in (-\infty,T]$, it attains its maximum value. The same argument as in the proof of Theorem 1.1.2 shows that the maximum its attained on $(-\infty,T] \times \partial\mathbb{R}^d_+$, so that v_n is nonpositive on $(-\infty,T] \times \overline{\mathbb{R}^d_+}$. Letting n tend to ∞ we deduce that $v \le 0$ on $(-\infty,T] \times \overline{\mathbb{R}^d_+}$, i.e., $u \le |c_0|^{-1}\|g\|_\infty + \|\psi\|_\infty$. Applying the same argument with u being replaced by $-u$, we get also the estimate $-u \le |c_0|^{-1}\|g\|_\infty + \|\psi\|_\infty$ and (11.14) follows. □

The following lemma will be used in the proof of the existence part of Theorem 11.2.1.

Lemma 11.2.3. *Set $K_+ = -2\chi_{\mathbb{R}^d_+}D_dK$, where K is the Gauss-Weierstrass kernel defined by* (4.3)*. Then, the following properties hold true:*

(i) *K_+ is infinitely many times differentiable in $\mathbb{R} \times \mathbb{R}^d_+$;*

(ii) *$D_tK_+ = \Delta K_+$ on $\mathbb{R} \times \mathbb{R}^d_+$;*

(iii) *the function $(t,x') \mapsto t^\gamma|x'|^\beta K_+(t,x',x_d)$ belongs to $L^1(\mathbb{R}^d)$ provided that $1-2\gamma-\beta > 0$. In such a case,*

$$\int_0^\infty dt\int_{\mathbb{R}^{d-1}} t^\gamma|x'|^\beta K_+(t,x',x_d)dx' = C_{\beta,\gamma}x_d^{\beta+2\gamma} \tag{11.15}$$

for each $x_d > 0$, where

$$C_{\beta,\gamma} = 4^\gamma\pi^{-\frac{d}{2}}\Gamma\bigg(\frac{2\gamma-\beta+1}{2}\bigg)\int_{\mathbb{R}^{d-1}}|y'|^\beta e^{-|y'|^2}dy';$$

(iv) *the functions K_+, D_tK_+, D_jK_+, $D_{ij}K_+$ are integrable on $\mathbb{R}^d$ (with respect to t and x') for any $x_d > 0$ and their L^1-norms are bounded on (ε,∞) for every $\varepsilon > 0$.*

Proof. (i) & (ii): By Lemma 4.1.2 we know that $K_+ \in C^\infty((0,\infty)\times\mathbb{R}^d_+)$. Moreover, a direct computation shows that $D_tK_+ = \Delta K_+$ on $(0,\infty)\times\mathbb{R}^d_+$. To conclude the proof, it suffices to observe that all the derivatives of K_+ vanish as (t,x) tends to $(0,x_0)$, for some $x_0>0$. This implies that $K_+\in C^\infty(\mathbb{R}\times\mathbb{R}^d_+)$ and it solves the heat equation in the whole $\mathbb{R}\times\mathbb{R}^d_+$.

(iii) The usual change of variables $y' = (2\sqrt{t})^{-1}x'$ and, then, the change of variable $s = x_d^2/4t$ show that

$$\begin{aligned}\int_0^\infty dt\int_{\mathbb{R}^{d-1}} t^\gamma|x'|^\beta K_+(t,x',x_d)dx' =&\frac{x_d}{(4\pi)^{\frac{d}{2}}}\int_0^\infty t^{\gamma-1-\frac{d}{2}}e^{-\frac{x_d^2}{4t}}dt\int_{\mathbb{R}^{d-1}}|x'|^\beta e^{-\frac{|x'|^2}{4t}}dx'\\ =&\frac{2^{\beta-1}}{\pi^{\frac{d}{2}}}x_d\int_0^\infty t^{\frac{\beta}{2}+\gamma-\frac{3}{2}}e^{-\frac{x_d^2}{4t}}dt\int_{\mathbb{R}^{d-1}}|y'|^\beta e^{-|y'|^2}dy'\\ =&4^\gamma C'_\beta x_d^{\beta+2\gamma}\int_0^\infty s^{-\gamma-\frac{\beta}{2}-\frac{1}{2}}e^{-s}dt.\end{aligned}$$

Hence, the integral is finite provided that $-2\gamma-\beta-1>-2$, i.e., $1-2\gamma-\beta>0$, and (11.15) follows.

(iv) It is a straightforward consequence of (iii). Indeed, an easy computation shows that

$$\begin{aligned}D_hK_+(t,x) =& -\frac{1}{2}t^{-1}x_hK_+(t,x)+\delta_{hd}x_d^{-1}K_+(t,x),\\ D_{ij}K_+(t,x) =& -\frac{1}{2}t^{-1}\delta_{ij}K_+(t,x)+\frac{1}{4}t^{-2}x_ix_jK_+(t,x),\\ D_{id}K_+(t,x) =& -\frac{1}{2}t^{-1}x_ix_d^{-1}K_+(t,x)+\frac{1}{4}t^{-1}x_ix_dK_+(t,x),\\ D_{dd}K_+(t,x) =& -\frac{3}{2}t^{-1}K_+(t,x)+\frac{1}{4}t^{-2}x_d^2K_+(t,x),\end{aligned}\tag{11.16}$$

for each $t>0$, $x\in\mathbb{R}^d_+$, $i,j=1,\ldots,d-1$ and $h=1,\ldots,d$. Hence, applying estimate (11.15) we conclude that the $L^1(\mathbb{R}^d)$-norm of each the first-order (resp. second-order) spatial derivatives of $K_+(\cdot,\cdot,x_d)$ is bounded from above by a constant times x_d^{-1} (resp. x_d^{-2}) for every $x_d>0$. Since $D_tK_+=\Delta K_+$ it follows that the $L^1(\mathbb{R}^d)$ norm of $D_tK_+(\cdot,\cdot,x_d)$ is bounded from above by a constant times x_d^{-2}. The assertion follows. □

Proof of Theorem 11.2.1 *when* $\mathcal{A}=\Delta-I$. Being rather long, we split the proof into two steps.

Step 1. Here, we prove that, for every $\widetilde{g}\in C_b^{\alpha/2,\alpha}((-\infty,T]\times\mathbb{R}^d)$ there exists a function $w\in C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^d)$ such that $D_tw=\Delta w-w+\widetilde{g}$ on $(-\infty,T]\times\mathbb{R}^d$. By setting $\widetilde{g}(t,\cdot)=\widetilde{g}(T,\cdot)$ for any $t>T$, we can extend $\widetilde{g}$ to the whole $\mathbb{R}^{d+1}$ with a function which belongs to $C_b^{\alpha/2,\alpha}(\mathbb{R}^{d+1})$ and satisfies the estimate $\|\widetilde{g}\|_{C_b^{\alpha/2,\alpha}(\mathbb{R}^{d+1})}=\|\widetilde{g}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\mathbb{R}^d)}$. We also consider a sequence $(\vartheta_n)\subset C_c^\infty(\mathbb{R})$, such that $\chi_{(-n,n)}\le\vartheta_n\le\chi_{(-2n,2n)}$ for every $n\in\mathbb{N}$.

For each $n\in\mathbb{N}$ we set

$$w_n(t,x)=\int_{-\infty}^t e^{s-t}(T(t-s)\widetilde{g}_n(s,\cdot))(x)ds,\qquad t\in\mathbb{R},\ x\in\mathbb{R}^d,$$

where $\{T(t)\}$ is the Gauss-Weierstrass semigroup and $\widetilde{g}_n=\vartheta_n\widetilde{g}$. Note that $\|\widetilde{g}_n\|_{C_b^{\alpha/2,\alpha}(\mathbb{R}^{d+1})}\le C_0\|\widetilde{g}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}$ for some positive constant C_0, independent of $\widetilde{g}$ and n. Since $\widetilde{g}_n$ is supported in $(-2n,2n)\times\mathbb{R}^d$, the integral defining function w_n converges for every $t\in\mathbb{R}$. Adapting the arguments in the proof of Theorem 6.1.4, we can easily show that $w_n\in C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})$, solves the equation $D_tw=\Delta w-w+\widetilde{g}_n$ on $\mathbb{R}^{d+1}$. Moreover, by (11.8) it satisfies the estimate $\|w_n\|_{C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})}\le C_1\|\widetilde{g}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\mathbb{R}^d)}$, C_1, being independent of n and $\widetilde{g}$. Thus, applying Arzelà-Ascoli theorem to the sequences (w_n), (D_tw_n), (D_jw_n) and $(D_{ij}w_n)$ $(i,j=1,\dots,d)$, we conclude that, up to a subsequence, w_n converges in $C^{1,2}([-r,r]\times\overline{B(0,r)})$ (for every $r>0$) to a function $w\in C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})$, which solves the equation $D_tw=\Delta w-w+\widetilde{g}$ and satisfies the estimate $\|D_tw\|_{C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})}\le C_2\|\widetilde{g}\|_{C^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}$, the constant C_2 being independent of $\widetilde{g}$.

Step 2. In view of Step 1, $u\in C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ solves the boundary value problem (11.12) if and only if the function $v=u-w\in C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ solves problem

$$\begin{cases} D_tv(t,x)=\Delta v(t,x)-v(t,x), & t\in(-\infty,T],\ \ x\in\mathbb{R}^d_+,\\ v(t,x',0)=\psi(t,x')-w(t,x',0), & t\in(-\infty,T],\ \ x'\in\mathbb{R}^{d-1},\end{cases}\tag{11.17}$$

where w is the solution to the equation $D_tw-\Delta w+w=\widetilde{g}$ on $(-\infty,T]\times\mathbb{R}^d$ provided by Step 1 and $\widetilde{g}$ is the even extension (with respect to the last variable x_d) of g. The function w satisfies the estimate

$$\|w\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^d)}\le C_2\|g\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}.\tag{11.18}$$

To ease the notation, we set $\widehat{\psi}=\psi-w(\cdot,\cdot,0)$. We will prove that the function $v:(-\infty,T]\times\overline{\mathbb{R}^d_+}\to\mathbb{R}$, defined by

$$v(t,x)=\int_{\mathbb{R}}ds\int_{\mathbb{R}^{d-1}}e^{s-t}K_+(t-s,x'-y',x_d)\widehat{\psi}(s,y')dy',\qquad t\in(-\infty,T],\ x\in\overline{\mathbb{R}^d_+},$$

belongs to $C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$, solves the boundary value problem (11.17) and satisfies the estimate

$$\|v\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\le C_3(\|\psi\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}+\|g\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})})\tag{11.19}$$

for some positive constant C_3, independent of v, ψ and g. An easy application of the dominated convergence theorem, as in the proof of Theorem 4.1.3, shows that $v\in C^{1,2}((-\infty,T]\times$

$\overline{\mathbb{R}^d_+}$) and it solves the heat equation. To prove estimate (11.19) and to show that $v(\cdot,\cdot,0)=\widehat{\psi}$ on $(-\infty,T]\times\overline{\mathbb{R}^d_+}$, we perform the change of variables $t-s=x_d^2r$, $x'-y'=x_dz'$, to write v in the much more convenient form

(11.20)

$$v(t,x)=\int_0^\infty e^{-rx_d^2}dr\int_{\mathbb{R}^{d-1}}K_+(r,z',1)\widehat{\psi}(t-rx_d^2,x'-x_dz')dz',\qquad t\in(-\infty,T],\ x\in\overline{\mathbb{R}^d_+}.$$

By applying the dominated convergence theorem, we easily see that

$$\begin{aligned}\lim_{(t,x)\to(t_0,x_0',0)}v(t,x)=&\int_0^\infty dr\int_{\mathbb{R}^{d-1}}K_+(r,z',1)\widehat{\psi}(t_0,x_0')dz'\\ =&\widehat{\psi}(t_0,x_0')\|K_+(\cdot,\cdot,1)\|_{L^1((0,+\infty)\times\mathbb{R}^{d-1})}.\end{aligned}$$

Lemma 11.2.3(iii) shows that $\|K_+(\cdot,\cdot,1)\|_{L^1((0,\infty)\times\mathbb{R}^{d-1})}=1$. Hence, the boundary condition in (11.17) is satisfied.

In view of Lemma 11.1.1, to prove (11.13) it suffices to prove that D_tv, $D_{ij}v$ ($i,j=1,\ldots,d$) belong to $C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ and their $C^{\alpha/2,\alpha}$-norms together with the sup-norm of v satisfy estimate (11.2). Throughout the rest of the proof, we denote by C a positive constant, which depends only on d and α, and may vary from line to line.

As it is immediately seen, from (11.20) it follows that $|v(t,x)|\le\|\widehat{\psi}\|_\infty$ for any $(t,x)\in(-\infty,T]\times\overline{\mathbb{R}^d_+}$.

We now estimate the $C_b^{\alpha/2,\alpha}((-\infty,T]\times\mathbb{R}^d_+)$-norm of D_tv and $D_{ij}v$ for $i,j=1,\ldots,d-1$. Since the arguments are just the same, we show in details how to estimate D_tv. Differentiating under the integral sign, from (11.20) it follows that

$$D_tv(t,x)=\int_0^\infty ds\int_{\mathbb{R}^{d-1}}K_+(s,y',1)e^{-sx_d^2}D_t\widehat{\psi}(t-sx_d^2,x'-x_dy')dy'$$

for each $t\in(-\infty,T]$ and $x\in\overline{\mathbb{R}^d_+}$. From this formula, we immediately deduce that $\|D_tv\|_\infty\le\|D_t\widehat{\psi}\|_\infty$. Moreover, for each $t_1<t_2\le T$ and $x\in\overline{\mathbb{R}^d_+}$ we can estimate

$$\begin{aligned}&|D_tv(t_2,x)-D_tv(t_1,x)|\\ \le&\int_{\mathbb{R}^{d-1}}K_+(s,y',1)e^{-sx_d^2}|D_t\widehat{\psi}(t_2-sx_d^2,x'-x_dy')-D_t\widehat{\psi}(t_1-sx_d^2,x'-x_dy')|dy'\\ \le&\sup_{x\in\mathbb{R}^{d-1}}[D_t\widehat{\psi}(\cdot,x)]_{C_b^{\alpha/2}((-\infty,T])}|t_2-t_1|^{\frac{\alpha}{2}}\int_0^\infty e^{-sx_d^2}ds\int_{\mathbb{R}^{d-1}}K_+(s,y',1)dy'\\ \le&\|D_t\widehat{\psi}\|_{C_b^{\alpha/2,0}((-\infty,T]\times\mathbb{R}^{d-1})}|t_2-t_1|^{\frac{\alpha}{2}},\end{aligned}$$

so that

$$[D_tv(\cdot,x)]_{C^{\alpha/2}((-\infty,T])}\le\|D_t\widehat{\psi}\|_{C^{\alpha/2,0}((-\infty,T]\times\mathbb{R}^{d-1})}\tag{11.21}$$

for any $x\in\overline{\mathbb{R}^d_+}$. Similarly, for each $t\le T$ it holds that

$$[D_tv(t,\cdot,x_d)]_{C_b^\alpha(\mathbb{R}^{d-1})}\le\|D_t\widehat{\psi}\|_{C^{0,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\tag{11.22}$$

for any $t\le T$ and $x_d\ge 0$. To estimate the α-Hölder seminorm of the function $D_tv(t,x',\cdot)$, we observe that

$$
\begin{aligned}
&|D_tv(t,x',x_d)-D_tv(t,x',\widetilde{x}_d)|\\
\le&\int_0^\infty |e^{-sx_d^2}-e^{-s\widetilde{x}_d^2}|ds\int_{\mathbb{R}^{d-1}}K_+(s,y',1)|D_t\widehat{\psi}(t-sx_d^2,x'-x_dy')|dy'\\
&+\int_0^\infty e^{-s\widetilde{x}_d^2}ds\int_{\mathbb{R}^{d-1}}K_+(s,y',1)|D_t\widehat{\psi}(t-sx_d^2,x'-x_dy')-D_t\widehat{\psi}(t-s\widetilde{x}_d^2,x'-\widetilde{x}_dy')|dy'\\
\le&C_\alpha|x_d^2-\widetilde{x}_d^2|^{\frac{\alpha}{2}}\|D_t\widehat{\psi}\|_\infty\int_0^\infty s^{\frac{\alpha}{2}}ds\int_{\mathbb{R}^{d-1}}K_+(s,y',1)dy'\\
&+\sup_{x\in\mathbb{R}^{d-1}}[D_t\widehat{\psi}(\cdot,x)]_{C_b^{\alpha/2}((-\infty,T])}|x_d^2-\widetilde{x}_d^2|^{\frac{\alpha}{2}}\int_0^\infty s^{\frac{\alpha}{2}}e^{-sx_d^2}ds\int_{\mathbb{R}^{d-1}}K_+(s,y',1)dy'\\
&+\sup_{t\le T}[D_t\widehat{\psi}(t,\cdot)]_{C_b^{\alpha}(\mathbb{R}^{d-1})}|x_d-\widetilde{x}_d|^\alpha\int_0^\infty e^{-sx_d^2}ds\int_{\mathbb{R}^{d-1}}|y'|^\alpha K_+(s,y',1)dy'\\
\le&C\|D_t\widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}(|x_d^2-\widetilde{x}_d^2|^{\frac{\alpha}{2}}+|x_d-\widetilde{x}_d|^\alpha)
\end{aligned}
\tag{11.23}
$$

for each $t\le T$, x_d, $\widetilde{x}_d\ge 0$, where we have taken advantage of (11.15).

Suppose that $x_d+\widetilde{x}_d\le 2|x_d-\widetilde{x}_d|$. Then, $|x_d^2-\widetilde{x}_d^2|\le 2|x_d-\widetilde{x}_d|^2$, so that from (11.23) it follows that

$$
|D_tv(t,x',x_d)-D_tv(t,x',\widetilde{x}_d)|\le C\|D_t\widehat{\psi}\|_{C^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}|x_d-\widetilde{x}_d|^\alpha. \tag{11.24}
$$

Formula (11.23) is of no help when $x_d+\widetilde{x}_d>2|x_d-\widetilde{x}_d|$. In such a case, we rewrite D_tv in the form

$$
D_tv(t,x)=\int_{\mathbb{R}}ds\int_{\mathbb{R}^{d-1}}e^{s-t}K_+(t-s,x'-y',x_d)D_t\widehat{\psi}(s,y')dy',\qquad t\le T,\ \ x\in\overline{\mathbb{R}^d_+}.
$$

Differentiating with respect to the variable x_d, taking (11.16) into account and observing that $\int_{\mathbb{R}^{d+1}}D_dK_+(s,y',x_d)dsdy'=0$ for every $x_d>0$, we get

$$
\begin{aligned}
D_dD_tv(t,x)=&\int_{\mathbb{R}}e^{s-t}ds\int_{\mathbb{R}^{d-1}}D_dK_+(t-s,x'-y',x_d)D_t\widehat{\psi}(s,y')dy'\\
=&\int_{\mathbb{R}}e^{s-t}ds\int_{\mathbb{R}^{d-1}}D_dK_+(t-s,x'-y',x_d)(D_t\widehat{\psi}(s,y')-D_t\widehat{\psi}(t,x'))dy'\\
=&-\frac{1}{2}x_d\int_0^\infty e^{-s}ds\int_{\mathbb{R}^{d-1}}s^{-1}K_+(s,y',x_d)(D_t\widehat{\psi}(t-s,x-y')-D_t\widehat{\psi}(t,x'))dy'\\
&+x_d^{-1}\int_0^\infty e^{-s}ds\int_{\mathbb{R}^{d-1}}K_+(s,y',x_d)(D_t\widehat{\psi}(t-s,x'-y')-D_t\widehat{\psi}(t,x'))dy'
\end{aligned}
$$

for every $t\in(-\infty,T]$ and $x\in\overline{\mathbb{R}^d_+}$. Hence, estimating $|D_t\widehat{\psi}(t-s,x'-y')-D_t\widehat{\psi}(t,x')|\le\|D_t\widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}(|s|^{\alpha/2}+|y'|^\alpha)$ we obtain

$$
|D_dD_tv(t,x)|\le\frac{1}{2}x_d\|D_t\widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\int_0^\infty ds\int_{\mathbb{R}^{d-1}}(s^{\frac{\alpha}{2}-1}+s^{-1}|y'|^\alpha)K_+(s,y',x_d)dy'
$$

$$+ x_d^{-1}\|D_t\widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\int_0^\infty ds\int_{\mathbb{R}^{d-1}}(s^{\frac{\alpha}{2}}+|y'|^\alpha)K_+(s,y',x_d)dy'.$$

Taking (11.15) once more into account, we conclude that

$$|D_dD_tv(t,x)|\le C''_\alpha\|D_t\widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}x_d^{\alpha-1},\qquad t\in(-\infty,T],\ x\in\mathbb{R}^d_+. \tag{11.25}$$

Hence, observing that the condition $x_d+\widetilde{x}_d>2|x_d-\widetilde{x}_d|$ implies that $2x_d>|x_d-\widetilde{x}_d|$, we can estimate

$$\begin{aligned}|D_tv(t,x',x_d)-D_tv(t,x',\widetilde{x}_d)|\le&C\|D_t\widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}x_d^{\alpha-1}|x_d-\widetilde{x}_d|\\ \le&C\|D_t\widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}|x_d-\widetilde{x}_d|^\alpha.\end{aligned} \tag{11.26}$$

From (11.24) and (11.26) it follows that

$$[D_tv(t,x',\cdot)]_{C_b^\alpha(\overline{\mathbb{R}_+})}\le C\|D_t\widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\mathbb{R}^{d-1})} \tag{11.27}$$

for every $t\le T$ and $x'\in\mathbb{R}^{d-1}$. So, (11.21), (11.22) and (11.27) imply that $D_tv\in C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ and $\|D_tv\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\le C\|\widehat{\psi}\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}$.

Since $D_tv=\Delta v-v$, then $D_{dd}v=D_tv-\sum_{j=1}^{d-1}D_{jj}v+v$ and we conclude that $D_{dd}v\in C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ and $\|D_{dd}v\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\le C\|\widehat{\psi}\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}$. Estimating the parabolic Hölder norm of second-order derivatives $D_{id}v$ $(i=1,\dots,d)$ trying to follow the same technique as above is not straightforward. To overcome this difficulty, we take advantage of Lemma 11.1.3. First of all, we observe that

$$D_{id}v(t,x)=\int_0^\infty ds\int_{\mathbb{R}^{d-1}}D_dK_+(s,y',x_d)D_i\widehat{\psi}(t-s,x'-y')dy',\qquad(t,x)\in(-\infty,T]\times\mathbb{R}^d_+.$$

Since $D_i\widehat{\psi}\in C_b^{\alpha/2,\alpha}((-\infty]\times\mathbb{R}^{d-1})$ and $\|D_i\widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty]\times\mathbb{R}^{d-1})}\le C\|\widehat{\psi}\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty]\times\mathbb{R}^{d-1})}$ as in the proof of (11.25), we can show that $|D_{id}v(t,x)|\le C\|\widehat{\psi}\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty]\times\mathbb{R}^{d-1})}x_d^{-1}$ for every $(t,x)\in(-\infty,T]\times\mathbb{R}^d_+$. It can also be checked that $D_{id}v\in C^{\alpha/2,\alpha}((-\infty,T]\times\mathbb{R}^{d-1}\times[\varepsilon,\infty))$ for every $\varepsilon>0$. As a byproduct, the function $v_\varepsilon=v(\cdot,\cdot,\cdot+\varepsilon)$ belongs to $C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ for every $\varepsilon>0$ and we can estimate, using Lemma 11.1.3,

$$\begin{aligned}&[D_{id}v_\varepsilon]_{\alpha,d}:=\sup_{t\le T}[D_{id}v_\varepsilon(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})}+\sup_{x\in\overline{\mathbb{R}^d_+}}[D_{id}v_\varepsilon(\cdot,x)]_{C^{\alpha/2}((-\infty,T])}\\ \le&C\bigg(\|D_tv_\varepsilon\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}+\sum_{i,j=1}^{d-1}\|D_{ij}v_\varepsilon\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}+\|D_{dd}v_\varepsilon\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\bigg)\\ \le&C\bigg(\|D_tv\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}+\sum_{i,j=1}^{d-1}\|D_{ij}v\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}+\|D_{dd}v\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\bigg)\end{aligned}$$

$$\leq C\|\widehat{\psi}\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}.$$

Letting ε tend to 0 gives $[D_{id}v]_\alpha \leq C\|\widehat{\psi}\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}$. Estimate (11.19) follows immediately.

To conclude the proof, it suffices to observe that (11.18) and (11.19) yield estimate (11.13). □

As a consequence of Theorem 11.2.1, we can now prove the following *a priori* estimates

Theorem 11.2.4. *There exists a positive constant C_0, depending only on the Hölder norms of the coefficients and the ellipticity constant μ in Hypothesis* 10.0.1*, such that*

$$\begin{aligned}\|u\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})} \leq &C_0(1-c_0^{-1})\|D_tu-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}\\ &+C_0\|u(\cdot,\cdot,0)\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}\end{aligned} \tag{11.28}$$

Proof. The weaker estimate

$$\begin{aligned}\|u\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})} \leq C_0(&\|D_tu-\mathcal{A}u\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})}+\|u\|_{C_b((-\infty,T]\times\overline{\mathbb{R}^d_+})}\\ &+\|u(\cdot,\cdot,0)\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})})\end{aligned} \tag{11.29}$$

can be obtained arguing as in Lecture 7, first considering the case when $\mathcal{A}=\mathrm{Tr}(QD^2)$ for some constant and positive definite matrix Q and then freezing the coefficients to handle the more general case. Since there are no sensible differences with the proof of Theorem 7.2.1 we omit the details.

Noting that u trivially solves the Cauchy problem (11.12) with $g = D_tu - \mathcal{A}u$ and $\psi = u(\cdot,\cdot,0)$, from (11.14) it follows that

$$\|u\|_{C_b((-\infty,T]\times\overline{\mathbb{R}^d_+})} \leq |c_0|^{-1}\|D_tu-\mathcal{A}u\|_{C_b((-\infty,T]\times\overline{\mathbb{R}^d_+})}+\|u(\cdot,\cdot,0)\|_{C_b((-\infty,T]\times\mathbb{R}^{d-1})}.$$

Replacing this estimate into (11.29), we immediately get (11.28). □

By applying the continuity method, using Theorems 11.2.1 and 11.2.4 we can complete the proof of Theorem 11.2.1.

Proof of Theorem 11.2.1*: the general case.* For $\sigma \in [0,1]$, let L_σ be the linear operator defined by $L_\sigma = (D_t - \mathcal{A}_\sigma u, u(\cdot,\cdot,0))$ for every $u \in X := C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$, where $\mathcal{A}_\sigma = \sigma\mathcal{A} + (1-\sigma)(\Delta - I)$. Clearly, each operator L_σ is bounded from X into $Y = C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+}) \times C^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})$ (endowed with the classical norm, i.e., $\|(g,\psi)\|_Y = \|g\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})} + \|\psi\|_{C^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}$). Let c'_σ denote the potential term of the operator $\mathcal{A}_\sigma$. Then, $c'_\sigma = \sigma - 1 + \sigma c \leq \sigma(1+c_0) - 1 \leq \max c_0, -1$ for every $\sigma \in [0,1]$. Hence, from (11.28) it follows that $\|L_\sigma u\|_Y \geq K_0\|u\|_X$ for a constant K_0 independent of σ. Since the operator L_1 is invertible by Theorem 11.2.1, applying Theorem 7.1.1 we conclude that also the operator L_1 is invertible and we are done. □

11.3 A refined version of Theorem 10.0.2

Now, we have all the tools to prove the main result of this lecture.

Theorem 11.3.1. *For every $f \in C_b^{2+\alpha}(\overline{\mathbb{R}^d_+})$ and $g \in C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$ such that $f = \mathcal{A}f + g(0,\cdot) = 0$ on $\partial\mathbb{R}^d_+$, there exists a unique classical solution u to problem* (10.1)*. In addition, u belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$ and there exists a positive constant $\overline{C}$, independent of f and g, such that*

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} \le \overline{C}(\|f\|_{C_b^{2+\alpha}(\overline{\mathbb{R}^d_+})} + \|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}). \tag{11.30}$$

Proof. Throughout the proof, we denote by C a positive constant, independent of the functions that we consider, which may vary from line to line. Moreover, we use the notation $\mathcal{P}(f,g)$ to denote the Cauchy problem (10.1) and we set $g_f = \mathcal{A}f + g$.

Without loss of generality we can assume that $c_0 < 0$. Indeed, suppose that $c_0 \ge 0$. Then, observe that the function u solves problem $\mathcal{P}(f,g)$ if and only if the function $(t,x)\mapsto v(t,x) = e^{-(c_0+1)t}u(t,x)$ solves the Cauchy problem

$$\begin{cases} D_t v(t,x) = \mathcal{A}'v(t,x) + \widehat{g}(t,x), & t\in[0,T], \quad x\in\overline{\mathbb{R}^d_+},\\ v(t,x',0) = 0, & t\in[0,T], \quad x\in\mathbb{R}^{d-1},\\ v(0,x) = f(x), & \qquad\qquad\quad\; x\in\overline{\mathbb{R}^d_+}, \end{cases}$$

where $\mathcal{A}' = \mathcal{A} - (c_0-1)I$ has potential term which is not greater than -1 on $\overline{\mathbb{R}^d}_+$ and $\widehat{g}(t,x) = e^{-(c_0+1)t}g(t,x)$ for every $(t,x)\in[0,T]\times\overline{\mathbb{R}^d_+}$. Moreover, $\|\widehat{g}\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} \le C_*\|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}$ and $\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} \le C_*\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}$ for some positive constant C_*. Hence, if v satisfies (11.30), then u satisfies (11.30) as well, with possibly a different constant $\overline{C}$.

So, in the rest of the proof, we assume that $c_0 < 0$. As a first step, we observe that $u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$ solves problem $\mathcal{P}(f,g)$ if and only if the function $v_1 = u - f$ solves the Cauchy problem $\mathcal{P}(0,g_f)$. The function g_f belongs to $C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$ and $\|g_f\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} \le \|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} + C\|f\|_{C_b^{2+\alpha}(\overline{\mathbb{R}^d}_+)}$. We extend $g_f(0,\cdot)$ to $\mathbb{R}^d$ by setting $g_f(0,x) = g_f(0,x',-x_d)$ for every $x'\in\mathbb{R}^{d-1}$ and $x_d<0$. Clearly, $g_f(0,\cdot)\in C_b^\alpha(\mathbb{R}^d)$ and

$$\|g_f(0,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \le 2\|g_f(0,\cdot)\|_{C_b^\alpha(\overline{\mathbb{R}^d_+})} \le C(\|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} + \|f\|_{C_b^{2+\alpha}(\overline{\mathbb{R}^d}_+)}).$$

Theorem 8.1.1 shows that there exists a unique function $v_2\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$ vanishing on $\{0\}\times\mathbb{R}^d$ and such that $D_tv_2 = \mathcal{A}v_2 + g_f(0,\cdot)$ on $[0,T]\times\mathbb{R}^d$. The function v_2 also satisfies the estimate

$$\|v_2\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \le C\|g_f(0,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \le C(\|f\|_{C_b^{2+\alpha}(\overline{\mathbb{R}^d_+})} + \|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}), \tag{11.31}$$

where C is the constant in (8.1), which is independent of v_2, f and g.

Note that v_1 solves the Cauchy problem $\mathcal{P}(0,g_f)$ if and only if the function $v_3 = v_1 - v_2 \in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$ solves the Cauchy problem

$$
\begin{cases}
D_t v_3(t,x) = \mathcal{A}v_3(t,x) + g_f(t,x) - g_f(0,x), & t\in(0,T], \quad x\in\mathbb{R}^d_+,\\
v_3(t,x',0) = -v_2(t,x',0), & t\in(0,T], \quad x'\in\mathbb{R}^{d-1},\\
v_3(0,x) = 0, & \qquad\qquad\quad x\in\mathbb{R}^d_+.
\end{cases}
\tag{11.32}
$$

Since $v_2(0,x',0) = 0$ and $D_t v_2(0,x',0) = g(0,x',0) + \mathcal{A}f(x',0) = 0$ for every $x'\in\mathbb{R}^{d-1}$, we can extend the function $v_2(\cdot,\cdot,0)$ to $(-\infty,T]\times\mathbb{R}^{d-1}$ by setting $v_2(t,x',0)=0$ for each $t<0$ and $x'\in\mathbb{R}^{d-1}$, obtaining a function which belongs to $C^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})$. Similarly, we can extend $g_f - g_f(0,\cdot)$ to $(-\infty,0]\times\overline{\mathbb{R}^d_+}$ in the trivial way obtaining a function which belongs to $C_b^{\alpha/2,\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$. Hence, by Theorem 11.2.1, there exists a unique function $v_3\in C^{1+\alpha/2,2+\alpha}((-\infty,T]\times\overline{\mathbb{R}^d_+})$ which solves the Cauchy problem (11.32). Moreover, v_3 vanishes on $(-\infty,0]\times\overline{\mathbb{R}^d_+}$ by Proposition 11.2.2 (with $T=0$) since on this set $D_t v_3 = \mathcal{A}v_3$ and $v_3(t,x',0)=0$ on $(-\infty,T]\times\mathbb{R}^{d-1}$. The quoted proposition also shows that

$$
\begin{aligned}
\|v_3\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} \le & C(\|v_2(\cdot,\cdot,0)\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^{d-1})} + \|g_f\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})})\\
\le & C(\|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} + \|f\|_{C_b^{2+\alpha}(\overline{\mathbb{R}^d_+})}).
\end{aligned}
\tag{11.33}
$$

We thus conclude that the function $u = v_2+v_3+f$, which belongs to $C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$, is a solution (actually, the unique solution by Proposition 11.0.1) of the Cauchy problem $\mathcal{P}(f,g)$. Moreover, from (11.31) and (11.33), estimate (11.30) follows at once. □

11.4 Notes

This lecture is based on [1, Chapters 8 & 9] and [2, Chapter 4]. We refer the reader to these monographs for further results.

11.5 Exercises

1. Complete the proof of Lemma 11.1.1.
2. Prove that if $u\in C_b^{1+\alpha/2,2+\alpha}(\mathbb{R}^{d+1})$ has support contained in $[a,b]\times\mathbb{R}^d$ for some $a,b\in\mathbb{R}$ with $a<b$, then

$$
u(t,x) = \int_{-\infty}^t (T(t-s)(D_t u(s,\cdot) - \Delta u(s,\cdot))(x)ds
$$

where $\{T(t)\}$ is the Gauss-Weierstrass semigroup.

3. Let $u \in C^3(\mathcal{C}_1(0,0)) \cap C(\overline{\mathcal{C}_1(0,0)})$ solve the homogeneous heat equation. Further, suppose that $D_j u \in C((-1,0] \times B(0,1))$ for every $j = 1, \ldots, d$ and let $\psi \in C_c^\infty((-1,1) \times B_1)$ satisfy $\psi(0,0) = 1$.
   (i) Prove that the function $v_\gamma = \gamma u^2 + \psi |\nabla_x u|^2$ solve the equation $D_t v_\gamma = \Delta v_\gamma + \Psi_\gamma$ on $\mathcal{C}_1(0,0)$, where
   $$\Psi_\gamma = 2(\gamma + \psi\Delta\psi + |\nabla_x \psi|^2 - \psi D_t \psi)|\nabla_x u|^2 + 2\psi^2 |D_x^2 u|^2 + 8\psi \sum_{i,j=1}^{d} D_{ij}^2 u D_i \psi D_j u$$
   (ii) Prove that $\Psi_\gamma \geq (2\gamma + 2\psi\Delta\psi + 2|\nabla_x\psi|^2 - 8|\nabla_x\psi|^2 - 2\psi D_t\psi)|\nabla_x u|^2$ and deduce that γ can be fixed large enough such that $\Psi_\gamma \geq 0$ on $\mathcal{C}_1(0,0)$.
   (iii) Show that there exists a constant $\beta_0 > 0$, independent of u, such that $|\nabla_x u(0,0)| \leq \beta_0 \|u\|_{C(\overline{\mathcal{C}_1(0,0)})}$.

4. (i) Prove that, if u is as in the previous exercise with $\mathcal{C}_1(0,0)$ being replaced by $\mathcal{C}_r(0,0)$, then $|\nabla_x u(0,0)| \leq \kappa r \|u\|_{C(\overline{\mathcal{C}_r(0,0)})}$ for some positive constant κ.
   (ii) By properly applying the above result to the function $D_j u$ ($j = 1, \ldots, d$) instead of u prove that, if $u \in C^4(\mathcal{C}_r(0,0))$ with spatial derivatives up to the second-order which are continuous on $(r^2, 0] \times B(0,r)$, then $|D_{ij} u(0,0)| \leq \beta_1 r^2 \|u\|_{C(\overline{\mathcal{C}_r(0,0)})}$ for every $i, j = 1, \ldots d$ and some positive constant β_1, independent of u.
   (iii) More generally, prove that if $u \in C^k(\mathcal{C}_r(0,0))$ has spatial derivatives up to the $(k-2)$-th order which are continuous on $(r^2, 0] \times B_r$, then the value at the origin of each $(k-2)$-th order spatial derivatives can be bounded in modulus by $\beta_h r^h \|u\|_{C(\overline{\mathcal{C}_r(0,0)})}$ for some positive constant β_h, independent of u.

5. Prove estimates (11.10) and (11.11).

6. Prove that, if $\widehat{\psi} \in C_b^{1+\alpha/2, 2+\alpha}((-\infty, T] \times \mathbb{R}^{d-1})$, then $D_i \widehat{\psi} \in C_b^{\alpha/2,\alpha}((-\infty, T] \times \mathbb{R}^{d-1})$ and
   $$\|D_i \widehat{\psi}\|_{C_b^{\alpha/2,\alpha}((-\infty,T]\times\mathbb{R}^{d-1})} \leq C \|\widehat{\psi}\|_{C_b^{1+\alpha/2,2+\alpha}((-\infty,T]\times\mathbb{R}^{d-1})}$$
   for every $i = 1, \ldots, d-1$ and some positive constant C, independent of $\widehat{\psi}$.

7. Prove that
   $$\int_{\mathbb{R}^{d+1}} D_d K_+(t,x)\, dt\, dx = 0.$$

8. Prove that the function $D_{id} v$ ($i = 1, \ldots, d$) in the proof of Theorem 11.2.1 belongs to $C_b^{\alpha/2,\alpha}((-\infty, T] \times \mathbb{R}^{d-1} \times [\varepsilon, \infty))$ for every $\varepsilon > 0$.

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Lecture 12

Parabolic equations in bounded smooth domains Ω with homogeneous Dirichlet boundary conditions.

In this lecture, we complete the analysis of parabolic Cauchy problems with bounded coefficients, dealing with the Cauchy-Dirichlet problem

$$
\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + g(t,x), & t \in [0,T], \quad x \in \Omega, \\ u(t,x) = 0, & t \in [0,T], \quad x \in \partial\Omega, \\ u(0,x) = f(x) & \qquad\qquad\quad x \in \overline{\Omega}, \end{cases} \tag{12.1}
$$

where $\Omega \subset \mathbb{R}^d$ is a bounded and smooth domain[1] (see the forthcoming Definition 12.0.2) and $\mathcal{A}$ is the second-order differential operator, defined by

$$
\mathcal{A}\zeta(x) = \sum_{i,j=1}^{d} q_{ij}(x) D_{ij}\zeta(x) + \sum_{j=1}^{d} b_j(x) D_j \zeta(x) + c(x)\zeta(x) \tag{12.2}
$$

on smooth enough functions $\zeta : \Omega \to \mathbb{R}$. We assume the following conditions on Ω and on the coefficients of the operator $\mathcal{A}$.

Hypotheses 12.0.1. (i) Ω is a bounded domain of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$ (see the forthcoming Definition 12.0.2);

(ii) the coefficients $q_{ij} = q_{ji}$, b_j $(i,j = 1,\dots,d)$ and c are bounded and α-Hölder continuous in $\overline{\Omega}$ for some $\alpha \in (0,1)$;

(iii) there exists a positive constant μ such that $\langle Q(x)\xi, \xi\rangle \geq \mu|\xi|^2$ for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^d$.

---

[1] By domain, we mean an open connected set.

Definition 12.0.2. A bounded domain $\Omega \subset \mathbb{R}^d$ is called a domain of class $C^{2+\alpha}$ ($\alpha \in (0,1)$) if there exist a positive constant r and, for every $x_0 \in \partial\Omega$, an open neighborhood $\mathcal{U}_{x_0}$ of x_0 and a diffeomorphism $\psi_{x_0} : \overline{\mathcal{U}_{x_0}} \to \overline{B(0,r)}$ of class $C^{2+\alpha}$ such that

$$\psi_{x_0}(\Omega \cap \mathcal{U}_{x_0}) = B_+(0,r) = \{x \in B(0,r) : x_d > 0\},\ \psi_{x_0}(\partial\Omega \cap \mathcal{U}_{x_0}) = \{x \in B(0,r) : x_d = 0\}.$$

The following proposition provides us with an equivalent condition for a bounded domain to be a domain of class $C^{2+\alpha}$. Its proof is deferred to Appendix C being rather technical.

Proposition 12.0.3. *A bounded domain Ω is a domain of class $C^{2+\alpha}$ ($\alpha \in (0,1)$) if and only if there exists a function $g \in C^{2+\alpha}(\mathbb{R}^d)$ such that*

$$\Omega = \{x \in \mathbb{R}^d : g(x) > 0\}, \qquad \partial\Omega = \{x \in \mathbb{R}^d : g(x) = 0\}, \qquad \mathbb{R}^d \setminus \overline{\Omega} = \{x \in \mathbb{R}^d : g(x) < 0\}$$

and $\nabla_x g \neq 0$ on $\partial\Omega$.

Remark 12.0.4. In view of the previous proposition examples of bounded domains of class $C^{2+\alpha}$ are easily provided. For instance every open ball has this property (actually, it is a C^∞-smooth domain. Indeed, $B(\overline{x},r) = \{x \in \mathbb{R}^d : r^2 - \sum_{i=1}^d (x_i - \overline{x}_i)^2 > 0\}$ for every $\overline{x} \in \mathbb{R}^d$ and the function $g : \mathbb{R}^d \to \mathbb{R}$, defined by $g(x) = r^2 - \sum_{i=1}^d (x_i - \overline{x}_i)^2 > 0$ for every $x \in \mathbb{R}^d$, belongs to $C^\infty(\mathbb{R}^d)$.

The main result of this lecture is the following counterpart of Theorems 8.1.1 and 11.3.1.

Theorem 12.0.5. *For every $f \in C^{2+\alpha}(\overline{\Omega})$ and $g \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$, such that f and $g + \mathcal{A}f$ vanish on $\partial\Omega$ and on $[0,T] \times \partial\Omega$, respectively, there exists a unique solution $u \in C^{1,2}((0,T] \times \Omega) \cap C([0,T] \times \overline{\Omega})$ to the Cauchy problem* (12.1). *In addition, $u \in C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$ and there exists a positive constant C, independent of u, f and g such that*

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})} \leq C(\|f\|_{C^{2+\alpha}(\overline{\Omega})} + \|g\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})}). \tag{12.3}$$

To solve the Cauchy problem (12.1) we will take advantage of the following result, commonly referred to as *partition of unity*.

Proposition 12.0.6. *Let $K \subset \mathbb{R}^d$ be a compact set and $\{\Omega_i : i = 1, \ldots, N\}$ be an open covering of K (i.e., $K \subset \bigcup_{i=1}^N \Omega_i$, Ω_i being open subsets of $\mathbb{R}^d$). Then, there exist functions $\eta_i \in C_c^\infty(\Omega_i)$ ($i = 1, \ldots, N$) such that $0 \leq \eta_i \leq 1$ on Ω_i and $\sum_{i=1}^N \eta_i \equiv 1$ on K.*

Proof. To begin with, we show that there exists a covering $\{\Omega_i' : i = 1, \ldots, N\}$ of K such that $\overline{\Omega_i'} \subset \Omega_i$ for every $i = 1, \ldots, N$. For this purpose, for every $x \in \Omega_i$, we consider a ball $B(x, r_x) \subset \Omega_i$. Clearly, $\{B(x, r_x/2) : x \in \Omega_i,\ i = 1, \ldots, N\}$ is an open covering of K, from which we can extract a finite subcovering $\{B(x_j, r_{x_j}/2) : j = 1, \ldots, M\}$. For every $i \in$

$\{1,\dots,N\}$, we set $I_i = \{j \in \{1,\dots,M\} : B(x_j, r_{x_j}/2) \subset \Omega_i\}$ and $\Omega_i' = \bigcup_{k\in I_i} B(x_k, r_{x_k}/2)$. Each Ω_i' is an open subset of $\mathbb{R}^d$ and

$$\overline{\Omega_i'} = \bigcup_{k\in I_i} \overline{B(x_k, r_{x_k}/2)} \subset \bigcup_{k\in I_i} B(x_k, r_{x_k}) \subset \Omega_i, \qquad i = 1,\dots,N.$$

By Exercise 7.3.1, for every $i = 1,\dots,N$ we can determine a function $\varrho_i \in C_c^\infty(\mathbb{R}^d)$ compactly supported in Ω_i and such that $\varrho_i \equiv 1$ in Ω_i'. Note that we can also assume that $0 \le \varrho_i \le 1$ on $\mathbb{R}^d$

To complete the proof, it suffices to set $\eta_1 = \varrho_1$, $\eta_2 = (1-\varrho_1)\varrho_2$ and, more generally, $\eta_k = \varrho_k \prod_{j=1}^{k-1}(1-\varrho_j)$ for $k = 2,\dots,N$. Each function η_i is compactly supported in Ω_i and its image is contained in $[0,1]$. Moreover[2],

$$\sum_{i=1}^N \eta_i(x) = 1 - \prod_{i=1}^N (1-\varrho_i(x)), \qquad x \in \mathbb{R}^d.$$

From this formula it follows immediately that $\sum_{i=1}^N \eta_i \equiv 1$ on K. Indeed, if $x \in K$, then $x \in \Omega_i'$ for some $i \in \{1,\dots,N\}$, and on Ω_i' the function $1-\varrho_i$ identically vanishes. $\square$

In Section 12.2 we shall make use of the following corollary of the Proposition 12.0.6.

Corollary 12.0.7. *Let $K \subset \mathbb{R}^d$ be a compact set and $\{\Omega_i : i = 1,\dots,N\}$ be an open covering of K. Then, there exist functions $\vartheta_i \in C_c^\infty(\Omega_i)$ $(i = 1,\dots,N)$ such that $0 \le \vartheta_i \le 1$ on Ω_i and $\sum_{i=1}^N \vartheta_i^2 \equiv 1$ on K.*

Proof. Let η_i $(i = 1,\dots,N)$ be the same functions as in the proof of Proposition 12.0.6. We claim that there exists a positive constant δ_0 such that $\sum_{i=1}^N \eta_i^2 \ge \delta_0$ on K. Indeed, the function $\sum_{i=1}^N \eta_i^2$ is continuous on K. Hence, it admits a minimum over K attained at some point x_0. If such a minimum were zero, then each function η_i $(i = 1,\dots,d)$ would vanish at x_0 and this would clearly contradict the condition $\sum_{i=1}^N \eta_i(x_0) = 1$.

Next, we claim that there exists $\varepsilon > 0$ such that $\sum_{i=1}^N \eta_i^2 \ge \delta_0/2$ on $K + B(0,\varepsilon)$. By contradiction, suppose that this is not the case. Then, for every $n \in \mathbb{N}$ we may find a point $x_n \in K + B(0,1/n)$ such that $\sum_{i=1}^N (\eta_i(x_n))^2 < \delta_0/2$. Since the sequence (x_n) is bounded, up to a subsequence it converges to some point $x \in K$, which satisfies the condition $\sum_{i=1}^N (\eta_i(x))^2 \le \delta_0/2$: a contradiction.

Now, we are almost done. Still by Exercise 7.3.1 we can determine a function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi_K \le \psi \le \chi_{K+B(0,\varepsilon)}$. Setting

$$\vartheta_i = \left(\sum_{j=1}^N \eta_j^2\right)^{-\frac{1}{2}} \eta_i \psi, \qquad i = 1,\dots,N,$$

we obtain the functions that we are looking for. $\square$

---

[2] This formula can be proved by induction on N

Now, we have all the tools to prove Theorem 12.0.5. The uniqueness part follows straightforwardly from Corollary 1.1.3. The proof of the remaining statements are the contents of the next two sections.

The following remark will be used in both the next sections.

Remark 12.0.8. Let $\{\mathcal{U}_x : x \in \partial\Omega\}$ be as in Definition 12.0.2. Since $\{\mathcal{U}_x : x \in \partial\Omega\}$ is an open covering of $\partial\Omega$, which is a compact set, there exists a finite subcovering $\{\mathcal{U}_{x_j} : j = 1, \dots, N-1\}$. Note that there exists $\delta > 0$ such that $\bigcup_{j=1}^{N-1} \mathcal{U}_{x_j} \supset \Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \le \delta\}$. Indeed, if this were not the case, we would find a sequence $(y_n) \subset \Omega$ such that $\mathrm{dist}(y_n, \partial\Omega) \le n^{-1}$ and $y_n \notin \bigcup_{j=1}^{N-1} \mathcal{U}_{x_j}$. The sequence (y_n) is bounded. Hence, it admits a subsequence (y_{n_k}) which converges to some $\overline{y} \in \partial\Omega \subset \bigcup_{j=1}^{N-1} \mathcal{U}_{x_j}$. Since $\bigcup_{j=1}^{N-1} \mathcal{U}_{x_j}$ is an open subset of $\mathbb{R}^d$ it should contain y_{n_k} for k sufficiently large, leading us to a contradiction. Therefore, if we set $\Omega_j = \mathcal{U}_{x_j}$ for each $j = 1, \dots, N-1$ and $\Omega_N = \Omega \setminus \Omega_{\delta/2}$, then the family $\{\Omega_j : j = 1, \dots, N\}$ is an open finite covering of $\overline{\Omega}$.

Finally, we introduce the following couple of notation: (i) given a function $\zeta : \Omega \to \mathbb{R}$ (resp. $\zeta : [0, T] \times \Omega \to \mathbb{R}$) we denote by $\overline{\zeta}$ the trivial extension of ζ to $\mathbb{R}^d$ (resp. to $[0, T] \times \mathbb{R}^d$), (ii) $\mathcal{A}_0 = \mathcal{A} - cI$.

12.1 Proof of estimate (12.2)

Let $u \in C_b^{1+\alpha/2, 2+\alpha}([0, T] \times \overline{\Omega})$ solve the Cauchy problem (12.1) and let $\{\Omega_j : j = 1, \dots, N\}$ be the covering of $\overline{\Omega}$ in Remark 12.0.8. By Proposition 12.0.6, we can determine N smooth functions $\eta_1, \dots, \eta_N$ such that $\mathrm{supp}(\eta_i) \subset \Omega_i$ $(i = 1, \dots, N)$ and $\sum_{i=1}^N \eta_i = 1$ on $\overline{\Omega}$.

We split

$$u = \sum_{k=1}^{N} u_k, \tag{12.4}$$

where $u_k = u\eta_k$ and analyze separately the cases $k < N$ (in which we are led to a Cauchy-Dirichlet problem on $\mathbb{R}^d_+$) and $k = N$ (in which we are led to a Cauchy problem in the whole $\mathbb{R}^d$). We begin by this latter case, being easier.

Note that each function u_k satisfies the differential equation

$$D_t u_k = \mathcal{A} u_k + g_k := \mathcal{A} u_k + g\eta_k - u\mathcal{A}_0\eta_k - 2\sum_{i,j=1}^{d} q_{ij} D_i u D_j \eta_k$$

on $[0, T] \times \Omega$ and identically vanishes on $[0, T] \times \partial\Omega$.

Throughout the section, we denote by C a positive constant, independent of f and g, which may vary from line to line.

12.1.1 The case $k = N$

As it is easily seen, the function u_N belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})$ and its support is contained in $[0,T]\times\Omega'$ where Ω' is an open set whose closure is contained in Ω. Therefore, $\overline{u_N}$ belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$. Again, by Exercise 7.3.1, we can determine a function $\vartheta\in C_c^\infty(\Omega)$ such that $\vartheta=1$ on Ω'. Let $\widetilde{q}_{ij}=\vartheta\overline{q_{ij}}+(1-\vartheta)\delta_{ij}$, $\widetilde{b}_j=\vartheta\overline{b_j}$ and $\widetilde{c}=\vartheta\overline{c}$ for every $i,j=1,\dots,d$. The coefficients $\widetilde{q}_{ij}$, $\widetilde{b}_j$ $(i,j=1,\dots,d)$ and $\widetilde{c}$ belong to $C_b^\alpha(\mathbb{R}^d)$ and

$$\begin{aligned}\sum_{i,j=1}^d\widetilde{q}_{ij}(x)\xi_i\xi_j=&\vartheta(x)\sum_{i,j=1}^d\overline{q_{ij}}(x)\xi_i\xi_j+(1-\vartheta(x))|\xi|^2\\ \geq&[\vartheta(x)\mu+1-\vartheta(x)]|\xi|^2\geq\min\{1,\mu\}|\xi|^2\end{aligned}$$

for every $x,\xi\in\mathbb{R}^d$. Hence, the operator

$$\mathcal{A}_N=\sum_{i,j=1}^d\widetilde{q}_{ij}D_{ij}+\sum_{j=1}^d\widetilde{b}_jD_j+\widetilde{c}\tag{12.5}$$

satisfies Hypotheses 7.0.1. Moreover, the function $\overline{u}_N$ solves the Cauchy problem

$$\begin{cases}D_t\overline{u_N}(t,x)=\mathcal{A}_N\overline{u_N}(t,x)+\overline{g_N}(t,x), & t\in[0,T],\quad x\in\mathbb{R}^d,\\ \overline{u_N}(0,x)=f_N(x), & x\in\mathbb{R}^d,\end{cases}$$

where $f_N=\overline{f\eta_N}$. Since $f_N\in C_b^{2+\alpha}(\mathbb{R}^d)$ and $\overline{g_N}\in C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)$, by Theorem 8.1.1

$$\|\overline{u_N}\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}\leq C_1(\|f_N\|_{C_b^{2+\alpha}(\mathbb{R}^d)}+\|\overline{g_N}\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}).\tag{12.6}$$

Note that $\|f_N\|_{C_b^{2+\alpha}(\mathbb{R}^d)}\leq C\|f\|_{C_b^{2+\alpha}(\overline{\Omega})}\|\eta_N\|_{C_b^{2+\alpha}(\mathbb{R}^d)}$ and, recalling that $u=u\vartheta$ on the support of the function η_N, we can estimate

$$\begin{aligned}\|\overline{g_N}\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\leq&\bigg(\|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})}\|\eta_N\|_{C_b^\alpha(\mathbb{R}^d)}+\|u\vartheta\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})}\|\mathcal{A}_0\eta_N\|_{C_b^\alpha(\overline{\Omega})}\\ &+2\sum_{i,j=1}^N\|q_{ij}\|_{C^\alpha(\overline{\Omega})}\|D_j(\vartheta u)\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})}\|D_i(\vartheta\eta_N)\|_{C_b^\alpha(\mathbb{R}^d)}\bigg)\\ \leq&C(\|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})}+\|\vartheta u\|_{C_b^{\alpha/2,1+\alpha}([0,T]\times\overline{\Omega})}).\end{aligned}\tag{12.7}$$

Applying the same arguments as in the proof of Theorem 7.2.1 to the function $\overline{\vartheta u}$, which belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$, we can estimate

$$\begin{aligned}\|\overline{\vartheta u}\|_{C_b^{\alpha/2,1+\alpha}([0,T]\times\mathbb{R}^d)}\leq&\varepsilon\|\overline{\vartheta u}\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)}+\widetilde{C}_\varepsilon\|\overline{\vartheta u}\|_{C_b([0,T]\times\overline{\mathbb{R}^d})}\\ \leq&\varepsilon\|u\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})}+\widetilde{C}_\varepsilon\|u\|_{C_b([0,T]\times\overline{\Omega})},\end{aligned}$$

for every $\varepsilon > 0$, where the constant $\widetilde{C}_\varepsilon$ blows up as ε tends to 0. Replacing this estimate in (12.7) we conclude that

$$\|\overline{g_N}\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} \le C(\|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})} + \varepsilon\|u\|_{C^{\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})} + \widetilde{C}_\varepsilon\|u\|_\infty) \tag{12.8}$$

for every $\varepsilon > 0$, and (12.6) yields

$$\|u_N\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})} \le C(\|f\|_{C_b^{2+\alpha}(\overline{\Omega})} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})} + \varepsilon\|u\|_{C^{\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})} + \widetilde{C}'_\varepsilon\|u\|_\infty) \tag{12.9}$$

for every $\varepsilon > 0$.

12.1.2 The case $k < N$

Let us fix such an index k and denote by $\psi_k := \psi_{x_k}$ the diffeomorphism in Definition 12.0.2 corresponding to the open neighborhood $\mathcal{U}_{x_k}$ of $x_k \in \partial\Omega$. Due to the smoothness of ψ_k, the function $v_k : [0,T] \times \overline{B_+(0,r)} \to \mathbb{R}$, defined by $v_k(t,y) = u_k(t, \psi_k^{-1}(y))$ for every $t \in [0,T]$ and $y \in \overline{B_+(0,r)}$, belongs to $C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{B_+(0,r)})$ and solves the Cauchy problem

$$\begin{cases} D_t v_k(t,y) = \mathcal{A}_k v_k(t,y) + \widehat{g}_k(t,y), & t \in [0,T], \quad x \in \overline{B_+(0,r)}, \\ v_k(t,y) = 0, & t \in [0,T], \quad y \in \overline{B_+(0,r)} \cap \partial\mathbb{R}^d_+, \\ v_k(0,y) = \widehat{f}_k(y), & \qquad\qquad y \in \overline{B_+(0,r)}, \end{cases}$$

where

$$\mathcal{A}_k = \sum_{i,j=1}^N q_{ij}^{(k)} D_{ij} + \sum_{j=1}^d b_j^{(k)} D_j + c^{(k)}, \tag{12.10}$$

with

$$\begin{aligned} q_{ij}^{(k)} &= [(\mathrm{Jac}\,\psi_k)(Q\circ\psi_k^{-1})(\mathrm{Jac}\,\psi_k)^*]_{ij}\circ\psi_k^{-1}, \\ b_j^{(k)} &= [\mathrm{Tr}(QD^2\psi_{k,j}) + \langle b, \nabla\psi_{k,j}\rangle]\circ\psi_k^{-1}, \\ c^{(k)} &= c\circ\psi_k^{-1}, \\ \widehat{f}_k &= (\eta_k f)\circ\psi_k^{-1}, \\ \widehat{g}_k &= g_k(\cdot, \psi_k^{-1}(\cdot)) \end{aligned}$$

for every $i,j = 1,\dots,d$, where $\psi_{k,j}$ denotes the j-th component of ψ_k. The functions $\widehat{f}_k$ and $\widehat{g}_k$ belong to $C^{2+\alpha}(\overline{B_+(0,r)})$ and to $C^{\alpha/2,\alpha}([0,T] \times \overline{B_+(0,r)})$, respectively. Moreover,

$$\|\widehat{f}_k\|_{C^{2+\alpha}(\overline{B_+(0,r)})} \le C\|f\|_{C^{2+\alpha}(\overline{\Omega})}$$

and

$$\|\widehat{g}_k\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B_+(0,r)})} \le C(\|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})} + \|u\|_{C^{\alpha/2,1+\alpha}([0,T]\times\overline{\Omega})}).$$

Note that

$$\begin{aligned}\sum_{i,j=1}^{d} q_{ij}^{(k)}(y)\xi_i\xi_j =&\sum_{i,j=1}^{d}[\mathrm{Jac}\,\psi_k(\psi_k^{-1}(y))Q(\psi_k^{-1}(y))(\mathrm{Jac}\,\psi_k(\psi_k^{-1}(y)))^*]_{ij}\xi_i\xi_j\\ =&\langle Q(\psi_k^{-1}(y))(\mathrm{Jac}\,\psi_k(\psi_k^{-1}(y)))^*\xi,(\mathrm{Jac}\,\psi_k(\psi_k^{-1}(y)))^*\xi\rangle\\ \ge&\mu|(\mathrm{Jac}\,\psi_k(\psi_k^{-1}(y)))^*\xi|^2.\end{aligned}$$

Since the matrix $\mathrm{Jac}\,\psi_k(x)$ is invertible at any $x\in\overline{\mathcal{U}_{x_k}}$, it follows that $\det(\mathrm{Jac}\,\psi_k(x))>0$ for every $x\in\overline{\mathcal{U}_{x_k}}$. Thus, the function $y\mapsto\|(\mathrm{Jac}\psi_k(y))^{-1}\|$ is bounded in $\overline{B_+(0,r)}$ by a positive constant M_0. As a byproduct, we deduce that $|(\mathrm{Jac}\psi_k(\psi_k^{-1}(y)))^*\xi|\ge M_0^{-1}|\xi|$ for every $y\in B_+(0,r)$ and, consequently, there exists a positive constant μ_k such that $\sum_{i,j=1}^d q_{ij}^{(k)}(y)\xi_i\xi_j\ge\mu_k|\xi|^2$ for every $\xi\in\mathbb{R}^d$ and $y\in B_+(0,r)$. Since η_k is compactly supported in $\mathcal{U}_{x_k}$, the function $\eta_k\circ\psi_k^{-1}$ is compactly supported in $B(0,r)$ and, consequently, the function v_k is supported in $[0,T]\times\overline{B_+(0,r')}$ for some $r'<r$. It thus follows immediately that the function $\overline{v_k}$ belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$. We now introduce a function $\vartheta'\in C_c^\infty(\mathbb{R}^d)$ such that $\chi_{B(0,r')}\le\vartheta'\le\chi_{B(0,r)}$. Clearly, $\overline{v_k}=\overline{v_k}\vartheta'$. Using the function ϑ' we "extend" the operator $\mathcal{A}_k$ to $\mathbb{R}^d_+$, by means of the operator $\widetilde{\mathcal{A}}_k$ defined on functions $\zeta\in C^2(\mathbb{R}^d_+)$ by

$$\widetilde{\mathcal{A}}_k\zeta=\sum_{i,j=1}^d\widetilde{q}_{ij}^{(k)}D_{ij}\zeta+\sum_{j=1}^d\widetilde{b}_j^{(k)}\zeta+\widetilde{c}^{(k)}\zeta, \tag{12.11}$$

where

$$\widetilde{q}_{ij}^{(k)}=\vartheta'\overline{q_{ij}^{(k)}}+(1-\vartheta')\delta_{ij},\qquad \widetilde{b}_j^{(k)}=\vartheta'\overline{b_j^{(k)}},\qquad \widetilde{c}^{(k)}=\vartheta'\overline{c^{(k)}}. \tag{12.12}$$

Clearly, the coefficients of the operator $\widetilde{\mathcal{A}}_k$ belongs to $C_b^\alpha(\overline{\mathbb{R}^d}_+)$. Moreover,

$$\sum_{i,j=1}^d\widetilde{q}_{ij}^{(k)}(x)\xi_i\xi_j\ge\min\{\mu_k,1\}|\xi|^2,\qquad x\in\mathbb{R}^d_+,\ \xi\in\mathbb{R}^d,$$

so that operator $\widetilde{\mathcal{A}}_k$ satisfies Hypotheses 10.0.1. Since the function $\overline{v_k}$ solves the Cauchy problem

$$\begin{cases}D_t\overline{v_k}(t,y)=\widetilde{\mathcal{A}}_k\overline{v_k}(t,y)+\overline{\widehat{g}_k}(t,y), & t\in[0,T],\quad y\in\overline{\mathbb{R}^d}_+,\\ \overline{v_k}(t,y)=0, & t\in[0,T],\quad y\in\partial\mathbb{R}^d_+,\\ \overline{v_k}(0,y)=\overline{\widehat{f}_k}(y), & \qquad\qquad y\in\overline{\mathbb{R}^d_+},\end{cases}$$

we can apply Theorem 11.3.1 to deduce that

$$\begin{aligned}\|v_k\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{B(0,r)})}\le&\|\overline{v_k}\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}\\ \le&C(\|f\|_{C^{2+\alpha}(\overline{\Omega})}+\|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})}+\|v_k\vartheta'\|_{C_b^{\alpha/2,1+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}).\end{aligned}$$

As in the case of the whole $\mathbb{R}^d$, we can show that

$$\|v_k\vartheta'\|_{C_b^{\alpha/2,1+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} \le \varepsilon\|v_k\vartheta'\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} + \widetilde{C}''_\varepsilon\|v_k\vartheta'\|_{C_b([0,T]\times\overline{\mathbb{R}^d_+})}.$$

Since $u_k(t,x) = v_k(t,\psi_k(x))$ for every $(t,x)\in[0,T]\times\overline{\Omega\cap\mathcal{U}_{x_k}}$, from the previous estimate, we conclude that

$$\begin{aligned}&\|u_k\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega\cap\mathcal{U}_{x_k}})}\\ \le{}&C(\|f\|_{C^{2+\alpha}(\overline{\Omega})} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})} + \varepsilon\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})} + \widetilde{C}'''_\varepsilon\|u\|_\infty).\end{aligned}\tag{12.13}$$

12.1.3 Conclusion

From (12.4), (12.9) and (12.13) we conclude that

$$\begin{aligned}\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})} \le{}& \sum_{k=1}^{N}\|u_k\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})}\\ \le{}& \sum_{k=1}^{N-1}\|u_k\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega\cap\mathcal{U}_{x_k}})} + \|u_N\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})}\\ \le{}&C(\|f\|_{C^{2+\alpha}(\overline{\Omega})} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})} + \varepsilon\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})} + C_\varepsilon\|u\|_\infty).\end{aligned}$$

Hence, choosing ε small enough, we deduce that

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})} \le C(\|f\|_{C^{2+\alpha}(\overline{\Omega})} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})} + \|u\|_\infty)\tag{12.14}$$

To remove the sup-norm of u from the right-hand side of (12.14), it suffices to observe that

$$\|u\|_\infty \le C(\|f\|_{C(\overline{\Omega})} + \|g\|_{C([0,T]\times\overline{\Omega})}),\tag{12.15}$$

(see Exercise 12.4.1).

12.2 The existence part

To prove the existence of a function $u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})$ which solves the Cauchy problem (12.1), we need some preliminary results.

Lemma 12.2.1. *The following properties are satisfied.*

(i) *Let $u\in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)$ be such that $u(0,\cdot)=0$. Then, there exists a positive constant C, independent of u and $\lambda\ge 1$, such that*

$$\begin{aligned}&\lambda^{1-\frac{\alpha}{2}}\|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} + \lambda^{\frac{1-\alpha}{2}}\sum_{i=1}^{d}\|D_iu\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\\ \le{}&C\|D_tu - \mathcal{A}u + \lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}.\end{aligned}\tag{12.16}$$

(ii) *Let $u \in C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^d_+})$ be such that $u(0,\cdot)=0$ on $\overline{\mathbb{R}^d_+}$ and $u(t,\cdot)=0$ on $\partial\mathbb{R}^d_+$ for every $t\in[0,T]$. Then, there exists a positive constant C, independent of u and $\lambda\geq 1$, such that*

$$\begin{aligned}&\lambda^{1-\frac{\alpha}{2}}\|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}+\lambda^{\frac{1-\alpha}{2}}\sum_{i=1}^d\|D_iu\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}\\ \leq &C\|D_tu-\mathcal{A}u+\lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}.\end{aligned}\tag{12.17}$$

Proof. We limit ourselves to proving estimate (12.16), since the proof of (12.17) can be obtained adapting the same arguments.

Let $v:[0,T]\times\mathbb{R}^{d+1}\to\mathbb{R}$ be the function defined by $v(t,y,x)=\cos(\sqrt{\lambda}y)u(t,x)$ for every $t\in[0,T]$, $y\in\mathbb{R}$ and $x\in\mathbb{R}^d$. As it is immediately seen v belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^{d+1})$. Let also denote by $\mathcal{A}'$ the operator defined on smooth enough functions $\zeta:\mathbb{R}^{d+1}\to\mathbb{R}$ by

$$\mathcal{A}'\zeta(y,x)=D_{yy}\zeta+\sum_{i,j=1}^d q_{ij}(x)D_{ij}\zeta(x,y),\qquad (y,x)\in\mathbb{R}^{d+1}.$$

It is easy to check that $\mathcal{A}'$ is an elliptic operator with ellipticity constant given by the minimum between 1 and the ellipticity constant μ of the operator $\mathcal{A}$. By Theorem 7.2.1 there exists a positive constant C, independent of v, such that

$$\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^{d+1})}\leq C\|D_tv-\mathcal{A}'v\|_{C_b^{\alpha}([0,T]\times\mathbb{R}^{d+1})}.\tag{12.18}$$

We observe that $D_tv(t,y,x)-\mathcal{A}'v(t,y,x)=(D_tu(t,x)-\mathcal{A}u(t,x)+\lambda u(t,x))\cos(\sqrt{\lambda}y)$ and

$$\begin{aligned}\|D_tv-\mathcal{A}'v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^{d+1})}\leq&\|D_tu-\mathcal{A}u+\lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^{d})}+C\lambda^{\frac{\alpha}{2}}\|D_tu-\mathcal{A}u+\lambda u\|_{\infty}\\ \leq&C\lambda^{\frac{\alpha}{2}}\|D_tu-\mathcal{A}u+\lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^{d})},\end{aligned}\tag{12.19}$$

(recall that $\lambda\geq 1$). Moreover,

$$\begin{aligned}&\lambda\|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+C_0\lambda^{\frac{\alpha}{2}}\sum_{i,j=1}^d\|D_{ij}u\|_{C_b([0,T]\times\mathbb{R}^d)}+\sqrt{\lambda}\sum_{i=1}^d\sup_{x\in\mathbb{R}^d}\|D_iu(\cdot,x)\|_{C^{\alpha/2}([0,T])}\\ \leq&\|D_{yy}v(\cdot,0,\cdot)\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+\sum_{i,j=1}^d\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}[D_{ij}v(t,\cdot,x)]_{C_b^{\alpha}(\mathbb{R})}\\ &+\sum_{i=1}^d\sup_{(y,x)\in\mathbb{R}^{d+1}}[D_{yi}v(\cdot,y,x)]_{C^{\alpha/2}([0,T])}\\ \leq&\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^{d+1})},\end{aligned}$$

(12.20)

where C_0 denotes the α-Hölder seminorm of the function cosine. Replacing (12.19) and (12.20) in (12.18), we conclude that

$$\lambda^{1-\frac{\alpha}{2}}\|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} + \lambda^{\frac{1-\alpha}{2}}\sum_{i=1}^d \sup_{x\in\mathbb{R}^d}\|D_iu(\cdot,x)\|_{C^{\alpha/2}([0,T])} + C_0\sum_{i,j=1}^d\|D_{ij}u\|_\infty$$

$$\le C\|D_tu-\mathcal{A}u+\lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}. \tag{12.21}$$

Since $\|\zeta\|_{C_b^{1+\alpha}(\mathbb{R}^d)} \le C_\alpha\|\zeta\|_{C_b^\alpha(\mathbb{R}^d)}^{\frac{1-\alpha}{2-\alpha}}\|\zeta\|_{C_b^2(\mathbb{R}^d)}^{\frac{1}{2-\alpha}}$ for every $\zeta\in C_b^{1+\alpha}(\mathbb{R}^d)$ and some positive constant C_α, from (12.21) we obtain that

$$\sum_{j=1}^d \sup_{t\in[0,T]}\|D_ju(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \le C\lambda^{-\frac{1-\alpha}{2}}\|D_tu-\mathcal{A}u+\lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}. \tag{12.22}$$

From (12.21) and (12.22), estimate (12.16) follows at once. □

We can now prove the existence of a solution $u\in C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})$ to the Cauchy problem (12.1). For this purpose, we consider the same covering $\{\Omega_1,\ldots,\Omega_N\}$ of $\overline{\Omega}$ in Remark 12.0.8 and functions $\vartheta_1,\ldots,\vartheta_N\in C_c^\infty(\mathbb{R}^d)$ such that the set $\{\vartheta_i^2 : i=1,\ldots,N\}$ is a partition of unity subordinated to the covering $\{\Omega_i : i=1,\ldots,N\}$ (see Corollary 12.0.7).

Note that $u\in C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})$ solves the Cauchy problem (12.1) if and only if the function $v=u-f$ solves the equation $D_tv=\mathcal{A}v+g+\mathcal{A}f$ on $[0,T]\times\overline{\Omega}$ and it vanishes on parabolic boundary of $(0,T)\times\overline{\Omega}$. For this reason, in the rest of the proof we will assume that $f\equiv 0$. In fact, we will show the existence of a solution $u_\lambda\in C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})$ to the Cauchy problem

$$\begin{cases} D_tu_\lambda(t,x)=\mathcal{A}u_\lambda(t,x)-\lambda u_\lambda(t,x)+g(t,x), & t\in[0,T], \quad x\in\overline{\Omega},\\ u_\lambda(t,x)=0, & t\in[0,T], \quad x\in\partial\Omega,\\ u_\lambda(0,x)=0, & \qquad\qquad\quad x\in\overline{\Omega}, \end{cases} \tag{12.23}$$

for λ sufficiently large. Note that a function u_λ solves problem (12.23) if and only if the function $u:[0,T]\times\overline{\Omega}\to\mathbb{R}$, defined by $u(t,x)=e^{\lambda t}u_\lambda(t,x)$ for every $(t,x)\in[0,T]\times\overline{\Omega}$, solves the Cauchy problem (12.1) with $f\equiv 0$.

In the rest of this section we denote by C a positive constant, independent of h and λ, which may vary from line to line.

For any $\lambda\ge 1$ and $h\in C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})$, we consider the Cauchy problems

$$\begin{cases} D_tv(t,y)=\widetilde{\mathcal{A}}_kv(t,y)-\lambda v(t,y)+\vartheta_k(\psi_{x_k}^{-1}(y))\overline{h}(t,\psi_{x_k}^{-1}(y)), & t\in[0,T], \quad y\in\overline{\mathbb{R}^d}_+,\\ v(t,y)=0, & t\in[0,T], \quad y\in\partial\mathbb{R}^d_+,\\ v(0,y)=0, & \qquad\qquad\quad y\in\overline{\mathbb{R}^d_+}, \end{cases} \tag{12.24}$$

for $k = 1, \ldots, N-1$, and

$$\begin{cases} D_t w(t,x) = \mathcal{A}_N w(t,x) - \lambda w(t,x) + \vartheta_N(x)\overline{h}(t,x), & t \in [0,T], \quad x \in \mathbb{R}^d, \\ w(0,x) = 0, & \qquad\qquad x \in \mathbb{R}^d, \end{cases} \tag{12.25}$$

where the operators $\widetilde{\mathcal{A}}_k$ ($k = 1, \ldots, N-1$) and $\mathcal{A}_N$ are defined by (12.11)-(12.12) and (12.5). Since, as it has been already pointed out, the coefficients of the operators $\widetilde{\mathcal{A}}_k$ and $\mathcal{A}_N$ satisfy Hypotheses 10.0.1 and 7.0.1, respectively, and the functions $\vartheta_j\overline{h}$ ($j = 1, \ldots, N$) belongs to $C_b^{\alpha/2,\alpha}([0,T] \times \overline{\mathbb{R}^d_+})$, if $k < N$, and to $C_b^{\alpha/2,\alpha}([0,T] \times \mathbb{R}^d)$ otherwise, Theorems 7.2.1 and 11.3.1 show that problems (12.24) ($k = 1, \ldots, N-1$) and (12.25) admit unique solutions $v_k \in C_b^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\mathbb{R}^d_+})$ and $u_N \in C_b^{1+\alpha/2,2+\alpha}([0,T] \times \mathbb{R}^d)$. Thus, we can introduce the operators $S_{\lambda,k}$ ($k = 1, \ldots, N$) defined on functions $h \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$ as follows:

- $(S_{\lambda,k}h)(t,x) = \overline{v_k}(t, \psi_k(x))$ for every $t \in [0,T]$ and $x \in \overline{\Omega \cap \mathcal{U}_{x_k}}$, if $k < N$;
- $S_{\lambda,N}h = u_N$.

A change of variables shows that, for every $h \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$, $S_{\lambda,k}h$ belongs to $C^{1+\alpha/2,\alpha}([0,T] \times \overline{\Omega \cap \mathcal{U}_{x_k}})$ and solves the Cauchy problem

$$\begin{cases} D_t u_k(t,x) = \mathcal{A}u_k(t,x) - \lambda u_k(t,x) + \vartheta_k(x)h(t,x), & t \in [0,T], \quad x \in \Omega \cap \mathcal{U}_{x_k}, \\ \widetilde{u}_k(t,x) = 0, & t \in [0,T], \quad x \in \overline{\mathcal{U}_{x_k}} \cap \partial\Omega, \\ \widetilde{u}_k(0,x) = 0, & \qquad\qquad x \in \overline{\Omega \cap \mathcal{U}_{x_k}}. \end{cases}$$

Since ϑ_k is compactly supported in $\mathcal{U}_{x_k}$, the function $\vartheta_k S_{\lambda,k}h$ ($k = 1, \ldots, N-1$) belongs to $C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$.

Based on all the above remarks, we can define a linear operator $S_\lambda : C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega}) \to C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$ setting $S_\lambda h = \sum_{k=1}^N \vartheta_k S_{\lambda,k}h$ for every $h \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$. As it is immediately seen, for every $h \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$, the function $S_\lambda h$ vanishes on the parabolic boundary of $(0,T) \times \Omega$. Moreover,

$$\begin{aligned} D_t S_\lambda h - \mathcal{A}S_\lambda h &= \sum_{k=1}^N \vartheta_k (D_t S_{\lambda,k}h - \mathcal{A}S_{\lambda,k}h) - \sum_{k=1}^N (\mathcal{A}_0\vartheta_k) S_{\lambda,k}h - 2\sum_{k=1}^N \langle Q\nabla_x S_{\lambda,k}h, \nabla\vartheta_k\rangle \\ &= \sum_{k=1}^N \vartheta_k^2 h - R_\lambda h = h - R_\lambda h, \end{aligned}$$

where $R_\lambda h := \lambda S_\lambda h + \sum_{k=1}^N (\mathcal{A}_0\vartheta_k) S_{\lambda,k}h + 2\sum_{k=1}^N \langle Q\nabla_x S_{\lambda,k}h, \nabla\vartheta_k\rangle$.

Clearly, R_λ is a bounded operator mapping $C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$ into itself. We claim that, for λ sufficiently large the operator $I - R_\lambda$ is invertible on $C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$. This will conclude the proof. Indeed, fix $g \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$, λ as above and take the unique $h_0 \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$ such that $h_0 - R_\lambda h_0 = g$. The function $u = S_\lambda h_0$ is the solution to the Cauchy problem (12.1) we are looking for, In view of Lemma 2.1.11 to prove the claim it suffices to show that $\|R_\lambda\|_{L(C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega}))}$ vanishes as λ tends to ∞. This follows from Lemma 12.2.1, which shows that

$$\|S_{\lambda,N}h\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}+\sum_{i=1}^d\|D_iS_{\lambda,N}h\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}\le C\lambda^{-\frac{1-\alpha}{2}}\|\vartheta_N\overline{h}\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} \tag{12.26}$$

for every $\lambda\ge 1$. The same lemma shows that the function v_k satisfies the estimate

$$\begin{aligned}&\|v_k\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}+\sum_{i=1}^d\|D_iv_k\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}\\ \le&C\lambda^{-\frac{1-\alpha}{2}}\|(\vartheta_k\circ\psi_{x_k}^{-1})\overline{h}(\cdot,\psi_{x_k}^{-1}(\cdot))\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}\\ \le&C\lambda^{-\frac{1-\alpha}{2}}\|h\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})}\end{aligned}$$

for every $k=1,\dots,N-1$, $\lambda\ge 1$. It follows that

$$\|S_{\lambda,k}h\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega\cap\mathcal{U}_{x_k}})}+\sum_{i=1}^d\|D_iS_{\lambda,k}h\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega\cap\mathcal{U}_{x_k}})}\le C\lambda^{-\frac{1-\alpha}{2}}\|h\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})} \tag{12.27}$$

for every $k=1,\dots,N-1$. From (12.26) and (12.27), we easily conclude that

$$\|R_\lambda\|_{L(C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega}))}\le C\lambda^{-\frac{1-\alpha}{2}},\qquad \lambda\ge 1.$$

The claim follows and this completes the proof of Theorem 12.0.5.

12.3 Notes

We stress that also more general homogeneous boundary conditions can be considered instead of the Dirichlet ones. More precisely, Theorem 12.0.5 holds true if we replace the boundary condition $u=0$ with the boundary condition $au+\langle\gamma,\nabla_xu\rangle=0$ on $[0,T]\times\partial\Omega$ for some functions $a\in C^{2+\alpha}(\partial\Omega)$ and $\gamma\in C^{1+\alpha}(\partial\Omega;\partial B(0,1))$ which satisfies the nontangential condition $\inf_{x\in\partial\Omega}|\langle\gamma(x),\nu(x)\rangle|>0$, where $\nu(x)$ denotes the exterior normal vector to $\partial\Omega$ at x. Since the boundary condition is represented by a first-order differential operator, there is only one compatibility condition which reads as follows: $af+\langle\gamma,\nabla_xf\rangle=0$. We do not enter too deeply into details since this could be the subject of a project of the second phase of the ISEM.

12.4 Exercises

1. Let $u\in C^{1,2}([0,T]\times\overline{\Omega})$ solve the Cauchy problem (12.1).

(i) By applying the maximum principle, prove that the function v, defined by $v(t,x) = e^{-(c_0+1)t}u(t,x)$ for every $(t,x) \in [0,T] \times \overline{\Omega}$, where c_0 denotes the maximum over $\overline{\Omega}$ of the potential term c of $\mathcal{A}$, satisfies the estimate

$$v(t,x) \le C_*(\|f\|_\infty + \|g\|_{C([0,T]\times\overline{\Omega})}), \qquad (t,x) \in [0,T] \times \overline{\Omega},$$

for some positive constant C_*, independent of f and g.

(ii) Use this result to prove estimate (12.15).

2. Let $\Omega_1 \subset \mathbb{R}^M$ and $\Omega_2 \subset \mathbb{R}^N$ be two open sets and let $u \in C_b(\overline{\Omega}_1)$, $v \in C_b^\theta(\overline{\Omega}_2)$ for some $\theta \in (0,1)$. Set $f(x,y) = u(x)v(y)$ for every $(x,y) \in \overline{\Omega_1} \times \overline{\Omega_2}$. Prove that the function $f(x,\cdot)$ belongs to $C_b^\theta(\overline{\Omega}_2)$ for every $x \in \overline{\Omega}_1$ and

$$\sup_{x\in\overline{\Omega}_1} [f(x,\cdot)]_{C_b^\theta(\overline{\Omega}_2)} = \|u\|_\infty [v]_{C_b^\theta(\overline{\Omega}_2)}.$$

3. Prove estimate (12.17).

Lecture 13

The Ornstein-Uhlenbeck operator and its associated semigroup in $C_b(\mathbb{R}^d)$.

In this lecture we study the Ornstein-Uhlenbeck operator, which is the prototype of an elliptic operator with unbounded coefficients. Such an operator is defined on smooth functions $\psi : \mathbb{R}^d \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{L}\psi(x) =& \sum_{i,j=1}^{d} q_{ij} D_{ij}\psi(x) + \sum_{i,j=1}^{d} b_{ij} x_j D_i\psi(x) \\ =& \mathrm{Tr}(QD^2\psi(x)) + \langle Bx, \nabla_x \psi(x)\rangle,\end{aligned}$$

where Q and B are $d \times d$ constant and real matrices, Q is positive definite and $B \neq 0$.

We will deal with the Ornstein-Uhlenbeck equation

$$\left\{\begin{array}{lll} D_t u(t,x) = \mathcal{L}u(t,x), & t > 0, & x \in \mathbb{R}^d, \\ u(0,x) = f(x), & & x \in \mathbb{R}^d, \end{array}\right. \tag{13.1}$$

where $f \in C_b(\mathbb{R}^d)$.

As in Lecture 4, we see that, for every $f \in C_b(\mathbb{R}^d)$, there exists an explicit formula for the classical solution to (13.1).

13.1 The Ornstein-Uhlenbeck equation in $\mathbb{R}^d$. Classical solutions: existence and uniqueness

For each $t > 0$ we consider the positive definite matrix

$$Q_t := \int_0^t e^{sB} Q e^{sB^*}\, ds,$$

where B^* denotes the adjoint matrix of B. Since Q_t is positive definite, we deduce that Q_t is invertible and $\det Q_t > 0$ for all $t > 0$.

We now introduce the so-called *Gaussian measure with mean* $e^{tB}x$ *and covariance* Q_t, i.e., the measure defined by

$$\mathcal{N}(e^{tB}x, Q_t)(dy) := (4\pi)^{-\frac{d}{2}}(\det Q_t)^{-\frac{1}{2}} e^{-\frac{1}{4}\langle Q_t^{-1}(e^{tB}x-y), e^{tB}x-y\rangle} dy, \quad t > 0. \tag{13.2}$$

We can now prove the existence and uniqueness of a classical solution to the Cauchy problem.

Theorem 13.1.1. *For each* $f \in C_b(\mathbb{R}^d)$, *the Cauchy problem* (13.1) *has a unique classical*[1] *solution* u *given by the following formula:*

$$u(t,x) = \begin{cases} f(x), & t = 0, \quad x \in \mathbb{R}^d, \\ \displaystyle\int_{\mathbb{R}^d} f(y)\mathcal{N}(e^{tB}x, Q_t)(dy), & t > 0, \quad x \in \mathbb{R}^d. \end{cases} \tag{13.3}$$

Moreover, $\|u(t,\cdot)\|_\infty \le \|f\|_\infty$ *for all* $t \ge 0$.

Proof. We observe first that, if u is given by (13.3), then

$$\begin{aligned} |u(t,x)| \le & \int_{\mathbb{R}^d} \left| f(y)\mathcal{N}(e^{tB}x, Q_t)(dy)\right| \\ \le & \|f\|_\infty \int_{\mathbb{R}^d} \mathcal{N}(e^{tB}x, Q_t)(dy) = \|f\|_\infty \int_{\mathbb{R}^d} \mathcal{N}(e^{tB}x, Q_t)(dy) = \|f\|_\infty \end{aligned}$$

for every $(t,x) \in (0,\infty) \times \mathbb{R}^d$, since $\int_{\mathbb{R}^d} \mathcal{N}(e^{tB}x, Q_t)(dy) = 1$ (see Exercise 13.4.1). So, $\|u(t,\cdot)\|_\infty \le \|f\|_\infty$ for all $t \ge 0$ as claimed.

To prove that u, given by (13.3), solves the Cauchy problem (13.1) we follow the approach given in Remark 4.1.4. For this purpose we look for a solution $v \in C([0,\infty) \times \mathbb{R}^d) \cap C^{1,2}((0,\infty) \times \mathbb{R}^d)$ of

$$\begin{cases} D_t v(t,y) = \operatorname{Tr}(e^{tB}Qe^{tB^*}D_y^2 v(t,y)), & t > 0, \quad y \in \mathbb{R}^d, \\ v(0,y) = f(y), & y \in \mathbb{R}^d. \end{cases} \tag{13.4}$$

Taking the Fourier transform of both the sides of the equation $D_t v = \operatorname{Tr}(e^{tB}Qe^{tB^*}D_y^2 v)$ with respect to y and interchanging the actions of $\mathcal{F}$ and the time derivative, we deduce that the function $\widehat{v}$, defined by $\widehat{v}(t,\xi) = (\mathcal{F}(v(t,\cdot)))(\xi)$ for every $t \ge 0$ and $\xi \in \mathbb{R}^d$, solves the Cauchy problem

$$\begin{cases} D_t \widehat{v}(t,\xi) = -\langle e^{tB}Qe^{tB^*}\xi, \xi\rangle \widehat{v}(t,\xi), & t \in (0,\infty), \quad \xi \in \mathbb{R}^d, \\ \widehat{u}(0,\xi) = \widehat{f}(\xi), & \xi \in \mathbb{R}^d. \end{cases}$$

---

[1] i.e., $u \in C([0,\infty) \times \mathbb{R}^d) \cap C^{1,2}((0,\infty) \times \mathbb{R}^d)$ and solves (13.1)

So, it is easy to deduce that

$$\widehat{v}(t,\xi) = e^{-\langle Q_t\xi,\xi\rangle}\widehat{f}(\xi), \qquad t > 0,\ \xi \in \mathbb{R}^d.$$

Taking the inverse Fourier transform one obtains

$$v(t,y) = (4\pi)^{-\frac{d}{2}}(\det Q_t)^{-\frac{1}{2}}\int_{\mathbb{R}^d} e^{-\frac{1}{4}\langle Q_t^{-1}(y-z),y-z\rangle}f(z)\,dz =: (G(t)f)(y), \qquad t > 0,\ y \in \mathbb{R}^d. \tag{13.5}$$

Note that $(G(t)f)(y) = \int_{\mathbb{R}^d}\widetilde{K}(t,z)f(y-z)\,dz$, where

$$\widetilde{K}(t,z) := (4\pi)^{-\frac{d}{2}}(\det Q_t)^{-\frac{1}{2}}e^{-\frac{1}{4}|Q_t^{-1/2}z|^2}, \quad t > 0,\ z \in \mathbb{R}^d. \tag{13.6}$$

As in Lemma 4.1.2 one can prove that $\widetilde{K} \in C^\infty((0,\infty)\times\mathbb{R}^d)$ and so, by the dominated convergence theorem one concludes that $v \in C^\infty((0,\infty)\times\mathbb{R}^d)$.

To prove now that v is a classical solution to problem (13.4) we need to show that v is continuous on $\{0\}\times\mathbb{R}^d$, where it equals the function f. To this purpose let us fix $y_0 \in \mathbb{R}^d$. Using the fact that $\int_{\mathbb{R}^d}\widetilde{K}(t,z)\,dz = 1$ for all $t > 0$, we can estimate

$$\begin{aligned}|v(t,y) - f(y_0)| =& \left|\int_{\mathbb{R}^d}\widetilde{K}(t,z)f(y-z)\,dz - \int_{\mathbb{R}^d}\widetilde{K}(t,z)f(y_0)\,dz\right|\\ \le& \int_{\mathbb{R}^d}\widetilde{K}(t,z)|f(y-z) - f(y)|\,dz + |f(y) - f(y_0)|\end{aligned}$$

for y, $y_0 \in \mathbb{R}^d$. Performing the change of variable $x = Q_t^{-1/2}z$ we get

$$\int_{\mathbb{R}^d}\widetilde{K}(t,z)|f(y-z) - f(y)|\,dz = (4\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\frac{|x|^2}{4}}|f(y - Q_t^{1/2}x) - f(y)|dx$$

for every $t > 0$ and $y \in \mathbb{R}^d$. We proceed now as in the proof of Theorem 4.1.3 to obtain that $v(t,y)$ tends to $f(y_0)$ as $(t,y)\to(0,y_0)$. So we have proved that $v \in C([0,\infty)\times\mathbb{R}^d)\cap C^{1,2}((0,\infty)\times\mathbb{R}^d)$ and solves (13.4). Thus, by an easy computation one can check that the function

$$u(t,x) = v(t,e^{tB}x), \qquad t \ge 0,\ x \in \mathbb{R}^d$$

belongs to $C([0,\infty)\times\mathbb{R}^d)\cap C^{1,2}((0,\infty)\times\mathbb{R}^d)$ and solves the Cauchy problem (13.1).

To prove the uniqueness part of the statement, we proceed as in the proof of Proposition 11.0.1. We consider the function $\varphi : \mathbb{R}^d \to \mathbb{R}$, defined by $\varphi(x) = 1 + |x|^2$ for every $x \in \mathbb{R}^d$. As easy computation yields

$$\mathcal{L}\varphi(x) = 2\mathrm{Tr}\,Q + 2\langle Bx, x\rangle, \qquad x \in \mathbb{R}^d.$$

So, there exists a positive constant λ such that $\mathcal{L}\varphi < \lambda\varphi$ on $\mathbb{R}^d$. Now, fix $T > 0$ and define the function $v_n(t,x) = e^{-\lambda t}u(t,x) - n^{-1}\varphi(x)$ for $n \in \mathbb{N}$ and $(t,x) \in [0,T]\times\mathbb{R}^d$, where u is a classical solution of (13.1) with $f \equiv 0$. Thus, the same proof as the one of Proposition 11.0.1 yields $u \le 0$ on $[0,T]\times\mathbb{R}^d$. The same arguments can be applied to $-u$ instead of u. So, one concludes that $u = 0$ on $[0,T]\times\mathbb{R}^d$ for every $T > 0$ and the uniqueness follows. □

For every $t > 0$, denote by $T_{OU}(t)$ the operator defined on $C_b(\mathbb{R}^d)$ by $T_{OU}(t)f = u(t, \cdot)$ where u is the solution to the Cauchy problem (13.1) provided by Theorem 13.1.1. This theorem also shows that $T_{OU}(t)$ is a bounded linear operator and $\|T_{OU}(t)\|_{L(C_b(\mathbb{R}^d))} \le 1$. Actually, one can show that $\|T_{OU}(t)\|_{C_b(\mathbb{R}^d)} = 1$ (see Exercise 13.4.2). Moreover, by the uniqueness of the classical solution to the Cauchy problem (13.1), the family $\{T_{OU}(t)\}$ is a semigroup of bounded linear operators on $C_b(\mathbb{R}^d)$, commonly referred to as the *Ornstein-Uhlenbeck semigroup*. Hence, two questions naturally arise:

- is the Ornstein-Uhlenbeck semigroup strongly continuous on $C_b(\mathbb{R}^d)$?
- is the Ornstein-Uhlenbeck semigroup analytic on $C_b(\mathbb{R}^d)$?

We will see that both the two previous questions have negative answer. As the Gauss-Weierstrass semigroup, $\{T_{OU}(t)\}$ is not strongly continuous on $C_b(\mathbb{R}^d)$ since it maps $C_b(\mathbb{R}^d)$ into $C_b^k(\mathbb{R}^d)$ for every $k \in \mathbb{N}$ as the following theorem shows. Hence, if $T_{OU}(t)f$ converges uniformly to f as $t \to 0^+$, then f should belong to $BUC(\mathbb{R}^d)$.

Theorem 13.1.2. *For every $t > 0$ and $f \in C_b(\mathbb{R}^d)$, the function $T_{OU}(t)f$ belongs to $C_b^k(\mathbb{R}^d)$ for each $k \in \mathbb{N}$. In particular, if $f \in C_b^1(\mathbb{R}^d)$,*

$$D_i T_{OU}(t)f = \sum_{j=1}^{d} (e^{tB^*})_{ij} T_{OU}(t) D_j f, \qquad t > 0,\ i, j = 1, \ldots, d. \tag{13.7}$$

Moreover, for each $\varepsilon > 0$ and $h \in \mathbb{N}$ with $h \le k$, there exists a positive constant $C = C_\varepsilon$ such that

$$\|D_x^\alpha T_{OU}(t)f\|_\infty \le C e^{k(s(B)+\varepsilon)t}(1 \vee t^{-\frac{k-h}{2}})\|f\|_{C_b^h(\mathbb{R}^d)}, \qquad t > 0, \tag{13.8}$$

for every $f \in C_b^h(\mathbb{R}^d)$ and $|\alpha| = k$, where $s(B)$ is the spectral bound[2] of the matrix B.

Proof. Formula (13.7) as well as estimate (13.8) with $h = 0$ and $k = 1$, follow easily differentiating (13.3) under the integral sign and observing that, for each $\delta > 0$, there exists a constant $\widetilde{C}_1 = \widetilde{C}_1(\delta)$ such that

$$\|e^{tB}\|_\infty \le \widetilde{C}_1 e^{(s(B)+\delta)t}, \qquad t > 0. \tag{13.9}$$

To prove (13.8) with $h = 0$ and $k \in \mathbb{N}$, we argue by induction. For this purpose, we suppose that $T_{OU}(t)f$ belongs to $C_b^k(\mathbb{R}^d)$ for every $t > 0$ and (13.8) is satisfied. Let us show that the $(k+1)$-th order derivatives $D_x^\alpha T_{OU}(t)f$ are bounded on $\mathbb{R}^d$ and satisfy (13.8). We fix any such derivative $D_x^\alpha = D_{i_1,\ldots,i_{k+1}}$ and observe that (13.7) yields the following formula:

$$\begin{aligned} D_{i_2,\ldots,i_{k+1}} T_{OU}(t)f =& D_{i_2,\ldots,i_{k+1}} T_{OU}((1-1/n)t) T_{OU}(t/n)f \\ =& D_{i_2,\ldots,i_k}\left\{ T_{OU}((1-1/n)t) \sum_{j_{k+1}=1}^{d} (e^{\frac{n-1}{n}B^* t})_{i_{k+1} j_{k+1}} D_{j_{k+1}} T_{OU}(t/n)f \right\} \end{aligned}$$

---

[2] The spectral bound of a matrix B is the supremum of the real parts of its eigenvalues

$$\begin{aligned}&=T_{OU}((1-k/n)t)\sum_{j_2,\dots,j_{k+1}=1}^{d}(e^{\frac{n-k}{n}tB^*})_{i_2j_2}\cdot\ldots\cdot(e^{\frac{n-1}{n}tB^*})_{i_{k+1}j_{k+1}}\\&\qquad\qquad\times D_{j_2}T_{OU}(t/n)\circ\ldots\circ D_{j_{k+1}}T_{OU}(t/n)f\end{aligned}\tag{13.10}$$

for every $n>k$ and $t>0$. Similarly,

$$\begin{aligned}&D_{j_2}T_{OU}(t/n)\circ\ldots\circ D_{j_{k+1}}T_{OU}(t/n)f\\&=T_{OU}((k/n-1/n)t)\sum_{l_2,\dots,l_k=1}^{d}(e^{\frac{1}{n}tB^*})_{j_kl_k}\cdot\ldots\cdot(e^{\frac{k-1}{n}tB^*})_{j_2l_2}D_{l_2,\dots,l_k,j_{k+1}}T_{OU}(t/n)f,\end{aligned}\tag{13.11}$$

for each $t>0$. From formulas (13.10), (13.11) and estimate (13.8) (with $k=1$) we conclude that $D_{i_2,\dots,i_{k+1}}T_{OU}(t)f$ belongs to $C^1_b(\mathbb{R}^d)$. Moreover,

$$\begin{aligned}D^\alpha_xT_{OU}(t)f=&\sum_{j_2,\dots,j_{k+1}=1}^{d}(e^{\frac{n-k}{n}tB^*})_{i_2j_2}\cdot\ldots\cdot(e^{\frac{n-1}{n}tB^*})_{i_{k+1}j_{k+1}}\\&\times\sum_{l_1,\dots,l_k=1}^{d}(e^{\frac{1}{n}tB^*})_{j_kl_k}\cdot\ldots\cdot(e^{\frac{k-1}{n}tB^*})_{j_2l_2}(e^{\frac{n-2}{n}tB^*})_{i_1l_1}\\&\times T_{OU}((1-2/n)t)D_{l_1}T_{OU}(t/n)D_{l_2,\dots,l_k,j_{k+1}}T_{OU}(t/n)f.\end{aligned}\tag{13.12}$$

Therefore, taking $\delta=1/n$ in (13.9) and using (13.8) with $|\alpha|=1$ and $|\alpha|=k$ and $\varepsilon=1/n$, we get

$$\|D^\alpha_xT_{OU}(t)f\|_\infty\le C_n\exp\left\{\left(k+1-\frac{1}{n}\right)\left(s(B)+\frac{1}{n}\right)t\right\}(1\vee t^{-\frac{k+1}{2}})\|f\|_\infty,$$

for every $t>0$ and some positive constant C_n. Since $(k+1-n^{-1})(s(B)+n^{-1})$ converges to $(k+1)s(B)$ as $n\to\infty$, estimate (13.8) follows taking n sufficiently large.

Estimate (13.8) with $h=k$ follows immediately from (13.9). Indeed, a straightforward computation shows that

$$D_{i_1,\dots,i_k}T_{OU}(t)f=\sum_{j_1,\dots,j_k=1}^{d}(e^{tB^*})_{i_1j_1}\cdot\ldots\cdot(e^{tB^*})_{i_kj_k}T_{OU}(t)D_{j_1,\dots,j_k}f,\tag{13.13}$$

for each $i_1,\dots,i_k\in\{1,\dots,d\}$, $k\in\mathbb{N}$ and $t>0$.

Finally, in the case when $0<h<k$, estimate (13.8) follows from (13.12) (with $|\alpha|=k$) observing that, thanks to (13.13), we have

$$\begin{aligned}&D_{l_1}T_{OU}(t/n)D_{l_2,\cdots,l_{k-1},j_k}T_{OU}(t/n)f\\&=\sum_{r_1,\dots,r_{h+1}=1}^{d}(e^{\frac{1}{n}tB^*})_{l_1r_1}(e^{\frac{1}{n}tB^*})_{l_{k-h+1}r_2}\cdot\ldots\cdot(e^{\frac{1}{n}tB^*})_{l_{k-1}r_h}(e^{\frac{1}{n}tB^*})_{j_kr_{h+1}}\\&\qquad\times T_{OU}(t/n)(D_{r_1,l_2,\dots,l_{k-h}}T_{OU}(t/n)D_{r_2,\dots,r_{h+1}}f)\end{aligned}\tag{13.14}$$

for each $t>0$. □

Remark 13.1.3. (i) From the proof of Theorem 13.1.2, it is clear that, if (13.9) holds with $\delta = 0$, then the estimate (13.8) can be written also with $\varepsilon = 0$. This is, for instance, the case when B is diagonalizable on $\mathbb{R}^d$.

(ii) We stress that estimate (13.8) is useful to estimate the long time behaviour (with respect to t) of the spatial derivatives of the function $T_{OU}(t)f$ when the spectral bound $s(B)$ of the matrix B is negative. Indeed, using the semigroup property, we can split $T_{OU}(t)f = T_{OU}(1)T_{OU}(t-1)f$ for each $t > 1$ so that, from (13.8) with $h = 0$ and $t = 1$, it follows that

$$\|D_x^\alpha T_{OU}(t)f\|_\infty \le Ce^{|\alpha|(s(B)+\varepsilon)}\|T_{OU}(t-1)f\|_\infty \le Ce^{|\alpha|(s(B)+\varepsilon)}\|f\|_\infty.$$

Corollary 13.1.4. *For every $t > 0$, $0 \le \alpha \le \theta \le 4$ there exists a positive constant $C_{\alpha,\theta,T}$ such that*

$$\|T_{OU}(t)f\|_{C_b^\theta(\mathbb{R}^d)} \le C_{\alpha,\theta,T}t^{-\frac{\theta-\alpha}{2}}\|f\|_{C_b^\alpha(\mathbb{R}^d)}, \qquad t \in (0,T],\ \ f \in C_b^\alpha(\mathbb{R}^d). \tag{13.15}$$

Proof. The proof can be obtained adapting the arguments in the proof of Theorem 4.1.12, so that the details are left to the reader. □

Remark 13.1.5. Note that in view of the estimates in Theorem 13.1.2, the restriction $\alpha, \theta \le 4$ can be removed in Corollary 13.1.4. Moreover, one can also easily show that, if $s(B) < 0$, then,

$$\|T_{OU}(t)f\|_{C_b^\theta(\mathbb{R}^d)} \le C_{\alpha,\theta}(1 \vee t^{-\frac{\theta-\alpha}{2}})e^{\omega_{\alpha,\theta}t}\|f\|_{C_b^\alpha(\mathbb{R}^d)}, \qquad t > 0,\ \ f \in C_b^\alpha(\mathbb{R}^d).$$

for some constants $C_{\alpha,\theta} > 0$ and $\omega_{\alpha,\theta} \le 0$.

As we have shown above, $f \in BUC(\mathbb{R}^d)$ is a necessary condition for $T_{OU}(t)f$ to converge to f, uniformly on $\mathbb{R}^d$ as $t \to 0^+$. Actually, this condition is not sufficient as the following proposition shows.

Proposition 13.1.6. *Let $f \in C_b(\mathbb{R}^d)$. Then, $T_{OU}(t)f$ converges to f uniformly in $\mathbb{R}^d$, as t tends to 0^+, if and only if $f \in BUC(\mathbb{R}^d)$ and $\|f(e^{tB}\cdot) - f\|_\infty$ tends to 0 as $t \to 0^+$.*

Proof. Fix a nontrivial $f \in BUC(\mathbb{R}^d)$. By the proof of Theorem 13.1.1 we know that $(T_{OU}(t)f)(x) = (G(t)f)(e^{tB}x)$ for every $t > 0$ and $x \in \mathbb{R}^d$, where

$$(G(t)f)(x) = (4\pi)^{-\frac{d}{2}}(\det Q_t)^{-\frac{1}{2}}\int_{\mathbb{R}^d} f(x-y)e^{-\frac{1}{4}\langle Q_t^{-1}y,y\rangle}dy, \qquad x \in \mathbb{R}^d.$$

We claim that $G(t)f$ tends to f uniformly in $\mathbb{R}^d$ as t tends to 0. Indeed, for every $n \in \mathbb{N}$, $t > 0$ and $x \in \mathbb{R}^d$ it holds that

$$|(G(t)f)(x) - f(x)| = \left|(4\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\frac{1}{4}|y|^2}\big(f(x+\sqrt{Q_t}y) - f(x)\big)dy\right|$$

$$
\begin{aligned}
&\leq (4\pi)^{-\frac{d}{2}}\int_{B(0,n)} e^{-\frac{1}{2}|y|^2}|f(x+\sqrt{Q_t}y)-f(x)|dy\\
&\quad + 2(4\pi)^{-\frac{d}{2}}\|f\|_\infty \int_{\mathbb{R}^d\setminus B(0,n)} e^{-\frac{1}{2}|y|^2}dy. \qquad (13.16)
\end{aligned}
$$

Since $|\sqrt{Q_t}y| \leq \|\sqrt{Q_t}\||y| \leq \|\sqrt{Q_t}\|n$ for every $t>0$ and $|y|\leq n$, and $\|\sqrt{Q_t}\|_\infty$ tends to 0 as $t\to 0^+$, it follows that $\sup_{x\in\mathbb{R}^d}\sup_{|y|\leq n}|f(x+\sqrt{Q_t}y)-f(x)|$ vanishes as $t\to 0^+$ for every $n\in\mathbb{N}$. Hence, letting t tend to 0^+ in the first and last side of (13.16) yields

$$\limsup_{t\to 0^+}\|G(t)f-f\|_\infty \leq 2(4\pi)^{-\frac{d}{2}}\|f\|_\infty \int_{\mathbb{R}^d\setminus B(0,n)} e^{-\frac{1}{2}|y|^2}dy$$

for every $n\in\mathbb{N}$. Letting n tend to ∞, we conclude that $\limsup_{t\to 0^+}\|G(t)f-f\|_\infty=0$, so that $G(t)f$ converges to f uniformly on $\mathbb{R}^d$ as $t\to 0^+$.

Writing

$$(T_{OU}(t)f)(x)-f(x) = (G(t)f)(e^{tB}x)-f(e^{tB}x)+f(e^{tB}x)-f(x), \qquad t>0,\ \ x\in\mathbb{R}^d,$$

the assertion follows at once. □

The following proposition proves that the Ornstein-Uhlenbeck semigroup is not analytic.

Proposition 13.1.7. *The Ornstein-Uhlenbeck semigroup is not analytic neither in $C_b(\mathbb{R}^d)$ nor in $BUC(\mathbb{R}^d)$.*

Proof. Indeed, if it were analytic in $C_b(\mathbb{R}^d)$, it would be analytic also in $BUC(\mathbb{R}^d)$, since the Ornstein-Uhlenbeck semigroup leaves $BUC(\mathbb{R}^d)$ invariant, as a byproduct of Theorem 13.1.2.

By contradiction, suppose that $\{T_{OU}(t)\}$ is analytic in $BUC(\mathbb{R}^d)$. Denote by A its infinitesimal generator. Then, $D_tT_{OU}(t)f = AT_{OU}(t)f$ for every $t>0$, by property (iv) in Theorem 3.2.2, whereas $D_tT_{OU}(t)f=\mathcal{L}T_{OU}(t)f$ due to Theorem 13.1.1, It thus follows that $AT_{OU}(t)f=\mathcal{L}T_{OU}(t)f$ for every $t>0$ and $f\in C_b(\mathbb{R}^d)$. The same theorem (see property (iii)) shows that $AT_{OU}(1)$ is a bounded operator in $C_b(\mathbb{R}^d)$. We will show that this is not the case, proving that there exists $h_0\in\mathbb{R}^d$ such that the function $\mathcal{L}T_{OU}(1)f_{h_0}$ in unbounded in $\mathbb{R}^d$, where $f_{h_0}(x)=\sin(\langle h_0,x\rangle)$ for each $x\in\mathbb{R}^d$. For this purpose, we observe that

$$(T_{OU}(1)f_{h_0})(x) = e^{-\frac{1}{2}\langle Q_1h_0,h_0\rangle}\sin(\langle e^Bx,h_0\rangle), \qquad x\in\mathbb{R}^d. \qquad (13.17)$$

From (13.17) it follows that the function $\mathrm{Tr}\,(QD^2_xT_{OU}(1)f_{h_0})$ is bounded in $\mathbb{R}^d$ for each $h_0\in\mathbb{R}^d$. On the other hand,

$$(B^*(\nabla_xT_{OU}(1)f_{h_0})(x))_i = (B^*e^{B^*}h)_ie^{-\frac{1}{2}\langle Q_1h_0,h_0\rangle}\cos(\langle e^Bx,h_0\rangle),$$

for every $x\in\mathbb{R}^d$ and $i=1,\dots,d$. Since $B\neq 0$, we can choose $h_0\in\mathbb{R}^d$ and $j\in\{1,\dots,d\}$ such that $(B^*e^{B^*}h_0)_j\neq 0$. Thus, it easily follows that

$$\sup_{x\in\mathbb{R}^d}|\langle Bx,(\nabla_xT_{OU}(1)f_{h_0})(x)\rangle| \geq \sup_{\sigma\in\mathbb{R}}|\sigma Be_j,(\nabla_xT_{OU}(1)f_{h_0})(\sigma e_j)\rangle|$$

$$=\sup_{\tau\in\mathbb{R}}|\tau(B^*e^{B^*}h_0)_j e^{-\frac{1}{2}\langle Q_1h_0,h_0\rangle}\cos(\tau(e^{B^*}h_0)_j)|=\infty.$$

As a byproduct, we conclude that the function $AT_{OU}(1)f_{h_0}=\mathcal{L}T_{OU}(1)f_{h_0}$ is unbounded on $\mathbb{R}^d$: a contradiction. □

13.2 Optimal spatial Schauder estimates

In this section, we want to prove the counterpart of Theorem 6.1.4, i.e., the following result.

Theorem 13.2.1. *Fix $T>0$, $\alpha\in(0,1)$, $f\in C_b^{2+\alpha}(\mathbb{R}^d)$ and $g\in C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)$. Then, the Cauchy problem*

$$\begin{cases} D_tu(t,x)=\mathcal{L}u(t,x)+g(t,x), & t\in[0,T], \quad x\in\mathbb{R}^d,\\ u(0,x)=f(x), & x\in\mathbb{R}^d,\end{cases}\tag{13.18}$$

admits a unique solution $u\in C^{1,2}((0,T]\times\mathbb{R}^d)\cap C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)$. Moreover, there exists a positive constant $C=C_T$ such that

$$\sup_{t\in[0,T]}\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}\le C\bigg(\|f\|_{C_b^{2+\alpha}(\mathbb{R}^d)}+\sup_{t\in[0,T]}\|g(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\bigg).\tag{13.19}$$

Remark 13.2.2. Differently from the case of elliptic operators with bounded coefficients, in general we can expect neither any additional smoothness properties on u, than we stated in Theorem 13.2.1 nor the boundedness of the time derivative. This is a typical feature of elliptic operators with unbounded coefficients, which do not exhibit a nice behaviour with respect to the time variable. In the case of the Ornstein-Uhlenbeck operator $\mathcal{L}$, this is easy to check. Indeed, the proof of Proposition 13.1.7 shows that the function $T(\cdot)f_{h_0}$, which solves the equation $D_tv=\mathcal{L}v$, has unbounded time derivative over $\mathbb{R}^d$.

In the proof of Theorem 13.2.1 we shall make use of the following result, see Exercise 13.4.6.

Lemma 13.2.3. *A function $f:\mathbb{R}^d\to\mathbb{R}$ belongs to $C_b^\theta(\mathbb{R}^d)$ for some $\theta\in(0,1)$ if and only if it is bounded and*

$$[[f]]_\theta=\sup_{x,y\in\mathbb{R}^d,\ x\neq y}\frac{|f(x)-2f(2^{-1}(x+y))+f(y)|}{|x-y|^\theta}<\infty.$$

Moreover, the classical norm of $C_b^\theta(\mathbb{R}^d)$ is equivalent to the norm $\|f\|_\infty+[[f]]_\theta$.

Proof of Theorem 13.2.1. The uniqueness of the solution $u\in C^{1,2}((0,T]\times\mathbb{R}^d)\cap C_b([0,T]\times\mathbb{R}^d)$ of the Cauchy problem (13.18), follows from Theorem 13.1.1.

So, let us prove the existence part. In view of Corollary 13.1.4, it suffices to prove that the function $v:[0,T]\times\mathbb{R}^d\to\mathbb{R}$, defined by

$$v(t,x)=\int_0^t(T_{OU}(t-s)g(s,\cdot))(x)ds, \qquad t\in[0,T]\times\mathbb{R}^d,$$

belongs to $C^{1,2}([0,T]\times\mathbb{R}^d)\cap C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)$, solves the Cauchy problem

$$\begin{cases} D_t v(t,x)=\mathcal{L}v(t,x)+g(t,x), & t\in[0,T], \quad x\in\mathbb{R}^d,\\ v(0,x)=0, & x\in\mathbb{R}^d \end{cases} \tag{13.20}$$

and satisfies the estimate

$$\|v\|_{C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)}\le C\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}, \tag{13.21}$$

where here and in the rest of the proof, C denotes a positive constant, independent of g, v (and in the sequel also of t), which may vary from line to line. Slightly modifying the arguments in the proof of Theorem 6.1.4. The only remarkable difference is in the way one can prove that v belongs to $C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)$ and satisfies estimate (13.21). Nevertheless, for the reader's convenience we provide an almost full proof of the remaining part of the theorem, just leaving the proof of the continuity on $[0,T]\times\mathbb{R}^d$ of the function v to the reader, as an exercise. The rest of the proof is split into two steps.

Step 1. Here, we prove that the function v belongs to $C_b^{0,2+\alpha}([0,T]\times\mathbb{R}^d)$ and satisfies estimate (13.21). For this purpose, we begin by observing that, by Corollary 13.1.4, for any $0\le s<t\le T$, the function $T_{OU}(t-s)g(s,\cdot)$ is differentiable in $\mathbb{R}^d$ and $\|\nabla_x T_{OU}(t-s)g(s,\cdot)\|_\infty\le c(t-s)^{\frac{\alpha-1}{2}}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}$. Hence, the dominated convergence theorem shows that $u_1(t,\cdot)$ is differentiable in $\mathbb{R}^d$ for any $t\in[0,T]$ and

$$\nabla_x v(t,x)=\int_0^t(\nabla_x T_{OU}(t-s)g(s,\cdot))(x)ds, \qquad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Similarly, since $\|D_x^2T_{OU}(t-s)g(s,\cdot)\|_\infty\le c(t-s)^{\frac{\alpha}{2}-1}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}$, we deduce that $u_1(t,\cdot)$ is twice differentiable in $\mathbb{R}^d$ and

$$D_x^2 v(t,x)=\int_0^t(D_x^2 T_{OU}(t-s)g(s,\cdot))(x)ds, \qquad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Moreover,

$$\begin{aligned}|D_x^j v(t,x)|\le&\int_0^t|(D_x^jT_{OU}(t-s)g(s,\cdot))(x)|ds\le\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\int_0^t(t-s)^{\frac{\alpha-j}{2}}ds\\ =&\frac{2}{\alpha-j+2}T^{\frac{\alpha-j+2}{2}}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\end{aligned}$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$ and $j=1,2$. We thus conclude that

$$\|v\|_{C_b^{0,2}([0,T]\times\mathbb{R}^d)}\le C\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}. \tag{13.22}$$

As in the case of the Laplacian, proving that the second order derivatives of v are in $C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)$ is much more tricky since the same arguments used to prove (13.22) lead us to a singular integral. We also can not adapt the arguments used for the Gauss-Weierstrass semigroup. We rather have to take advantage of a different argument, which is a hidden use of interpolation[3]. For this purpose, we fix $i, j \in \{1, \ldots, d\}$, $t \in (0, T]$ and split $D_{ij}v(t, \cdot) = a_\xi(t, \cdot) + b_\xi(t, \cdot)$, where,

$$a_\xi(t,x) = \begin{cases} \displaystyle\int_{t-\xi}^{t} (D_{ij}T_{OU}(t-s)g(s,\cdot))(x)ds, & \text{if } \xi \le t, \\ \displaystyle\int_0^t (D_{ij}T_{OU}(t-s)g(s,\cdot))(x)ds, & \text{if } \xi > t, \end{cases}$$

$$b_\xi(t,x) = \begin{cases} \displaystyle\int_0^{t-\xi} (D_{ij}T_{OU}(t-s)g(s,\cdot))(x)ds, & \text{if } \xi \le t, \\ 0, & \text{if } \xi > t, \end{cases}$$

for each $x \in \mathbb{R}^d$ and $\xi \in (0,1)$. Taking estimate (13.15) into account, we easily deduce that $a_\xi(t, \cdot) \in C_b(\mathbb{R}^d)$ and $b_\xi(t, \cdot) \in C_b^2(\mathbb{R}^d)$ for every $t \in [0, T]$. Moreover,

(13.23)
$$\|a_\xi(t,\cdot)\|_\infty \le C\int_{\max(t-\xi,0)}^{t} (t-s)^{-1+\frac{\alpha}{2}}ds\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)} \le C\xi^{\frac{\alpha}{2}}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)},$$

(13.24)
$$\|b_\xi(t,\cdot)\|_{C_b^2(\mathbb{R}^d)} \le C\int_0^{\max(t-\xi,0)} (t-s)^{-2+\frac{\alpha}{2}}ds\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)} \le C\xi^{-1+\frac{\alpha}{2}}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)},$$

the constant C being independent of ξ. Next we observe that

$$\begin{aligned} &|D_{ij}v(t,x) - 2D_{ij}v(t,2^{-1}(x+y)) + D_{ij}v(t,y)| \\ \le &|a_\xi(t,x) - 2a_\xi(t,2^{-1}(x+y)) + a_\xi(t,y)| + |b_\xi(t,x) - 2b_\xi(t,2^{-1}(x+y)) + b_\xi(t,y)| \end{aligned}$$

for every $(t,x) \in [0,T] \times \mathbb{R}^d$. As it is easily seen,

$$|a_\xi(t,x) - 2a_\xi(t,2^{-1}(x+y)) + a_\xi(t,y)| \le 4\|a_\xi(t,\cdot)\|_\infty, \qquad t \in [0,T],\ x \in \mathbb{R}^d. \tag{13.25}$$

On the other hand,

$$\begin{aligned} &|b_\xi(t,x) - 2b_\xi(t,2^{-1}(x+y)) + b_\xi(t,y)| \\ =&\left|\int_0^1 \frac{d}{ds}b_\xi(t,sx+(1-s)2^{-1}(x+y))ds + \int_0^1 \frac{d}{ds}b_\xi(t,sy+(1-s)2^{-1}(x+y))ds\right| \\ =&\frac{1}{2}\left|\int_0^1 \langle \nabla b_\xi(t,sx+(1-s)2^{-1}(x+y)) - \nabla b_\xi(t,sy+(1-s)2^{-1}(x+y)), x-y\rangle ds\right| \end{aligned}$$

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[3] We do not think convenient here to introduce the basic results from interpolation theory which should be used. We rather prefer to adapt some of those techniques to our situation

$$\leq |x-y|\int_0^1 |\nabla b_\xi(t, sx+(1-s)2^{-1}(x+y)) - \nabla b_\xi(t, sy+(1-s)2^{-1}(x+y))|ds$$
$$\leq |x-y|\sum_{h=1}^d\int_0^1 |D_h b_\xi(t, sx+(1-s)2^{-1}(x+y)) - D_h b_\xi(t, sy+(1-s)2^{-1}(x+y))|ds$$
$$\leq |x-y|^2\|b_\xi(t,\cdot)\|_{C^2_b(\mathbb{R}^d)} \tag{13.26}$$

for every $t \in [0,T]$ and $x \in \mathbb{R}^d$. From (13.25) and (13.26) we conclude that

$$|D_{ij}v(t,x)-2D_{ij}v(t,2^{-1}(x+y))+D_{ij}v(t,y)| \leq 4\|a_\xi(t,\cdot)\|_\infty+|x-y|^2\|b_\xi(t,\cdot)\|_{C^2_b(\mathbb{R}^d)} \tag{13.27}$$

Choosing $\xi = |x-y|^2$ and using (13.23) and (13.24) we get

$$|D_{ij}v(t,x) - 2D_{ij}v(t, 2^{-1}(x+y)) + D_{ij}v(t,y)| \leq C\|g\|_{C^{0,\theta}_b([0,T]\times\mathbb{R}^d)}|x-y|^\alpha$$

for every $t \in [0,T]$ and $x \in \mathbb{R}^d$, so that Lemma 13.2.3 allows us to conclude that $D_{ij}v(t,\cdot)$ belongs to $C^\alpha_b(\mathbb{R}^d)$ and

$$\sup_{t\in[0,T]}\|D_{ij}v(t,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)} \leq C\|g\|_{C^{0,\alpha}_b([0,T]\times\mathbb{R}^d)}. \tag{13.28}$$

From (13.22) and (13.28), estimate (13.21) follows at once.

Step 2. Here, we prove that v is continuously differentiable with respect to t in $[0,T]\times\mathbb{R}^d$. For notational convenience, we denote by D^+_t (resp. D^-_t) the right (resp. left) time derivative.

For each $\varepsilon \in (0,1)$ we introduce the function $v_\varepsilon : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ defined by

$$v_\varepsilon(t,x) = \int_0^{\varepsilon t}(T_{OU}(t-s)g(s,\cdot))(x)ds, \qquad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Note that v_ε converges to v as $\varepsilon \to 1^-$, uniformly in $[0,T]\times\mathbb{R}^d$. Indeed,

$$|v(t,x)-v_\varepsilon(t,x)| = \left|\int_{\varepsilon t}^t (T_{OU}(t-s)g(s,\cdot))(x)ds\right| \leq \|g\|_{C_b([0,T]\times\mathbb{R}^d)}(1-\varepsilon)T$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$ and all $\varepsilon\in(0,1)$.

To prove that v_ε is differentiable in $[0,T]\times\mathbb{R}^d$ with respect to t, we fix $t\in[0,T)$, $h\in(0,T-t)$ and split

$$\begin{aligned}\frac{v_\varepsilon(t+h,x)-v_\varepsilon(t,x)}{h} =& \frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)}(T_{OU}(t+h-s)g(s,\cdot))(x)ds\\ &+\int_0^{\varepsilon t}\frac{(T_{OU}(t+h-s)g(s,\cdot))(x)-(T_{OU}(t-s)g(s,\cdot))(x)}{h}ds\\ =&: J^+_1(h)+J^+_2(h).\end{aligned}$$

We claim that $J_1^+(h)$ converges to $\varepsilon(T_{OU}((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)$ as $h\to 0^+$. Indeed,

$$\begin{aligned}
&J_1^+(h)-\varepsilon(T_{OU}((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)\\
=&\frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)}(T_{OU}(t+h-s)g(s,\cdot))(x)ds-\varepsilon(T_{OU}((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)\\
=&\frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)}[(T_{OU}(t+h-s)g(s,\cdot))(x)-(T_{OU}((1-\varepsilon)t)g(s,\cdot))(x)]ds\\
&+\int_{\varepsilon t}^{\varepsilon(t+h)}\frac{1}{h}(T_{OU}((1-\varepsilon)t)(g(s,\cdot)-g(\varepsilon t,\cdot)))(x)\,ds.
\end{aligned}$$

It is clear that the second integral term in the last side of the previous chain of equalities converges to 0 as $h\to 0^+$. As far as the first integral term is concerned, we observe that the function $(r,s)\mapsto (T_{OU}(r)g(s,\cdot))(x)$ is continuous on $[0,T]\times[0,T]$ (see Exercise 13.4.7). Therefore, it is therein uniformly continuous. Hence, for each $\rho>0$ there exists $\delta>0$ such that $|(T_{OU}(r_2)g(s,\cdot))(x)-(T_{OU}(r_1)g(s,\cdot))(x)|\le\rho$ if $|r_2-r_1|\le\delta$ and $s\in[0,T]$. Consequently, if $|h|\le\delta$, then $|(T_{OU}(t+h-s)g(s,\cdot))(x)-(T_{OU}((1-\varepsilon)t)g(s,\cdot))(x)|\le\rho$ for $s\in[\varepsilon t,\varepsilon(t+h)]$ and

$$\frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)}[(T_{OU}(t+h-s)g(s,\cdot))(x)-(T_{OU}((1-\varepsilon)t)g(s,\cdot))(x)]ds\le\varepsilon\rho$$

Therefore,

$$\lim_{h\to 0}\frac{1}{h}\int_{\varepsilon t}^{\varepsilon(t+h)}[(T_{OU}(t+h-s)g(s,\cdot))(x)-(T_{OU}((1-\varepsilon)t)g(s,\cdot))(x)]ds=0.$$

As far as the term $J_2^+(h)$ is concerned, we observe that the function under the integral sign converges to $(D_tT_{OU}(t-s)g(s,\cdot))(x)=(\mathcal{L}T_{OU}(t-s)g(s,\cdot))(x)$ as $h\to 0^+$. Moreover, using the mean value theorem, we estimate

$$\begin{aligned}
\left|\frac{(T_{OU}(t+h-s)g(s,\cdot))(x)-(T_{OU}(t-s)g(s,\cdot))(x)}{h}\right|=&|(\mathcal{L}T_{OU}(t+\xi-s)g(s,\cdot))(x)|\\
\le&C_x(t-s)^{\frac{\alpha}{2}-1}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}
\end{aligned}$$

for all $s\in[0,\varepsilon t]$, where ξ is a suitable point on the line joining 0 and h and we have taken advantage of (13.15). Hence, we can apply the dominated convergence theorem and conclude that

$$\lim_{h\to 0^+}J_2^+(h)=\int_0^{\varepsilon t}(\mathcal{L}T_{OU}(t-s)g(s,\cdot))(x)ds.$$

We have so proved that v_ε is differentiable from the right in $[0,T)\times\mathbb{R}^d$, with respect to t, and

$$D_t^+v_\varepsilon(t,x)=\varepsilon(T_{OU}((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)+\int_0^{t\varepsilon}(\mathcal{L}T_{OU}(t-s)g(s,\cdot))(x)ds$$

$$=\varepsilon(T_{OU}((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)+\mathcal{L}v_\varepsilon(t,x).$$

In a similar way, it can be proved that v_ε is differentiable from the left in $(0,T]\times\mathbb{R}^d$, with respect to t and

$$D_t^- v_\varepsilon(t,x)=\varepsilon(T_{OU}((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)+\mathcal{L}v_\varepsilon(t,x).$$

Summing up, v_ε is differentiable with respect to t in $[0,T]\times\mathbb{R}^d$ and

$$D_t v_\varepsilon(t,x)=\varepsilon(T_{OU}((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)+\int_0^{t\varepsilon}(\mathcal{L}T_{OU}(t-s)g(s,\cdot))(x)ds$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$. Letting ε tend to 1^-, we conclude that D_tv_ε converges locally uniformly on $[0,T]\times\mathbb{R}^d$ to the function $g+\mathcal{L}v$. Indeed, for any $R>0$, any $t\in[0,T]$ and $x\in\overline{B(0,R)}$, we can estimate

$$\begin{aligned}
&\left|\varepsilon(T_{OU}((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)+\int_0^{t\varepsilon}(\mathcal{L}T_{OU}(t-s)g(s,\cdot))(x)ds-g(x)-\mathcal{L}v(t,x)\right|\\
\le&|\varepsilon(T_{OU}((1-\varepsilon)t)g(\varepsilon t,\cdot))(x)-g(t,x)|+\left|\int_{t\varepsilon}^t(\mathcal{L}T_{OU}(t-s)g(s,\cdot))(x)ds\right|\\
\le&|\varepsilon(T_{OU}((1-\varepsilon)t)[g(\varepsilon t,\cdot)-g(t,\cdot)])(x)|+(1-\varepsilon)|(T_{OU}((1-\varepsilon)t)g(t,\cdot))(x)|\\
&+|(T_{OU}((1-\varepsilon)t)g(t,\cdot))(x)-g(t,x)|+C\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\int_{t\varepsilon}^t(t-s)^{\frac{\alpha}{2}-1}ds\\
\le&(1-\varepsilon)^{\frac{\alpha}{2}}T^{\frac{\alpha}{2}}\|g\|_{C_b^{\alpha/2,0}([0,T]\times\mathbb{R}^d)}+(1-\varepsilon)\|g\|_{C_b([0,T]\times\mathbb{R}^d)}+\int_0^{(1-\varepsilon)t}|(D_sT_{OU}(s)g(t,\cdot))(x)|ds\\
&+2\alpha^{-1}C(1-\varepsilon)^{\frac{\alpha}{2}}T^{\frac{\alpha}{2}}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)}\\
\le&2(1-\varepsilon)^{\frac{\alpha}{2}}T^{\frac{\alpha}{2}}\|g\|_{C^{\alpha/2,0}([0,T]\times\mathbb{R}^d)}+(1-\varepsilon)\|g\|_{C_b([0,T]\times\mathbb{R}^d)}+4\alpha^{-1}C(1-\varepsilon)^{\frac{\alpha}{2}}T^{\frac{\alpha}{2}}\|g\|_{C_b^{0,\alpha}([0,T]\times\mathbb{R}^d)},
\end{aligned}$$

where the constant C is independent of $x\in\overline{B(0,R)}$. Letting ε tend to 1^- we conclude that $\varepsilon(T_{OU}(1-\varepsilon)\cdot)g(\varepsilon\cdot,\cdot)$ converges to $g+\mathcal{L}v$ uniformly on $[0,T]\times\overline{B(0,R)}$.

Since v_ε converges to v uniformly in $[0,T]\times\mathbb{R}^d$, it follows that v is differentiable with respect to t in $[0,T]\times\mathbb{R}^d$ and therein $D_tv=\mathcal{L}v+g$. The proof is now complete. □

13.3 Notes

As we have proved, the Ornstein-Uhlenbeck semigroup is neither strongly continuous nor analytic in $C_b(\mathbb{R}^d)$ and in $BUC(\mathbb{R}^d)$, so that we cannot define the concept of infinitesimal generator in the sense described in Lectures 2 and 3. Nevertheless, we can still associate a "generator" with the Ornstein-Uhlenbeck semigroup (and more generally with semigroups associated with elliptic operators with unbounded coefficients), the so-called weak generator, which has properties similar to those of the infinitesimal generator. The weak generator A may be defined in three equivalent ways.

(i) The family $\{R(\lambda) : \lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$, defined by

$$(R(\lambda)f)(x) = \int_0^{+\infty} e^{-\lambda t}(T_{OU}(t)f)(x)dt, \qquad x \in \mathbb{R}^d, \ \mathrm{Re}\lambda > 0, \ f \in C_b(\mathbb{R}^d)$$

is a resolvent family. Hence, there exists a closed operator A such that $R(\lambda) = R(\lambda, A)$ for any λ as above. This is the approach in [1] and [4],

(ii) The second definition is based on the bounded pointwise convergence: a sequence $\{f_n\} \subset C_b(\mathbb{R}^d)$ is said to be boundedly and pointwise convergent to $f \in C_b(\mathbb{R}^d)$ if there exists a positive constant C such that $\|f_n\|_\infty \le C$ for any $n \in \mathbb{N}$ and if $f_n(x)$ converges to $f(x)$ for any $x \in \mathbb{R}^d$. This notion of convergence leads to the following definition of the weak generator which was introduced in [6, 7]:

(13.29)

$$\begin{cases} D(A) = \Big\{ f \in C_b(\mathbb{R}^d) : \ \sup_{t \in (0,1)} \frac{\|T_{OU}(t)f - f\|_\infty}{t} < +\infty \ \text{ and } \ \exists g \in C_b(\mathbb{R}^d) : \\ \qquad\qquad \displaystyle\lim_{t \to 0^+} \frac{(T_{OU}(t)f)(x) - f(x)}{t} = g(x) \ \ \forall x \in \mathbb{R}^d \Big\}, \\ (A_2 f)(x) = \lim_{t \to 0^+} \frac{(T_{OU}(t)f)(x) - f(x)}{t}, \qquad x \in \mathbb{R}^d, \qquad f \in D(A). \end{cases}$$

(iii) The third definition is based on the notion of mixed topology introduced in [8]. The mixed topology τ^M is the finest locally convex topology which agrees on every norm–bounded subsets of $C_b(\mathbb{R}^d)$ with the topology of the uniform convergence on compact sets (see e.g., [3]). See also [5] for a similar approach.

Operator A can be defined as the infinitesimal generator of the semigroup in the mixed topology, that is

$$\begin{cases} D(A) = \Big\{ f \in C_b(\mathbb{R}^d) : \ \exists g \in C_b(\mathbb{R}^d) : \ \tau^M\text{–}\displaystyle\lim_{t \to 0^+} \frac{T_{OU}(t)f - f}{t} = g \Big\}, \\ Af = \tau^M\text{–}\displaystyle\lim_{t \to 0^+} \frac{T_{OU}(t)f - f}{t}, \qquad f \in D(A). \end{cases} \tag{13.30}$$

We also underline another crucial difference between bounded and unbounded coefficients in the study of nonhomogeneous Cauchy problems. In the first case, starting from the heat equation and, roughly speaking, freezing the coefficients and using the continuity method, we have analyzed more general operators. For elliptic operators with unbounded coefficients the method of freezing the coefficients does not work in general. Hence, different strategies should be used to study such problems and typically some conditions on the growth at infinity of the coefficients of the elliptic operator should be assumed.

Again roughly speaking, we can say that the Ornstein-Uhlenbeck semigroup can be used as a "test" to see what properties one can expect and what properties in general one can not expect from a semigroup associated with an elliptic operator with unbounded coefficients. For instance, as we have seen in this lecture, we can not expect nice properties with respect to the time variable of solutions to Cauchy problems associated with elliptic operators with unbounded coefficients.

For further details on the Ornstein-Uhlenbeck semigroup we refer the interested reader to [2], and to the notes of the 19th Internet Seminar.

13.4 Exercises

1. Prove that

   $$\int_{\mathbb{R}^d} \mathcal{N}(e^{tB}x, Q_t)(dy) = 1$$

   for every $t > 0$.

2. Prove that $\|T_{OU}(t)\|_{L(C_b(\mathbb{R}^d))} = 1$ for each $t > 0$.

3. (i) Exhibit a function $f \in BUC(\mathbb{R}^d)$ such that $\|f(e^{tB}\cdot) - f\|_\infty$ does not converge to 0 as $t \to 0^+$.

   (ii) Prove that, if $f \in C_0(\mathbb{R}^d)$ (the space of continuous functions over $\mathbb{R}^d$ which vanish at infinity), then $f(e^{tB}\cdot)$ converges to f uniformly on $\mathbb{R}^d$ as $t \to 0$.

   (iii) Taking advantage of the previous exercise prove that the restriction of the Ornstein-Uhlenbeck semigroup to $C_0(\mathbb{R}^d)$ defines a strongly continuous semigroup.

4. Prove formulas (13.13) and (13.14).

5. Prove formula (13.17).

6. Given a function $f : \mathbb{R}^d \to \mathbb{R}$ and $\theta \in (0,1)$. Define

   $$[[f]]_\theta := \sup_{x,y\in\mathbb{R}^d,\ x\neq y} \frac{|f(x) - 2f(2^{-1}(x+y)) + f(y)|}{|x-y|^\theta}.$$

   (a) Prove that if $f \in C_b^\theta(\mathbb{R}^d)$ then

   $$[[f]]_\theta \le 2^{1-\theta}[f]_{C_b^\theta(\mathbb{R}^d)}.$$

   (b) Fix $x_1 \in \mathbb{R}^d$ and define the function $g_{x_1}(y) := f(x_1 + y) - f(x_1)$ for $y \in \mathbb{R}^d$. Prove that

   $$|g_{x_1}(y) - 2g_{x_1}(y/2)| \le [[f]]_\theta |y|^\theta, \qquad y \in \mathbb{R}^d,$$

   and deduce that

   $$|g_{x_1}(y) - 2^n g_{x_1}(2^{-n}y)| \le \frac{2^{n(1-\theta)}}{2^{1-\theta} - 1}|y|^\theta [[f]]_\theta.$$

(c) Prove that

$$|f(x_1) - f(x_2)| \leq (4\|f\|_\infty + (2^{1-\theta} - 1)^{-1}[[f]]_\theta)|x_2 - x|^\theta$$

for $x_2 \in \mathbb{R}^d$ such that $|x_1 - x_2| < 1$.

(d) Deduce that $f \in C_b^\theta(\mathbb{R}^d)$ if and only if f is bounded and $[[f]]_\theta < \infty$.

(e) Prove that the norms $\| \cdot \| + [\,\cdot\,]_{C_b^\theta(\mathbb{R}^d)}$ and $\| \cdot \| + [\,\cdot\,]_{C_b^\theta(\mathbb{R}^d)}$ are equivalent.

7. Prove that, for any bounded and continuous function $g : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, the function $(t,s,x) \mapsto (T_{OU}(t)g(s,\cdot))(x)$ is bounded and continuous on $[0,\infty) \times [0,T] \times \mathbb{R}^d$.

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Lecture 14

More general elliptic operators with unbounded coefficients

In this lecture, we will consider more general elliptic operators $\mathcal{A}$ with unbounded coefficients and study the homogeneous Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x), & t \in (0,\infty), \quad x \in \mathbb{R}^d, \\ u(0,x) = f(x), & \qquad\qquad\quad\ \ x \in \mathbb{R}^d, \end{cases} \tag{14.1}$$

where

$$\mathcal{A}\zeta(x) = \sum_{i,j=1}^{d} q_{ij}(x) D_{ij}\zeta(x) + \sum_{j=1}^{d} b_j(x) D_j \zeta(x) + c(x)\zeta(x), \qquad x \in \mathbb{R}^d$$

on smooth enough functions $\zeta : \mathbb{R}^d \to \mathbb{R}$. On the coefficients of the operator $\mathcal{A}$ we assume the following conditions.

Hypotheses 14.0.1.
- (i) $q_{ij} \equiv q_{ji}$ for every $i,j = 1,\ldots,d$ and there exists a continuous function $\kappa : \mathbb{R}^d \to (0,\infty)$ such that $\langle Q(x)\xi,\xi\rangle \geq \kappa(x)|\xi|^2$ for every $\xi, x \in \mathbb{R}^d$, where $Q = (q_{ij})$;
- (ii) q_{ij}, b_i $(i,j = 1,\ldots,d)$ and c belong to $C^{\alpha}_{\rm loc}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$;
- (iii) there exists $c_0 \in \mathbb{R}$ such that $c(x) \leq c_0$ for every $x \in \mathbb{R}^d$.

Remark 14.0.2.
- (i) Note that the previous assumptions cover also the case when the matrix $Q(x)$ degenerates at ∞. So, we are not assuming that the operator $\mathcal{A}$ is uniformly elliptic on $\mathbb{R}^d$.
- (ii) At this stage we are not assuming any growth assumption of the diffusion and drift coefficients of the operator $\mathcal{A}$. The only (algebraic) condition that we assume is on the potential c which should be bounded from above on $\mathbb{R}^d$.

14.1 Existence of a classical solution to the Cauchy problem (14.1)

In this section we prove the existence of a classical solution to the Cauchy problem (14.1), which is bounded in each strip $[0,T]\times\mathbb{R}^d$.

Theorem 14.1.1. *For every $f\in C_b(\mathbb{R}^d)$, problem* (14.1) *admits a solution $u\in C([0,\infty)\times\mathbb{R}^d)\cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,\infty)\times\mathbb{R}^d)$ which satisfies the estimate*

$$|u(t,x)|\le e^{c_0t}\|f\|_\infty,\qquad t>0,\quad x\in\mathbb{R}^d. \tag{14.2}$$

Proof. We split the proof into two steps. First, in Step 1, we prove the assertion for functions $f\in C^{2+\alpha}_c(\mathbb{R}^d)$. Then, in Step 2, based on this result, we prove the statement in its full generality. We adapt some arguments from the proof of Theorem 9.0.1.

Step 1. Let us first assume that $f\in C^{2+\alpha}_c(\mathbb{R}^d)$. Let $n_0\in\mathbb{N}$ be the smallest integer such that $\operatorname{supp}(f)\subset B(0,n_0)$. By Theorem 12.0.5, for every $n\in\mathbb{N}$ such that $n\ge n_0$, there exists a unique function $v_n:[0,\infty)\times\overline{B(0,n)}\to\mathbb{R}$, which belongs to $C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{B(0,n)})$ and solves the Cauchy problem

$$\begin{cases} D_tv_n(t,x)=\mathcal{A}v_n(t,x), & t\in[0,\infty),\quad x\in\overline{B(0,n)},\\ v_n(t,x)=0, & t\in[0,\infty),\quad x\in\partial B(0,n),\\ v_n(0,x)=f(x), & \qquad\qquad\quad\; x\in\overline{B(0,n)}.\end{cases}$$

From Corollary 1.1.4, for each $n\ge n_0$ as above, it holds that

$$\|v_n(t,\cdot)\|_{C(\overline{B(0,n)})}\le e^{c_0t}\|f\|_\infty,\qquad t>0. \tag{14.3}$$

This estimate and Theorem B.0.6 show that, for every $m\in\mathbb{N}$, there exists a positive constant C_m, independent of n, such that

$$\|v_n\|_{C^{1+\alpha/2,2+\alpha}(\overline{\Omega_m})}\le C_m\|f\|_\infty, \tag{14.4}$$

for each $n\ge m\vee n_0$, where $\Omega_m=(0,m)\times B(0,m)$.

Take $m=1$. Since the sequence (v_n) is bounded in $C^{1+\alpha/2,2+\alpha}(\overline{\Omega_1})$, there exists a subsequence $(v_{n_k^{(1)}})$ which converges in $C^{1,2}(\overline{\Omega_1})$ to a function $u^{(1)}$ which belongs to $C^{1+\alpha/2,2+\alpha}(\overline{\Omega_1})$. Estimate (14.4) shows that sequence $(v_{n_k^{(1)}})$ is bounded in $C^{1+\alpha/2,2+\alpha}(\overline{\Omega_2})$. Hence, it admits a subsequence $(v_{n_k^{(2)}})$ which converges in $C^{1,2}(\overline{\Omega_2})$ to a function $u^{(2)}\in C^{1+\alpha/2,2+\alpha}(\overline{\Omega_2})$. Clearly, by uniqueness, the functions $u^{(1)}$ and $u^{(2)}$ coincide on Ω_1. By induction, we can prove that for any $m\in\mathbb{N}$ there exists an increasing sequence $(n_k^{(m)})\in\mathbb{N}$ with the following properties:

(i) $(n_k^{(m)})$ is a subsequence of $(n_k^{(m-1)})$;

(ii) the sequence $(v_{n_k^{(m)}})$ converges in $C^{1,2}(\overline{\Omega_m})$ to a function $u^{(m)}\in C^{1+\alpha/2,2+\alpha}(\overline{\Omega_m})$ such that $u^{(m)}=u^{(m-1)}$ on Ω_{m-1}.

Now, we introduce the diagonal sequence $(\widetilde{n}_k) = (n_k^{(k)})$. For each $m \in \mathbb{N}$, $(\widetilde{n}_k)$ is definitively a subsequence of $(n_k^{(m)})$. Moreover, in view of the previous property (ii), we can define a function $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ by setting $u(t,x) = u^{(m)}(t,x)$, where m is the smallest integer such that $(t,x) \in \Omega_m$. This function clearly belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,\infty) \times \mathbb{R}^d)$ and the sequence $(v_{\widetilde{n}_k})$ converges to u in $C^{1,2}(\overline{[0,T_*] \times K})$ for every compact set $K \subset \mathbb{R}^d$ and $0 < T_*$. Since $D_t v_{\widetilde{n}_k} = \mathcal{A} v_{\widetilde{n}_k}$ on $[0,\infty) \times \overline{B(0,\widetilde{n}_k)}$, letting k tend to ∞ we conclude that $D_t u - \mathcal{A} u = 0$ on $[0,\infty) \times \mathbb{R}^d$ and, from (14.3), we also obtain estimate (14.2). Since $v_{\widetilde{n}_k}(0,\cdot) = f$ on $\mathbb{R}^d$ for every $k \in \mathbb{N}$, it follows that $u(0,\cdot) = f$. Hence, u is a solution to the Cauchy problem (14.1) as smooth as in the statement of the theorem.

Step 2. Let us now fix $f \in C_b(\mathbb{R}^d)$ and consider a sequence $(f_n) \subset C^{2+\alpha}_c(\mathbb{R}^d)$, bounded with respect to the sup norm and converging to f locally uniformly on $\mathbb{R}^d$. By Step 1, for every $n \in \mathbb{N}$, the Cauchy problem (14.1), with f being replaced by f_n, admits a solution $u_n \in C^{1+\alpha/2,2+\alpha}_{\rm loc}([0,\infty) \times \mathbb{R}^d)$ such that $\|u_n(t,\cdot)\|_\infty \le e^{c_0 t}\|f_n\|_\infty \le C e^{c_0 t}$ for every $t > 0$ and some positive constant C, independent of t and $n \in \mathbb{N}$.

Applying estimate (B.2) we conclude that $\|u_n\|_{C^{1+\alpha/2,2+\alpha}(\overline{\Omega_m})} \le C_m$ for every $n \ge n_0$ and some $n_0 = n_0(m)$. As in Step 1, Arzelà-Ascoli theorem and a diagonal argument show that there exists a subsequence (u_{n_k}) which converges in $C^{1,2}(\overline{\Omega_m})$ for every $m \in \mathbb{N}$ to a function $u_f \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,\infty) \times \mathbb{R}^d)$. Clearly, u_f solves the differential equation $D_t u - \mathcal{A} u = 0$ on $(0,\infty) \times \mathbb{R}^d$.

To complete the proof, let us show that u can be extended by continuity to $[0,\infty) \times \mathbb{R}^d$ by setting $u(0,\cdot) = f$. For this purpose, we take advantage of Theorems 8.2.2 and B.0.7.

We fix $m \in \mathbb{N}$ and a smooth cut-off function ϑ such that $\chi_{B(0,m)} \le \vartheta \le \chi_{B(0,2m)}$. For each $k \in \mathbb{N}$, the function $v_k = \vartheta u_{n_k}$ belongs to $C([0,\infty) \times \mathbb{R}^d)$ and solves the Cauchy-Dirichlet problem

$$
\left\{
\begin{array}{lll}
D_t v_k(t,x) = \widetilde{\mathcal{A}} v_k(t,x) + \psi_k(t,x), & t > 0, & x \in \mathbb{R}^d, \\
v_k(0,x) = \vartheta(x) f(x), & & x \in \mathbb{R}^d,
\end{array}
\right.
\tag{14.5}
$$

where $\psi_k = -2\langle Q\nabla_x u_{n_k}, \nabla\vartheta\rangle - u_{n_k}{\rm Tr}(QD_{ij}\vartheta) - u_{n_k}\langle b, \nabla\vartheta\rangle$, and $\widetilde{\mathcal{A}}$ is any elliptic operator on $\mathbb{R}^d$ whose coefficients belong to $C^\alpha_b(\mathbb{R})$ and agree with the coefficients of the operator $\mathcal{A}$ on $B(0,2m)$.

By the uniqueness of the classical solution to the Cauchy problem (14.5) it follows that v_k splits into the sum of the classical solution w of the Cauchy problem

$$
\left\{
\begin{array}{lll}
D_t w(t,x) = \widetilde{\mathcal{A}} w(t,x), & t > 0, & x \in \mathbb{R}^d, \\
w(0,x) = \vartheta(x) f(x), & & x \in \mathbb{R}^d,
\end{array}
\right.
$$

and the classical solution to the Cauchy problem

$$
\left\{
\begin{array}{lll}
D_t z_k(t,x) = \widetilde{\mathcal{A}} z_k(t,x) + \psi_k(t,x), & t > 0, & x \in \mathbb{R}^d, \\
z_k(0,x) = 0, & & x \in \mathbb{R}^d.
\end{array}
\right.
$$

By Theorem 9.0.1, the function w belongs to $C([0,\infty) \times \mathbb{R}^d)$.

Let us consider the function z_k. For this purpose, we observe that Theorem B.0.7 and estimate (14.2) show that there exists a positive constant $\widetilde{C}_1$, depending on m but being independent of n such that

$$t\|D_x^2 u_{n_k}(t,\cdot)\|_{C(\overline{B(0,2m)})}+\sqrt{t}\,\|\nabla_x u_{n_k}(t,\cdot)\|_{C(\overline{B(0,2m)})}\le \widetilde{C}_1,\qquad t\in(0,1],\ \ k\in\mathbb{N}.$$

From this estimate and (4.16) it follows immediately that

$$\|u_{n_k}(t,\cdot)\|_{C^{1+\alpha}(\overline{B(0,2m)})}\le \widetilde{C}_2 t^{-\frac{1+\alpha}{2}},\qquad t\in(0,1],\ \ k\in\mathbb{N}$$

with $\widetilde{C}_2$ also being independent of k. Hence, $\psi_{n_k}(t,\cdot)$ belongs to $C^\alpha_b(\mathbb{R}^d)$ for every $t\in(0,1]$ and $\|\psi_{n_k}(t,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)}\le \widetilde{C}_3 t^{-\frac{1+\alpha}{2}}$ for every $t\in(0,1]$, $k\in\mathbb{N}$ and some positive constant $\widetilde{C}_3$, being independent of k. Therefore, applying Theorem 8.2.2 we conclude that

$$t^{\frac{1+\alpha}{2}}\|D_t z_{n_k}(t,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)}\le \widetilde{C}_4,\qquad t\in(0,1],\ \ k\in\mathbb{N}. \tag{14.6}$$

Now, writing

$$z_n(t,x)=\int_0^t D_t z_n(\sigma,x)d\sigma,\qquad t\in(0,1],\qquad x\in\mathbb{R}^d,$$

and using (14.6) we conclude that

$$\|z_n(t,\cdot)\|_\infty\le \widetilde{C}_4\int_0^t \sigma^{-\frac{1+\alpha}{2}}d\sigma = C_5 t^{\frac{1-\alpha}{2}}.$$

Summing up, we have proved that

$$|u_{n_k}(t,x)-f(x)|\le |w(t,x)-\vartheta(x)f(x)|+|z_k(t,x)|\le |w(t,x)-\vartheta(x)f(x)|+\widetilde{C}_5 t^{\frac{1-\alpha}{2}}$$

for every $t\in(0,1]$ and $x\in\overline{B(0,m)}$. Letting n tend to ∞, we conclude that

$$\|u(t,\cdot)-f\|_{C(\overline{B(0,m)})}\le \|w(t,\cdot)-f\|_{C(\overline{B(0,m)})}+\widetilde{C}_5 t^{\frac{1-\alpha}{2}}.$$

As t tends to 0^+ this estimate shows that u can be extended by continuity on $\{0\}\times\overline{B(0,m)}$ by setting $u(0,\cdot)=f$. The arbitrariness of $m>0$ allows us to complete the proof. □

14.2 Uniqueness and nonuniqueness

In this section, we address the problem of the uniqueness of the solution $u\in C([0,\infty)\times\mathbb{R}^d)\cap C^{1,2}((0,\infty)\times\mathbb{R}^d)$, which is bounded on $[0,T]\times\mathbb{R}^d$ for every $T>0$. In the case of bounded coefficients, the uniqueness of the classical solution to the Cauchy problem (14.1) was a consequence of the classical maximum principle. As a matter of fact, without any additional conditions on the unbounded coefficients of the operator $\mathcal{A}$, no maximum principle holds. In fact, we will provide examples of operators $\mathcal{A}$ whose associated Cauchy problem (14.1) admits infinitely many solutions $u \in C([0,\infty) \times \mathbb{R}^d) \cap C^{1,2}((0,\infty) \times \mathbb{R}^d)$, which are bounded on $[0,T] \times \mathbb{R}^d$ for every $T > 0$.

The maximum principle can be proved assuming the existence of a so-called *Lyapunov function*, i.e., a smooth enough function, diverging to $+\infty$ as $|x| \to \infty$ and satisfying an algebraic condition. More precisely.

Theorem 14.2.1. *Suppose that there exists a positive function* $\varphi \in C^2(\mathbb{R}^d)$ *blowing up as* $|x| \to \infty$ *and such that* $\mathcal{A}\varphi \le \lambda\varphi$ *on* $\mathbb{R}^d$ *for some* $\lambda > 0$. *Then, if* $u \in C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$ *solve the differential inequality* $D_t u - \mathcal{A}u \le 0$ *on* $(0,T] \times \mathbb{R}^d$. *If* $u(0,\cdot) \le 0$, *then* $u \le 0$ *on* $[0,T] \times \mathbb{R}^d$. *As a byproduct, there exists a unique solution* $u \in C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$ *to the Cauchy problem* (14.1).

Proof. The proof can be obtained adapting the arguments in the proof of Proposition 11.0.1. For the reader convenience we provide the full proof. Without loss of generality, we can assume that $\lambda > c_0$.

Let u be as in the first part of the statement. As it is immediately seen, the function $v : [0,T] \times \mathbb{R}^d$ defined by $v(t,x) = e^{-2\lambda t}u(t,x)$ for every $(t,x) \in [0,T] \times \mathbb{R}^d$ is as smooth as u is and solves the differential inequality $D_t v - (\mathcal{A} - 2\lambda)v \le 0$ on $(0,T] \times \mathbb{R}^d$. Since φ is positive on $\mathbb{R}^d$, it follows immediately that $\mathcal{A}\varphi < 2\lambda\varphi$ on $\mathbb{R}^d$. Hence, for every $n \in \mathbb{N}$ the function $v_n = v - n^{-1}\varphi$ satisfies the differential inequality $D_t v_n - (\mathcal{A}v_n - 2\lambda v_n) < 0$ and tends to $-\infty$ as $|x| \to \infty$, uniformly with respect to $t \in [0,T]$. Since it is continuous on $[0,T] \times \mathbb{R}^d$, it follows immediately that v_n admits a maximum value on $[0,T] \times \mathbb{R}^d$. This maximum cannot be achieved on $(0,T] \times \mathbb{R}^d$, due to the fact that the potential term of the operator $\mathcal{A} - 2\lambda$ is negative on $\mathbb{R}^d$ (see the proof of Theorem 1.1.2). Hence, it is achieved on $\{0\} \times \mathbb{R}^d$, so that each function v_n is nonpositive on $[0,T] \times \mathbb{R}^d$. Letting n tend to ∞, we conclude that u is nonpositive on $[0,T] \times \mathbb{R}^d$, as well.

To complete the proof, we observe that, by Theorem 14.1.1 we know that a solution $u \in C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$ to the Cauchy problem (14.1) actually exists. If $v \in C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$ is another solution. then the function $w = u - v$ vanishes on $\{0\} \times \mathbb{R}^d$ and satisfies the differential equality $D_t w - \mathcal{A}w = 0$ on $(0,T] \times \mathbb{R}^d$. From the first part of the proof, applied to w and $-w$, we conclude that w identically vanishes on $[0,T] \times \mathbb{R}^d$. $\square$

Remark 14.2.2. Typically, one takes as φ polynomials or exponential type functions. For instance, if $\mathcal{A} = \Delta + \langle b, \nabla\rangle$, then the function $\varphi(x) = 1 + |x|^2$, for every $x \in \mathbb{R}^d$, is a Lyapunov function for $\mathcal{A}$ if and only if the function $x \mapsto 2d + 2\langle b(x), x\rangle$ grows at most quadratically at infinity.

14.2.1 The one-dimensional case

To begin with we prove, for the one-dimensional elliptic operator

$$\mathcal{A}\zeta(x) = q(x)\zeta''(x) + b(x)\zeta'(x) + c(x)\zeta(x), \quad x \in \mathbb{R},$$

on smooth functions $\zeta : \mathbb{R} \to \mathbb{R}$, the following result.

Proposition 14.2.3. *The following properties are equivalent:*

(i) *for each $\lambda > c_0$ and $f \in C_b(\mathbb{R})$, the elliptic equation $\lambda v - \mathcal{A}v = f$ admits a unique solution $v \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$;*

(ii) *for each $f \in C_b(\mathbb{R})$, there exists a unique solution u to problem*

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x), & t \in (0,\infty), \quad x \in \mathbb{R}, \\ u(0,x) = f, & x \in \mathbb{R}, \end{cases} \tag{14.7}$$

which belongs to $C([0,\infty) \times \mathbb{R}) \cap C^{1,2}((0,\infty) \times \mathbb{R})$ for every $T > 0$ and satisfies the estimate $\|u(t,\cdot)\|_\infty \le Me^{\rho t}\|f\|_\infty$ for some constants $M > 0$ $\rho \ge c_0$ and every $t > 0$.

Proof. We split the proof into two steps.

Step 1. Let us assume that property (i) holds true and let $u \in C([0,\infty) \times \mathbb{R}) \cap C^{1,2}((0,\infty) \times \mathbb{R})$ be a solution of the parabolic problem (14.7) with $u(0,\cdot) = 0$, such that

$$|u(t,x)| \le Me^{\rho t}, \qquad t > 0, \ x \in \mathbb{R}, \tag{14.8}$$

for some $\rho \ge c_0$ We fix $\lambda > \rho$, and consider the functions

$$v_n = \int_{1/n}^n e^{-\lambda t} u(t,\cdot)dt, \quad n \in \mathbb{N}.$$

Estimate (14.8) shows that v_n belongs to $C_b(\mathbb{R})$ for each $n \in \mathbb{N}$ and $\|v_n\|_\infty \le M(\lambda - \rho)^{-1}$ for each $n \in \mathbb{N}$. Moreover, v_n converges, uniformly on $\mathbb{R}$, to the function

$$v = \int_0^\infty e^{-\lambda t} u(t,\cdot)dt$$

as n tend to ∞.

A straightforward computation shows that $v_n \in C^2(\mathbb{R})$ for each $n \in \mathbb{N}$ and

$$\mathcal{A}v_n = \int_{1/n}^n e^{-\lambda t}\mathcal{A}u(t,\cdot)dt = \int_{1/n}^n e^{-\lambda t} D_t u(t,\cdot)dt = e^{-\lambda n}u(n,\cdot) - e^{-\lambda/n}u(1/n,\cdot) + \lambda v_n, \tag{14.9}$$

which implies that $\mathcal{A}v_n \in C_b(\mathbb{R})$ and that $\mathcal{A}v_n$ converges to λv, locally uniformly on $\mathbb{R}$, as n tends to ∞. From Exercise 14.4.2ii it follows that the sequence v_n is bounded in $C^2([-M,M])$ for every $M > 0$. Applying Arzelà-Ascoli theorem and a diagonal argument as in the proof of Theorem 14.1.1 we deduce that, up to a subsequence, v_n and v_n' converge locally uniformly on $\mathbb{R}$ as n tends to ∞. As a byproduct, $v \in C^1(\mathbb{R})$. To conclude that v actually belongs to $C^2(\mathbb{R})$ it suffices to write $v_n'' = q^{-1}\mathcal{A}v_n - bq^{-1}v_n' - cq^{-1}v_n$ and let $n \to \infty$. Now, from (14.9), it follows that $\lambda v - \mathcal{A}v = 0$ just letting n tend to ∞. Since $v \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$, from (i) it follows that v identically vanishes on $\mathbb{R}$.

Finally, the arbitrariness of $\lambda > \rho$ and the uniqueness of the Laplace transform imply that $u \equiv 0$ on $(0,\infty) \times \mathbb{R}$.

Step 2. Let $v \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ be a solution of the equation $\lambda v - \mathcal{A}v = 0$ with $\lambda > c_0$. Let us prove that $v \equiv 0$. For this purpose, we observe that the function u, defined by $u(t,x) = e^{\lambda t}v(x)$ for every $t \geq 0$ and $x \in \mathbb{R}$, belongs to $C([0,\infty) \times \mathbb{R}) \cap C^{1,2}((0,\infty) \times \mathbb{R})$ and solves problem (14.7), with $u(0,\cdot) = v$, and $\|u(t,\cdot)\| = e^{\lambda t}\|v\|_\infty$ for each $t > 0$. By property (ii) this Cauchy problem admits a unique solution with the above properties. Hence, by Theorem 14.1.1 it follows that $e^{\lambda t}\|v\|_\infty = \|u(t,\cdot)\|_\infty \leq e^{c_0 t}\|v\|_\infty$ for every $t > 0$. From this chain of inequalities it follows at once that $\|v\|_\infty = 0$, that is v identically vanishes on $\mathbb{R}$. □

In view of the previous proposition, in the rest of the subsection we study the uniqueness of the solution $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ to the equation $\lambda u - \mathcal{A}u = f \in C_b(\mathbb{R})$, when $\lambda > c_0$.

This topic has been studied at the beginning of previous century by W. Feller, in the particular case when the potential c of the operator $\mathcal{A}$ identically vanishes on $\mathbb{R}$. Here, we present a part of those results. So, from now on, we assume that $c(x) = 0$ for all $x \in \mathbb{R}$.

To begin with, we prove the following crucial result.

Lemma 14.2.4. *There exist a positive, decreasing function* $\overline{u}_1$ *and a positive, increasing function* $\overline{u}_2$*, which solve equation* $\lambda u - \mathcal{A}u = 0$*.*

Proof. As a first step, we observe that it is enough to prove the existence of a positive decreasing solution u to the equation $\lambda u - \mathcal{A}u = 0$. Indeed, once this property is checked, it suffices to set $\overline{u_2}(x) = v(-x)$ for every $x \in \mathbb{R}$, where $v \in C^2(\mathbb{R})$ is positive, decreasing and $\lambda v(x) - q(-x)v''(x) + b(-x)v'(x) = 0$ for every $x \in \mathbb{R}$, to obtain the increasing positive solution to the equation $\lambda u - \mathcal{A}u = 0$ we are looking for.

Being rather long, we split the rest of the proof into some steps. Moreover, for notational convenience, we denote by $u_b \in C^2(\mathbb{R})$ the solution to the equation $\lambda u - \mathcal{A}u = 0$ which satisfies the conditions $u_b(0) = 1$, $u_b'(0) = b$.

Step 1. Here, we introduce the set $\mathcal{B}$ of all $b \in \mathbb{R}$ such that $u_b(x) = 0$ at some $x > 0$, and prove that it is an interval. To begin with, we observe that $\mathcal{B}$ is not empty. To this end, we recall that the more general solution to the equation $\lambda v - \mathcal{A}v = 0$ is given by $v = c_1v_1 + c_2v_2$, where c_1 and c_2 are arbitrary real constants and v_1 and v_2 are two independent solutions to that equation. By Exercise 14.4.3 the matrix whose rows are $(v_1(0), v_2(0))$ and $(v_1(x_0), v_2(x_0))$ is invertible for every $x_0 > 0$, since $v \equiv 0$ is the unique solution to the equation $\lambda v - \mathcal{A}v = 0$ such that $v(0) = v(x_0) = 0$. This implies that for every $x_0 > 0$ there exists a unique solution $v \in C^2(\mathbb{R})$ such that $v(0) = 1$, $v(x_0) = 0$ and $\lambda v - \mathcal{A}v = 0$ on $\mathbb{R}$. Hence, $b = v'(0)$ belongs to $\mathcal{B}$.

Now, we show that, if $b \in \mathcal{B}$, then $(-\infty, b] \subset \mathcal{B}$. This will imply that $\mathcal{B}$ is an interval. Fix $c < b$. Since $u_c'(0) < u_b'(0)$, it follows that $u_c < u_b$ in $(0, x_1)$ for some $x_1 > 0$. We claim that $x_1 = \infty$. By contradiction, suppose that $x_1 < \infty$. Then, by continuity, $u_b(x_1) = u_c(x_1)$. This would imply that the function $v = u_b - u_c$, which solves the equation $\lambda v - \mathcal{A}v = 0$ on $\mathbb{R}$, should have two zeroes. By Exercise 14.4.3 this is a contradiction. Since $u_c < u_b$ in $(0,\infty)$, it follows that u_c should vanish at some point $x \in (0,\infty)$. Consequently, $c \in \mathcal{B}$.

Step 2. Here, we show that $\mathcal{B} \subset (-\infty, 0]$. For this purpose, we prove that, the set $\{u_b, b \in \mathcal{B}\}$ consists of decreasing functions.

Fix $b \in \mathcal{B}$ and let $x_0 > 0$ be the unique zero of the function u_b. Then, u_b is decreasing in $(-\infty, x_0)$, otherwise it should have a positive maximum which is not possible by Exercise 14.4.3. Similarly, u_b is decreasing in $[x_0, \infty)$. Indeed, $u_b'(x_0) < 0$, so that u_b is strictly decreasing in a neighborhood of x_0. If u_b were not decreasing in the whole (x_0, ∞) it should have a negative minimum, which is again a contradiction, by Exercise 14.4.3.

Step 3. Here, we complete the proof, showing that $\overline{u}_1 = u_{\overline{b}}$ is the function that we are looking for. Here, $\overline{b} = \sup B$ and $\overline{u}_1 = u_{\overline{b}}$.

Since $u_b' \leq 0$ for each $b < \overline{b}$, by the continuous dependence of u_b on b, it follows that $\overline{u}_1' \leq 0$. Actually, $\overline{u}_1$ is decreasing, otherwise it would be constant on some interval (see again Exercise 14.4.3). To prove that $\overline{u}_1$ is positive in $\mathbb{R}$, we show that $\overline{b} \notin \mathcal{B}$. By contradiction, assume that $\overline{b} \in \mathcal{B}$ and denote by $x_0 > 0$ the positive zero of $\overline{u}_1$. Further, let $u \in C^2(\mathbb{R})$ be such that $\lambda u - \mathcal{A}u = 0$ on $\mathbb{R}$, $u(0) = 1$ and $u(2x_0) = 0$. The function $v = u - \overline{u}_1$ vanishes at zero and is positive at x_0. Hence, it is positive on $[0, x_0]$, otherwise it would have a negative minimum, which could not be the case. Hence, $u'(0) \geq \overline{u}_1'(0) = \overline{b}$. Actually, the strict inequality holds, otherwise, if $u'(0) = \overline{u}_1'(0)$, u and $\overline{u}_1$ should coincide, which, of course, is not the case. Hence, $u'(0) \in \mathcal{B}$, but this is a contradiction since $\overline{b} = \sup \mathcal{B}$. □

We now introduce the functions $\mathcal{W}, \mathcal{P}, \mathcal{R} : \mathbb{R} \to \mathbb{R}$, defined by

$$\mathcal{W}(x) = \exp\left(-\int_0^x \frac{b(s)}{q(s)}ds\right), \qquad \mathcal{P}(x) = \frac{1}{q(x)\mathcal{W}(x)}\int_0^x \mathcal{W}(s)ds,$$
$$\mathcal{R}(x) = \mathcal{W}(x)\int_0^x \frac{1}{q(s)\mathcal{W}(s)}ds,$$

for every $x \in \mathbb{R}$.

Proposition 14.2.5. *The following properties are satisfied:*

(i) *all the solutions to the equation $\lambda u - \mathcal{A}u = 0$ admit finite limit at $+\infty$ if and only if the function $\mathcal{R}$ belongs to $L^1((0,\infty))$;*

(ii) *if $\mathcal{P} \in L^1((0,\infty))$ and $\mathcal{R} \notin L^1((0,\infty))$ then, every positive decreasing function such that $\lambda u - \mathcal{A}u = 0$ satisfies $\lim_{x\to+\infty} \frac{u'(x)}{\mathcal{W}(x)} = 0$;*

(iii) *if $\mathcal{P}$, $\mathcal{R}$ belong to $L^1((0,\infty))$ and $u \in C^2(\mathbb{R})$ satisfies the equation $\lambda u - \mathcal{A}u = 0$ on $\mathbb{R}$, then the functions u and $u'/\mathcal{W}$ admit finite limits at $+\infty$. Moreover, there exist two decreasing functions u_1 and u_2 such that $\lambda u_j - \mathcal{A}u_j = 0$ on $\mathbb{R}$ $(j = 1, 2)$ and*

$$\lim_{x\to+\infty} u_j(x) = j - 1, \qquad \lim_{x\to+\infty} \frac{u_j'(x)}{\mathcal{W}(x)} = -2 + j, \qquad j = 1, 2; \tag{14.10}$$

(iv) *the equation $\lambda u - \mathcal{A}u = 0$ admits a decreasing solution, with $\lim_{x\to+\infty} u(x) > 0$, if and only if $\mathcal{P} \in L^1((0,\infty))$.*

Proof. (i) Since each solution to the equation $\lambda u - \mathcal{A}u = 0$ is given by a linear combination of the functions $\overline{u}_1$, $\overline{u}_2$ and $\overline{u}_1$ is decreasing (by Lemma 14.2.4), it suffices to show that $\lim_{x\to+\infty}\overline{u}_2(x) \in \mathbb{R}$ if and only if $\mathcal{R} \in L^1((0,\infty))$. Recalling that $\overline{u}_2$ is increasing and $\overline{u}_2(0) = 1$, we can estimate

$$\mathcal{R}(x) \le \mathcal{W}(x)\int_0^x \frac{\overline{u}_2(s)}{q(s)\mathcal{W}(s)}ds \le \overline{u}_2(x)\mathcal{R}(x), \qquad x > 0. \tag{14.11}$$

Moreover, by (14.21) we can write

$$\overline{u}_2'(x) = \mathcal{W}(x)\left(\overline{u}_2'(0) + \lambda\int_0^x \frac{\overline{u}_2(s)}{q(s)\mathcal{W}(s)}ds\right), \qquad x > 0. \tag{14.12}$$

Suppose that $\overline{u}_2$ is bounded in a neighborhood of $+\infty$. Then, the two terms in the right-hand side of (14.12) are in $L^1((0,\infty))$, since they are both positive. Therefore, from (14.11) it follows that $\mathcal{R} \in L^1((0,\infty))$.

Vice versa, suppose that $\mathcal{R} \in L^1((0,\infty))$. Plugging (14.11) into (14.12) we deduce that $\overline{u}_2' \le \overline{u}_2'(0)\mathcal{W} + \lambda\mathcal{R}\overline{u}_2$ on $(0,\infty)$. Therefore, by Gronwall's lemma,

$$\overline{u}_2(x) \le \exp\left(\lambda\int_0^x \mathcal{R}(t)dt\right)\left[1 + \overline{u}_2'(0)\int_0^x \mathcal{W}(t)\exp\left(-\lambda\int_0^t \mathcal{R}(s)ds\right)dt\right] \tag{14.13}$$

for every $x > 0$. From the definition of the function $\mathcal{W}$ it follows that

$$\mathcal{W}(x) \le \left(\int_0^1 \frac{1}{q(s)\mathcal{W}(s)}ds\right)^{-1}\mathcal{R}(x), \qquad x \ge 1,$$

and since $\mathcal{R} \in L^1((0,\infty))$, we conclude that $\mathcal{W}$ belongs to $L^1((0,\infty))$. Thus, from (14.13) it follows that $\overline{u}_2$ is bounded in $(0,\infty)$, so that its limit at $+\infty$ is finite.

(ii) Let u be a positive decreasing solution to the equation $\lambda u - \mathcal{A}u = 0$. Then, from (14.20) we deduce that $u'/\mathcal{W}$ is a negative and increasing function on $\mathbb{R}$. Therefore, it converges to a nonpositive number, which we denote by k, as x tends to $+\infty$. Let us prove that $k = 0$. Integrating (14.20) from x to c and, then, letting c tend to $+\infty$, gives

$$u'(x) = \mathcal{W}(x)\left(k - \lambda\int_x^\infty \frac{u(s)}{q(s)\mathcal{W}(s)}ds\right), \qquad x \in \mathbb{R}. \tag{14.14}$$

Since $\mathcal{P} \in L^1((0,\infty))$, it follows that the function $x \mapsto \mathcal{W}(x)\int_x^\infty (q(s)\mathcal{W}(s))^{-1}ds$ belongs to $L^1((0,\infty))$, by the Fubini theorem. The boundedness of u in $(0,\infty)$ yields the summability in $(0,\infty)$ of the function

$$x \mapsto \int_x^\infty \frac{u(s)}{q(s)\mathcal{W}(s)}ds.$$

Therefore, from formula (14.14) it follows that, if $k \ne 0$, then the function $\mathcal{W}$ is integrable in $(0,\infty)$. Since

$$\frac{1}{q(x)\mathcal{W}(x)} \le \left(\int_0^1 \mathcal{W}(s)ds\right)^{-1}\mathcal{P}(x), \qquad x \ge 1, \tag{14.15}$$

the function $1/(q\mathcal{W})$ is integrable in $(0,\infty)$. As a byproduct, function $\mathcal{R}$ is integrable in $(0,\infty)$ as well. Hence, we get a contradiction.

(iii) Let us prove that, if $\mathcal{P},\mathcal{R} \in L^1((0,\infty))$, then, for every solution u to the equation $\lambda u - \mathcal{A}u = 0$, the functions $u'/\mathcal{W}$ and u admit finite limits at $+\infty$. By (i) it is enough to deal with the first function. For this purpose we observe that (14.15) and the boundedness of u at ∞ imply that the function $u/(q\mathcal{W})$ is integrable in $(0,\infty)$. Therefore, dividing both the sides of (14.21) by $\mathcal{W}(x)$ and letting x tend to $+\infty$, we conclude that $u'(x)/\mathcal{W}(x)$ admits finite limit at $+\infty$.

Let us now set $\overline{u} = \overline{u}_1 - c\overline{u}_2$ where c is a constant such that $\lim_{x\to+\infty}\overline{u}(x) = 0$. Note that $c \in (0,1)$. Checking that c is positive is immediate since $\overline{u}_1$ and $\overline{u}_2$ are both positive in $\mathbb{R}$. Thus, $\overline{u}$ is decreasing in $\mathbb{R}$ and, since $\overline{u}(0) = 1-c$, it follows that $c<1$. The same arguments as in the proof of (ii) show that $\overline{u}'(x)/\mathcal{W}(x)$ tends to a nonpositive limit k as x tends to $+\infty$. Suppose that $k=0$. Then (14.14) yields

$$\frac{\overline{u}'(x)}{\overline{u}(x)} = -\lambda\frac{\mathcal{W}(x)}{\overline{u}(x)}\int_x^{+\infty}\frac{\overline{u}(s)}{q(s)\mathcal{W}(s)}ds \geq -\lambda\mathcal{W}(x)\int_x^{+\infty}\frac{1}{q(s)\mathcal{W}(s)}ds \tag{14.16}$$

for each $x \in \mathbb{R}$. By the proof of (ii), the last side of (14.16) belongs to $L^1((0,\infty))$ since $\mathcal{P} \in L^1((0,\infty))$. It follows that the function $\log(\overline{u})$ is bounded from below in $(0,\infty)$ and, consequently, $\lim_{x\to\infty}\overline{u}(x) > 0$: a contradiction. Thus, we conclude that k is negative. Therefore, setting $u_1 = -\overline{u}/k$, we obtain a solution to the equation $\lambda u - \mathcal{A}u = 0$ which satisfies (14.10), with $j=1$.

To complete the proof of property (iii), let us prove the existence of the function u_2. For this purpose, we consider the function $w \in C^2(\mathbb{R})$ which solves the equation $\lambda w - \mathcal{A}w = 0$ on $\mathbb{R}$ and fulfills the conditions $w(0)=0$, $w'(0)=1$. By Exercise 14.4.3, w is increasing in $\mathbb{R}$. Therefore, (14.20) implies that $w'/\mathcal{W}$ is increasing as well and, consequently, $\lim_{x\to\infty} w'(x)/\mathcal{W}(x) \in (1,\infty)$.

Let us set $v = u_1 + dw$, where $d>0$ is chosen to let $v'/\mathcal{W}$ converge to zero as x tends to ∞. Since v is nonnegative, again by (14.20) we conclude that v is decreasing in $\mathbb{R}$. Moreover, since $\lim_{x\to+\infty} v'(x)/\mathcal{W}(x) = 0$, the previous arguments show that $\ell := \lim_{x\to+\infty} v(x)$ is positive. Hence, if we set $u_2 = v/\ell$, we find the function we are looking for.

(iv) Let u be a decreasing solution to the equation $\lambda u - \mathcal{A}u = 0$ with positive limit ℓ at $+\infty$. From the proof of the property (ii), it follows that $u'/\mathcal{W}$ has nonpositive limit k at $+\infty$ and u' is given by (14.14). Hence, the function $u/(q\mathcal{W})$ belon $L^1((0,\infty))$. Since all the terms in the right-hand side of (14.14) are nonpositive, and the left-hand side is summable in $(0,\infty)$, the function $x \mapsto \mathcal{W}(x)\int_x^{+\infty} u(s)/(q(s)\mathcal{W}(s))ds$ belongs to $L^1((0,\infty))$ as well. Using the condition $u \geq \ell$ on $(0,\infty)$, we immediately conclude that the function $x \mapsto \mathcal{W}(x)\int_x^{+\infty}(q(s)\mathcal{W}(s))^{-1}ds$ belongs to $L^1((0,\infty))$. Now, the Fubini Theorem yields $\mathcal{P} \in L^1((0,\infty))$.

Conversely, let us suppose that $\mathcal{P} \in L^1((0,\infty))$ and let us prove that the equation $\lambda u - \mathcal{A}u = 0$ admits a positive decreasing solution u with $\lim_{x\to\infty} u(x) = \ell > 0$. If $\mathcal{R} \in L^1((0,\infty))$, then the property (iii) gives us the wished function u. If $\mathcal{R} \notin L^1((0,\infty))$, then every positive and decreasing solution u to the equation $\lambda u - \mathcal{A}u = 0$ is given by (14.14) and $k = 0$ by the property (ii). Hence u'/u satisfies (14.16) and, consequently, arguing as in the proof of the property (iii), we conclude that $\lim_{x\to+\infty} u(x) = \ell > 0$. $\square$

Concerning the behaviour of the solutions of the equation $\lambda u - \mathcal{A}u = 0$ in a neighborhood of $-\infty$, the following result holds true.

Proposition 14.2.6. *The following properties are satisfied:*

(i) *all the solutions $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ to the equation $\lambda u - \mathcal{A}u = 0$ admit finite limit at $-\infty$ if and only if the function $\mathcal{R}$ belongs to $L^1((-\infty, 0))$;*

(ii) *if $\mathcal{P} \in L^1((-\infty, 0))$ and $\mathcal{R} \notin L^1((-\infty, 0))$ then, for every positive function $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ such that $\lambda u - \mathcal{A}u = 0$, it holds $\lim_{x\to-\infty} \frac{u'(x)}{\mathcal{W}(x)} = 0$*

(iii) *if $\mathcal{P}$, $\mathcal{R}$ belong to $L^1((-\infty, 0))$ then every solution $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ to the equation $\lambda u - \mathcal{A}u = 0$ is such that u and $u'/\mathcal{W}$ admit finite limits at $-\infty$. Moreover, there exist two increasing solutions u_1 and u_2 of the equation $\lambda u - \mathcal{A}u = 0$ such that*

$$\lim_{x\to-\infty} u_j(x) = j - 1, \qquad \lim_{x\to-\infty} \frac{u_j(x)}{\mathcal{W}(x)} = 2 - j, \qquad j = 1, 2;$$

(iv) *there exists a solution $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ to the equation $\lambda u - \mathcal{A}u = 0$, which is increasing on $\mathbb{R}$ and admits positive limit at $-\infty$, if and only if $\mathcal{P} \in L^1((-\infty, 0))$.*

We now introduce a few nomenclature. For this purpose, we set $I_+ = (0, \infty)$ and $I_- = (-\infty, 0)$

Definition 14.2.7. The point $\pm\infty$ is said

$$\text{accessible if} \qquad \begin{cases} \text{regular, i.e.,} & \mathcal{P} \in L^1(I_\pm),\ \mathcal{R} \in L^1(I_\pm), \\ \text{exit, i.e.,} & \mathcal{P} \notin L^1(I_\pm),\ \mathcal{R} \in L^1(I_\pm), \end{cases}$$

$$\text{unaccessible if} \qquad \begin{cases} \text{entrance, i.e.,} & \mathcal{P} \in L^1(I_\pm),\ \mathcal{R} \notin L^1(I_\pm), \\ \text{natural, i.e.,} & \mathcal{P} \notin L^1(I_\pm),\ \mathcal{R} \notin L^1(I_\pm). \end{cases}$$

Combining Propositions 14.2.5 and 14.2.6 we can now show the following result.

Proposition 14.2.8. *The following properties are satisfied:*

(i) $+\infty$ (*resp.* $-\infty$) *is regular if and only if the differential equation $\lambda u - \mathcal{A}u = 0$ admits two positive decreasing* (*resp. increasing*) *solutions u_1 and u_2 such that*

$$\lim_{x\to\infty} u_j(x) = j - 1, \qquad \lim_{x\to\infty} \frac{u_j'(x)}{\mathcal{W}(x)} = -2 + j, \qquad j = 1, 2,$$

$$\left(\textit{resp. } \lim_{x\to-\infty} u_j(x) = j-1, \qquad \lim_{x\to-\infty}\frac{u_j'(x)}{\mathcal{W}(x)} = 2-j, \quad j=1,2\right).$$

In this case all the solutions $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ to the equation $\lambda u - \mathcal{A}u = 0$ are bounded in $(0,\infty)$ (resp. in $(-\infty,0)$);

(ii) *$+\infty$ (resp. $-\infty$) is an exit if and only if all the solutions of the equation $\lambda u - \mathcal{A}u = 0$ are bounded in $(0,\infty)$ (resp. in $(-\infty,0)$) and every positive decreasing (resp. increasing) solution u vanishes at $+\infty$ (resp. at $-\infty$);*

(iii) *$+\infty$ (resp. $-\infty$) is an entrance if and only if the differential equation $\lambda u - \mathcal{A}u = 0$ admits a positive decreasing (resp. increasing) solution u such that the functions u and $u'/\mathcal{W}$ converge, respectively, to 1 and 0 as x tends to $+\infty$ (resp. as x tends to $-\infty$), and any other solution of the above equation, which is independent of u, is unbounded in $(0,\infty)$ (resp. in $(-\infty,0)$);*

(iv) *$+\infty$ (resp. $-\infty$) is natural if and only the differential equation $\lambda u - \mathcal{A}u = 0$ admits a positive decreasing (resp. increasing) solution u such that the functions u and $u'/\mathcal{W}$ vanishes as x tends to $+\infty$ (resp. as x tends to $-\infty$), and any other solution of the above equation, which is independent of u, is unbounded in $(0,\infty)$ (resp. in $(-\infty,0)$).*

Proof. We just prove the property (iv) since the other properties follow easily from Propositions 14.2.5 and 14.2.6. Moreover, we limit ourselves to dealing with the point $+\infty$ since the other case can be deduced from this one using Proposition 14.2.6 instead of Proposition 14.2.5.

Suppose that $+\infty$ is natural. Then, according to Proposition 14.2.5(iv), $\overline{u}_1$ vanishes at $+\infty$. To prove that $\overline{u}_1'/\mathcal{W}$ vanishes at $+\infty$ as well, we begin by observing that (14.20), written with $\overline{u}_1$ instead of u, show that $\overline{u}_1'/\mathcal{W}$ admits finite (and nonpositive) limit as x tends to $+\infty$ since it is negative and increasing. By contradiction, we assume that the previous limit k is negative. Since every solution of the equation $\lambda u - \mathcal{A}u = 0$ is a linear combination of $\overline{u}_1$ and $\overline{u}_2$ and, according to Proposition 14.2.5(i), the above equation admits solutions which are unbounded in a neighborhood of $+\infty$, it follows that $\overline{u}_2(x)$ diverges as x tends to $+\infty$. By Exercise 14.4.5, we can write

$$1 = \frac{1}{w_0}\left(\frac{\overline{u}_2'}{\mathcal{W}}\overline{u}_1 - \frac{\overline{u}_1'}{\mathcal{W}}\overline{u}_2\right)$$

on $\mathbb{R}$. Recalling that $\overline{u}_2'$ and $\overline{u}_1$ are positive in $\mathbb{R}$ and taking the limit as x tends to $+\infty$, we are led to a contradiction.

Vice versa, let us assume that there exists a positive decreasing solution $\widetilde{u}$ to the equation $\lambda u - \mathcal{A}u = 0$ vanishing at $+\infty$ together with the function $\widetilde{u}'/\mathcal{W}$ and that any other solution to the above equation, independent of $\widetilde{u}$, is unbounded at $+\infty$. According to Proposition 14.2.5(i), the function $\mathcal{R}$ does not belong to $L^1((0,\infty))$. To show that also $\mathcal{P}$ does not belong to $L^1((0,\infty))$, we take a solution v to the equation $\lambda u - \mathcal{A}u = 0$ linearly independent of $\widetilde{u}$. As a byproduct, if $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ solves the equation $\lambda u - \mathcal{A}u = 0$, then $u = c_1\widetilde{u} + c_2 v$, for some $c_1, c_2 \in \mathbb{R}$. Therefore, u is bounded at $+\infty$ if and only if $c_2 = 0$. But in such a case, u vanishes at $+\infty$. Therefore, Proposition 14.2.5(iv) implies that $\mathcal{P} \notin L^1((0,\infty))$ and this completes the proof. □

Remark 14.2.9. Let $\mathcal{A}_+$ be the one-dimensional differential operator defined on smooth functions ζ by $\mathcal{A}_+\zeta(x) = \zeta''(x) + x^3\zeta'(x)$ for every $x \in \mathbb{R}$. In this case $\mathcal{P} \notin L^1((-\infty,0)) \cup L^1((0,\infty))$ and $\mathcal{R} \in L^1((-\infty,0)) \cap L^1((0,\infty))$. Therefore, $+\infty$ and $-\infty$ are both exit points, i.e., they are accessible, and, according to Proposition 14.2.8, the equation $\lambda u - \mathcal{A}_+ u = 0$ admits a nontrivial solution $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$. In view of Proposition 14.2.3, we thus conclude that, for every $f \in C_b(\mathbb{R})$, the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}_+ u(t,x), & t \in (0,\infty), \quad x \in \mathbb{R}, \\ u(0,x) = f(x), & \qquad\qquad\quad\;\; x \in \mathbb{R} \end{cases}$$

admits several bounded solutions $u \in C([0,\infty) \times \mathbb{R}) \cap C^{1,2}((0,\infty) \times \mathbb{R})$.

Now, consider the elliptic operator $\mathcal{A}_-$ defined on smooth functions ζ by $\mathcal{A}_-\zeta(x) = \zeta''(x) - x^3\zeta'(x)$ for every $x \in \mathbb{R}$. In this case $-\infty$ and ∞ are entrance points and the elliptic equation $\lambda u - \mathcal{A}_- u = 0$ admits just the trivial solution. In view of Proposition 14.2.3, we thus conclude that, for every $f \in C_b(\mathbb{R})$, the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}_- u(t,x), & t \in (0,\infty), \quad x \in \mathbb{R}, \\ u(0,x) = f(x), & \qquad\qquad\quad\;\; x \in \mathbb{R} \end{cases}$$

admits a unique bounded solution $u \in C([0,\infty) \times \mathbb{R}) \cap C^{1,2}((0,\infty) \times \mathbb{R})$.

Note that the operators $\mathcal{A}_-$ and $\mathcal{A}_+$ differ only in the sign of the drift coefficient, but this difference is crucial.

The same situation occurs in the d-dimensional case where

$$\mathcal{A}_- = \Delta - |x|^2\langle x, \nabla\rangle, \qquad \mathcal{A}_+ = \Delta + |x|^2\langle x, \nabla\rangle,$$

(see Remark 14.2.2). Hence, for elliptic operators with unbounded coefficients the longterm behaviour of the coefficients is crucial: an information on their moduli is not enough.

14.3 Notes

Even if the classical solutions of the Cauchy problem 14.1 might be infinitely many, when the datum f is nonnegative there exists a minimal nonnegative solution. Unfortunately, the techniques used in the proof of Theorem **??** do not allow to check this property, since more refined results on the Cauchy-Dirichlet problem on balls are needed. For this result and other further details on elliptic operators with unbounded coefficients on $C_b(\mathbb{R}^d)$, we refer the reader to [1–3]

14.4 Exercises

1. (i) Prove that there exists a positive constant C_1 such that

$$\|u'\|_{C([0,1])} \le C_1(\|u\|_{C([0,1])} + \|u\|^{1/2}_{C([0,1])}\|u''\|_{C([0,1])}) \tag{14.17}$$

for every $u \in C^2([0,1])$. (For this purpose it might be useful to check that the previous estimate is satisfied if $[0,1]$ is replaced by $[0,\infty)$ and by $(-\infty,1]$ and to split $u = \eta u + (1-\eta)u$ where η is any smooth function supported in $[0,3/4]$ and such that $\eta = 1$ on $[0,1/2]$.

(ii) Use estimate (14.17) to deduce that, for every $M > 0$, there exists a positive constant C_2 such that

$$\|u'\|_{C([-M,M])} \le C_2(\|u\|_{C([-M,M])} + \|u\|^{1/2}_{C([-M,M])}\|u''\|_{C([-M,M])}). \tag{14.18}$$

2. Let $A = qD_{xx} + bD_x + c$, where q, b and c are continuous functions over $\mathbb{R}$, q being everywhere positive.

(i) Using estimate (14.18) prove that

$$\|u'\|_{C([-M,M])} \le C_3(\|u\|_{C([-M,M])} + \|\mathcal{A}u\|_{C([-M,M])}). \tag{14.19}$$

(ii) Prove that there exists a positive constant K, depending on M such that

$$\|u\|_{C^2([-M,M])} \le K(\|u\|_{C([-M,M])} + \|\mathcal{A}u\|_{C([-M,M])})$$

for every $u \in C^2([-M,M])$ and every $M > 0$.

3. Let $u \in C^2(\mathbb{R})$ be a solution to the equation $\lambda u - \mathcal{A}u = 0$. Prove that u can attain neither a positive maximum nor a negative minimum. Conclude that, if u vanishes at two different points x_0 and x_1, then u identically vanishes on $\mathbb{R}$.

4. Prove that $u \in C^2(\mathbb{R})$ solves equation $\lambda u - \mathcal{A}u = 0$ if and only if

$$\left(\frac{u'}{\mathcal{W}}\right)' = \lambda\frac{u}{q\mathcal{W}}. \tag{14.20}$$

Deduce that all the solutions u of the equation $\lambda u - \mathcal{A}u = 0$ satisfy the equation

$$u'(x) = \mathcal{W}(x)\left(u'(0) + \lambda\int_0^x \frac{u(s)}{q(s)\mathcal{W}(s)}ds\right), \qquad x \in \mathbb{R}. \tag{14.21}$$

5. Prove that $\overline{u}_1\overline{u}_2' - \overline{u}_1'\overline{u}_2 = w_0\mathcal{W}$ on $\mathbb{R}$ for some positive constant w_0.

6. Deduce Proposition 14.2.6 from Proposition 14.2.5.

7. Prove that if operator $\mathcal{A}$ is as in Remark 14.2.9, then ∞ and $-\infty$ are both exit points.

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Lecture 15

A class of non analytic Markov semigroups in $C_b(\mathbb{R}^d)$

In Lecture 13 we have seen that the Ornstein-Uhlenbeck semigroup is neither analytic nor strongly continuous in $BUC(\mathbb{R}^d)$. In this chapter we deal with the semigroup, called *Markov semigroup*, associated with the operator $\mathcal{A}$ defined on smooth functions by

$$\mathcal{A}\zeta(x) = \Delta\zeta(x) + \sum_{j=1}^{d} b_j(x)D_j\zeta(x), \qquad x \in \mathbb{R}^d, \tag{15.1}$$

where we assume that the corresponding homogeneous Cauchy problem (14.1) admits a unique solution and b_j $(j = 1, \ldots, d)$ are unbounded locally Lipschitz continuous functions in $\mathbb{R}^d$. We recall that the existence and uniqueness of solutions to (14.1) are given by Theorem 14.1.1 and Theorem 14.2.1. Let us denote by $(T(t)f)(x)$ for each $(t, x) \in [0, \infty) \times \mathbb{R}^d$, the unique solution to (14.1) provided by Theorem 14.1.1. By uniqueness it is easy to see that $\{T(t)\}$ defines a semigroup of linear bounded operators on $C_b(\mathbb{R}^d)$. Moreover, by (14.7), the semigroup $\{T(t)\}$ satisfies the condition

$$\lim_{t\to 0^+} \sup_{x\in K} |(T(t)f)(x) - f(x)| = 0$$

for every $f \in C_b(\mathbb{R}^d)$ and every compact subset $K \subset \mathbb{R}^d$.

As it was observed for the Ornstein-Uhlenbeck semigroup, see Proposition 13.1.6, the semigroup $\{T(t)\}$ is not strongly continuous on $C_b(\mathbb{R}^d)$. So, it is not possible to define its infinitesimal generator. Nevertheless, as we noticed in the Notes of Lecture 13, the weak generator exists and it can be defined as follows:

$$\begin{aligned} D(\widehat{A}) &= \Big\{ f \in C_b(\mathbb{R}^d) : \sup_{t>0} \frac{\|T(t)f - f\|_\infty}{t} < \infty \text{ and } \exists g \in C_b(\mathbb{R}^d) \text{ s.t.} \\ &\qquad \lim_{t\to 0^+} \frac{(T(t)f)(x) - f(x)}{t} = g(x),\ x \in \mathbb{R}^d \Big\} \\ \widehat{A}f(x) &= \lim_{t\to 0^+} \frac{(T(t)f)(x) - f(x)}{t}, \quad x \in \mathbb{R}^d,\ f \in D(\widehat{A}). \end{aligned}$$

For some useful properties of the weak generator $\widehat{A}$ of $\{T(t)\}$ we refer to Exercise 15.3.1.

For sectorial operators we refer to Chapter 3 and in particular to Proposition 3.2.8, where one can easily see that (3.18) is also a necessary condition for an operator to be sectorial in the sense of Definition 3.2.1

In this lecture we provide conditions on $b = (b_1, \ldots, b_d)$ implying that $\{T(t)\}$ is not analytic in $C_b(\mathbb{R}^d)$. The results of this lecture are taken from [4].

15.1 Non analytic semigroups in $C_b(\mathbb{R}^d)$

The main result of this section is Theorem 15.1.1.

Theorem 15.1.1. *Assume that the coefficients* b_j $(j = 1, \ldots, d)$ *in* (15.1) *are locally Lipschitz continuous. Further, assume that there exist three sequences* (r_n), $(\lambda_n) \subset (0, \infty)$ *and* $(\sigma_n) \subset \mathbb{R}^d$ *such that*

$$r_n \le M, \qquad n \in \mathbb{N}, \tag{15.2}$$

for some positive constant M*;*

$$\lim_{n\to\infty} \frac{r_n}{\lambda_n^2} = 0; \tag{15.3}$$

$$\lim_{n\to\infty} \frac{r_n}{\lambda_n} b(\lambda_n x + \sigma_n) = h \in \mathbb{R}^d, \qquad h \neq 0, \tag{15.4}$$

uniformly with respect to x *on compact subsets of* $\mathbb{R}^d$*. Then,* $(\widehat{A}, D(\widehat{A}))$ *is not sectorial in* $C_b(\mathbb{R}^d)$*. In particular, if* $r_n = 1$ *for each* $n \in \mathbb{N}$*, then the spectrum of* $(\widehat{A}, D(\widehat{A}))$ *contains the imaginary axis.*

Proof. To prove the assertion we argue by contradiction. We show that, if $(\widehat{A}, D(\widehat{A}))$ were sectorial, then the semigroup $\{S(t)\} \subset L(C_0(\mathbb{R}^d))$, defined by $(S(t)f)(x) = f(x + th)$ for each $t > 0$, $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$, should be analytic. But this cannot be the case since the infinitesimal generator of $\{S(t)\}$, is the operator $B : D(B) = \{u \in C_0(\mathbb{R}^d) : \langle h, Du\rangle \in C_0(\mathbb{R}^d)\} \to C_0(\mathbb{R}^d)$ defined by $Bu = \langle h, Du\rangle$ for each $u \in D(B)$, whose spectrum contains the imaginary axis, see Exercise 15.3.2. Here, h is as in (15.4).

So, let us assume that $(\widehat{A}, D(\widehat{A}))$ is sectorial and let us introduce, for each $n \in \mathbb{N}$, the isometries $I_n : C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ defined by

$$(I_n u)(x) = u\left(\frac{x - \sigma_n}{\lambda_n}\right), \qquad x \in \mathbb{R}^d, \quad u \in C_b(\mathbb{R}^d), \quad n \ge 1.$$

For $n \in \mathbb{N}$ we denote by $\mathcal{A}_n$ the second-order differential operator defined on smooth functions by

$$(\mathcal{A}_n u)(x) = \frac{r_n}{\lambda_n^2}\Delta u(x) + \frac{r_n}{\lambda_n}\sum_{j=1}^d b_j(\lambda_n x + \sigma_n) D_j u(x), \qquad x \in \mathbb{R}^d,$$

and we denote by $\{T_n(t)\}$ the semigroup of contractions in $C_b(\mathbb{R}^d)$ associated with the operator $\mathcal{A}_n$. Since $\mathcal{A}_n = r_n I_n^{-1}\mathcal{A}I_n$, it is easy to check that, for each $n \in \mathbb{N}$, the weak generator $\widehat{A}_n$ of the semigroup $\{T_n(t)\}$ is the operator $\widehat{A}_n : D(\widehat{A}_n) := I_n^{-1}(D(\widehat{A})) \to C_b(\mathbb{R}^d)$ defined by $\widehat{A}_n = r_n(I_n^{-1}\widehat{A}I_n)$. Therefore, $\rho(\widehat{A}_n) = r_n\rho(\widehat{A})$ and

$$R(\lambda, \widehat{A}_n) = \frac{1}{r_n} I_n^{-1} R(r_n^{-1}\lambda, \widehat{A}) I_n, \qquad n \in \mathbb{N}. \tag{15.5}$$

Since we are assuming that $(\widehat{A}, D(\widehat{A}))$ is sectorial, it follows that there exist two positive constants C and K such that

$$\|R(\lambda, \widehat{A})\|_{L(C_b(\mathbb{R}^d))} \le \frac{C}{|\lambda|}, \qquad \operatorname{Re}\lambda > K. \tag{15.6}$$

Using (15.5), (15.6) and the assumptions on the sequence (r_n), we can easily show that

$$\|R(\lambda, \widehat{A}_n)\|_{L(C_b(\mathbb{R}^d))} \le \frac{C}{|\lambda|}, \qquad \operatorname{Re}\lambda > MK. \tag{15.7}$$

Now, we fix $f \in C_c^\infty(\mathbb{R}^d)$ and check that, for every $\lambda \in \mathbb{C}$, with $\operatorname{Re}\lambda > MK$, and all $x \in \mathbb{R}^d$, it holds that

$$\lim_{n\to\infty} (R(\lambda, \widehat{A}_n)f)(x) = (R(\lambda, B)f)(x). \tag{15.8}$$

Of course, once (15.8) is proved, letting n tend to ∞ in (15.7) we will get

$$\|R(\lambda, B)f\|_\infty \le \frac{C}{|\lambda|}\|f\|_\infty, \qquad \operatorname{Re}\lambda > MK,$$

first for $f \in C_c^\infty(\mathbb{R}^d)$ and then, by density, for $f \in C_0(\mathbb{R}^d)$. Theorem 3.2.8 then will imply that B is sectorial: a contradiction.

As a first step we prove that, for each $f \in C_c^\infty(\mathbb{R}^d)$ and $T > 0$, the function $T_n(\cdot)f$ converges to $S(\cdot)f$ in $[0,T] \times \mathbb{R}^d$ as n tends to ∞. For this purpose, we begin by observing that, since $S(t)$ maps $C_c^\infty(\mathbb{R}^d)$ into itself for each $t \ge 0$, and $C_c^\infty(\mathbb{R}^d)$ is contained in $D(\widehat{A}_n)$, by Exercise 15.3.1,

$$\frac{d}{dt}(T_n(t)S(s)f)(x) = (T_n(t)\widehat{A}_n S(s)f)(x), \qquad s, t > 0, \;\; x \in \mathbb{R}^d,$$

for every $f \in C_c^\infty(\mathbb{R}^d)$. Moreover, since $\{S(t)\}$ is a strongly continuous semigroup of contractions and the domain of its infinitesimal generator contains $C_c^\infty(\mathbb{R}^d)$, then the map $t \mapsto S(t)f$ is differentiable on $[0,\infty)$ with values in $C_0(\mathbb{R}^d)$ and its derivative at t is $BS(t)f$. Therefore, we have

$$(T_n(t)f)(x) - (S(t)f)(x) = -\int_0^t \frac{d}{ds}[(T_n(t-s)S(s)f)(x)]ds$$

$$=\int_0^t (T_n(t-s)(\widehat{A}_n - B)S(s)f)(x)ds,$$

for $t>0$ and $x\in\mathbb{R}^d$, which implies that

$$\|T_n(t)f - S(t)f\|_\infty \le T\sup_{s\in[0,T]}\|(\widehat{A}_n - B)S(s)f\|_\infty, \tag{15.9}$$

for all $t\in[0,T]$ and all $T>0$. Let us observe that the right-hand side of (15.9) converges to 0 as n tends to ∞. Indeed, recalling that $\widehat{A}_nS(s)f=\mathcal{A}_nS(s)f$ (see again Exercise 15.3.1), we can estimate

$$\begin{aligned}&|(\widehat{A}_nS(s)f)(x)-(BS(s)f)(x)|\\ \le&\frac{r_n}{\lambda_n^2}\|\Delta S(s)f\|_\infty+\left|\frac{r_n}{\lambda_n}b(\lambda_n x+\sigma_n)-h\right|\|\nabla S(s)f\|_\infty\chi_{\mathrm{supp}(S(s)f)}(x),\end{aligned}$$

for all $s\in[0,T]$ and $x\in\mathbb{R}^d$. Since $\mathrm{supp}(S(s)f)\subset\overline{B(0,|h|T)}+\mathrm{supp}(f)$ for every $s\in[0,T]$, using the conditions (15.3) and (15.4), we easily deduce that $\|(\widehat{A}_n-B)S(s)f\|_\infty$ tends to 0 as n tends to ∞. Therefore, from (15.9) it follows that $T_n(\cdot)$ tends to $S(\cdot)f$ uniformly in $[0,T]\times\mathbb{R}^d$.

Now, by Exercise 15.3.1, we can write

$$(R(\lambda,\widehat{A}_n)f)(x)-(R(\lambda,B)f)(x)=\int_0^\infty e^{-\lambda t}(T_n(t)f-S(t)f)(x)dt,$$

for all $x\in\mathbb{R}^d$. Consequently,

$$|(R(\lambda,\widehat{A}_n)f)(x)-(R(\lambda,B)f)(x)|\le\int_0^\infty e^{-\mathrm{Re}\,\lambda t}|(T_n(t)f)(x)-(S(t)f)(x)|dt,$$

for each $x\in\mathbb{R}^d$, and the right-hand side of (15.10) tends to 0 as n tends to ∞, by the dominated convergence theorem. Therefore, (15.8) follows.

To conclude the proof, let us prove that, if $r_n=1$ for every $n\in\mathbb{N}$, then the spectrum of $(\widehat{A},D(\widehat{A}))$ contains the imaginary axis. For this purpose, we observe that (15.5) yields $\|R(\lambda,\widehat{A})\|_{L(C_b(\mathbb{R}^d))}=\|R(\lambda,\widehat{A}_n)\|_{L(C_b(\mathbb{R}^d))}$ for every $\lambda\in\rho(\widehat{A}_n)=\rho(\widehat{A})$. Therefore, since $\rho(\widehat{A})$ contains the halfplane $\{\lambda\in\mathbb{C}:\mathrm{Re}\,\lambda>0\}$, by (15.8) we conclude that

$$\|R(\lambda,B)\|_{L(C_b(\mathbb{R}^d))}\le\limsup_{n\to\infty}\|R(\lambda,\widehat{A}_n)\|_{L(C_b(\mathbb{R}^d))}=\|R(\lambda,\widehat{A})\|_{L(C_b(\mathbb{R}^d))}.$$

But, since $i\mathbb{R}\subset\sigma(B)$, it follows that

$$\lim_{\substack{\lambda\to is\\ \mathrm{Re}\lambda>0}}\|R(\lambda,A)\|_{L(C_b(\mathbb{R}^d))}=\infty,$$

for all $s>0$, so that $i\mathbb{R}\subset\sigma(\widehat{A})$, see Exercise 15.3.3. □

Now we look for sufficient conditions under which (15.2)-(15.4) are satisfied.

Corollary 15.1.2. *Assume that there exist two sequences* $(\tau_n) \subset \mathbb{R}^d$ *and* $(\lambda_n) \subset (0,\infty)$ *such that* $b(\tau_n) \neq 0$ *for each* $n \in \mathbb{N}$, (λ_n) *is bounded,* $\lambda_n|b(\tau_n)|$ *tends to* ∞ *as* n *tends to* ∞ *and*

$$\lim_{n\to\infty} \frac{|b(\tau_n + \lambda_n x) - b(\tau_n)|}{|b(\tau_n)|} = 0, \tag{15.10}$$

uniformly on compact subsets of $\mathbb{R}^d$. *Then,* $(\widehat{A}, D(\widehat{A}))$ *is not sectorial.*

Proof. Let (σ_n) be a subsequence of $\{\tau_n\}$ such that $b(\sigma_n)/|b(\sigma_n)|$ converges to some $h \in \mathbb{R}^d$ as n tends to ∞. If we set $r_n = \lambda_n |b(\sigma_n)|^{-1}$ for $n \in \mathbb{N}$, then it is immediate to check that the three sequences (r_n), (λ_n) and (σ_n) satisfy the conditions (15.2)-(15.4). □

Corollary 15.1.3. *Let* $b_j \in C^1(\mathbb{R}^d)$ *for each* $j = 1, \ldots, d$. *Further, assume that there exists a sequence* $\{\tau_n\} \subset \mathbb{R}^d$, *diverging to* ∞, *such that* $|b(\tau_n)| \geq K|\tau_n|^\alpha$ *for some* $K, \alpha > 0$ *and all* $n \in \mathbb{N}$. *If*

$$\sum_{i,j=1}^{d} |D_i b_j(x)|^2 \leq K(1 + |x|^\beta)^2, \qquad x \in \mathbb{R}^d, \tag{15.11}$$

for some $\beta < \alpha$, *then the assumptions of Corollary* 15.1.2 *are satisfied.*

Proof. Take $\lambda_n = 1$ for all $n \in \mathbb{N}$. The mean value theorem yields

$$|b(\tau_n + x) - b(\tau_n)| \leq R \sup_{0\leq\theta\leq 1} \left(\sum_{i,j=1}^{d} |D_i b_j(\tau_n + \theta x)|^2 \right)^{\frac{1}{2}} \leq KR(1 + (|\tau_n| + R)^\beta),$$

for each $x \in \overline{B(0,R)}$. Dividing by $|b(\tau_n)|$, we easily see that condition (15.10) is satisfied since, by assumption, $|b(\tau_n)|$ tend to ∞ faster than $|\tau_n|^\beta$. □

Remark 15.1.4. The conditions in Corollary 15.1.3 are satisfied, for instance, if b_j are polynomials for each $j = 1, \ldots, N$. Indeed, let b_i be the polynomial of maximal degree and define $\tau_n = n x_0$ for $n \in \mathbb{N}$, where $x_0 \in \mathbb{R}^d$ is such that $|x_0| = 1$ and the homogeneous part of b_i of maximum degree does not vanish at x_0. It is immediate to check that $|b(\tau_n)| \geq |b_i(\tau_n)| \geq K n^{\deg(b_i)}$ for some positive constant $K > 0$ and every n sufficiently large. Moreover, since $\sum_{i,j=1}^{d} |D_i b_j|^2$ is a polynomial of degree $2(\deg(b_i) - 1)$, the condition (15.11) is satisfied with $\beta = \deg(b_i) - 1 < \alpha := \deg(b_i)$.

Example 15.1.5. Let $\mathcal{A}$ be the operator defined on smooth functions by

$$\mathcal{A}\zeta(x) = \Delta\zeta(x) - |x|^r \langle x, \nabla\zeta(x)\rangle, \qquad x \in \mathbb{R}^d,$$

where $r > 0$. Then the function $\varphi : \mathbb{R}^d \to \mathbb{R}$, defined by $\varphi(x) = 1 + |x|^2$ for any $x \in \mathbb{R}^d$, satisfies the condition $\mathcal{A}\varphi \leq 2d\varphi$. So, by Theorem 14.2.1, there exists a unique bounded classical solution to the Cauchy problem (14.1) associated to $\mathcal{A}$. Thus, Remark 15.1.4 and Corollary 15.1.3 imply that $(\widehat{A}, D(\widehat{A}))$ is not sectorial.

Corollary 15.1.6. *Let $b_j \in C^1(\mathbb{R}^d)$ $(j = 1, \dots, d)$. Assume that there exist a sequence $(\tau_n) \subset \mathbb{R}^d$ such that $|b(\tau_n)|$, diverges to ∞ as n tends to ∞, and positive numbers γ, δ, C such that*

$$\left(\sum_{i,j=1}^{d} |D_i b_j(x)|^2\right)^{\frac{1}{2}} \le \gamma |b(x)|^{\frac{3}{2}} + C,$$

for every $x \in \mathbb{R}^d$ with $|x - \tau_n| \le \delta |b(\tau_n)|^{-1/2}$. Then $(\widehat{A}, D(\widehat{A}))$ is not sectorial in $C_b(\mathbb{R}^d)$.

Proof. Without loss of generality, we can assume that $\delta < (4\gamma)^{-1}$ and that $b(\tau_n) \neq 0$ for all $n \in \mathbb{N}$. We fix an arbitrary $n \in \mathbb{N}$ and observe that, for $s \in [0, \delta)$ and $|x| \le s|b(\tau_n)|^{-1/2}$, we can estimate

$$\begin{aligned}\frac{|b(\tau_n + x) - b(\tau_n)|}{|b(\tau_n)|} \le& \frac{|x|}{|b(\tau_n)|} \sup_{0 \le \theta \le 1} \left(\sum_{i,j=1}^{d} |D_i b_j(\tau_n + \theta x)|^2\right)^{\frac{1}{2}} \\ \le& \gamma s \sup_{0 \le \theta \le 1} \frac{|b(\tau_n + \theta x)|^{\frac{3}{2}}}{|b(\tau_n)|^{\frac{3}{2}}} + Cs|b(\tau_n)|^{-\frac{3}{2}}.\end{aligned} \tag{15.12}$$

Set

$$F_n(s) = \sup_{|x| \le s|b(\tau_n)|^{-\frac{1}{2}}} \frac{|b(\tau_n + x)|}{|b(\tau_n)|}, \qquad s \in [0, \delta).$$

From (15.12) we get

$$F_n(s) \le 1 + \gamma s (F_n(s))^{\frac{3}{2}} + Cs|b(\tau_n)|^{-\frac{3}{2}}, \qquad s \in [0, \delta).$$

We claim that $F_n(s) < 4$ for all $s \in [0, \delta)$ and all n sufficiently large. To prove the claim, we begin by noting that, for n large enough, we have

$$1 + Cs|b(\tau_n)|^{-\frac{3}{2}} \le 1 + C\delta |b(\tau_n)|^{-\frac{3}{2}} \le 2.$$

It follows that $F_n(s) \le 2 + \gamma s (F_n(s))^{\frac{3}{2}}$ for every $s \in [0, \delta)$. Recalling that $4\delta\gamma < 1$, we get

$$F_n(s) < 2 + \frac{1}{4}(F_n(s))^{\frac{3}{2}}, \qquad s \in [0, \delta).$$

Since F_n is continuous in $[0, \infty)$ with $F_n(0) = 1$ for every $n \in \mathbb{N}$, and $[0, 4)$ is the biggest interval containing 1 in which the inequality $x < 2 + x^{3/2}/4$ holds, we easily deduce that $F_n(s) \in [0, 4)$ for all $s \in [0, \delta)$, and the claim follows. Thus, for n large enough, the equation (15.12) yields, for $|x| \le s|b(\tau_n)|^{-1/2}$,

$$\frac{|b(\tau_n + x) - b(\tau_n)|}{|b(\tau_n)|} \le 8\gamma s + Cs|b(\tau_n)|^{-\frac{3}{2}}. \tag{15.13}$$

Now, fix $R>0$ and let $\lambda_n=|b(\tau_n)|^{-\frac{3}{4}}$. For $y\in B(0,R)$ and n sufficiently large such that $R|b(\tau_n)|^{-1/4}<\delta$, we have $|\lambda_n y|\le s|b(\tau_n)|^{-1/2}$ with $s=R|b(\tau_n)|^{-1/4}$. Applying (15.13), we get

$$\frac{|b(\tau_n+\lambda_n y)-b(\tau_n)|}{|b(\tau_n)|}\le 8R\gamma|b(\tau_n)|^{-\frac{1}{4}}+CR|b(\tau_n)|^{-\frac{7}{4}}.$$

Corollary 15.1.2 now allows us to conclude the proof. □

Next corollary gives suitable conditions on the drift term of the operator $\mathcal{A}$ implying that the spectrum of $\widehat{A}$ contains the imaginary axis.

Corollary 15.1.7. *Assume that $b_j\in C^1(\mathbb{R})$ for each $j=1,\dots,N$ and there exists a sequence $(\tau_n)\subset\mathbb{R}^d$, diverging to infinity, such that $|b(\tau_n)|$ tends to ∞ as n tends to ∞. Further, assume that $\sum_{i,j=1}^d|D_ib_j(x)|^2$ and $|b(x)|/|x|$ tend to 0 as $|x|$ tends to ∞. Then, the spectrum of $(\widehat{A},D(\widehat{A}))$ contains the imaginary axis. As a consequence, $(\widehat{A},D(\widehat{A}))$ is not sectorial.*

Proof. Let (σ_n) be a subsequence of (τ_n) such that $\lim_{n\to\infty}\frac{b(\sigma_n)}{|b(\sigma_n)|}=h$ for some $h\in\mathbb{R}^d$. Fix $R>0$. By the mean value theorem we have

$$|b(\sigma_n+\lambda_n x)-b(\sigma_n)|\le R\lambda_n\sup_{0\le\theta\le1}\left(\sum_{i,j=1}^d|D_ib_j(\sigma_n+\theta\lambda_n x)|^2\right)^{\frac{1}{2}},$$

for all $x\in\overline{B(0,R)}$ and every sequence $(\lambda_n)\subset(0,\infty)$.

Now, we take $\lambda_n=|b(\sigma_n)|$ and $r_n=1$ for all $n\in\mathbb{N}$, and we show that

$$\lim_{n\to\infty}\sup_{0\le\theta\le1}\left(\sum_{i,j=1}^d|D_ib_j(\sigma_n+\theta\lambda_n x)|^2\right)^{\frac{1}{2}}=0,$$

uniformly with respect to $x\in\overline{B(0,R)}$. For this purpose it suffices to show that $|\sigma_n+\theta\lambda_n x|$ diverges to ∞ as n tends to ∞, uniformly with respect to $\theta\in[0,1]$ and $x\in\overline{B(0,R)}$. Since b is sublinear, it follows that

$$|\sigma_n+\theta\lambda_n x|\ge|\sigma_n|\left(1-\frac{R|b(\sigma_n)|}{|\sigma_n|}\right)\ge\frac{1}{2}|\sigma_n|$$

for all $\theta\in[0,1]$, $x\in\overline{B(0,R)}$ and n sufficiently large. Therefore,

$$\lim_{n\to\infty}\inf_{\substack{\theta\in[0,1]\\ x\in\overline{B(0,R)}}}|\sigma_n+\theta\lambda_n x|=\infty.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{\lambda_n}b(\sigma_n+\lambda_n x)=\lim_{n\to\infty}\frac{b(\sigma_n+\lambda_n x)-b(\sigma_n)}{|b(\sigma_n)|}+\lim_{n\to\infty}\frac{b(\sigma_n)}{|b(\sigma_n)|}=h.$$

So, Theorem 15.1.1 allows us to conclude. □

Up to now we have shown sufficient conditions implying that $(\widehat{A}, D(\widehat{A}))$ is not sectorial. In the next theorem we provide suitable conditions on the drift coefficient of the operator $\mathcal{A}$ which guarantee that $(\widehat{A}, D(\widehat{A}))$ is sectorial in $C_b(\mathbb{R}^d)$.

Theorem 15.1.8. *Let $b_j \in C^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, for some $p \in (d, \infty)$ and every $j = 1, \ldots, d$, be such that* $\operatorname{div} b$ *is bounded. Then, $(\widehat{A}, D(\widehat{A}))$ is sectorial in $C_b(\mathbb{R}^d)$.*

Proof. As a first step we introduce the formal adjoint of $\mathcal{A}$, i.e. the operator

$$\mathcal{A}^* = \Delta - \sum_{i=1}^{d} b_i D_i - \operatorname{div} b,$$

and prove that its realization A^* in $L^1(\mathbb{R}^d)$ with domain $D(A^*) = \{u \in L^1(\mathbb{R}^d) : \Delta u \in L^1(\mathbb{R}^d)\}$ (Δu being meant in the sense of distributions[1]) generates an analytic semigroup in $L^1(\mathbb{R}^d)$. Then, we use this result to show that $(\widehat{A}, D(\widehat{A}))$ is sectorial.

Let us observe that, since $\operatorname{div} b$ is bounded, according to the bounded perturbation theorem [2, Theorem 1.3, Chap. III] the operator $B = \Delta - \operatorname{div} b$, with domain $D(B) = D(A^*)$, generates an analytic semigroup in $L^1(\mathbb{R}^d)$.

We now recall that if $u \in L^1(\mathbb{R}^d)$ is such that $\Delta u \in L^1(\mathbb{R}^d)$, then $u \in W^{1,q}(\mathbb{R}^d)$ for each $q \in [1, d/(d-1))$ and there exists a positive constant $C = C(d,q)$, independent of u, such that

$$\|\nabla u\|_{L^q(\mathbb{R}^d)} \le C(d,q) \left(\|u\|_{L^1(\mathbb{R}^d)} + \|\Delta u\|_{L^1(\mathbb{R}^d)} \right), \tag{15.14}$$

see [5, Theorem 5.8]. Applying the estimate (15.14), with $q = p/(p-1)$, to the function $v_\lambda : \mathbb{R}^d \to \mathbb{R}$ $(\lambda > 0)$, defined by $v(x) = u(\lambda x)$ for $x \in \mathbb{R}^d$, and minimizing with respect to $\lambda > 0$, we get the following inequality:

$$\|\nabla u\|_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} \le 2C(d,p) \|u\|_{L^1(\mathbb{R}^d)}^{\frac{p-d}{2p}} \|\Delta u\|_{L^1(\mathbb{R}^d)}^{\frac{d+p}{2p}}.$$

Hölder's and Young's inequality then give

$$\begin{aligned} \left\| \sum_{i=1}^{d} b_i D_i u \right\|_{L^1(\mathbb{R}^d)} \le & \|b\|_{L^p(\mathbb{R}^d)} \|\nabla u\|_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} \\ \le & 2C(d,p) \|b\|_{L^p(\mathbb{R}^d)} \|u\|_{L^1(\mathbb{R}^d)}^{\frac{p-d}{2p}} \|\Delta u\|_{L^1(\mathbb{R}^d)}^{\frac{p+d}{2p}} \\ \le & C(d,p,\varepsilon) \|b\|_{L^p(\mathbb{R}^d)} \|u\|_{L^1(\mathbb{R}^d)} + \varepsilon \|\Delta u\|_{L^1(\mathbb{R}^d)} \end{aligned} \tag{15.15}$$

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[1] i.e., Δu is the unique function $g \in L^1(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} u \Delta \psi dx = \int_{\mathbb{R}^d} g \psi dx, \qquad \psi \in C_c^\infty(\mathbb{R}^d).$$

for every $\varepsilon > 0$ and $u \in L^1(\mathbb{R}^d)$ such that $\Delta u \in L^1(\mathbb{R}^d)$. From (15.15) it follows that the term $u \mapsto \langle b, \nabla u\rangle$ is a small perturbation of $(B, D(B))$. Therefore, according to [2, Theorem 2.10, Chap. III], $(A^*, D(A^*))$ generates an analytic semigroup in $L^1(\mathbb{R}^d)$.

Now, we can prove that $(\widehat{A}, D(\widehat{A}))$ is sectorial in $C_b(\mathbb{R}^d)$. Possibly replacing A^* with $A^* - \omega$, with ω sufficiently large, we can assume that

$$\|R(\lambda, A^*)\|_{L(L^1(\mathbb{R}^d))} \le \frac{M}{|\lambda|},$$

for every $\lambda \in \mathbb{C}$ with positive real part. Let us set $D_\lambda = (\lambda - A^*)(C_c^\infty(\mathbb{R}^d))$ for such λ's. Since $C_c^\infty(\mathbb{R}^d)$ is a core of A^*, by Exercise 15.3.4, then D_λ is dense in $L^1(\mathbb{R}^d)$ for every $\operatorname{Re}\lambda > 0$. Therefore, if $u \in D(\widehat{A}) \subset C_b(\mathbb{R}^d)$ and λ is as above, then

$$\begin{aligned}
\|u\|_\infty =& \sup\left\{\int_{\mathbb{R}^d} u\varphi\, dx : \varphi \in D_\lambda,\ \|\varphi\|_{L^1(\mathbb{R}^d)} \le 1\right\}\\
\le& \sup\left\{\int_{\mathbb{R}^d} u(\lambda - A^*)v\, dx : v \in C_c^\infty(\mathbb{R}^d),\ \|v\|_{L^1(\mathbb{R}^d)} \le \frac{M}{|\lambda|}\right\}\\
=& \sup\left\{\int_{\mathbb{R}^d} v(\lambda - \widehat{A})u\, dx : v \in C_c^\infty(\mathbb{R}^d),\ \|v\|_{L^1(\mathbb{R}^d)} \le \frac{M}{|\lambda|}\right\}\\
\le& \frac{M}{|\lambda|}\|(\lambda - \widehat{A})u\|_\infty,
\end{aligned}$$

and we conclude that $\|R(\lambda, \widehat{A})\|_{L(C_b(\mathbb{R}^d))} \le M/|\lambda|$ and the assertion follows from Proposition 3.2.8. □

Example 15.1.9. Let $\mathcal{A}$ be the operator defined on smooth functions of two variables by

$$\mathcal{A}\zeta(x,y) = \Delta\zeta(x,y) + b_0(\sqrt{x^2+y^2})(xD_y\zeta - yD_x\zeta), \qquad (x,y) \in \mathbb{R}^2,$$

where $b_0 \in C^1(\mathbb{R})$. Note that the drift $b(x,y) = b_0(\sqrt{x^2+y^2})(-y,x)$ has divergence 0. By an easy computation one can see that the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, defined by $\varphi(x,y) = 1 + x^2 + y^2$ for every $(x,y) \in \mathbb{R}^2$, satisfies the inequality $\mathcal{A}\varphi \le 4\varphi$. So, the uniqueness of the bounded classical solution to the homogeneous Cauchy problem (14.1), associated to the operator $\mathcal{A}$, follows from Theorem 14.2.1. Moreover the assumptions of Theorem 15.1.8 hold if

$$\int_0^\infty r^{p+1}|b_0(r)|^p dr < \infty,$$

for some $p > 2$. In such a case, the semigroup associated with the operator $\mathcal{A}$ is analytic in $C_b(\mathbb{R}^d)$.

15.2 Notes

By knowing the concept of *minimal semigroup*, the assumption on the uniqueness of the bounded classical solution to the homogeneous Cauchy problem (14.1)can be removed. Indeed, it can be shown, that if $f \in C_b(\mathbb{R}^d)$ is nonnegative, then it always exists a minimal nonnegative (actually positive if f does not identically vanishes on $\mathbb{R}^d$) classical solution to the Cauchy problem (14.1), which is bounded on $[0,T] \times \mathbb{R}^d$ for every $T > 0$.

We also stress that the assumption on the boundedness of $\operatorname{div} b$ in Theorem 15.1.8 can be relaxed, just requiring the boundedness from below of $\operatorname{div} b$. For more details we refer to [4], see also [3, Chapter 10]. Finally, we remark that the technique used to prove Theorem 15.1.1 is inspired by the paper [1].

15.3 Exercises

1. Fix $f \in D(\widehat{A})$. Prove the following properties:
   - (i) $T(t)f \in D(\widehat{A})$ for all $t \geq 0$ and $\widehat{A}T(t)f = T(t)\widehat{A}f$;
   - (ii) for each $x \in \mathbb{R}^d$, the function $(T(\cdot)f)(x)$ is of class C^1 and $\frac{d}{dt}(T(t)f)(x) = (T(t)\widehat{A}f)(x)$;
   - (iii) $D(\widehat{A})$ is dense in $C_b(\mathbb{R}^d)$ with respect the dominated pointwise convergence, i.e. for every $f \in C_b(\mathbb{R}^d)$, there is a sequence of functions $(f_n) \subseteq D(\widehat{A})$ uniformly bounded and converges pointwise to f;
   - (iv) $(\widehat{A}, D(\widehat{A}))$ is closed in $C_b(\mathbb{R}^d)$ with respect to the dominated pointwise convergence.
   - (v) $(0,\infty) \subset \rho(\widehat{A})$ and for each $\lambda > 0$

     $$R(\lambda, \widehat{A})f(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)\,dt, \quad f \in C_b(\mathbb{R}^d),\ x \in \mathbb{R}^d;$$

   - (vi) $C_c^\infty(\mathbb{R}^d) \subset D(\widehat{A})$ and $\mathcal{A}f = \widehat{A}f$ for every $f \in C_c^\infty(\mathbb{R}^d)$.

2. Fix $h \in \mathbb{R}^d$ and consider the family of operators $\{S(t) : t > 0\}$ defined on $C_0(\mathbb{R}^d)$ by

   $$(S(t)f)(x) = f(x + th), \qquad t > 0,\ x \in \mathbb{R}^d$$

   for $f \in C_0(\mathbb{R}^d)$. Prove that $\{S(t)\}$ is a C_0-semigroup on $C_0(\mathbb{R}^d)$ whose generator is given by

   $$Bu = \langle h, Du\rangle,\ u \in D(B) = \big\{u \in C_0(\mathbb{R}^d) : \langle h, Du\rangle \in C_0(\mathbb{R}^d)\big\}$$

   and that $i\mathbb{R} \subset \sigma(B)$.

3. Let A be a linear operator with domain $D(A)$ on a Banach space X. Assume that $\rho(A) \neq \varnothing$ prove that

$$\sigma(R(\lambda, A)) \setminus \{0\} = \left\{ \frac{1}{\lambda - \mu} : \mu \in \sigma(A) \right\}$$

and deduce that

$$\operatorname{dist}(\lambda, \sigma(A)) = \frac{1}{r(R(\lambda, A))} \geq \frac{1}{\|R(\lambda, A)\|_{L(X)}}$$

for $\lambda \in \rho(A)$, where $r(R(\lambda, A))$ denotes the spectral radius of $R(\lambda, A)$.

4. Consider the Laplacian Δ on $L^1(\mathbb{R}^d)$ with domain

$$D(\Delta) = \{u \in L^1(\mathbb{R}^d) : \Delta u \in L^1(\mathbb{R}^d)\},$$

where Δu is meant in the sense of distributions. Prove that $C_c^\infty(\mathbb{R}^d)$ is a core for Δ, i.e., $C_c^\infty(\mathbb{R}^d)$ is dense in $D(\Delta)$ with respect to the graph norm:

$$\|f\|_\Delta = \|f\|_{L^1(\mathbb{R}^d)} + \|\Delta f\|_{L^1(\mathbb{R}^d)}, \quad f \in D(\Delta).$$

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Appendix A

Basic notions of Functional Analysis in Banach spaces

In this appendix we collect a few basic results on linear operators, and on elementary spectral theory that we use in this lecture. For more details and for the proofs of the results that we present, we refer the reader mainly to [4, 5].

A.1 Bounded and closed linear operators

Let X and Y be two Banach spaces. We denote by $L(X,Y)$ the vector space of linear and bounded operators $T : X \to Y$. We endow it with the norm

$$\|T\|_{L(X,Y)} = \sup_{\substack{x\in X\\ \|x\|=1}} \|Tx\|_Y = \sup_{x\in X\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X}. \tag{A.1}$$

If $Y = X$, we write $L(X)$ instead of $L(X,X)$.

The norm in (A.1) makes $L(X,Y)$ a Banach space.

Now, we introduce another class of linear operators which we use in these lectures. If $D(A)$ is a vector subspace of X and $A : D(A) \subset X \to Y$ is a linear operator, we say that A is *closed* if its graph $\mathcal{G}_A = \{(x,y) \in X \times Y : x \in D(A),\ y = Ax\}$ is a closed subset of $X \times Y$. As it is easily seen, A is closed if and only if, for any sequence $\{x_n\} \subset D(A)$ such that x_n and Ax_n converge, respectively, to some elements $x \in X$ and $y \in Y$, as n tends to $+\infty$, then $x \in D(A)$ and $y = Ax$.

In general, a closed operator is not bounded in $(X, \|\cdot\|)$. It turns out to be bounded if we endow $D(A)$ with the *graph norm*

$$\|x\|_{D(A)} = \|x\|_X + \|Ax\|_Y, \qquad x \in D(A). \tag{A.2}$$

Note that $D(A)$ is a Banach space when it is endowed with the graph norm.

An operator $A : D(A) \subset X \to Y$ is said to be *closable* if there exists a (closed) operator $\overline{A}$ whose graph coincides with the closure of $\mathcal{G}_A$. The operator $\overline{A}$ is called the *closure* of A.

In such a case every $x \in D(\overline{A})$ is the limit of a sequence $\{x_n\} \subset D(A)$ which is a Cauchy sequence with respect to the graph norm (A.2). Moreover, $\overline{A}x := \lim_{n \to +\infty} Ax_n$.

Equivalently, A is closable if, for each sequence $\{x_n\} \in D(A)$ converging to 0 in X and such that Ax_n converges to some $y \in Y$ as n tends to $+\infty$, then $y = 0$.

A.2 Vector valued Riemann integral

In this section we define the Riemann integral for vector-valued functions. For more details and for the proof of the results that we present here, we refer the reader to [1, Chapter 2], [2, Chapter 3] and [3, Chapter 3].

Definition A.2.1. A bounded function $f : [a, b] \mapsto X$ $(-\infty < a < b < +\infty)$ is said to be *integrable* on $[a, b]$ if there exists $x \in X$ with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for every partition $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a, b]$, with $\max_{i=1,\ldots,n}(t_i - t_{i-1}) < \delta$, and for every choice of the points $\xi_i \in [t_{i-1}, t_i]$, we have

$$\left\| x - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right\| < \varepsilon.$$

In this case we define

$$\int_a^b f(t)dt = x.$$

Arguing as in the real-valued case, one can easily check that the set of integrable functions on $[a, b]$ is a vector space and that the integral over $[a, b]$ is a linear operator on this vector space. In particular, if $f : [a, b] \to X$ is continuous, then it is integrable. Moreover, if f is integrable on $[a, b]$, then the map $t \mapsto \|f(t)\|_X$ is integrable on $[a, b]$ as well. Finally, if f is integrable on $[a, b]$, then it is integrable on every $[c, d] \subset [a, b]$ and

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^d f(t)dt$$

for every $c \in (a, b)$.

As in the real-valued case, the definition of the Riemann integral can be easily extended to the case of unbounded intervals or unbounded functions

Definition A.2.2. Let $I \subset \mathbb{R}$ be an interval with endpoints a and b $(-\infty \leq a < b \leq +\infty)$ with a and b not necessarily in I. Moreover, let $f : I \to X$ be Riemann integrable on $[c, d]$ for every $a < c < d < b$. We say that f admits an improper integral on I if, for each $t_0 \in I$, the limits

$$\lim_{c \to a^+} \int_c^{t_0} f(t)dt, \qquad \lim_{d \to b^-} \int_{t_0}^d f(t)dt$$

exist in X. In this case we set

$$\int_I f(x)dx = \lim_{c\to a^+}\int_c^{t_0} f(t)dt + \lim_{d\to b^-}\int_{t_0}^d f(t)dt.$$

Note that the previous definition is independent of t_0.

To simplify the notation in the following proposition we simply denote by I the interval in which the function f is defined. When we say that "f is integrable on I" we mean indifferently, that f is Riemann integrable on I or that it admits improper integral on I.

Proposition A.2.3. *Let $f : I \to X$ be integrable on I. Then, the following properties are met:*

(*i*) *for each bounded operator $T \in L(X, Y)$, the function Tf is Riemann integrable on $[a, b]$ and*

$$T\int_I f(t)dt = \int_I Tf(t)dt;$$

(*ii*) *if $A : D(A) \subset X \to Y$ is a closed operator such that Af is integrable on I, then*

$$\int_I f(t)dt \in D(A) \qquad \text{and} \qquad A\int_I f(t)dt = \int_I Af(t)dt.$$

Proof. We limit ourselves to proving the last part of the proposition, since the first one is similar and even simpler.

Proof of (ii). We first assume that $I = [a, b]$ for some $a, b \in \mathbb{R}$, $a < b$ and f is Riemann integrable on $[a, b]$. Set $x = \int_a^b f(t)dt$. We fix $n \in \mathbb{N}$ and consider the partition $\mathcal{P}_n = \{a = t_0 < \ldots < t_n = b\}$ where $t_k = a + (b - a)k/n$ for every $k = 0, \ldots, n$. Moreover, for every $k = 0, \ldots, n - 1$ fix $\xi_i \in [t_i, t_{i+1}]$ and set

$$S_n = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}), \qquad n \in \mathbb{N}.$$

Of course S_n belongs to $D(A)$ for each $n \in \mathbb{N}$ and

$$AS_n = \sum_{i=1}^n Af(\xi_i)(t_i - t_{i-1}), \qquad n \in \mathbb{N}.$$

Since both f and Af are integrable, S_n and AS_n converge, respectively, to x and $y = \int_a^b Af(t)dt$ as n tends to $+\infty$. Since A is closed, it follows that x belongs to $D(A)$ and $Ax = y$.

Now, suppose that f admits improper integral on I. To fix the ideas we assume that $I = [a, +\infty)$ and that f is Riemann integrable on $[a, b]$ for each $b > a$. Then,

$$A\int_a^b f(t)dt = \int_a^b Af(t)dt$$

By hypothesis

$$\lim_{b\to+\infty}\int_a^b Af(t)dt = \int_a^{+\infty} Af(t)dt \qquad \text{and} \qquad \lim_{b\to+\infty}\int_a^b f(t)dt = \int_a^{+\infty} f(t)dt.$$

Since A is closed,

$$\int_a^{+\infty} f(t)dt \in D(A) \qquad \text{and} \qquad A\int_a^{+\infty} f(t)dt = \int_a^{+\infty} Af(t)dt.$$

Next we recall the *fundamental theorem of calculus* for X-valued functions. For this purpose we first recall the definition of *Fréchet derivative.* Let $I \subset \mathbb{R}$ be an (open) interval and let $t_0 \in I$. The function $f : I \to X$ is *Fréchet differentiable* at $t_0 \in I$ if the limit

$$\lim_{t\to t_0}\frac{f(t)-f(t_0)}{t-t_0}$$

exists in X. Such a limit, when existing, is denoted by $f'(t_0)$ and is called the Fréchet derivative of f at t_0. In an analogous way the right and left derivatives can be defined.

Theorem A.2.4. *Let $f : [a,b] \to X$ be continuous. Then, the integral function $F : [a,b] \to X$ defined by*

$$F(t) = \int_a^t f(s)ds, \qquad t \in [a,b],$$

is (Fréchet) differentiable, and $F'(t) = f(t)$ for each $t \in [a,b]$.

Now, we recall the definition of the integral of vector-valued functions of a complex variable, along a smooth curve γ.

Definition A.2.5. Let Ω be an open subset of $\mathbb{C}$, $f : \Omega \to X$ be a continuous function and $\gamma : [a,b] \to \Omega$ be a piecewise C^1-curve. The integral of f along γ is defined as follows:

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

As in the case of vector-valued functions defined on a real interval, we can define the *improper complex integrals* in an obvious way.

Definition A.2.6. Let $\Omega \subset \mathbb{C}$ be a (possibly) unbounded open set. Moreover, let $I = (a,b)$ be a (possibly unbounded) interval and $\gamma : I \to \mathbb{C}$ be a (piecewise) C^1 curve in Ω. We say that f admits an improper integral along γ if for each $t_0 \in (a,b)$ the limits

$$\lim_{s\to a^+}\int_s^{t_0} f(\gamma(\tau))\gamma'(\tau)d\tau \qquad \text{and} \qquad \lim_{s\to b^-}\int_{t_0}^b f(\gamma(\tau))\gamma'(\tau)d\tau$$

exist in X. In such a case, we set

$$\int_\gamma f(z)dz = \lim_{s\to a^+}\int_s^{t_0} f(\gamma(\tau))\gamma'(\tau)d\tau + \lim_{s\to b^-}\int_{t_0}^s f(\gamma(\tau))\gamma'(\tau)d\tau.$$

Note that the definition of the improper integral is independent of the choice of t_0. Moreover, if I is bounded and the integral of f along γ exists, then f admits an improper integral along γ and the two integrals coincide.

A.3 Resolvent families

In this section, we prove the following important result.

Proposition A.3.1. *Let* $\Omega \subset \mathbb{C}$ *be an open set, and let* $\{F(\lambda) : \lambda \in \Omega\} \subset L(X)$ *be a family of linear operators verifying the resolvent identity*

$$F(\lambda) - F(\mu) = (\mu - \lambda)F(\lambda)F(\mu), \quad \lambda, \mu \in \Omega.$$

If the operator $F(\lambda_0)$ *is invertible, for some* $\lambda_0 \in \Omega$, *then there exists a closed linear operator* $A : D(A) \subset X \to X$ *such that* $\rho(A)$ *contains* Ω, *and* $R(\lambda, A) = F(\lambda)$ *for each* $\lambda \in \Omega$.

Proof. Fix $\lambda_0 \in \Omega$, and set

$$D(A) = \text{Range } F(\lambda_0), \;\; Ax = \lambda_0 x - F(\lambda_0)^{-1}x, \quad x \in D(A).$$

For $\lambda \in \Omega$ and $y \in X$ the resolvent equation $\lambda x - Ax = y$ is equivalent to $(\lambda - \lambda_0)x + F(\lambda_0)^{-1}x = y$. Applying $F(\lambda)$ we obtain $(\lambda - \lambda_0)F(\lambda)x + F(\lambda)F(\lambda_0)^{-1}x = F(\lambda)y$, and using the resolvent identity it is easily seen that

$$F(\lambda)F(\lambda_0)^{-1} = F(\lambda_0)^{-1}F(\lambda) = (\lambda_0 - \lambda)F(\lambda) + I.$$

Hence, if x is solution of the resolvent equation, then $x = F(\lambda)y$. Let us check that $x = F(\lambda)y$ is actually a solution. In fact, $(\lambda - \lambda_0)F(\lambda)y + F(\lambda_0)^{-1}F(\lambda)y = y$, and therefore λ belongs to $\rho(A)$ and the equality $R(\lambda, A) = F(\lambda)$ holds. □

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Appendix B

Interior Schauder estimates and optimal Schauder estimates for parabolic problems with parabolic problems associated with elliptic operators with unbounded coefficients

B.1 Interior Schauder estimates

As a byproduct of Theorem 8.1.1, we can prove some interesting interior Schauder estimates satisfied by the solutions to the differential equation $D_t u = \mathcal{A}u + g$ in $(0,T] \times \Omega$, $\Omega \subset \mathbb{R}^d$ being an open set. Such estimates are of local type, i.e., they allow to estimate the parabolic Hölder norm of u and its derivatives in every compact subset of $(0,T] \times \Omega$ in terms of the sup-norm of the solution itself in a larger compact set. We assume the following conditions on the coefficients of the operator $\mathcal{A}$.

Hypotheses B.1.1. The coefficients $q_{ij} = q_{ji}$, b_j and c $(i,j = 1,\ldots,d)$ belong to $C^{\alpha}_{\rm loc}(\Omega)$ [1] and there exists a positive continuous function $\mu : \Omega \to \mathbb{R}$ such that

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \geq \mu(x)|\xi|^2, \qquad \xi \in \mathbb{R}^d,\ x \in \Omega. \tag{B.1}$$

Remark B.1.2. Note that condition (B.1) implies that the matrix $Q(x)$ is positive definite

---

[1] $C^{\alpha}_{\rm loc}(\Omega)$ denotes the set of all functions $f : \Omega \to \mathbb{R}$ which are α-Hölder continuous in each compact subset of Ω.

in Ω and, for every compact set $K \subset \Omega$,

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \geq \mu_K|\xi|^2, \qquad \xi \in \mathbb{R}^d, \; x \in K,$$

where $\mu_K = \inf_{x\in K}\mu(x)$. Note also that, under the previous assumptions, it may happen that $\inf_{x\in\Omega}\mu(x) = 0$.

Now, we can state precisely the result of this subsection.

Theorem B.1.3. *Let $u \in C^{1,2}((0,T)\times\Omega)$ be a classical solution to the differential equation $D_t u = \mathcal{A}u + g$, corresponding to $g \in C^{\alpha/2,\alpha}_{\rm loc}((0,T)\times\Omega)$[2]. Then, $u \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,T)\times\Omega)$ and for every pair of bounded closed intervals I_1 and I_2, such that $I_1 \Subset I_2 \Subset (0,T)$ [3], and every pair of compact sets $K_1 \Subset K_2 \Subset \Omega$, there exists a positive constant $C = C(\mu, I_1, I_2, K_1, K_2)$ such that*

$$\|u\|_{C^{1+\alpha/2,2+\alpha}(I_1\times K_1)} \leq C(\|u\|_{C(I_2\times K_2)} + \|g\|_{C^{\alpha/2,\alpha}(I_2\times K_2)}). \tag{B.2}$$

In the proof of Theorem B.1.3 we shall use of the following result

Lemma B.1.4. *There exists a positive constant C such that for each $\varepsilon \in (0,1)$ and each function $v \in C^{1+\alpha/2,2+\alpha}_b([0,T_0]\times\mathbb{R}^d)$ it holds that*

$$\|v\|_{C^{\alpha/2,1+\alpha}_b([0,T_0]\times\mathbb{R}^d)} \leq C(\varepsilon\|v\|_{C^{1+\alpha/2,2+\alpha}_b([0,T_0]\times\mathbb{R}^d)} + \varepsilon^{-(1+\alpha)}\|v\|_\infty). \tag{B.3}$$

Proof. Throughout the proof, C denotes a positive constant, independent of ε and the functions that we consider.

Using the interpolation estimates (5.22) and (5.23) together with the Young inequality, we get

$$\|\zeta\|_{C^{\alpha}_b(\mathbb{R}^d)} \leq C(\varepsilon^{-\frac{\alpha}{2}}\|\zeta\|_\infty + \varepsilon\|\zeta\|_{C^{2+\alpha}_b(\mathbb{R}^d)}),$$
$$\|\zeta\|_{C^{1+\alpha}_b(\mathbb{R}^d)} \leq C(\varepsilon^{-(1+\alpha)}\|\zeta\|_\infty + \varepsilon\|\zeta\|_{C^{2+\alpha}_b(\mathbb{R}^d)}),$$

for every $\zeta \in C^{2+\alpha}_b(\mathbb{R}^d)$, and

$$\|\varphi\|_{C^{\alpha/2}([0,T_0])} \leq C(\varepsilon^{-\frac{\alpha}{2}}\|\varphi\|_\infty + \varepsilon\|\varphi\|_{C^{1+\alpha/2}([0,T_0])}), \qquad \varphi \in C^{1+\alpha/2}([0,T_0])$$

for every $\varepsilon \in (0,1)$ and some positive constant C independent of ε and ζ. Applying these estimates with $\zeta = v(t,\cdot)$ and $\varphi = v(\cdot,x)$, we deduce that

$$\|v\|_{C^{\alpha/2,\alpha}_b([0,T_0]\times\mathbb{R}^d)} + \sum_{j=1}^{d}\|D_j v\|_{C^{0,\alpha}_b([0,T_0]\times\mathbb{R}^d)}$$

---

[2] $C^{\alpha/2,\alpha}_{\rm loc}((0,T)\times\Omega)$ is the space of all functions $f \in C^{\alpha/2,\alpha}((a,b)\times K)$ for each $(a,b)\times K$ with compact closure in $(0,T)\times\Omega$.

[3] $\Omega_1 \Subset \Omega_2$ means that $\overline{\Omega_1} \subset \Omega_2$

$$\leq C(\varepsilon\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)}+\varepsilon^{-(1+\alpha)}\|v\|_\infty) \tag{B.4}$$

for each $\varepsilon \in (0,1)$. To estimate the $\alpha/2$-Hölder norm of the functions $D_jv(\cdot,x)$ ($j = 1,\ldots,d$) for every $x\in\mathbb{R}^d$, we use Remark 6.1.7 which shows that

$$\|D_jv(t_2,\cdot)-D_jv(t_1,\cdot)\|_\infty \leq C\|v\|_{C_b^{1,2+\alpha}([0,T_0]\times\mathbb{R}^d)}|t_2-t_1|^{\frac{1+\alpha}{2}}$$

for every $t_1,t_2\in[0,T_0]$ and $j=1,\ldots,d$. Hence, D_jv belongs to $C_b^{(1+\alpha)/2,0}([0,T_0]\times\mathbb{R}^d)$ for each $j=1,\ldots,d$ and

$$\sum_{j=1}^d\|D_jv\|_{C_b^{(1+\alpha)/2,0}([0,T_0]\times\mathbb{R}^d)}\leq C\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)}. \tag{B.5}$$

Using the interpolation estimate (5.24) and, again, Young's inequality, we deduce that

$$\|\varphi\|_{C^{\alpha/2}([0,T_0])}\leq C(\varepsilon\|\varphi\|_{C^{(1+\alpha)/2}([0,T_0])}+\varepsilon^{-\alpha}\|\varphi\|_\infty),\qquad \varphi\in C^{(1+\alpha)/2}([0,T_0]),\ \ \varepsilon\in(0,1).$$

This inequality, (B.4) and (B.5) show that

$$\begin{aligned}\sum_{j=1}^d\|D_jv\|_{C_b^{\alpha/2,\alpha}([0,T_0]\times\mathbb{R}^d)}\leq C\bigg(&\varepsilon\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)}\\ &+\varepsilon^{-(1+\alpha)}\|v\|_{C_b([0,T_0]\times\mathbb{R}^d)}+\varepsilon^{-\alpha}\sum_{j=1}^d\|D_jv\|_{C_b([0,T_0]\times\mathbb{R}^d)}\bigg).\end{aligned} \tag{B.6}$$

Now, from (5.25) and Young's inequality, we can estimate

$$\sum_{j=1}^d\|D_jv\|_{C_b([0,T_0]\times\mathbb{R}^d)}\leq\delta\|v\|_{C_b^{0,2+\alpha}([0,T_0]\times\mathbb{R}^d)}+\delta^{-\frac{1}{1+\alpha}}\|v\|_{C_b([0,T_0]\times\mathbb{R}^d)}$$

for each $\delta\in(0,1)$. Choosing $\delta=\varepsilon^{1+\alpha}$ and replacing this estimate in (B.6), we get

$$\sum_{j=1}^d\|D_jv\|_{C_b^{\alpha/2,\alpha}([0,T_0]\times\mathbb{R}^d)}\leq C(\varepsilon\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)}+\varepsilon^{-(1+\alpha)}\|v\|_{C_b([0,T_0]\times\mathbb{R}^d)}). \tag{B.7}$$

From (B.4) and (B.7), estimate (B.3) follows at once. □

Proof of Theorem B.1.3. The main step of the proof consists in showing that, for each $x_0\in\Omega$, $r>0$, such that $2r<\mathrm{dist}(x_0,\partial\Omega)$, $0<\tau_0<\tau<T$, estimate (B.2) is satisfied with $I_1=[\tau,T]$, $I_2=[\tau_0,T_0]$ $K_1=\overline{B(x_0,r)}$ and $K_2=\overline{B(x_0,2r)}$. This is the content of Step 1. Indeed, once this latter estimate is proved, a covering argument will allow us, in Step 2, to obtain estimate (B.2) in its full generality.

Step 1. Fix $0 < \tau < T_0$, $r > 0$ and let us prove (B.2) with I_1, I_2, K_1 and K_2 as above. Throughout the proof, we denote by C a positive constant, independent of u, g and n, which may vary from line to line.

We introduce the real sequences (r_n) and (t_n), defined by $r_n = (2 - 2^{-n})r$ and $t_n = \tau_0 + 2^{-n}(\tau - \tau_0)$, for every $n \in \mathbb{N} \cup \{0\}$. Note that the previous sequences are, respectively, increasing and decreasing. Next, we consider two functions $\varphi, \vartheta \in C^\infty(\mathbb{R})$ which satisfy the conditions $\chi_{[2,\infty)} \le \varphi \le \chi_{[1,\infty)}$ and $\chi_{(-\infty,1]} \le \vartheta \le \chi_{(-\infty,2]}$. For every $n \in \mathbb{N} \cup \{0\}$, we set

$$\varphi_n(t) = \varphi\left(1 + \frac{t - t_{n+1}}{t_n - t_{n+1}}\right), \qquad t \in \mathbb{R}, \qquad \vartheta_n(x) = \vartheta\left(1 + \frac{|x - x_0| - r_n}{r_{n+1} - r_n}\right), \qquad x \in \mathbb{R}^d.$$

As it is easily seen, $\varphi_n(t) = 1$ if $t \ge t_n$ and vanishes if $t \le t_{n+1}$. Similarly, $\vartheta_n(x) = 1$ if $x \in \overline{B(x_0, r_n)}$ and $\vartheta_n(x) = 0$ if $x \notin B(x_0, r_{n+1})$. Moreover,

We introduce the operator $\widetilde{\mathcal{A}} = \sum_{i,j=1}^d \widetilde{q}_{ij} D_{ij} + \sum_{j=1}^d \widetilde{b}_j D_j + \widetilde{c}$, where $\widetilde{q}_{ij} = \varrho q_{ij} + (1-\varrho)\delta_{ij}$, $\widetilde{b}_j = \varrho b_j$ $(i,j = 1, \ldots, d)$, $\widetilde{c} = \varrho c$, and $\varrho \in C_c^\infty(\mathbb{R}^d)$ is compactly supported in Ω with $0 \le \varrho \le 1$ and $\varrho \equiv 1$ in $B(x_0, 2r)$. Clearly, the coefficients of the operator $\widetilde{\mathcal{A}}$, extended in the trivial way to the whole $\mathbb{R}^d$, belong to $C_b^\alpha(\mathbb{R}^d)$ and coincide with the coefficients of the operator $\mathcal{A}$ in $B(x_0, 2r)$. Moreover,

$$\sum_{i,j=1}^d \widetilde{q}_{ij}(x)\xi_i\xi_j = \varrho(x)\sum_{i,j=1}^d q_{ij}(x)\xi_i\xi_j + (1-\varrho)|\xi|^2 \ge [\varrho(x)\mu(x) + 1 - \varrho(x)]|\xi|^2 \ge \mu_0|\xi|^2$$

for all $x, \xi \in \mathbb{R}^d$, where $\mu_0 = \min\{1, \inf_{x \in \operatorname{supp}\varrho} \mu(x)\} > 0$. From the above arguments (and in particular the fact that $\vartheta_n(x) = 0$ for $x \notin B(x_0, 2r)$) it follows that each function $v_n = u\varphi_n\vartheta_n$ belongs to $C_b^{1,2}([0,T] \times \mathbb{R}^d) \cap C_b([0,T] \times \mathbb{R}^d)$, vanishes at $t = 0$ and satisfies the equation $D_t v_n = \widetilde{\mathcal{A}} v_n + g_n$ in $(0,\infty) \times \mathbb{R}^d$, where

$$g_n = \varphi_n\vartheta_n g - \varphi_n u(\widetilde{\mathcal{A}}\vartheta_n - c\vartheta_n) - 2\varphi_n\langle \widetilde{Q}\nabla\vartheta_n, \nabla_x u\rangle + \varphi_n'\vartheta_n u$$

and $\widetilde{Q} = (\widetilde{q}_{ij})$. Since the coefficients of the operator $\widetilde{\mathcal{A}}$ satisfy Hypotheses 7.0.1, the Cauchy problem

$$\begin{cases} D_t u(t,x) = \widetilde{\mathcal{A}} u(t,x) + g_n(t,x), & t \in (0,T], \quad x \in \mathbb{R}^d, \\ u(0,x) = 0, & \qquad\qquad\quad x \in \mathbb{R}^d, \end{cases}$$

admits a unique solution which belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T_0] \times \mathbb{R}^d)$. Hence, the function v_n belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T_0] \times \mathbb{R}^d)$ and $\|v_n\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)} \le C\|g_n\|_{C_b^{\alpha/2,\alpha}([0,T_0]\times\mathbb{R}^d)}$ for every $n \in \mathbb{N}$, where the constant C is independent of n. In particular, taking $n = 0$, we deduce that u belongs to $C^{1+\alpha/2,2+\alpha}([\tau, T_0] \times \overline{B(x_0, r)})$. The arbitrariness of τ and r imply that $u \in C^{1+\alpha/2,2+\alpha}(J \times K)$ for each closed interval $J \Subset (0, T_0]$ and each compact set $K \subset \mathbb{R}^d$.

Estimating the $C_b^{\alpha/2,\alpha}([0,T_0]\times\mathbb{R}^d)$-norm of the function g_n, we get

$$\begin{aligned}
&\|v_n\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)}\\
\le& C\|\vartheta_n\|_{C_b^3(\mathbb{R}^d)}\|\varphi\|_{C_b^2(\mathbb{R})}\big(\|u\|_{C^{\alpha/2,1+\alpha}([t_{n+1},T_0]\times\overline{B(x_0,r_{n+1})})}+\|g\|_{C^{\alpha/2,\alpha}([t_{n+1},T_0]\times\overline{B(x_0,r_{n+1})})}\big)\\
\le& 2^{5n}C\big(\|v_{n+1}\|_{C_b^{\alpha/2,\alpha}([0,T_0]\times\mathbb{R}^d)}+\sum_{j=1}^d\|D_jv_{n+1}\|_{C_b^{\alpha/2,\alpha}([0,T_0]\times\mathbb{R}^d)}+\|g\|_{C^{\alpha/2,\alpha}([\tau_0,T_0]\times\overline{B(x_0,2r)})}\big),
\end{aligned}$$

where we have used the inequalities $\|\vartheta_n\|_{C_b^3(\mathbb{R}^d)}\le 2^{3n}C$ and $\|\varphi\|_{C_b^2(\mathbb{R})}\le 2^{2n}C$, which hold true for all $n\in\mathbb{N}\cup\{0\}$, and observed that $v_{n+1}\equiv u$ in $[t_{n+1},T_0]\times\overline{B(x_0,r_{n+1})}$. This estimate and (B.3) yield

$$\begin{aligned}
&\|v_n\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)}\\
\le& 2^{5n}C(\varepsilon\|v_{n+1}\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)}+\varepsilon^{-(1+\alpha)}\|v_{n+1}\|_\infty+\|g\|_{C^{\alpha/2,\alpha}([\tau_0,T_0]\times\overline{B(x_0,2r)})})
\end{aligned}$$

for each $\varepsilon\in(0,1)$. We fix $\eta\in(0,2^{-5(2+\alpha)})$ and choose $\varepsilon=\varepsilon_n=2^{-5n}C^{-1}\eta$; from the previous estimate we obtain

$$\zeta_n\le\eta\zeta_{n+1}+2^{5n(2+\alpha)}C_\eta\|u\|_{C([\tau_0,T_0]\times\overline{B(x_0,2r)})}+2^{5n}C\|g\|_{C^{\alpha/2,\alpha}([\tau_0,T_0]\times\overline{B(x_0,2r)})},\tag{B.8}$$

where $\zeta_n=\|v_n\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)}$. Multiplying both sides of (B.19) by η^n and summing from 1 to $m\in\mathbb{N}$, we get

$$\begin{aligned}
\zeta_0-\eta^{m+1}\zeta_{m+1}\le& C_\eta\|u\|_{C([\tau_0,T_0]\times\overline{B(x_0,2r)})}\sum_{n=0}^m(2^{5(2+\alpha)}\eta)^n+C\|g\|_{C^{\alpha/2,\alpha}([\tau_0,T_0]\times\overline{B(x_0,2r)})}\sum_{n=0}^m(32\eta)^n\\
\le& C(\|u\|_{C([\tau_0,T_0]\times\overline{B(x_0,2r)})}+\|g\|_{C^{\alpha/2,\alpha}([\tau_0,T_0]\times\overline{B(x_0,2r)})}),
\end{aligned}\tag{B.9}$$

since both the two series converge, due to the choice of η. To conclude, we observe that $\eta^{n+1}\zeta_{n+1}$ tends to 0 as m tends to ∞. Indeed, we have shown that $u\in C^{1+\alpha/2,2+\alpha}(\overline{J}\times\overline{K})$ for every $J\Subset(0,\infty)$ and $K\Subset\mathbb{R}^d$. In particular, $u\in C^{1+\alpha/2,2+\alpha}([\tau_0,T_0]\times\overline{B(x_0,2r)})$. Hence,

$$\zeta_n=\|v_n\|_{C_b^{1+\alpha/2,2+\alpha}([0,T_0]\times\mathbb{R}^d)}\le C\|u\|_{C_b^{1+\alpha/2,2+\alpha}([\tau_0,T_0]\times\mathbb{R}^d)}\|\varphi_n\|_{C^2([0,T_0])}\|\vartheta_n\|_{C_b^3(\mathbb{R}^d)}\le 2^{5n}C.$$

The choice of η implies that $\eta^{m+1}\zeta_{m+1}$ vanishes as $m\to\infty$. Letting n tend to ∞ in (B.9) we conclude that $\zeta_0\le C(\|u\|_\infty+\|g\|_{C^{\alpha/2,\alpha}([\tau_0,T_0]\times\overline{B(x_0,2r)})})$ or, equivalently,

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([\tau,T_0]\times\overline{B(x_0,r)})}\le c(\|u\|_{C([\tau_0,T_0]\times\overline{B(x_0,2r)})}+\|g\|_{C^{\alpha/2,\alpha}([\tau_0,T_0]\times\overline{B(x_0,2r)})}),$$

i.e., estimate (B.2) with the above choices of I_j and K_j $(j=1,2)$ follows.

Step 2. Let δ denote the minimum between $\text{dist}(\partial I_1, \partial I_2)$ and $\text{dist}(\partial K_1, \partial K_2)$. As it is immediately,

$$I_1 \subset \bigcup_{t_0 \in I_1} (t_0 - \delta/2, t_0 + \delta/2) \subset I_2, \qquad K_1 \subset \bigcup_{x_0 \in K_1} B(x_0, \delta/2) \subset K_2.$$

Since I_1 and K_1 are compact sets, we can find out $m \in \mathbb{N}$ and points $t_1, \ldots, t_m \in I_1$, $x_1, \ldots, x_m \in K_1$ such that

$$I_1 \subset \bigcup_{j=1}^{m} (t_j - \delta/2, t_j + \delta/2), \qquad K_1 \subset \bigcup_{j=1}^{m} B(x_j, \delta/2).$$

We claim that

$$\|u\|_{C^{1+\alpha/2,2+\alpha}(I_1 \times K_1)} \le \sum_{j,k=1}^{m} \|u\|_{C^{1+\alpha/2,2+\alpha}([t_j-\delta/2,t_j+\delta/2]\times\overline{B(x_k,\delta/2)})}. \tag{B.10}$$

It is clear that $\|u\|_{C^{1,2}(I_1\times K_1)} \le \max_{j,k=1,\ldots,m} \|u\|_{C^{1,2}([t_j-\delta/2,t_j+\delta/2]\times\overline{B(x_k,\delta/2)})}$. As far as the Hölder seminorms of $D_t u$ and $D_{ij}u$ $(i, j = 1, \ldots, d)$ are concerned, we fix $x \in K_1$, $s, t \in I_1$, with $s < t$, and estimate $|D_t u(t, x) - D_t u(s, x)|$. We denote by i and j, respectively, the largest and the smallest indices such that $s \in (t_i - \delta/2, t_i + \delta/2)$ and $t \in (t_j - \delta/2, t_j + \delta/2)$. By definition $i \le j$. If $i = j$, then we are done, since

$$|D_t u(t,x) - D_t u(s,x)| \le [D_t u]_{C^{\alpha/2,0}([t_i-\delta/2,t_i+\delta/2]\times\overline{B(x_k,\delta/2)})} |t-s|^{\frac{\alpha}{2}},$$

where k is chosen such that $x \in B(x_k, \delta/2)$.

So, we suppose that $i < j$ and consider all the indices $i_1, \ldots, i_h$ such that $t_i < t_{i_1} < \ldots < t_{i_h} < t_j$. As it is immediately seen, $t_{i_1} \in (t_i - \delta/2, t_i + \delta/2)$, $t_{i_h} \in (t_j - \delta/2, t_j + \delta/2)$ and $t_{i_p} \in (t_{i_{p-1}} - \delta/2, t_{i_{p-1}} + \delta/2)$ if $p \in \{2, \ldots, h-1\}$. Hence, we can split

$$\begin{aligned}
|D_t u(t,x) - D_t u(s,x)| \le & |D_t u(s,x) - D_t u(t_i + \delta/2, x)| \\
& + |D_t u(t_{i_1} + \delta/2, x) - D_t u(t_i + \delta/2, x)| \\
& + \sum_{p=1}^{h-1} |D_t u(t_{i_{p+1}}, x) - D_t u(t_{i_p}, x)| + |D_t u(t_{i_h}, x) - D_t u(t, x)| \\
\le & [D_t u]_{C^{\alpha/2,0}([t_i-\delta/2,t_i+\delta/2]\times\overline{B(x_k,\delta/2)})} |t_i + \delta - s|^{\frac{\alpha}{2}} \\
& + \sum_{p=1}^{h-1} [D_t u]_{C^{\alpha/2,0}([t_{i_p}-\delta/2,t_{i_p}+\delta/2]\times\overline{B(x_k,\delta/2)})} |t_{i_{p+1}} - t_{i_p}|^{\frac{\alpha}{2}} \\
& + [D_t u]_{C^{\alpha/2,0}([t_j-\delta/2,t_j+\delta/2]\times\overline{B(x_k,\delta/2)})} |t - t_{i_h}|^{\frac{\alpha}{2}} \\
\le & \sum_{h=1}^{m} [D_t u]_{C^{\alpha/2,0}([t_h-\delta/2,t_h+\delta/2]\times\overline{B(x_k,\delta/2)})} |t-s|^{\frac{\alpha}{2}}
\end{aligned}$$

and this shows that

$$\|D_t u\|_{C^{\alpha/2,0}(I_1\times K_1)} \le \sum_{j,k=1}^{m} \|D_t u\|_{C^{\alpha/2,0}([t_j-\delta/2,t_j+\delta/2]\times \overline{B(x_k,\delta/2)})}.$$

The same arguments show that

$$\|D_{ij} u\|_{C^{\alpha/2,0}(I_1\times K_1)} \le \sum_{h,k=1}^{m} \|D_{ij} u\|_{C^{\alpha/2,0}([t_h-\delta/2,t_h+\delta/2]\times \overline{B(x_k,\delta/2)})}, \qquad i,j=1,\dots,d,$$

and that

$$\begin{aligned}&\|D_t u\|_{C^{0,\alpha}(I_1\times K_1)} + \|D_{ij} u\|_{C^{0,\alpha}(I_1\times K_1)}\\ \le &\sum_{h,k=1}^{m} (\|D_t u\|_{C^{0,\alpha}([t_h-\delta/2,t_h+\delta/2]\times \overline{B(x_k,\delta/2)})} + \|D_{ij} u\|_{C^{0,\alpha}([t_h-\delta/2,t_h+\delta/2]\times \overline{B(x_k,\delta/2)})})\end{aligned}$$

for $i,j=1,\dots,d$. The proof of (B.10) follows.

Finally, by Step 1, with $\tau_0 = t_j - 3\delta/4$, $\tau = t_j - \delta_2$, $T_0 = t_j + \delta/2$ and $r=\delta$, we get

$$\begin{aligned}&\|u\|_{C^{1+\alpha/2,2+\alpha}([t_j-\delta/2,t_j+\delta/2]\times \overline{B(x_k,\delta/2)})}\\ \le &C_{jk}(\|u\|_{C([t_j-3\delta/4,t_j+\delta/2]\times \overline{B(x_k,\delta)})} + \|g\|_{C^{\alpha/2,\alpha}([t_j-3\delta/4,t_j+\delta/2]\times \overline{B(x_k,\delta)})})\\ \le &C_{jk}(\|u\|_{C_b(I_2\times K_2)} + \|g\|_{C^{\alpha/2,\alpha}(I_2\times K_2)})\end{aligned} \tag{B.11}$$

for every $j,k=1,\dots,m$. Replacing (B.11) into (B.10) we easily obtain (B.2). ☐

The same arguments as in the proof of Theorem B.1.3 can be used to prove the following result. For the reader's convenience, we point out the differences.

Theorem B.1.5. *Let $u\in C^{1,2}([0,T)\times\Omega)$ be a classical solution to the differential equation $D_t u = \mathcal{A}u + g$, corresponding to $g\in C^{\alpha/2,\alpha}_{\rm loc}([0,T)\times\Omega)$[4] and such that $u(0,\cdot)\in C^{2+\alpha}_{\rm loc}(\Omega)$. Then, $u\in C^{1+\alpha/2,2+\alpha}_{\rm loc}([0,T)\times\Omega)$ and for every pair of compact sets $K_1\Subset K_2\Subset\Omega$, there exists a positive constant $C=C(\mu,K_1,K_2)$ such that*

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times K_1)} \le C(\|u(0,\cdot)\|_{C^{2+\alpha}(K_2)} + \|u\|_{C([0,T]\times K_2)} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times K_2)}). \tag{B.12}$$

Proof. The only slight differences with respect to the proof of Theorem B.1.3 are in the proof of (B.12) with $K_1 = \overline{B(x_0,r)}$ and $K_2 = \overline{B(x_0,2r)}$. Hence, here we deal also with this case.

Throughout the proof, we denote by C a positive constant, independent of u, g and n, which may vary from line to line, and consider the same sequence (ϑ_n) and the same operator $\widetilde{\mathcal{A}}$ as in the proof of Theorem B.1.3.

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[4] $C^{\alpha/2,\alpha}_{\rm loc}([0,T)\times\Omega)$ is the space of all functions $f\in C^{\alpha/2,\alpha}([0,b)\times K)$ for each $0<b<T$ and each compact set $K\subset\mathbb{R}^d$. The definition of the space $C^{1+\alpha/2,2+\alpha}_{\rm loc}([0,T)\times\Omega)$ is completely similar.

For every $n \in \mathbb{N}$ the function $v_n = u\vartheta_n$ belongs to $C_b^{1,2}([0,b] \times \mathbb{R}^d)$, vanishes at $t = 0$ and satisfies the equation $D_t v_n = \widetilde{\mathcal{A}} v_n + g_n$ in $(0,\infty) \times \mathbb{R}^d$, where

$$g_n = \vartheta_n g - u(\widetilde{\mathcal{A}}\vartheta_n - c\vartheta_n) - 2\langle \widetilde{Q}\nabla\vartheta_n, \nabla_x u\rangle \tag{B.13}$$

and $\widetilde{Q} = (\widetilde{q}_{ij})$. Since the coefficients of the operator $\widetilde{\mathcal{A}}$ satisfy Hypotheses 7.0.1, the Cauchy problem

$$\begin{cases} D_t w(t,x) = \widetilde{\mathcal{A}} w(t,x) + g_n(t,x), & t \in [0,b], \quad x \in \mathbb{R}^d, \\ w(0,x) = \vartheta_n(x) u(0,x), & \qquad\qquad\quad x \in \mathbb{R}^d, \end{cases}$$

admits a unique solution which belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T] \times \mathbb{R}^d)$. Hence, the function v_n belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T] \times \mathbb{R}^d)$ and

$$\|v_n\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \le C(\|\vartheta_n u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|g_n\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)}), \qquad n \in \mathbb{N}.$$

In particular, taking $n = 0$, we deduce that u belongs to $C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{B(x_0,r)})$. The arbitrariness of r imply that $u \in C^{1+\alpha/2,2+\alpha}([0,T] \times K)$ for each compact set $K \subset \mathbb{R}^d$.

Estimating the $C_b^{\alpha/2,\alpha}([0,T] \times \mathbb{R}^d)$-norm of the function g_n and observing that

$$\|\vartheta_n u(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} \le 2^{3n} C \|u(0,\cdot)\|_{C^{2+\alpha}(B(x_0,2r))}, \qquad n \in \mathbb{N},$$

we get

$$\begin{aligned}
&\|v_n\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \\
\le& C\|\vartheta_n\|_{C_b^3(\mathbb{R}^d)}\big(\|u(0,\cdot)\|_{C^{2+\alpha}(\overline{B(x_0,2r)})} + \|u\|_{C^{\alpha/2,1+\alpha}([0,T]\times\overline{B(x_0,r_{n+1})})} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,2r)})}\big) \\
\le& 2^{3n} C\Big(\|v_{n+1}\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} + \sum_{j=1}^d \|D_j v_{n+1}\|_{C_b^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)} \\
&\qquad + \|u(0,\cdot)\|_{C^{2+\alpha}(\overline{B(x_0,2r)})} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,2r)})}\Big).
\end{aligned}$$

This estimate and (B.3) yield

$$\begin{aligned}
\|v_n\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \le& 2^{3n} C(\varepsilon\|v_{n+1}\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^d)} \\
&+ \varepsilon^{-(1+\alpha)}\|v_{n+1}\|_\infty + \|u(0,\cdot)\|_{C^{2+\alpha}(\overline{B(x_0,2r)})} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,2r)})})
\end{aligned}$$

for each $\varepsilon \in (0,1)$. Fixing $\eta \in (0, 2^{-3(2+\alpha)})$, choosing $\varepsilon = \varepsilon_n = 2^{-3n}C^{-1}\eta$ and arguing as in the proof of Theorem B.1.3 we can easily show that

$$\begin{aligned}
&\|u\|_{C^{1+\alpha/2,2+\alpha}([0,b]\times\overline{B(x_0,r)})} \\
\le\;& c(\|u(0,\cdot)\|_{C^{2+\alpha}(\overline{B(x_0,2r)})} + \|u\|_{C([0,b]\times\overline{B(x_0,2r)})} + \|g\|_{C^{\alpha/2,\alpha}([0,b]\times\overline{B(x_0,2r)})}),
\end{aligned}$$

and we are done. □

As a consequence of Theorems B.1.3 and Remark B.1.5 we can prove the following result.

Proposition B.1.6. *Suppose that the coefficients of the operator $\mathcal{A}$ belong to $C^{1+\alpha}_{\rm loc}(\Omega)$ and* (B.1) *is satisfied. Then, the following properties are satisfied.*

(i) *Let $g\in C^{\alpha/2,1+\alpha}_{\rm loc}([0,T]\times\Omega)$ and $u\in C^{1+\alpha/2,2+\alpha}_{\rm loc}([0,T]\times\Omega)$ solve the equation $D_tu=\mathcal{A}u+g$ on $(0,T]\times\mathbb{R}^d$, with $u(0,\cdot)\in C^{3+\alpha}_{\rm loc}(\Omega)$. Then, $D_ku\in C^{1+\alpha/2,2+\alpha}_{\rm loc}([0,T]\times\Omega)$ for every $k=1,\dots,d$, solves the differential equation $D_tD_ku=\mathcal{A}D_ku+\sum_{i,j=1}^d D_kq_{ij}D_{ij}D_ku+\langle D_kb,\nabla_xu\rangle+D_kcu$ and, for every pair of compact sets $K_1\Subset K_2\Subset\Omega$, there exists a positive constant C such that*

(B.14)

$$\|D_ku\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times K_1)}\le C(\|u(0,\cdot)\|_{C^{3+\alpha}(K_2)}+\|u\|_{C([0,T]\times K_2)}+\|g\|_{C^{\alpha/2,1+\alpha}([0,T]\times K_2)}).$$

(ii) *Let $g\in C^{\alpha/2,1+\alpha}_{\rm loc}((0,T)\times\Omega)$ and $u\in C^{1,2}((0,T)\times\Omega)$ solve the equation $D_tu=\mathcal{A}u+g$ on $(0,T]\times\mathbb{R}^d$. Then, $D_ku\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,T)\times\Omega)$, solves the differential equation $D_tD_ku=\mathcal{A}D_ku+\sum_{i,j=1}^d D_kq_{ij}D_{ij}D_ku+\langle D_kb,\nabla_xu\rangle+D_kcu$ and, for every $k=1,\dots,d$ and, for every pair of intervals $I_1\Subset I_2\Subset\Omega$ and compact sets $K_1\Subset K_2\Subset\Omega$, there exists a positive constant C such that*

$$\|D_ku\|_{C^{1+\alpha/2,2+\alpha}(I_1\times K_1)}\le C(\|u\|_{C(I_2\times K_2)}+\|g\|_{C^{\alpha/2,1+\alpha}(I_2\times K_2)}. \tag{B.15}$$

Proof. (i) We fix $K_1\Subset\widehat{K}\Subset\widehat{K}\Subset K_2\Subset\Omega$. For each $h\in\mathbb{R}\setminus\{0\}$, such that

$$h<\min\{{\rm dist}(K_1,\partial\widehat{K}),{\rm dist}(\widehat{K},\partial K_2),{\rm dist}(K_2,\partial\Omega)\},$$

and $k\in\{1,\dots,d\}$, we introduce the operator $\tau_{h,k}$, defined by $\tau_{h,k}\psi=h^{-1}(\psi(\cdot+he_k)-\psi)$ for every function ψ. Finally, we denote by $v_{h,k}:[0,T]\times K_1\to\mathbb{R}$ the function defined by $v_{h,k}(t,\cdot)=\tau_{h,k}(u(t,\cdot))$ for any $t\in[0,T]$. Clearly, $v_{j,h}$ satisfies the differential equation

$$D_tv_{h,k}=\mathcal{A}_{h,k}v_{h,k}+\sum_{i,j=1}^d(\tau_{h,k}q_{ij})D_{ij}u+\sum_{j=1}^d\tau_{h,k}b_j,\nabla_xu\rangle+\tau_{h,k}g+(\tau_{h,k}c)u$$

where $\mathcal{A}_{h,k}v_{h,k}={\rm Tr}(Q(\cdot+he_k)D^2_xv_{h,k})+\langle b(\cdot+he_k),\nabla_xv_{h,k}\rangle+c(\cdot+he_k)v_{h,k}$. Let us observe that

$$\tau_{h,k}q_{ij}(x)=\frac{1}{h}\int_0^h D_kq_{ij}(x+\sigma e_k)d\sigma,\qquad x\in\widehat{K}$$

so that $\|\tau_{h,k}q_{ij}\|_{C(\widehat{K})}\le\|D_kq_{ij}\|_{C(K_2)}$ and

$$|\tau_{h,k}q_{ij}(x)-\tau_{h,k}q_{ij}(y)|\le\frac{1}{h}\int_0^{he_j}|D_kq_{ij}(x+\sigma e_j)-D_kq_{ij}(y+\sigma e_j)|d\sigma$$

$$\leq [D_k q_{ij}]_{C^\alpha(K_3)} |x-y|^\alpha$$

for each $x, y \in \widehat{K}$. It thus follows that $\tau_{hk} q_{ij} \in C^\alpha(\widehat{K})$ and $\|\tau_{h,k} q_{ij}\|_{C^\alpha(\widehat{K})} \leq \|D_k q_{ij}\|_{C(K_2)}$. Similarly, $\|\tau_{h,k} b_j\|_{C^\alpha(\widehat{K})} \leq \|D_k b_j\|_{C(K_2)}$ and $\|\tau_{h,k} c\|_{C^\alpha(\widehat{K})} \leq \|D_k c\|_{C(K_2)}$. Applying estimate B.12 with K_2 being replaced by $\widehat{K}$, we thus conclude that

$$\begin{aligned}
\|v_{h,k}\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times K_1)} \leq & C\bigg(\|\tau_{h,k} u(0,\cdot)\|_{C^{2+\alpha}(\widehat{K})} + \|\tau_{h,k} u\|_{C_b([0,T]\times\widehat{K})} \\
& \qquad + C_1 \|u\|_{C_b^{0,2+\alpha}([0,T]\times\widehat{K})} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times K_2)} \bigg) \\
\leq & C\bigg(\|u(0,\cdot)\|_{C_b^{3+\alpha}(K_2)} + C\|u\|_{C_b^{0,2+\alpha}([0,T]\times\widetilde{K})} \\
& \qquad + \|u\|_{C_b^{0,1}([0,T]\times\widetilde{K})} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times K_2)} \bigg),
\end{aligned}$$

where

$$C_1 = \sum_{i,j=1}^d \|D_k q_{ij}\|_{C_b^\alpha([0,T]\times K_2)} + \sum_{i=1}^d \|D_j b_i\|_{C_b^\alpha([0,T]\times K_2)} + \|c\|_{C_b^\alpha([0,T]\times K_2)}$$

Using once more estimate B.12, with K_1 being replaced by $\widetilde{K}$, we can infer that

$$\|v_h\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times K_1)} \leq C(\|u(0,\cdot)\|_{C_b^{3+\alpha}(K_2)} + \|g\|_{C_b^{\alpha/2,\alpha}([0,T]\times K_2)})$$

Letting h tend to 0 we conclude that $D_k u \in C_b^{1+\alpha/2,2+\alpha}([0,T]\times K_1)$. and $D_t D_k u = \mathcal{A} D_j u + \sum_{i,j=1}^d D_k q_{ij} D_{ij} D_k u + \langle D_k b, \nabla_x u\rangle + D_k c u$ on $[0,T]\times K_1$ for each $j = 1, \ldots, d$. The arbitrariness of K_1 yields the assertion.

(ii) The proof of (B.15) can be obtained adapting the arguments in (i) and using estimate (B.2) instead of (B.12). □

To conclude this appendix we prove the following weighted estimates.

Theorem B.1.7. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $u \in C_b([0,T]\times\overline{\Omega}) \cap C^{1,2}((0,T]\times\Omega)$ solves the equation $D_t u = \mathcal{A}u + g$ in $(0,T)\times\Omega$ for some $g \in C_{\rm loc}^{\alpha/2,\alpha}((0,T]\times\Omega)$. Further, assume that the function $t \mapsto t\|u(t,\cdot)\|_{C_b^2(\Omega)}$ is bounded in $(0,T)$. Then, for every pair of compact sets $K_1 \Subset K_2 \Subset \Omega$, there exists a positive constant $C = C(\mu, T, K_1, K_2)$ such that, for every $t \in (0,T)$,*

$$t\|D_x^2 u(t,\cdot)\|_{C(K_1)} + \sqrt{t}\,\|\nabla_x u(t,\cdot)\|_{C(K_1)} \leq C(\|u\|_{C_b([0,T]\times\overline{K}_2)} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times K_2)}). \tag{B.16}$$

Proof. Throughout the proof, we denote by C a positive constant independent of n, which can vary from line to line. The main part of the proof consists of proving estimate (B.16) when $K_1 = \overline{B(x_0,r)}$ and $K_2 = \overline{B(x_0,2r)}$. We also consider the same sequence of cut-off functions (ϑ_n) as in the proof of Theorem B.1.3.

Let us set $u_n := \vartheta_n u$ and observe that each function u_n vanishes on $[s,T]\times\partial B(0,r_{n+1})$ and $D_t u_n = \widetilde{\mathcal{A}}u_n + g_n$ in $(0,T)\times\mathbb{R}^d$, where the operator $\widetilde{\mathcal{A}}$ is the same as in the proof of Theorem B.1.3 and the function g_n is given by (B.13).

In view of the variation-of-constants-formula it thus follows that

$$u_n(t,x) = (T(t)(\vartheta_n u(0,\cdot)))(x) + \int_0^t (T(t-s)g_n(s,\cdot))(x)ds, \qquad t\in[0,T],\ \ x\in\mathbb{R}^d,$$

where $\{T(t)\}$ is the semigroup associated in $C_b(\mathbb{R}^d)$ with the operator $\widetilde{\mathcal{A}}$ (see Theorem 9.0.1). That theorem also shows that $t\|T(t)\psi\|_{C^2_b(\mathbb{R}^d)} \le C\|\psi\|_{C_b(\mathbb{R}^d)}$ and $\sqrt{t}\|T(t)\zeta\|_{C^2_b(\mathbb{R}^d)} \le C\|\zeta\|_{C^1_b(\mathbb{R}^d)}$ for every $t\in(0,T)$, $\psi\in C_b(\mathbb{R}^d)$ and $\zeta\in C^1_b(\mathbb{R}^d)$.

Using (4.16) from the previous estimates we deduce that $t^{1-\frac{\alpha}{2}}\|T(t)\psi\|_{C^2_b(\mathbb{R}^d)} \le C\|\psi\|_{C^\alpha_b(\mathbb{R}^d)}$ for every $\psi\in C^\alpha_b(\mathbb{R}^d)$ and $t\in(0,T)$. Since $g_n(s,\cdot)\in C^\alpha_b(\mathbb{R}^d)$ for every $s\in(0,T)$, we can estimate

$$t\|u_n(t,\cdot)\|_{C^2_b(\mathbb{R}^d)} \le C\|u\|_\infty + C\int_0^t (t-s)^{-1+\frac{\alpha}{2}}\|g_n(s,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)}ds, \qquad t\in(0,T). \tag{B.17}$$

Note that $\|g_n(s,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)} \le C\|\vartheta_n\|_{C^{2+\alpha}_b(\mathbb{R}^d)}(\|u(s,\cdot)\|_{C^{1+\alpha}_b(\mathbb{R}^d)} + \|g(s,\cdot)\|_{C^\alpha_b(\mathbb{R}^d)})$ for every $s\in(0,T)$. Moreover, for every $\sigma\in(s,T)$, $\varepsilon>0$ and $n\in\mathbb{N}$ it holds that

$$\|u(s,\cdot)\|_{C^1(B(x_0,r_{n+1}))} \le Cs^{-\frac{1}{2}}\|u\|_\infty^{\frac{1}{2}}\zeta_{n+1}^{\frac{1}{2}} \le s^{-\frac{1}{2}}(C\varepsilon^{-1}\|u\|_\infty + \varepsilon\zeta_{n+1}),$$

$$\|\nabla_x u(s,\cdot)\|_{C^\alpha(B(x_0,r_{n+1}))} \le Cs^{-\frac{\alpha+1}{2}}\|u\|_\infty^{\frac{1-\alpha}{2}}\zeta_{n+1}^{\frac{1+\alpha}{2}} \le s^{-\frac{\alpha+1}{2}}(C\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|u\|_\infty + \varepsilon\zeta_{n+1}),$$

where $\zeta_n := \sup_{s\in(0,T)} s\|u(s,\cdot)\|_{C^2(\overline{B(x_0,r_n)})}$. Since $\|\vartheta_n\|_{C^{2+\alpha}_b(\mathbb{R}^d)} \le C8^n$ for any $n\in\mathbb{N}$, from these last three estimates we conclude that

$$\|g_n(s,\cdot)\|_{C^\alpha(B(x_0,r_{n+1}))} \le 8^n s^{-\frac{\alpha+1}{2}}(C\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|u\|_\infty + \varepsilon\zeta_{n+1}) + C8^n\|g(s,\cdot)\|_{C^\alpha(B(x_0,r_{n+1}))}, \tag{B.18}$$

for any $s\in(0,T)$ and $\varepsilon>0$. Replacing (B.18) into (B.17) yields

$$\zeta_n \le C\|u\|_\infty + 8^n C\left(\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|u\|_\infty + \varepsilon\zeta_{n+1} + \|g\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{B(x_0,2r)})}\right), \qquad n\in\mathbb{N},\ \ \varepsilon\in(0,1). \tag{B.19}$$

Let us fix $\eta\in(0,64^{-1/(1-\alpha)})$ and $\varepsilon = 8^{-n}C^{-1}\eta$. Multiplying both sides of (B.19) by η^n and summing from 1 to $m\in\mathbb{N}$, and arguing as in the last part of the proof of Theorem B.1.3, we get the assertion. □

B.2 Optimal Schauder estimates

In this section we extend Theorem 13.2.1 to more general elliptic operators with unbounded coefficients. For the sake of simplicity, we assume that

$$\mathcal{A} = \Delta + \sum_{j=1}^d b_j(x)D_j,$$

under the following conditions on its coefficients

Hypotheses B.2.1. (i) $b_j \in C^{3+\alpha}_{\rm loc}(\mathbb{R}^d)$ for each $j = 1, \dots, N$, and there exist two functions $b_0, r : \mathbb{R}^d \to \mathbb{R}$ and constants L_1, L_2 such that $\sup_{\mathbb{R}^d}(d + L_1 r) < \infty$, $\langle b(x), x\rangle \leq L_2(1 + |x|^2)$, for every $x \in \mathbb{R}^d$,

$$\sum_{i,j=1}^{d} D_i b_j(x)\xi_i\xi_j \leq b_0(x)|\xi|^2, \qquad x, \xi \in \mathbb{R}^d; \tag{B.20}$$

and $|D^\beta b_j(x)| \leq r(x)$ for every $x \in \mathbb{R}^d$, $j = 1, \dots, d$ and $|\beta| = 2, 3$;

(ii) there exist a positive function $\varphi \in C^2(\mathbb{R}^d)$, diverging to ∞ as $|x|$ tends to ∞, and a positive constant λ_0 such that $\mathcal{A}\varphi \leq \lambda_0\varphi$ on $\mathbb{R}^d$;

As in the proof of Theorems 6.1.4 and 13.2.1, we first prove some uniform estimates for the derivatives of the function $T(t)f$, when $f \in C_b(\mathbb{R}^d)$. More precisely, we prove that, for every $T > 0$ and $\alpha, \beta \in \mathbb{R}$, with $0 \leq \alpha \leq \beta \leq 3$, there exists a positive constant C_T such that

$$\|T(t)f\|_{C^\beta_b(\mathbb{R}^d)} \leq C_T t^{-\frac{\beta-\alpha}{2}}\|f\|_{C^\alpha_b(\mathbb{R}^d)}, \qquad t \in (0, T], \ f \in C^\alpha_b(\mathbb{R}^d). \tag{B.21}$$

To prove (B.21), with $\alpha, \beta \in \mathbb{N}$, we use the well celebrated Bernstein method (see [1]) and approximate $T(t)f$ by solutions of Cauchy problems in bounded domains. We assume weak dissipativity-type (see (B.20)) and growth conditions on the coefficients b_j ($j = 1, \dots, d$). Note that, without condition (B.20), estimate (B.21) does not hold in general, see Example B.2.4. An interpolation argument allow us to prove estimate (B.21) in the general case. Then, the argument, in the proof of Theorem 13.2.1 can be repeated with slight differences to prove the following result.

Theorem B.2.2. *For each* $f \in C^{2+\alpha}_b(\mathbb{R}^d)$ *and* $g \in C^{0,\alpha}_b([0, T] \times \mathbb{R}^d)$*, there exists a unique classical solution* $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ *to the Cauchy problem*

$$\begin{cases} D_t u(t, x) = \mathcal{A}u(t, x) + g(t, x), & t \in [0, T], \quad x \in \mathbb{R}^d, \\ u(0, x) = f(x), & \qquad\qquad\quad\; x \in \mathbb{R}^d. \end{cases}$$

Function u *belongs to* $C^{1,2}([0, T] \times \mathbb{R}^d) \cap C^{0,2+\alpha}_b([0, T] \times \mathbb{R}^d)$ *and there exists a positive constant* $C > 0$*, independent of* u*, such that*

$$\|u\|_{C^{0,\alpha}_b([0,T]\times\mathbb{R}^d)} \leq C(\|f\|_{C^{2+\alpha}_b(\mathbb{R}^d)} + \|g\|_{C^{0,\alpha}_b([0,T]\times\mathbb{R}^d)}).$$

B.2.1 Uniform estimates

Theorem B.2.3. *For each* $T > 0$ *and* $0 \leq \alpha \leq \beta \leq 3$*, there exists a positive constant* C_T *such that estimate* (B.21) *holds true.*

Proof. The arguments in the proof of Theorem 4.1.12 show that it suffices to prove (B.21) when $\alpha = h$, $\beta = k$ and $h, k \in \mathbb{N}$. Being rather long, we split the proof into two steps.

Step 1. Here, we deal with the case when $h = 0$, $k = 3$ and first assume that $f \in C^{5+\alpha}_c(\mathbb{R}^d)$. For each $n \in \mathbb{N}$, let $\vartheta_n : \mathbb{R}^d \to \mathbb{R}$ be the function defined by $\vartheta_n(x) = \vartheta(|x|/n)$ for every $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $\vartheta \in C^\infty_c(\mathbb{R})$ being a nonincreasing function such that $\chi_{(-1/2,1/2)} \le \vartheta \le \chi_{(-1,1)}$. For each $n \in \mathbb{N}$ such that $\mathrm{supp}(f) \subset B(0,n)$, and $T > 0$, we introduce the function $v_{0,3,n} : [0,T] \times \overline{B(0,n)} \to \mathbb{R}$, defined by[5]

$$v_{0,3,n}(t,x) = |u_n(t,x)|^2 + at\vartheta_n^2|\nabla_x u_n(t,x)|^2 + a^2t^2\vartheta_n^4|D^2_x u_n(t,x)|^2 + a^3t^3\vartheta_n^6|D^3_x u_n(t,x)|^2,$$

for each $t \in [0,T]$ and $x \in \overline{B(0,n)}$, where u_n denotes the (unique) classical solution of the Cauchy-Dirichlet problem

$$\begin{cases} D_t u_n(t,x) = \mathcal{A}u_n(t,x), & t > 0, \quad x \in B(0,n), \\ u_n(t,x) = 0, & t > 0, \quad x \in \partial B(0,n), \\ u_n(0,x) = f(x), & \qquad\quad x \in B(0,n). \end{cases}$$

By repeatedly applying Proposition B.1.6, we can infer that $D_j u_n$, $D_{ij}u_n$ and $D_{ijh}u_n$ $(i,j,h = 1,\dots,d)$ belong to $C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{B(0,R)})$ for every $R < n$. Function $v_{0,3}$[6] belongs to $C^{1,2}([0,T] \times \overline{B(0,n)})$ and, with some computations, one can see that $D_t v_{0,3} \le \mathcal{A}v_{0,3} + g$ on $[0,T] \times \overline{B(0,n)}$, where $g = \sum_{j=1}^4 g_j$ with

$$\begin{aligned} g_1 &= (a\vartheta^2 - 2)|\nabla_x u|^2 + 2at\vartheta^2(a\vartheta^2 - 1)|D^2_x u|^2 + a^2t^2\vartheta^4(3a\vartheta^2 - 2)|D^3_x u|^2 - 2a^3t^3\vartheta^6|D^4_x u|^2, \\ g_2 &= -2at\vartheta\mathcal{A}(\vartheta)|\nabla_x u|^2 - 4a^2t^2\vartheta^3\mathcal{A}(\vartheta)|D^2_x u|^2 - 6a^3t^3\vartheta^5\mathcal{A}(\vartheta)|D^3_x u|^2 \\ &\quad - 8at\vartheta\mathcal{Q}_0(u) - 16a^2t^2\vartheta^3\sum_{i=1}^d \mathcal{Q}_0(D_i u) - 24a^3t^3\vartheta^5\sum_{i,j=1}^d \mathcal{Q}_0(D_{ij}u), \\ g_3 &= 2at\vartheta^2\mathcal{B}_1(u) + 4a^2t^2\vartheta^4\sum_{i=1}^d \mathcal{B}_1(D_i u) + 6a^3t^3\vartheta^6\sum_{i,j}^d \mathcal{B}_1(D_{ij}u), \\ g_4 &= 2a^2t^2\vartheta^4\mathcal{B}_2(u) + 6a^3t^3\vartheta^6\sum_{i=1}^d \mathcal{B}_2(D_i u) + 2a^3t^3\vartheta^6\sum_{i,j,h,k=1}^d D_{ihk}b_j D_j u D_{ihk}u. \end{aligned}$$

and

$$\mathcal{Q}_0(v) = \langle D^2 v \nabla v, \nabla\vartheta\rangle, \qquad \mathcal{B}_1(v) = \sum_{j,h=1}^d D_h b_j D_j v D_h v, \quad \mathcal{B}_2(v) = \sum_{j,h,k=1}^d D_{hk} b_j D_j v D_{hk} v$$

on smooth enough functions $v : \mathbb{R}^d \to \mathbb{R}$. To estimate the function g_2 we first observe that $n\|\nabla_x \vartheta_n\|_\infty + n^2\|\Delta\vartheta\|_\infty \le C_0$ for each $n \in \mathbb{N}$ and some positive constant C_0. Using

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[5] For each smooth enough function, we set $|D^2 v|^2 = \sum_{i,j=1}^d |D_{ij}v|^2$ and $|D^3 v|^2 = \sum_{i,j,h=1}^d |D_{ijh}v|^2$.

[6] For notational convenience, throughout the rest of the proof, we drop out the dependence on n and the dependence of the functions that we consider on the variables, when there is no damage of confusion.

Hypothesis B.2.1(i) and recalling that ϑ is nonincreasing in $\mathbb{R}$, we obtain that the function $x \mapsto \langle b(x), x\rangle$ is bounded from below on $\mathbb{R}^d$. Therefore, $-\mathcal{A}\vartheta_n \le C_1$ on $\mathbb{R}^d$ for some positive constant C_1. Similarly, we can show that $\mathcal{Q}_0(v) \le C_0 n^{-1}|\nabla_x v||D^2_x v|$ for every $v \in C^2(\mathbb{R}^d)$. From all the above estimates and recalling that $\beta\gamma < 2^{-1}(\beta^2+\gamma^2)$ for each $\beta,\gamma \in \mathbb{R}$, we deduce that

$$\begin{aligned}
g_2 \le& 2aC_1t\vartheta|\nabla_x u|^2 + 4a^2C_1t^2\vartheta^3|D^2_x u|^2 + 6a^3C_1t^3\vartheta^5|D^3_x u|^2 + 8aC_0n^{-1}t|\nabla_x u|(\vartheta|D^2_x u|)\\
&+16a^2C_0n^{-1}t^2\sum_{i=1}^d(\vartheta|\nabla_x D_i u|)(\vartheta^2|D^2_x D_i u|)+24a^3C_0n^{-1}t^3\sum_{i,j=1}^d(\vartheta^2|\nabla_x D_{ij}u|)(\vartheta^3|D^2_x D_{ij}u|)\\
\le& 2aC_1t\vartheta|\nabla_x u|^2 + 4a^2C_1t^2\vartheta^3|D^2_x u|^2 + 6a^3C_1t^3\vartheta^5|D^3_x u|^2 + 4aC_0n^{-1}t\left(|\nabla_x u|^2+\vartheta^2|D^2_x u|^2\right)\\
&+ 8a^2C_0n^{-1}t^2\left(\vartheta^2|D^2_x u|^2+\vartheta^4|D^3_x u|^2\right) + 12a^3C_0n^{-1}t^3\left(\vartheta^4|D^3_x u|^2+\vartheta^6|D^4_x u|^2\right)\\
\le& 2a(C_1+2C_0n^{-1})t|\nabla_x u|^2 + 4a(aC_1t + C_0n^{-1} + 2aC_0n^{-1}t)t\vartheta^2|D^2_x u|^2\\
&+ 2a^2(3aC_1t + 4C_0n^{-1} + 6aC_0n^{-1}t)t^2\vartheta^4|D^3_x u|^2 + 12a^3C_0n^{-1}t^3\vartheta^6|D^4_x u|^2.
\end{aligned}$$

Taking advantage of Hypothesis B.2.1(i), we deduce that

$$g_3 \le b_0(2at\vartheta^2|\nabla_x u|^2 + 4a^2t^2\vartheta^4|D^2_x u|^2 + 6a^3t^3\vartheta^6|D^3_x u|^2).$$

To estimate the term g_4, we first observe that

$$\mathcal{B}_2(v) \le \sum_{j=1}^d |D^2 b_j||D_j v||D^2 v| \le d^{\frac{3}{2}} r|\nabla v||D^2 v| \le d^2 r|\nabla v||D^2 v|, \qquad v \in C^2(\mathbb{R}^d).$$

Hence,

$$\begin{aligned}
g_4 \le& 2t^2\vartheta^4d^2r(a^{\frac{3}{4}}|\nabla_x u|)(a^{\frac{5}{4}}|D^2_x u|^2) + 6t^3\vartheta^6d^2r\sum_{i=1}^d(a^{\frac{5}{4}}|\nabla_x D_i u|)(a^{\frac{7}{4}}|D^2_x D_i u|^2)\\
&+ 2t^3\vartheta^6d^2r(a|\nabla_x u|)(a^2|D^3_x u|^2)\\
\le& a^{\frac{3}{2}}t^2\vartheta^4d^2(1+\sqrt{a}t)r|\nabla_x u|^2 + a^{\frac{5}{2}}t^2\vartheta^4(1+3t)d^2r|D^2_x u|^2 + a^{\frac{7}{2}}t^3\vartheta^6d^2(3+\sqrt{a})d^2r|D^3_x u|^2.
\end{aligned}$$

Combining all the above estimates, we easily see that

$$\begin{aligned}
g \le& \big[-2+2a(C_1+2C_0n^1)T + a + a\vartheta^2t(2b_0+\sqrt{a}Td^2(1+\sqrt{a}T)r)\big]|\nabla_x u|^2\\
&+ at\vartheta^2\big[2a-2+4(aC_1T+C_0n^{-1}+2aC_0T) + at\vartheta^2(4b_0+\sqrt{a}(1+3T)d^2r)\big]|D^2_x u|^2\\
&+ a^2t^2\vartheta^4\big[3a-2+6aC_1T+8C_0n^{-1}+12aC_0T + at\vartheta^2(6b_0+\sqrt{a}d^2(3+\sqrt{a})d^2r)\big]|D^3_x u|^2\\
&+ 2a^3t^3\vartheta^6(-1+6C_0n^{-1})|D^4_x u|^2
\end{aligned}$$

on $(0,T]\times B(0,n)$. We now choose $n > 8C_0$. With this choice the above estimate simplifies to

$$g \le \big[-2+2a(C_1+2C_0)T + a + a\vartheta^2t(2b_0+\sqrt{a}Td^2(1+\sqrt{a}T)r)\big]|\nabla_x u|^2$$

$$
\begin{aligned}
&+ at\vartheta^2\big[-1+2a+4aT(C_1+2C_0)+at\vartheta^2(4b_0+\sqrt{a}(1+3T)d^2r)\big]|D_x^2u|^2\\
&+ a^2t^2\vartheta^4\big[-1+3a+2aT(3C_1+6C_0)+at\vartheta^2(6b_0+\sqrt{a}d^2(3+\sqrt{a})d^2r)\big]|D_x^3u|^2.
\end{aligned}
$$

Next, we choose a_0 such that

$$\sqrt{a_0}Td^2(1+\sqrt{a_0}T)\le 2L_2,\qquad \sqrt{a_0}(1+3T)d^2\le 4L_2,\qquad \sqrt{a_0}d^2(3+\sqrt{a_0})d^2\le 6L_2$$

and denote by M the supremum over $\mathbb{R}^d$ of the function $d+L_2r$. Then, for each $a\le a_0$, we can estimate

$$
\begin{aligned}
g \le& \big[-2+2a(C_1+2C_0)T+a+2aM^+T\big]|\nabla_xu|^2\\
&+ at\vartheta^2\big[-1+2a+4aT(C_1+2C_0)+4aTM^+\big]|D_x^2u|^2\\
&+ a^2t^2\vartheta^4\big[-1+3a+2aT(3C_1+6aC_0)+6aTM^+\big]|D_x^3u|^2.
\end{aligned}
$$

Since the coefficients in front of $|\nabla_xu|^2$, $|D_x^2u|^2$ and $|D_x^3u|^2$ have negative limits as $a\to 0^+$, we can fix a small enough such that $g\le 0$ on $[0,T]\times\overline{B(0,n)}$. Recalling that v vanishes on $[0,T]\times\partial B(0,n)$, from the classical maximum principle we now deduce that $|v_{0,3,n}(t,x)|\le\|\vartheta_nf\|_\infty^2\le\|f\|_\infty^2$ for every $(t,x)\in[0,T]\times\overline{B(0,n)}$.

Note that, (B.14) and a diagonal argument show, that up to a subsequence, $D_x^\beta u_n$ converges to $D_x^\beta u$ pointwise on $[0,T]\times B(0,n)$. Hence, taking the limit as n tends to ∞, in the inequality $|v_{0,3,n}(t,x)|\le\|f\|_\infty^2$, formula (B.21) follows in this case for functions $f\in C_c^{5+\alpha}(\mathbb{R}^d)$. To extend it to every $f\in C_b(\mathbb{R}^d)$, it suffices to approximate f by a convolution with a sequence $(f_n)\subset C_c^{5+\alpha}(\mathbb{R}^d)$, converging to f locally uniformly on $\mathbb{R}^d$ and such that $\|f_n\|_\infty\le\|f\|_\infty$ for any $n\in\mathbb{N}$. Clearly, it holds that

$$\|T(t)f_n\|_{C_b^3(\mathbb{R}^d)}\le C_Tt^{-\frac{3}{2}}\|f\|_\infty,\qquad t\in(0,T],\ n\in\mathbb{N}. \tag{B.22}$$

Since $D_tT(\cdot)f_n=\mathcal{A}T(\cdot)f_n$ on $(0,T)\times\mathbb{R}^d$ and $\|T(t)f_n\|_\infty\le\|f\|_\infty$, from (B.15) and a compactness argument, we can easily show that, for every $t\in(0,T]$ there exists $g_t\in C^3(\mathbb{R}^d)$ such that, up to a subsequence $T(t)f_n$ converges to g_t in $C^3(\overline{B(0,R)})$ for every $R>0$. The same arguments used in the proof of Theorem 4.1.12, and Hypothesis B.2.1(ii), which yields the uniqueness of the solution to the Cauchy problem (14.1), show that, still up to a subsequence, $T(t)f_n$ converges to $T(t)f$ locally uniformly on $\mathbb{R}^d$. Hence, $g_t=T(t)f$ and from (B.22), it follows immediately that $\|T(t)f\|_{C_b^3(\mathbb{R}^d)}\le C_Tt^{-3/2}\|f\|_\infty$ for every $t\in(0,T]$, and we are done.

Step 2. The proof of estimate (B.21) for the other values of the pair (h,k) is very similar. It suffices to apply the quoted arguments to the function $v_{h,k,n}:[0,T]\times\overline{B(0,n)}\to\mathbb{R}$, defined by

$$v_{h,k,n}(t,x)=\sum_{j=0}^{l}a^jt^{(j-k)^+}(\vartheta_n(x))^{2j}|D^ju_n(t,x)|^2,\qquad t\in(0,T],\ x\in B(0,n),$$

first assuming that $f\in C_c^{5+\alpha}(\mathbb{R}^d)$ and, then, using the same density argument to cover the general case. The details are left to the reader. □

Finally, we show that the estimates (B.21) may fail to hold without any dissipativity assumption.

Example B.2.4. Consider the one-dimensional operator $\mathcal{A}$, defined on smooth functions $\zeta : \mathbb{R} \to \mathbb{R}$ by $\mathcal{A}\zeta(x) = \zeta''(x) + p'(x)\zeta'(x) = e^{-p(x)}\left(e^{p(x)}\zeta'(x)\right)'$ for every $x \in \mathbb{R}$, where $p \in C^1(\mathbb{R})$ is defined by $p(x) = -x^4 + \log(h(x))$ for every $x \in \mathbb{R}$, where $h(x) = \varepsilon_n$ if $x = n - 2^{-1}\delta_n$ and $n \in \mathbb{N}$, $h(x) \in [\varepsilon_n, 1]$ if $n - \delta_n < x < n$ and $n \in \mathbb{N}$, $h(x) = 1$ otherwise, $\varepsilon_n = n^{-1}e^{(n-1/2)^4-(n+1/2)^4}$, $\delta_n = n^{-2}e^{-n^4}\varepsilon_n$. We also consider a functions $f \in C_b(\mathbb{R}^d)$ such that $f \equiv 1$ on $(0,\infty)$ and $\int_{\mathbb{R}} f(t)e^{p(t)}dt = 0$.

A straightforward computation shows that the solutions of the equation $\mathcal{A}u = f$ are given by

$$u(x) = C_1 + \int_0^x e^{-p(t)}\left(C_2 + \int_0^t f(s)e^{p(s)}ds\right)dt, \qquad x \in \mathbb{R}, \tag{B.23}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary real constants. We take

$$C_2 = -\int_0^\infty f(t)e^{p(t)}dt = \int_{-\infty}^0 f(t)e^{p(t)}dt.$$

and observe that, if $x > 0$ then from (B.23) and Fubini Theorem we get

$$u(x) = C_1 - \int_0^x e^{-p(t)}dt\int_t^\infty f(s)e^{p(s)}ds = C_1 - \int_0^\infty e^{p(s)}f(s)ds\int_0^{s\wedge x} e^{-p(t)}dt. \tag{B.24}$$

Consider the function $Q : \mathbb{R} \to \mathbb{R}$ defined by $Q(x) = e^{p(x)}\int_0^x e^{-p(t)}dt$ for each $x \in \mathbb{R}$. Since

$$\begin{aligned}Q(x) \le & e^{-x^4}\int_0^x \frac{e^{t^4}}{h(t)}dt \le e^{-x^4}\int_0^x e^{t^4}dt + e^{-x^4}\sum_{n=1}^{[x]+1}\int_{n-\delta_n}^n \frac{e^{n^4}}{\varepsilon_n}dt\\ \le & e^{-x^4}\int_0^x e^{t^4}dt + e^{-x^4}\sum_{n=1}^\infty \frac{\delta_n e^{n^4}}{\varepsilon_n} = e^{-x^4}\int_0^x e^{t^4}dt + e^{-x^4}\sum_{n=1}^\infty \frac{1}{n^2},\end{aligned}$$

Q belongs to $L^1((0,\infty))$. Hence, from (B.24) it follows that u is bounded on $(0,\infty)$ and $\|u\|_{C_b([0,\infty))} \le |C_1| + \|f\|_\infty\|Q\|_{L^1((0,\infty))}$. Using the fact that $Q \in L^1((-\infty,0))$, it can be proved that u is bounded also on $(-\infty,0)$. Thus, $u \in C_b(\mathbb{R})$. On the other hand, since $f = 1$ on $(0,\infty)$ it follows that

$$u'(x) = -\frac{e^{x^4}}{h(x)}\int_x^\infty h(t)e^{-t^4}dt, \qquad x > 0,$$

so that at $x = n - \delta_n/2$

$$|u'(n-\delta_n/2)| = \frac{e^{(n-\delta_n/2)^4}}{\varepsilon_n}\int_{n-\frac{\delta_n}{2}}^\infty h(t)e^{-t^4}dt \ge \frac{e^{(n-1/2)^4}}{\varepsilon_n}\int_n^{n+1/2} e^{-t^4}dt \ge \frac{n}{2},$$

which shows that u is unbounded.

Let $v : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be the function defined by $v(t,x) = e^t u(x)$ for every $(t,x) \in [0,\infty) \times \mathbb{R}$. It solves the equation $D_t v = \mathcal{A}v$ and is bounded on each strip $[0,T] \times \mathbb{R}$ On the other hand, $D_x v(t,x) = e^t u'(x)$ for each $t \geq 0$ and $x \in \mathbb{R}$. Consequently, $D_x v$ is unbounded on $[1/2,1] \times \mathbb{R}^d$ and estimate (B.21) with $\alpha = 0$, $beta = 1$ fails to hold.

We note that in this case the dissipativity assumption (B.20) fails since p'' is unbounded from above. Indeed, let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = \log(h(x))$ for each $x \in \mathbb{R}$. Since $g(n - \delta_n) = g(n) = 0$ and $g(n - \delta_n/2) = \log(\varepsilon_n)$ for every $n \in \mathbb{N}$, then, by the mean value theorem, there exist two points $y_n \in (n - \delta_n, n - \delta_n/2)$ and $z_n \in (n - \delta_n/2, n)$ such that $g'(y_n) = -g'(z_n) = 2\log(\varepsilon_n)\delta_n^{-1}$ for every $n \in \mathbb{N}$. Applying again the mean value theorem, it follows that there exists $x_n \in (y_n, z_n)$ such that

$$g''(x_n) = -\frac{4\log(\varepsilon_n)}{\delta_n(z_n - y_n)} \geq -\frac{4\log(\varepsilon_n)}{\delta_n^2} = -\frac{4\log(\varepsilon_n)}{\varepsilon_n^2} n^4 e^{2n^4}, \qquad n \in \mathbb{N}.$$

Since ε_n vanishes as n tends to ∞, then, for n large enough it holds that

$$p''(x_n) = -12x_n^2 + g''(x_n) \geq -12n^2 + n^4 e^{2n^4},$$

which shows that p'' is unbounded from above.

Bibliography

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Appendix C

Proof of Proposition 12.0.3

This appendix is devoted to the proof of Proposition 12.0.3.

Remark C.0.6. The proof of Proposition 12.0.6 actually shows that $\sum_{i=1}^N \eta_i = 1$ on a neighborhood of K. Indeed, $\sum_{i=1}^N \eta_i = 1$ on $\bigcup_{j=1}^M B(x_j, r_{x_j}/2)$ and this set contains the set $K_\delta = \{x \in \mathbb{R}^d : \operatorname{dist}(x, K) \leq \delta\}$. By contradiction, suppose that there exists a sequence $(y_n) \in \mathbb{R}^d$ such that $\operatorname{dist}(y_n, K) \leq n^{-1}$ and $y_n \notin \bigcup_{j=1}^M B(x_j, r_{x_j}/2)$ for every $n \in \mathbb{N}$. Since this sequence is bounded, up to a subsequence, it converges to some $\overline{y} \in K$, which belongs to the open set $\bigcup_{j=1}^M B(x_j, r_{x_j}/2)$. Hence, for large n, y_n belongs to $\bigcup_{j=1}^M B(x_j, r_{x_j}/2)$: a contradiction.

Now, if we introduce a function $\zeta \in C_c^\infty(\mathbb{R}^d)$ such that $\chi_{K_{\delta/2}} \leq \zeta \leq \chi_{K_\delta}$ (whose existence follows from Exercise 7.3.1) and set $\widetilde{\eta}_i = \eta_i \zeta$ for every $i = 1, \ldots, N$, then we can claim that

(i) $\widetilde{\eta}_i \in C_c^\infty(\mathbb{R}^d)$, $\widetilde{\eta}_i \leq \chi_{\Omega_i}$ for every $i = 1, \ldots, N$;

(ii) $0 \leq \sum_{i=1}^N \widetilde{\eta}_i \leq 1$ on $\mathbb{R}^d$ and $\sum_{i=1}^N \widetilde{\eta}_i = 1$ on $K_{\delta/2}$.

We will use this remark in the proof of Proposition 12.0.3.

Proof of Proposition 12.0.3. We split the proof into two steps.

Step 1. Here, we prove that, if Ω is of class $C^{2+\alpha}$ then there exists a function g as in the statement. For this purpose, for every $x \in \partial\Omega$ we denote by φ_x the last component of the function ψ_x in Definition 12.0.2. It is easy to check that

$$\begin{aligned} \Omega \cap \mathcal{U}_x &= \{y \in \mathcal{U}_x : \varphi_x(y) > 0\}, \\ \partial\Omega \cap \mathcal{U}_x &= \{y \in \mathcal{U}_x : \varphi_x(y) = 0\}, \\ (\mathbb{R}^d \setminus \overline{\Omega}) \cap \mathcal{U}_x &= \{y \in \mathcal{U}_x : \varphi_x(y) < 0\}. \end{aligned}$$

We now consider a finite subcovering $\{\mathcal{U}_{x_i} : i = 1, \ldots, N-1\}$ of $\partial\Omega$ and observe that, by the same arguments in Remark C.0.6, we can show that there exists $\delta > 0$ such that

$$K_\delta = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) \leq \delta\} \subset \bigcup_{i=1}^{N-1} \mathcal{U}_{x_i}.$$

This remark also shows that we can determine nonnegative functions $\widetilde{\eta}_i \in C_c^\infty(\mathbb{R}^d)$ ($i = 1,\ldots,N-1$) such that $\operatorname{supp}(\widetilde{\eta}_i) \subset \mathcal{U}_i$ for every i and $\chi_{K_\delta} \le \widetilde{\eta} := \sum_{i=1}^{N-1} \widetilde{\eta}_i \le 1$ on $\mathbb{R}^d$.

Let $g : \mathbb{R}^d \to \mathbb{R}$ be the function defined on $\mathbb{R}^d$ by

$$g(x) = (\widetilde{\eta}(x) - 1)(1 - 2\chi_\Omega(x)) + \sum_{i=1}^{N-1} \widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x), \qquad x \in \mathbb{R}^d.$$

It is clear that each function $\widetilde{\eta}_i\overline{\varphi_{x_i}}$ belongs to $C_b^{2+\alpha}(\mathbb{R}^d)$ since it is compactly supported in $\mathcal{U}_{x_i}$. Similarly, the function $(\widetilde{\eta} - 1)(1 - 2\chi_\Omega)$ belongs to $C_b^{2+\alpha}(\mathbb{R}^d)$ since it vanishes on a neighborhood of $\partial\Omega$ and it is trivially smooth elsewhere.

We are going to prove that function g has all the properties in the statement of the proposition. To begin with, we prove that it vanishes on $\partial\Omega$. For this purpose, we observe that on $\partial\Omega$, $\widetilde{\eta} \equiv 1$ (since $\partial\Omega \subset K_\delta$). Hence,

$$g(x) = \sum_{i=1}^{N-1} \widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x), \qquad x \in \partial\Omega.$$

Fix $x \in \partial\Omega$ and $i = 1,\ldots,N-1$. We claim that $\widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x) = 0$. Indeed, if $x \in \mathcal{U}_{x_i}$, then $\varphi_{x_i}(x) = 0$, otherwise $\widetilde{\eta}_i(x) = 0$.

Suppose now that $x \in \Omega$. Then,

$$g(x) = 1 - \widetilde{\eta}(x) + \sum_{i=1}^{N-1} \widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x), \qquad x \in \mathbb{R}^d.$$

Note that $\widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x) \ge 0$ for every i. Indeed, if $x \in \mathcal{U}_{x_i}$, then $\varphi_{x_i}(x) > 0$ and $\widetilde{\eta}_i(x) \ge 0$, whereas, if $x \notin \mathcal{U}_{x_i}$, then $\widetilde{\eta}_i(x) = 0$, so that $\widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x) = 0$. We have two possibilities:

(i) $\widetilde{\eta}_i(x) = 0$ for every $i = 1,\ldots,N-1$;

(ii) there exists $i \in \{1,\ldots,N-1\}$ such that $\widetilde{\eta}_i(x) \neq 0$.

In the first case $0 = \widetilde{\eta}(x) = \sum_{i=1}^{N-1} \widetilde{\eta}_i(x)$ and $g(x) \ge 1$, On the other hand, if (ii) holds, then $x \in \mathcal{U}_{x_i}$. Since $\varphi_{x_i}(x) > 0$, it follows that $g(x) \ge \widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x) > 0$.

Finally, we suppose that $x \notin \overline{\Omega}$. In such a case,

$$g(x) = \widetilde{\eta}(x) - 1 + \sum_{i=1}^{N-1} \widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x).$$

Arguing as above, we can easily show that $\widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x) \le 0$ for every $i = 1,\ldots,N-1$. We have also the same two cases (i) and (ii) as above. In the first case $g(x) = -1$, whereas in the latter one, $g(x) \le \widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x) < 0$, since $\varphi_{x_i} < 0$ on $(\mathbb{R}^d \setminus \Omega) \cap \mathcal{U}_{x_i}$.

To complete the proof, let us show that the gradient of the function g nowhere vanishes on $\partial\Omega$. For this purpose, we fix $x \in \partial\Omega$ and observe that the function $(1 - \widetilde{\eta})(1 - 2\chi_\Omega)$ identically vanishes on a neighborhood of x. Hence,

$$\nabla g(x) = \sum_{i=1}^{N-1} \nabla\widetilde{\eta}_i(x)\overline{\varphi_{x_i}}(x) + \sum_{i=1}^{N-1} \widetilde{\eta}_i(x)\overline{\nabla\varphi_{x_i}}(x).$$

Note that if $\nabla\widetilde{\eta}_i(x) \neq 0$, then $\varphi_{x_i}(x) = 0$. Based on this remark, we simplify the previous formula obtaining

$$\nabla g(x) = \sum_{i\in J_x} \widetilde{\eta}_i(x)\overline{\nabla\varphi_{x_i}}(x) = \sum_{i\in J_x} \widetilde{\eta}_i(x)|\nabla\varphi_{x_i}(x)|\nu_i(x).$$

where $J_x = \{i \in \{1, \dots, N-1\} : x \in \mathcal{U}_{x_i}\}$ and $\nu_i(x) = |\nabla\varphi_{x_i}(x)|^{-1}\nabla\varphi_{x_i}(x)$ We claim that $\nu_i(x) = \nu_j(x)$ for every $i, j \in J_x$. Denoting by $\nu(x)$ the common value, from the claim it will follow that

$$\nabla g(x) = \nu(x)\sum_{i\in J_x} \widetilde{\eta}_i(x)|\nabla\varphi_{x_i}(x)|.$$

Since $\sum_{i\in J_x} \widetilde{\eta}_i(x) = 1$ and $|\nabla\varphi_{x_i}(x)| > 0$ for every $i \in J_x$, we will easily conclude that $\nabla_x g(x) > 0$.

So, let us prove the claim. Fix $i, j \in J_x$. Since $\nu_i(x)$ and $\nu_j(x)$ are the normal vectors to $\Omega \cap \mathcal{U}_{x_i}$ at x, $\nu_i(x) = \nu_j(x)$ or $\nu_i(x) = -\nu_j(x)$. To prove that the second possibility can not occur, we apply Taylor formula to write

$$\begin{aligned}
&\varphi_{x_i}(x + t\nu_i(x)) = \varphi_{x_i}(x) + t\langle\nabla\varphi_{x_i}(x), \nu_i(x)\rangle + o(t, 0) = t(|\nabla\varphi_{x_i}(x)| + o(1, 0)),\\
&\varphi_{x_j}(x + t\nu_j(x)) = \varphi_{x_j}(x) + t\langle\nabla\varphi_{x_j}(x), \nu_j(x)\rangle + o(t, 0) = t(|\nabla\varphi_{x_j}(x)| + o(1, 0)),
\end{aligned}$$

for $t \in (-\delta_0, \delta_0)$ and δ_0 small enough. From these formulas it follows that, up to replacing δ_0 with a smaller value, if needed, $x + t\nu_i(x)$ belongs to Ω for every $t \in [0, \delta_0)$ whereas $x - t\nu_j(x)$ does not belong to Ω if $t \in [0, \delta_0)$. We thus conclude that $\nu_j(x) = \nu_i(x)$.

Step 2. We now show that if $\Omega = \{x \in \mathbb{R}^d : g(x) > 0\}$ for some function $g \in C^{2+\alpha}(\mathbb{R}^d)$ such that $\nabla_x g \neq 0$ on $\partial\Omega = \{x \in \mathbb{R}^d : g(x) = 0\}$ and Ω is bounded, then it is a domain of class $C^{2+\alpha}$. For this purpose, we fix $x_0 \in \partial\Omega$ and observe that since $\nabla g \neq 0$ on $\partial\Omega$, it follows that there exists $i \in \{1, \dots, d\}$ such that $D_i g(x_0) \neq 0$. Let $\widetilde{\psi}_{x_0} : \mathbb{R}^d \to \mathbb{R}^d$ be the function defined by

$$\widetilde{\psi}_{x_0}(x) = (x_1 - x_{0,1}, \dots, x_{i-1} - x_{0,i-1}, g(x), x_{i+1} - x_{0,i+1}, \dots, x_d - x_{0,d}), \qquad x \in \mathbb{R}^d.$$

As it is immediately seen, $\widetilde{\psi}_{x_0}$ belongs to $C^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$ and its Jacobian determinant at x_0 is positive. Hence, there exists $r > 0$ such that the function $\widetilde{\psi}_{x_0}$ is invertible in $\overline{B(x_0, r)}$. Since $\widetilde{\psi}_{x_0}(x_0) = 0$, we can fix $\delta > 0$ such that $B(0, \delta) \subset \widetilde{\psi}_{x_0}(B(x_0, r))$. Hence, if we set $\mathcal{U}_{x_0} = \widetilde{\psi}_{x_0}^{-1}(B(0, \delta))$, then $\widetilde{\psi}_{x_0}$ is a diffeomorphism of class $C^{2+\alpha}$ between $\overline{\mathcal{U}_{x_0}}$ and $\overline{B(0, \delta)}$. We are going to prove that $\widetilde{\psi}_{x_0}(\Omega \cap \mathcal{U}_{x_0}) = \{x \in B(0, \delta) : x_i > 0\}$ and $\widetilde{\psi}_{x_0}(\partial\Omega \cap \mathcal{U}_{x_0}) = \{x \in B(0, \delta) : x_i = 0\}$. For this purpose, we recall that $x \in \Omega$ (resp. $x \in \partial\Omega$) if and only if $g(x) > 0$ (resp. $g(x) = 0$). Hence, if $x \in \Omega \cap \mathcal{U}_{x_0}$ then $y = \widetilde{\psi}_{x_0}(x) \in B(0, \delta)$ and $y_i > 0$. Similarly, if $x \in \partial\Omega \cap \mathcal{U}_{x_0}$ then $y = \widetilde{\psi}_{x_0}(x) \in B(0, \delta)$ and $y_i = 0$. Viceversa, let us assume that $y \in B(0, \delta)$ is such that $y_d > 0$. Then, $y = \widetilde{\psi}_{x_0}(y)$, with $x = \widetilde{\psi}_{x_0}^{-1}(y) \in \mathcal{U}_{x_0}$. Moreover, $g(x) = y_d > 0$. Hence, x belongs also to Ω. In the same way it can be proved that each

$y \in B(0,\delta)$ such that $y_d = 0$ is the image under $\widetilde{\psi}_{x_0}$ of an element of $\Omega \cap \mathcal{U}_{x_0}$ and we are done.

Now, we set $\psi_{x_0} = T\widetilde{\psi}_{x_0}$, where

$$T(y) = (\delta^{-1}y_1, \ldots, \delta^{-1}y_{i-1}, \delta^{-1}y_d, \delta^{-1}y_{i+1}, \ldots, \delta^{-1}y_{d-1}, \delta^{-1}y_i)$$

for every $y \in \mathbb{R}^d$. Function ψ_{x_0} belongs to $C^{2+\alpha}(\overline{\mathcal{U}}_{x_0})$, $\psi_{x_0}(\Omega \cap \mathcal{U}_{x_0}) = B_+(0,1)$ and $\psi_{x_0}(\partial\Omega \cap \mathcal{U}_{x_0}) = \{y \in B(0,1) : y_d = 0\}$. This completes the proof. $\square$