

18th Internet Seminar on Evolution Equations

Form Methods for Evolution Equations, and Applications

Lecture Notes

WOLFGANG ARENDT RALPH CHILL CHRISTIAN SEIFERT
HENDRIK VOGT JÜRGEN VOIGT

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Lecture 1

C_0 -semigroups

C_0 -semigroups serve to describe the time evolution of autonomous linear systems. The objective of the present lecture is to introduce the notion of C_0 -semigroups and their generators, and to derive some basic properties. In an “interlude” we provide some fundamental facts concerning operators as well as integration and differentiation of Banach space valued functions.

1.1 Motivation

Let X be a Banach space. A C_0 -semigroup is a function $T: [0, \infty) \rightarrow \mathcal{L}(X)$ associated with the solutions of the initial value problem for a linear autonomous differential equation on $[0, \infty)$,

$$u' = Au, \quad u(0) = x.$$

Here, A should be a suitable (usually unbounded) linear operator in X . If $x \in \text{dom}(A)$, then the function $u: [0, \infty) \rightarrow X$ given by $u(t) := T(t)x$ should be the unique solution of the initial value problem given above.

At the first glance, the differential equation looks like an ordinary differential equation. However, the right hand side $x \mapsto Ax$ of the differential equation will, in general, not be continuous; in fact, in the intended applications it is not even defined for all $x \in X$. Typically, the Banach space will be a space of functions, and the operator A will be a partial differential operator. For example, the heat equation

$$\partial_t u = \Delta u$$

will be treated in the context of C_0 -semigroups. This equation, in fact, is a paradigmatic example of a parabolic partial differential equation.

1.2 Definition and some basic properties

Let X be a (real or complex) Banach space.

A **one-parameter semigroup** on X is a function $T: [0, \infty) \rightarrow \mathcal{L}(X)$ (where $\mathcal{L}(X)$ denotes the space of bounded linear operators in X , with domain all of X), satisfying

- (i) $T(t + s) = T(t)T(s)$, for all $t, s \geq 0$.

If additionally

(ii) $\lim_{t \rightarrow 0^+} T(t)x = x$ for all $x \in X$,

then T is called a C_0 -semigroup (on X) (also a **strongly continuous semigroup**).

If T is defined on \mathbb{R} instead of $[0, \infty)$, and (i) holds for all $t, s \in \mathbb{R}$, then T is called a **one-parameter group**, and if additionally (ii) holds, then T is called a C_0 -group.

1.1 Remarks. (a) Property (i) implies that for $t, s \geq 0$ the operators $T(t), T(s)$ commute; also, if $t_1, t_2, \dots, t_n \geq 0$, then $T(\sum_{j=1}^n t_j) = \prod_{j=1}^n T(t_j)$.

(b) Property (i) implies that $T(0) = T(0)^2$ is a projection.

(c) If T is a C_0 -semigroup, then $T(0)x = \lim_{t \rightarrow 0^+} T(t)T(0)x = \lim_{t \rightarrow 0^+} T(t)x = x$ for all $x \in X$, i.e., $T(0) = I$.

In property (i) one immediately recognises the functional equation for the exponential function, and in fact this will be our first example for a C_0 -group.

1.2 Example. Let $A \in \mathcal{L}(X)$, and define

$$T(t) := e^{tA} = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!}$$

for $t \in \mathbb{R}$. Then T is a C_0 -group. (More strongly, $\mathbb{R} \ni t \mapsto T(t) \in \mathcal{L}(X)$ is continuous in the operator norm.)

We leave this as an exercise (cf. Exercise 1.1).

1.3 Lemma. Let T be a one-parameter semigroup on X , and assume that there exists $\delta > 0$ such that $M := \sup_{0 \leq t < \delta} \|T(t)\| < \infty$. Then there exists $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

Proof. If $T(0) = 0$, then $M = 0$, and the assertion is trivial. Otherwise $T(0)$ is a non-zero projection, and therefore $M \geq \|T(0)\| \geq 1$. Let $\omega := \frac{1}{\delta} \ln M$; then $M = e^{\omega \delta}$. For $t \geq 0$ there exists $n \in \mathbb{N}_0$ such that $n\delta \leq t < (n+1)\delta$. The semigroup property (i) implies $T(t) = T(\frac{t}{n+1})^{n+1}$, and therefore

$$\|T(t)\| \leq \|T(\frac{t}{n+1})\|^{n+1} \leq M^{n+1} = Me^{\omega \delta n} \leq Me^{\omega t}. \quad \square$$

1.4 Proposition. Let T be a C_0 -semigroup on X .

(a) Then there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

(b) For all $x \in X$ the function $[0, \infty) \ni t \mapsto T(t)x \in X$ is continuous. In other words, the function T is strongly continuous.

(c) If T is a C_0 -group on X , then there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{|\omega| |t|} \text{ for all } t \in \mathbb{R}.$$

For all $x \in X$ the function $\mathbb{R} \ni t \mapsto T(t)x \in X$ is continuous.

Proof. (a) In view of Lemma 1.3 it is sufficient to show that there exists $\delta > 0$ such that $\sup_{0 \leq t < \delta} \|T(t)\| < \infty$. Assuming that this is not the case we can find a null sequence (t_n) in $(0, \delta)$ such that $\|T(t_n)\| \rightarrow \infty$ as $n \rightarrow \infty$. However, for all $x \in X$ the sequence $(T(t_n)x)$ is convergent (to x), by property (ii) of C_0 -semigroups. Therefore the uniform boundedness theorem (for which we refer to [Yos68; II,1 Corollary 1] or [Bre83; Théorème II.1]) implies that $\sup_{n \in \mathbb{N}} \|T(t_n)\| < \infty$; a contradiction.

(b) Let $x \in X$, $t > 0$. Then $T(t+h)x - T(t)x = T(t)(T(h)x - x) \rightarrow 0$ as $h \rightarrow 0+$, which proves the right-sided continuity of $T(\cdot)x$. In order to prove the left-sided continuity we let $-t \leq h < 0$ and write $T(t+h)x - T(t)x = T(t+h)(x - T(-h)x)$. Then we obtain

$$\|T(t+h)x - T(t)x\| \leq \left(\sup_{0 \leq s \leq t} \|T(s)\| \right) \|x - T(-h)x\| \rightarrow 0$$

as $h \rightarrow 0-$.

(c) First we show that, given $x \in X$, the orbit $T(\cdot)x$ is continuous. As the restriction of T to $[0, \infty)$ is a C_0 -semigroup it follows from (b) that $T(\cdot)x$ is continuous on $[0, \infty)$. Let $t \leq 0$. Then $T(t+h)x - T(t)x = T(t-1)(T(1+h)x - T(1)x) \rightarrow 0$ ($h \rightarrow 0$), and this implies that $T(\cdot)x$ is continuous on \mathbb{R} .

As a consequence, the function $[0, \infty) \ni t \mapsto T(-t) \in \mathcal{L}(X)$ is a C_0 -semigroup, and therefore satisfies an estimate as in (a). Putting the estimates for the C_0 -semigroups $t \mapsto T(t)$ and $t \mapsto T(-t)$ together one obtains the asserted estimate. \square

In examples it is sometimes not immediately clear how to prove the strong continuity property (ii) of a one-parameter semigroup, whereas the boundedness condition of Lemma 1.3 is easy to verify. The following condition is useful for such examples.

1.5 Lemma. *Let T be a one-parameter semigroup on X . Assume that $\sup_{0 \leq t \leq 1} \|T(t)\| < \infty$ and that there exists a dense subset D of X such that $\lim_{t \rightarrow 0+} T(t)x = x$ for all $x \in D$. Then T is a C_0 -semigroup.*

This lemma is an immediate consequence of the following fundamental fact of functional analysis, which we insert here, also for further reference.

1.6 Proposition. *Let X, Y be Banach spaces over the same field, and let $(B_n)_{n \in \mathbb{N}}$ be a bounded sequence in $\mathcal{L}(X, Y)$ (the space of bounded linear operators from X to Y , with domain all of X). Let $D \subseteq X$ be a dense subset of X , and assume that for all $x \in D$ the sequence $(B_n x)_{n \in \mathbb{N}}$ is convergent.*

*Then $Bx := \lim_{n \rightarrow \infty} B_n x$ exists for all $x \in X$, and $B: X \rightarrow Y$ thus defined is an operator $B \in \mathcal{L}(X, Y)$. In other words, there exists $B \in \mathcal{L}(X, Y)$ such that (B_n) is **strongly convergent to B** , abbreviated $B = \text{s-lim}_{n \rightarrow \infty} B_n$.*

Proof. A standard 3ε -argument shows that $(B_n x)$ is a Cauchy sequence in Y , for all $x \in X$, and therefore convergent. The linearity and boundedness of B are then easy to show. \square

We note that in applications of this proposition it often happens that the limiting operator is already known, but the pointwise convergence is only known on a dense subset. (In the application to Lemma 1.5, for instance, one has $B = I$.)

1.7 Examples. Right translation on $L_p(\mathbb{R})$, $L_p(-\infty, 0)$, $L_p(0, \infty)$ and $L_p(0, 1)$, for $1 \leq p < \infty$.

(a) On $L_p(\mathbb{R})$: For $t \in \mathbb{R}$ we define $T(t) \in \mathcal{L}(L_p(\mathbb{R}))$ by

$$T(t)f(x) := f(x - t) \quad (x \in \mathbb{R}, f \in L_p(\mathbb{R})).$$

It is clear that $T(t)$ is an isometric isomorphism for all $t \in \mathbb{R}$. Also, it is easy to show that T is a one-parameter group. Let $f = \mathbf{1}_{[a,b]}$ be the indicator function of the interval $[a, b]$, where $a, b \in \mathbb{R}$, $a < b$. Then it is easy to see that $T(t)f \rightarrow f$ as $t \rightarrow 0$. This carries over to all linear combinations of such indicator functions. Now the linear span D of such indicator functions is dense in $L_p(\mathbb{R})$. Therefore Lemma 1.5 implies that T is a C_0 -group.

(b) On $L_p(-\infty, 0)$: The operator $T(t)$, for $t \geq 0$, is defined by the same formula as in (a),

$$T(t)f(x) := f(x - t) \quad (x \in (-\infty, 0), f \in L_p(-\infty, 0)).$$

In this case the operators $T(t)$ are not isometric, for $t > 0$, but they are contractions, even $\|T(t)\| = 1$. Again it is easy to see that T is a one-parameter semigroup (not a group), and as in (a) one shows that T is a C_0 -semigroup.

(c) On $L_p(0, \infty)$: Denote by S the C_0 -semigroup of right translations on $L_p(\mathbb{R})$, defined in (a), but with time parameter t restricted to $[0, \infty)$. Consider $L_p(0, \infty)$ as the subspace $\{f \in L_p(\mathbb{R}); f|_{(-\infty, 0)} = 0\}$ of $L_p(\mathbb{R})$. It is easy to see that $S(t)$, for all $t \geq 0$, leaves this subspace invariant, and that this implies that the restriction T of S to this subspace is a C_0 -semigroup. The operator $T(t)$ is isometric for all $t \geq 0$. However, for $t > 0$ the operator $T(t)$ is not surjective.

(d) On $L_p(0, 1)$: Analogously to part (b) we can define the C_0 -semigroup of right translations S on $L_p(-\infty, 1)$. Similarly to (c) one shows that $S(t)$ leaves the subspace $L_p(0, 1)$ of $L_p(-\infty, 1)$ invariant, and therefore the restriction T of S to $L_p(0, 1)$ is a C_0 -semigroup. This semigroup has the property that $\|T(t)\| = 1$ for $0 \leq t < 1$, and that $T(t) = 0$ for $t \geq 1$. Because of the latter property it is called the *nilpotent* right shift semigroup on $L_p(0, 1)$.

1.3 Interlude: operators, integration and differentiation

1.3.1 Operators

Let X, Y be two vector spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For a linear relation in $X \times Y$, i.e., a subspace $A \subseteq X \times Y$, we define the **domain of A** ,

$$\text{dom}(A) := \{x \in X; \text{there exists } y \in Y \text{ such that } (x, y) \in A\},$$

the **range of A** ,

$$\text{ran}(A) := \{y \in Y; \text{there exists } x \in X \text{ such that } (x, y) \in A\},$$

and the **kernel (or null space) of A** ,

$$\text{ker}(A) := \{x \in X; (x, 0) \in A\}.$$

The linear relation

$$A^{-1} := \{(y, x); (x, y) \in A\}$$

in $Y \times X$ is the **inverse relation of A** . If B is another linear relation in $X \times Y$, satisfying $A \subseteq B$, then B is called an **extension of A** , and A a **restriction of B** .

In this setting, a **linear operator from X to Y** is a linear relation in $X \times Y$ satisfying additionally

$$A \cap (\{0\} \times Y) = \{(0, 0)\}.$$

Then, for all $x \in \text{dom}(A)$, there exists a unique $y \in Y$, such that $(x, y) \in A$, and we will write $Ax = y$. In this sense, A is also a mapping $A: \text{dom}(A) \rightarrow Y$. As we will consider only linear operators we will mostly drop ‘linear’ and simply speak of ‘operators’. If the spaces X and Y coincide, then we call A an **operator in X** .

Next, let X and Y be Banach spaces. We define a norm on $X \times Y$ by

$$\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y \quad ((x, y) \in X \times Y),$$

which makes $X \times Y$ a Banach space. In this context an operator A from X to Y is called **closed** if A is a closed subset of $X \times Y$, and A is **closable** if the closure \bar{A} of A in $X \times Y$ is an operator.

For a subspace $D \subseteq \text{dom}(A)$, the **restriction of A to D** is the operator $A|_D := A \cap (D \times Y)$. The set D is called a **core for A** if A is a restriction of the closure of $A|_D$, i.e., $A \subseteq \overline{A|_D}$.

Finally, if A and B are operators from X to Y , then the **sum of A and B** is the operator defined by

$$\text{dom}(A + B) := \text{dom}(A) \cap \text{dom}(B), \quad (A + B)x := Ax + Bx \quad (x \in \text{dom}(A + B)),$$

or, expressed differently,

$$A + B = \{(x, Ax + Bx); x \in \text{dom}(A) \cap \text{dom}(B)\} \subseteq X \times Y.$$

We mention that in most cases of the use of a sum of two operators the domain of one of the operators is a subset of the domain of the other, or more specially, one of the operators is defined everywhere.

1.3.2 Integration of continuous functions

Let $a, b \in \mathbb{R}$, $a < b$, and let X be a Banach space. We define the **space of step functions** from $[a, b]$ to X ,

$$T([a, b]; X) := \text{lin}\{\mathbf{1}_{[s,t]}x; a \leq s \leq t \leq b, x \in X\}.$$

(The letter ‘T’ stands for the German ‘Treppenfunktion’.) For $f \in T([a, b]; X)$, $f = \sum_{j=1}^n \mathbf{1}_{[s_j, t_j]}x_j$, we define the integral

$$\int_a^b f(t) dt := \sum_{j=1}^n (t_j - s_j)x_j.$$

It is standard to show that this integral is well-defined, that the mapping $f \mapsto \int_a^b f(t) dt$ is linear, and that

$$\left\| \int_a^b f(t) dt \right\| \leq (b-a) \sup_{a \leq t \leq b} \|f(t)\| \quad (f \in T([a, b]; X)). \quad (1.1)$$

In short, the mapping $T([a, b]; X) \ni f \mapsto \int_a^b f(t) dt \in X$ is a continuous linear mapping, where $T([a, b]; X)$ is provided with the supremum norm. This implies that there is a unique continuous extension of the integral to functions in the closure of $T([a, b]; X)$ in the Banach space

$$\ell_\infty([a, b]; X) := \{f: [a, b] \rightarrow X; f \text{ bounded}\},$$

provided with the supremum norm. We denote this closure by $R([a, b]; X)$ (the regulated functions!) and observe that $R([a, b]; X)$ contains the subspace of continuous functions $C([a, b]; X)$. Moreover, the extension of the integral is linear, and the inequality (1.1) carries over to all $f \in R([a, b]; X)$.

We will need properties how operators act on integrals.

1.8 Theorem. *Let X, Y be Banach spaces over the same field, and let $a, b \in \mathbb{R}$, $a < b$.*

(a) *Let $f: [a, b] \rightarrow X$ be continuous, and let $A \in \mathcal{L}(X, Y)$. Then*

$$A \int_a^b f(t) dt = \int_a^b Af(t) dt.$$

(b) *(Hille's theorem) Let A be a closed operator from X to Y . Let $f: [a, b] \rightarrow X$ be continuous, $f(t) \in \text{dom}(A)$ for all $t \in [a, b]$, and $t \mapsto Af(t) \in Y$ continuous. Then $\int_a^b f(t) dt \in \text{dom}(A)$, and*

$$A \int_a^b f(t) dt = \int_a^b Af(t) dt.$$

Proof. (a) The equality is clear for step functions and carries over to continuous functions (in fact, even to regulated functions) by continuous extension.

(b) The hypotheses are just a complicated way to express that one is given a continuous function $t \mapsto (f(t), g(t)) \in A \subseteq X \times Y$ (where $g(t) = Af(t)$). Because A is a closed subspace of $X \times Y$, therefore a Banach space, it follows that $\int_a^b (f(t), g(t)) dt \in A$. As the canonical projections from $X \times Y$ to X, Y are bounded linear operators, one concludes from part (a) that

$$\left(\int_a^b f(t) dt, \int_a^b g(t) dt \right) = \int_a^b (f(t), g(t)) dt \in A,$$

and this shows the assertions. □

The last issue in this interlude is the connection between differentiation and integration, i.e., the fundamental theorem of differential and integral calculus for Banach space valued functions.

1.9 Theorem. Let $a, b \in \mathbb{R}$, $a < b$.

(a) Let $f: [a, b] \rightarrow X$ be continuous,

$$F(t) := \int_a^t f(s) ds \quad (a \leq t \leq b).$$

Then F is continuously differentiable, and $F' = f$.

(b) Let $g: [a, b] \rightarrow X$ continuously differentiable. Then

$$\int_a^b g'(t) dt = g(b) - g(a). \quad (1.2)$$

Proof. (a) is proved in the same way as for scalar valued functions.

(b) For $\eta \in X'$ ($= \mathcal{L}(X, \mathbb{K})$, the dual space of X) the function $\eta \circ g$ is continuously differentiable, and one has $(\eta \circ g)' = \eta \circ g'$. The fundamental theorem of differential and integral calculus for \mathbb{K} -valued functions then implies that

$$\begin{aligned} \eta\left(\int_a^b g'(t) dt\right) &= \int_a^b \eta(g'(t)) dt = \int_a^b (\eta \circ g)'(t) dt \\ &= (\eta \circ g)(b) - (\eta \circ g)(a) = \eta(g(b) - g(a)). \end{aligned}$$

As this holds for all $\eta \in X'$, the equation (1.2) follows from an application of the theorem of Hahn-Banach. \square

1.4 The generator of a C_0 -semigroup

Let X be a Banach space.

For a C_0 -semigroup T we define the **generator** (also called the **infinitesimal generator**) A , an operator in X , by

$$A := \{(x, y) \in X \times X; y = \lim_{h \rightarrow 0^+} h^{-1}(T(h)x - x) \text{ exists}\}.$$

In other words, the domain of A consists of those $x \in X$ for which the orbit $t \mapsto T(t)x$ is (right-sided) differentiable at $t = 0$, and the image of x under A is the derivative of this orbit at $t = 0$.

Similarly to the property that for the ordinary exponential function the derivative at 0 determines the function, we will see that the generator determines the C_0 -semigroup.

First we derive some fundamental properties of the generator.

1.10 Theorem. Let T be a C_0 -semigroup on X , with generator A . Then:

(a) For $x \in \text{dom}(A)$ one has $T(t)x \in \text{dom}(A)$ for all $t \geq 0$, the function $t \mapsto T(t)x$ is continuously differentiable on $[0, \infty)$, and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax \quad (t \geq 0).$$

(b) For all $x \in X$, $t > 0$ one has $\int_0^t T(s)x \, ds \in \text{dom}(A)$,

$$A \int_0^t T(s)x \, ds = T(t)x - x.$$

(c) $\text{dom}(A)$ is dense in X , and A is a closed operator.

In the proof we need a small fact on strong convergence of operators which we insert here.

1.11 Lemma. *Let X, Y be Banach spaces, and let (B_n) be a sequence in $\mathcal{L}(X, Y)$, $B \in \mathcal{L}(X, Y)$, and $B = \text{s-lim}_{n \rightarrow \infty} B_n$. Let (x_n) in X , $x_n \rightarrow x \in X$ ($n \rightarrow \infty$).*

Then $B_n x_n \rightarrow Bx$ as $n \rightarrow \infty$.

Proof. The uniform boundedness theorem (see [Yos68; II,1 Corollary 1], [Bre83; Théorème II.1]) implies that $\sup_{n \in \mathbb{N}} \|B_n\| < \infty$. Therefore

$$\begin{aligned} \|Bx - B_n x_n\| &\leq \|Bx - B_n x\| + \|B_n(x - x_n)\| \\ &\leq \|Bx - B_n x\| + \left(\sup_{j \in \mathbb{N}} \|B_j\|\right) \|x - x_n\| \rightarrow 0 \quad (n \rightarrow \infty). \quad \square \end{aligned}$$

Proof of Theorem 1.10. (a) For $t \geq 0$, $h > 0$ one has

$$h^{-1}(T(t+h)x - T(t)x) = h^{-1}(T(h) - I)T(t)x = T(t)h^{-1}(T(h)x - x).$$

As $h \rightarrow 0$, the third of these expressions converges to $T(t)Ax$. Looking at the second term, one obtains $T(t)x \in \text{dom}(A)$, and looking at the first term one concludes that $t \mapsto T(t)x$ is right-sided differentiable, with right-sided derivative

$$\left(\frac{d}{dt}\right)_r T(t)x = AT(t)x = T(t)Ax.$$

On the other hand, let $t > 0$, $h \in (0, t)$. Then

$$\frac{1}{-h}(T(t-h)x - T(t)x) = T(t-h)\frac{1}{h}(T(h)x - x),$$

and this converges to $T(t)Ax$ as $h \rightarrow 0$, by Lemma 1.11.

So we have shown that the continuous function $t \mapsto AT(t)x = T(t)Ax$ is the derivative of $t \mapsto T(t)x$.

(b) Let $0 < h < t$. Then Theorem 1.8(a) implies

$$\begin{aligned} &h^{-1}(T(h) - I) \int_0^t T(s)x \, ds \\ &= h^{-1} \left(\int_h^{t+h} - \int_0^t \right) T(s)x \, ds = h^{-1} \left(\int_t^{t+h} - \int_0^h \right) T(s)x \, ds \\ &= (T(t) - I)h^{-1} \int_0^h T(s)x \, ds \rightarrow (T(t) - I)x \quad (h \rightarrow 0+), \end{aligned}$$

where the last convergence follows from Theorem 1.9(a) (see also Exercise 1.4(a)). This shows $\int_0^t T(s)x \, ds \in \text{dom}(A)$ as well as the asserted equation.

(c) Let $x \in X$. Then $h^{-1} \int_0^h T(s)x \, ds \in \text{dom}(A)$ for all $h > 0$, by part (b). Theorem 1.9(a) implies that $h^{-1} \int_0^h T(s)x \, ds \rightarrow x$ ($h \rightarrow 0+$). This shows that $\text{dom}(A)$ is dense in X .

Let $((x_n, y_n))_{n \in \mathbb{N}}$ be a sequence in A , $(x_n, y_n) \rightarrow (x, y)$ in $X \times X$. From (a) and Theorem 1.9(b) we obtain

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n \, ds$$

for all $t > 0$, $n \in \mathbb{N}$. For $n \rightarrow \infty$ one concludes that

$$T(t)x - x = \int_0^t T(s)y \, ds,$$

and then

$$t^{-1}(T(t)x - x) = t^{-1} \int_0^t T(s)y \, ds \rightarrow y \quad (t \rightarrow 0+).$$

This shows that $(x, y) \in A$. □

Theorem 1.10(a) implies in particular that the function $t \mapsto T(t)x$ solves the initial value problem $u' = Au$, $u(0) = x$. We now show that this solution is unique.

1.12 Theorem. *Let T be a C_0 -semigroup on X , with generator A .*

(a) *Let $t_0 > 0$, and let $u: [0, t_0) \rightarrow X$ be continuous, $u(t) \in \text{dom}(A)$ for all $t \in (0, t_0)$, u continuously differentiable on $(0, t_0)$, and $u'(t) = Au(t)$ for all $t \in (0, t_0)$. Then $u(t) = T(t)u(0)$ for all $t \in [0, t_0)$.*

(b) *Let S be a C_0 -semigroup on X , with generator $B \supseteq A$. Then $S = T$, $B = A$.*

Proof. (a) Let $0 < t < t_0$. Lemma 1.11 implies that the function $[0, t] \ni s \mapsto T(t-s)u(s) \in X$ is continuous, and it is not difficult to see that this function is differentiable on $(0, t)$, with derivative

$$\frac{d}{ds} T(t-s)u(s) = -T(t-s)Au(s) + T(t-s)u'(s) = 0$$

(a kind of product rule). Therefore Theorem 1.9(b) implies

$$u(t) = T(t-t)u(t) = T(t)u(0).$$

(b) Let $x \in \text{dom}(A)$, $u(t) := T(t)x$ ($t \geq 0$). Then u satisfies the equation $u'(t) = Au(t) = Bu(t)$ ($t \geq 0$). From part (a) it follows that then $u(t) = S(t)u(0) = S(t)x$, for all $t \geq 0$. This shows that $S(t) = T(t)$ on $\text{dom}(A)$. As $\text{dom}(A)$ is dense in X one obtains $S(t) = T(t)$ for all $t \geq 0$. The equality of the semigroups implies equality of the generators. □

1.13 Example. If $A \in \mathcal{L}(X)$, then A is the generator of the C_0 -group $(e^{tA})_{t \in \mathbb{R}}$; see Example 1.2 and Exercise 1.1.

For determining the generator of the C_0 -semigroups described in Examples 1.7 the following result will be useful.

1.14 Proposition. *Let T be a C_0 -semigroup on X , and let A be its generator. Let D be a subspace of $\text{dom}(A)$ that is dense in X , and assume that D is invariant under T (i.e., $T(t)(D) \subseteq D$ for all $t \geq 0$).*

Then D is a core for A .

Proof. Let $x \in \text{dom}(A)$. We have to show that $(x, Ax) \in \overline{A|_D}$.

Let (x_n) be a sequence in D , $x_n \rightarrow x$ ($n \rightarrow \infty$) in X . Let $n \in \mathbb{N}$, $t > 0$. The function $[0, t] \ni s \mapsto (T(s)x_n, AT(s)x_n) \in A|_D$ is continuous; therefore (recall Theorem 1.10(a) and Theorem 1.9(b) for the first equality)

$$\begin{aligned} \left(\int_0^t T(s)x_n \, ds, T(t)x_n - x_n \right) &= \left(\int_0^t T(s)x_n \, ds, \int_0^t AT(s)x_n \, ds \right) \\ &= \int_0^t (T(s)x_n, AT(s)x_n) \, ds \in \overline{A|_D}. \end{aligned}$$

Taking $n \rightarrow \infty$ we conclude that

$$\left(\int_0^t T(s)x \, ds, T(t)x - x \right) \in \overline{A|_D}.$$

Dividing by t and taking $t \rightarrow 0+$ one obtains $(x, Ax) \in \overline{A|_D}$. □

1.15 Remark. We note that, for the C_0 -group T of right translations on $L_p(\mathbb{R})$ (see Example 1.7(a)) it is not difficult to show that $D := C_c^1(\mathbb{R})$ (the continuously differentiable functions with compact support) is a subspace of $\text{dom}(A)$, and $Af = -f'$ for all $f \in C_c^1(\mathbb{R})$. Also, $C_c^1(\mathbb{R})$ obviously is invariant under T , and is a dense subspace of $L_p(\mathbb{R})$. By Proposition 1.14, D is a core for A . For participants familiar with Sobolev spaces: This implies that $\text{dom}(A) = W_p^1(\mathbb{R})$. We will come back to this example.

Also we refer to Exercise 1.2 for the case of translation semigroups on continuous functions.

Notes

The classical treatise on one-parameter semigroups is [HP57]. The notation ‘ C_0 -semigroup’ goes back to Phillips [Phi55]; in [HP57; Section 10.6] it is put into context with other properties of one-parameter semigroups. Here is a – by far incomplete – list of treatises on C_0 -semigroups: [Dav80], [Paz83], [Gol85], [Nag86], [EN00]. Theorem 1.8(b) is the version of [HP57; Theorem 3.7.12], for the case of continuous functions.

Exercises

1.1 Let X be a Banach space, and let $A, B \in \mathcal{L}(X)$ satisfy $AB = BA$.

(a) Show that $e^A := \sum_{j=0}^{\infty} \frac{1}{j!} A^j$ is absolutely convergent in $\mathcal{L}(X)$ and that $e^{A+B} = e^A e^B = e^B e^A$.

(b) Show that $t \mapsto e^{tA}$ is a one-parameter group, that the function is continuously differentiable as an $\mathcal{L}(X)$ -valued function, and that $\frac{d}{dt} e^{tA} = A e^{tA}$ ($t \in \mathbb{R}$). (In particular, A is the generator of $(e^{tA})_{t \in \mathbb{R}}$.)

1.2 For a locally compact subset G of \mathbb{R}^n we define

$$C_0(G) := \{f \in C(G); \text{for all } c > 0 \text{ the set } [|f| \geq c] \text{ is compact}\}$$

(where $C(G)$ denotes the space of continuous \mathbb{K} -valued functions, and $[|f| \geq c] := \{x \in G; |f(x)| \geq c\}$). We recall that a topological space is called **locally compact** if every point possesses a compact neighbourhood.

As in Example 1.7 we define the one-parameter semigroup of right translations on the spaces $C_0(\mathbb{R})$, $C_0(-\infty, 0]$, $C_0(0, \infty)$, $C_0(0, 1]$.

(a) Show that $C_0(G)$, provided with the supremum norm, is a Banach space. Show further that

$$C_c(G) := \{f \in C(G); \text{spt } f \text{ compact}\}$$

is a dense subspace of $C_0(G)$ (where the **support of** f is defined by $\text{spt } f := \overline{[f \neq 0]}^G$).

(b) Assuming that it is shown that T on $C_0(\mathbb{R})$ is strongly continuous, show that its generator A is given by

$$\text{dom}(A) = C_0^1(\mathbb{R}) := \{f \in C^1(\mathbb{R}); f, f' \in C_0(\mathbb{R})\}, \quad Af = -f'.$$

(c) Again assuming that it is shown that T on $C_0(0, 1]$ is strongly continuous, show that its generator A is given by

$$\text{dom}(A) = C_0^1(0, 1] := \{f \in C^1(0, 1]; f, f' \in C_0(0, 1]\}, \quad Af = -f'.$$

1.3 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $1 \leq p < \infty$. Let $a: \Omega \rightarrow \mathbb{K}$ be a measurable function, and assume that $\text{Re } a(x) \leq 0$ ($x \in \Omega$).

For $t \geq 0$, define $T(t) \in \mathcal{L}(L_p(\mu))$ by

$$T(t)f := e^{ta} f \quad (f \in L_p(\mu)).$$

(a) Show that T is a C_0 -semigroup on $L_p(\mu)$.

(b) Show that the **maximal multiplication operator** M_a by the function a , defined by

$$M_a = \{(f, g) \in L_p(\mu) \times L_p(\mu); g = af\},$$

is closed and densely defined.

(c) Show that M_a is the generator of T .

1.4 Let X be a Banach space.

(a) Let $f: [0, 1] \rightarrow X$ be continuous. Show that $\lim_{h \rightarrow 0^+} h^{-1} \int_0^h f(t) dt = f(0)$. (This is the main step for proving Theorem 1.9(a).)

(b) Let T be an operator norm continuous one-parameter semigroup on X . Show that then the generator A of T is an operator $A \in \mathcal{L}(X)$. (Hint: Show that, for small t , the operator $\int_0^t T(s) ds$ is invertible in $\mathcal{L}(X)$. Further, note that Theorem 1.10(b) implies that the operator $x \mapsto A \int_0^t T(s)x ds$ belongs to $\mathcal{L}(X)$.)

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Lecture 2

Characterisation of generators of C_0 -semigroups

Generators of C_0 -semigroups have special spectral properties. They will be studied in the Section 2.2. The main goal of this lecture is the Hille-Yosida theorem which characterises generators of C_0 -semigroups. The exponential formula for C_0 -semigroups presented in the Section 2.3 will be useful for applications. We start with an interlude on spectral theory of operators as well as on more integration.

2.1 Interlude: the resolvent of operators, and some more integration

2.1.1 Resolvent set, spectrum and resolvent

Let X be a Banach space over \mathbb{K} , and let A be an operator in X .

We define the **resolvent set** of A ,

$$\rho(A) := \{\lambda \in \mathbb{K}; \lambda I - A: \text{dom}(A) \rightarrow X \text{ bijective, } (\lambda I - A)^{-1} \in \mathcal{L}(X)\}.$$

The operator $R(\lambda, A) := (\lambda I - A)^{-1}$ is called the **resolvent of A at λ** , and the mapping

$$R(\cdot, A): \rho(A) \rightarrow \mathcal{L}(X)$$

is called the **resolvent of A** . The set

$$\sigma(A) := \mathbb{K} \setminus \rho(A)$$

is called the **spectrum of A** .

2.1 Remarks. (a) If $\rho(A) \neq \emptyset$ and $\lambda \in \rho(A)$, then $(\lambda I - A)^{-1} \in \mathcal{L}(X)$ is closed – note that every operator belonging to $\mathcal{L}(X)$ is closed. Hence $\lambda I - A$ is closed, and therefore A is closed, by the reasoning presented subsequently in part (b).

(b) If A is a closed operator and $B \in \mathcal{L}(X)$, then the sum $A + B$ is a closed operator. Indeed, if $((x_n, y_n))$ is a sequence in $A + B$, $(x_n, y_n) \rightarrow (x, y)$ in $X \times X$ ($n \rightarrow \infty$), then $Bx_n \rightarrow Bx$, and therefore $Ax_n = (A + B)x_n - Bx_n \rightarrow y - Bx$, and the hypothesis that A is closed implies that $(x, y - Bx) \in A$, i.e., $x \in \text{dom}(A)$ and $y = Ax + Bx$.

(c) Let A be a closed operator. Assume that $\lambda \in \mathbb{K}$ is such that $\lambda I - A: \text{dom}(A) \rightarrow X$ is bijective. Then the inverse $(\lambda I - A)^{-1}$ is a closed operator which is defined on all of X . Therefore the closed graph theorem (for which we refer to [Yos68; II.6, Theorem 1], [Bre83; Theorem II.7]) implies that $(\lambda I - A)^{-1} \in \mathcal{L}(X)$. This implies that

$$\rho(A) = \{\lambda \in \mathbb{K}; \lambda I - A: \text{dom}(A) \rightarrow X \text{ bijective}\}.$$

(d) Usually, in treatments of operator theory the above notions are only defined for the case of complex Banach spaces. The reason is that many important results of spectral theory depend on complex analysis of one variable. For our purpose it is – for the moment – possible and convenient to include the case of real scalars.

Before proceeding we include a piece of notation that will be used in different contexts. If (M, d) is a metric space, $x \in M$ and $r \in (0, \infty]$, then

$$B(x, r) := \{y \in M; d(y, x) < r\} \quad \text{and} \quad B[x, r] := \{y \in M; d(y, x) \leq r\}$$

are the **open ball** and **closed ball** with centre x and radius r , respectively. If necessary, we might also write $B_M(x, r)$ and $B_M[x, r]$ for making it clear in which metric space we are.

The following theorem contains the basic results concerning the resolvent.

2.2 Theorem. *Let A be a closed operator in X .*

- (a) *If $\lambda \in \rho(A)$, $x \in \text{dom}(A)$, then $AR(\lambda, A)x = R(\lambda, A)Ax$.*
- (b) *For all $\lambda, \mu \in \rho(A)$ one has the **resolvent equation***

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A).$$

- (c) *For $\lambda \in \rho(A)$ one has $B(\lambda, \frac{1}{\|R(\lambda, A)\|}) \subseteq \rho(A)$, and for $\mu \in B(\lambda, \frac{1}{\|R(\lambda, A)\|})$ one has*

$$R(\mu, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1}. \quad (2.1)$$

As a consequence, $\rho(A)$ is an open subset of \mathbb{K} , and $R(\cdot, A): \rho(A) \rightarrow \mathcal{L}(X)$ is analytic.

2.3 Remarks. (a) For the proof of Theorem 2.2 we recall the **Neumann series**: If $B \in \mathcal{L}(X)$ satisfies $\|B\| < 1$, then $I - B$ is invertible in $\mathcal{L}(X)$, and the inverse is given by $(I - B)^{-1} = \sum_{n=0}^{\infty} B^n$, with absolute convergence of the series.

(b) If $A \in \mathcal{L}(X)$ and $\lambda \in \mathbb{K}$ with $|\lambda| > \|A\|$, then part (a) implies that $\lambda I - A = \lambda(I - \frac{1}{\lambda}A)$ is invertible in $\mathcal{L}(X)$, with inverse

$$(\lambda I - A)^{-1} = \sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}} A^j.$$

As a consequence, $\{\lambda \in \mathbb{K}; |\lambda| > \|A\|\} \subseteq \rho(A)$.

Proof of Theorem 2.2. (a) $AR(\lambda, A)x - \lambda R(\lambda, A)x = -x = R(\lambda, A)Ax - R(\lambda, A)\lambda x$.

(b) Multiplying the equation

$$(\mu I - A) - (\lambda I - A) = (\mu - \lambda)I|_{\text{dom}(A)}$$

from the right by $R(\lambda, A)$ and from the left by $R(\mu, A)$, one obtains the resolvent equation.

(c) Let $\lambda \in \rho(A)$ and $\mu \in B(\lambda, \frac{1}{\|R(\lambda, A)\|})$. Then the operator $I - (\lambda - \mu)R(\lambda, A)$ is invertible in $\mathcal{L}(X)$ since $|\lambda - \mu|\|R(\lambda, A)\| < 1$ (Neumann series). Therefore the equality

$$\mu I - A = (\lambda I - A) - (\lambda - \mu)I = (I - (\lambda - \mu)R(\lambda, A))(\lambda I - A)$$

shows that the mapping $\mu I - A: \text{dom}(A) \rightarrow X$ is bijective, and hence $\mu \in \rho(A)$. Moreover,

$$R(\mu, A) = R(\lambda, A)(I - (\lambda - \mu)R(\lambda, A))^{-1},$$

and the formula (2.1) for the resolvent is then a consequence of the Neumann series. Now it follows that $\rho(A)$ is an open subset of \mathbb{K} , and the analyticity of $R(\cdot, A)$ (meaning that $R(\cdot, A)$ can be written as a power series about every point of $\rho(A)$) is a consequence of (2.1) as well. \square

2.4 Remarks. (a) Theorem 2.2(c) shows that $\|R(\lambda, A)\| \geq \text{dist}(\lambda, \sigma(A))^{-1}$ for all $\lambda \in \rho(A)$. This implies that the norm of the resolvent has to blow up if λ approaches $\sigma(A)$.

(b) As in first year analysis, the analyticity of $R(\cdot, A)$ implies that $R(\cdot, A)$ is infinitely differentiable, and from the power series (2.1) one can read off the derivatives,

$$\left(\frac{d}{d\lambda}\right)^n R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1} \quad (\lambda \in \rho(A), n \in \mathbb{N}_0).$$

2.1.2 Integration of operator valued functions, and improper integrals

2.5 Proposition. *Let X, Y be Banach spaces, $a, b \in \mathbb{R}$, $a < b$. Let $F: [a, b] \rightarrow \mathcal{L}(X, Y)$ be strongly continuous, and assume that $h: [a, b] \rightarrow [0, \infty)$ is an integrable function such that $\|F(t)\| \leq h(t)$ ($a \leq t \leq b$). Then the mapping*

$$X \ni x \mapsto \int_a^b F(t)x \, dt \in Y$$

belongs to $\mathcal{L}(X, Y)$ and has norm less or equal $\int_a^b h(t) \, dt$.

Some comments: Writing $\mathcal{L}(X, Y)$ we tacitly assume that the two Banach spaces are over the same scalar field. **Strongly continuous** means that $t \mapsto F(t)x$ is continuous for all $x \in X$. (In other words, it means that F is continuous with respect to the **strong operator topology** on $\mathcal{L}(X, Y)$, which is defined as the initial topology with respect to the family of mappings $(\mathcal{L}(X, Y) \ni A \mapsto Ax \in Y)_{x \in X}$.)

Proof of Proposition 2.5. The linearity of the mapping is obvious. For $x \in X$ we estimate

$$\left\| \int_a^b F(t)x \, dt \right\| \leq \int_a^b \|F(t)x\| \, dt \leq \int_a^b h(t) \, dt \|x\|,$$

and this shows the norm estimate for the mapping. \square

Abbreviating, we will write $\int_a^b F(t) \, dt$ for the mapping defined in Proposition 2.5. This integral is called the **strong integral**; one has to keep in mind that, in general, it is not an integral of the $\mathcal{L}(X, Y)$ -valued function as treated in Subsection 1.3.2.

We will also need ‘improper integrals’ of continuous Banach space valued functions. For simplicity we restrict our attention to integrals over $[0, \infty)$ (because this is what will be needed next).

2.6 Proposition. *Let X be a Banach space, $f: [0, \infty) \rightarrow X$ continuous, and assume that there exists an integrable function $g: [0, \infty) \rightarrow [0, \infty)$ such that $\|f(t)\| \leq g(t)$ ($0 \leq t < \infty$). Then*

$$\int_0^\infty f(t) \, dt := \lim_{c \rightarrow \infty} \int_0^c f(t) \, dt$$

exists.

We omit the (easy) proof of this proposition and mention that Proposition 2.5 has its analogue for these improper integrals.

2.2 Characterisation of generators of C_0 -semigroups

In this section let X be a Banach space.

2.7 Theorem. *Let T be a C_0 -semigroup on X , and let A be its generator. Let $M \geq 1$, $\omega \in \mathbb{R}$ be such that*

$$\|T(t)\| \leq Me^{\omega t} \quad (t \geq 0)$$

(see Proposition 1.4).

Then $\{\lambda \in \mathbb{K}; \operatorname{Re} \lambda > \omega\} \subseteq \rho(A)$, and for all $\lambda \in \mathbb{K}$ with $\operatorname{Re} \lambda > \omega$ one has

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) \, dt \quad (\text{strong improper integral; see Subsection 2.1.2}),$$

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re} \lambda - \omega)^n} \quad (n \in \mathbb{N}).$$

In the proof we will use the concept of rescaling. If T is a C_0 -semigroup on X with generator A , and $\lambda \in \mathbb{K}$, then it is easy to see that T_λ , defined by

$$T_\lambda(t) := e^{-\lambda t} T(t) \quad (t \geq 0),$$

is also a C_0 -semigroup, called a **rescaled semigroup**, and that the generator of T_λ is given by $A - \lambda I$; see Exercise 2.2.

Proof of Theorem 2.7. Let $\lambda \in \mathbb{K}$, $\operatorname{Re} \lambda > \omega$. Observe that the rescaled semigroup T_λ obeys the estimate

$$\|T_\lambda(t)\| \leq M e^{(\omega - \operatorname{Re} \lambda)t} \quad (t \geq 0)$$

and that the resolvent of A at λ corresponds to the resolvent of $A - \lambda I$ at 0. This means that it is sufficient to prove the existence and the formula of the resolvent for the case $\lambda = 0$ and $\omega < 0$.

The estimate $\|T(t)\| \leq M e^{\omega t}$ ($t \geq 0$) implies that the strong improper integral

$$R := \int_0^\infty T(t) dt$$

defines an operator $R \in \mathcal{L}(X)$. Let $x \in \operatorname{dom}(A)$. Then

$$RAx = \int_0^\infty T(t)Ax dt = \lim_{c \rightarrow \infty} \int_0^c \frac{d}{dt} T(t)x dt = \lim_{c \rightarrow \infty} (T(c)x - x) = -x.$$

Further, $\|T(t)x\| \leq M e^{\omega t} \|x\|$ and $\|AT(t)x\| \leq M e^{\omega t} \|Ax\|$ ($t \geq 0$), and therefore Theorem 1.8(b) (Hille's theorem), which also holds in the present context, implies that $Rx \in \operatorname{dom}(A)$ and

$$ARx = \int_0^\infty AT(t)x dt = \int_0^\infty T(t)Ax dt = RAx = -x.$$

If $x \in X$, and (x_n) is a sequence in $\operatorname{dom}(A)$ with $x = \lim_{n \rightarrow \infty} x_n$, then $Rx_n \rightarrow Rx$ and $ARx_n = -x_n \rightarrow -x$ ($n \rightarrow \infty$), and because A is closed we conclude that $Rx \in \operatorname{dom}(A)$ and $ARx = -x$. The two equations $RA = -I|_{\operatorname{dom}(A)}$, $AR = -I$ imply that $0 \in \rho(A)$ and $R = (-A)^{-1}$.

For the powers of $R(\lambda, A)$ we now obtain (recall Remark 2.4(b))

$$\begin{aligned} R(\lambda, A)^n &= (-1)^{n-1} \frac{1}{(n-1)!} \left(\frac{d}{d\lambda} \right)^{n-1} \int_0^\infty e^{-\lambda t} T(t) dt \\ &= \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} T(t) dt. \end{aligned} \tag{2.2}$$

(The last equality is obtained by differentiation under the integral; Exercise 2.1. See also the subsequent Remark 2.8.) By Proposition 2.5 this yields the estimate

$$\begin{aligned} \|R(\lambda, A)^n\| &\leq \frac{1}{(n-1)!} M \int_0^\infty t^{n-1} e^{(\omega - \operatorname{Re} \lambda)t} dt \\ &= \frac{1}{(n-1)!} M \left(\frac{d}{d\omega} \right)^{n-1} \int_0^\infty e^{(\omega - \operatorname{Re} \lambda)t} dt \\ &= \frac{1}{(n-1)!} M \left(\frac{d}{d\omega} \right)^{n-1} \frac{1}{\operatorname{Re} \lambda - \omega} = \frac{M}{(\operatorname{Re} \lambda - \omega)^n}. \end{aligned} \quad \square$$

2.8 Remark. In the following we will mainly be interested in the case where $M = 1$ in the estimate for the C_0 -semigroup, in which case the C_0 -semigroup is called **quasi-contractive**. For such semigroups it is sufficient to prove the estimate for the resolvent

in Theorem 2.7 for $n = 1$ (because then taking powers one obtains the estimate for all $n \in \mathbb{N}$). For $n = 1$ the second equality in (2.2) is trivial.

In Exercise 2.3 one can find a method how to reduce the proof for the estimates for the resolvents to the case of contractive C_0 -semigroups.

Next, we are going to show that the necessary conditions for the generator are also sufficient. We restrict ourselves to the quasi-contractive case and delegate the general case to Exercise 2.4.

2.9 Theorem. (*Theorem of Hille-Yosida, quasi-contractive case*) *Let A be a closed, densely defined operator in X . Assume that there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega} \quad (\lambda \in (\omega, \infty)).$$

Then A is the generator of a C_0 -semigroup T satisfying the estimate

$$\|T(t)\| \leq e^{\omega t} \quad (t \geq 0).$$

As a preliminary remark we note that it is sufficient to treat the case $\omega = 0$. Indeed, defining $\tilde{A} := A - \omega I$ we see that \tilde{A} satisfies the conditions of Theorem 2.9 with $\omega = 0$. Having obtained the contractive C_0 -semigroup \tilde{T} with generator \tilde{A} one obtains the C_0 -semigroup generated by $A = \tilde{A} + \omega I$ as the rescaled semigroup $\tilde{T}_{-\omega}$.

We now define the **Yosida approximations**

$$A_n := A \left(I - \frac{1}{n} A \right)^{-1} = nAR(n, A) = n^2R(n, A) - nI \in \mathcal{L}(X) \quad (n \in \mathbb{N})$$

of A . The proof of Theorem 2.9 will consist of three steps:

In the first step we show that the semigroups generated by A_n are contractive.

In the second step we show that these semigroups converge strongly to a C_0 -semigroup.

In the third step we show that A is the generator of the limiting semigroup.

The following lemma justifies that the operators A_n can be considered as approximations of A .

2.10 Lemma. *Let A be a closed, densely defined operator in X . Assume that there exists $\lambda_0 > 0$ such that $(\lambda_0, \infty) \subseteq \rho(A)$ and that $M := \sup_{\lambda > \lambda_0} \|\lambda R(\lambda, A)\| < \infty$. Then:*

(a) $\lambda R(\lambda, A)x \rightarrow x$ ($\lambda \rightarrow \infty$) for all $x \in X$.

(b) $A(\lambda R(\lambda, A))x \rightarrow Ax$ ($\lambda \rightarrow \infty$) for all $x \in \text{dom}(A)$.

Proof. (a) If $x \in \text{dom}(A)$, then

$$\lambda R(\lambda, A)x = (\lambda I - A + A)R(\lambda, A)x = x + R(\lambda, A)Ax \rightarrow x \quad (\lambda \rightarrow \infty).$$

As $\|\lambda R(\lambda, A)\| \leq M$ ($\lambda > \lambda_0$) and $\text{dom}(A)$ is dense, the convergence carries over to all $x \in X$, by Proposition 1.6.

(b) For $x \in \text{dom}(A)$ the convergence proved in part (a) implies

$$A(\lambda(\lambda I - A)^{-1})x = \lambda(\lambda I - A)^{-1}Ax \rightarrow Ax \quad (\lambda \rightarrow \infty). \quad \square$$

Proof of Theorem 2.9. Recall that, without loss of generality, we only treat the case $\omega = 0$.

(i) For $n \in \mathbb{N}$, $t \geq 0$ we obtain the estimate

$$\|e^{tA_n}\| = \|e^{t(n^2R(n,A)-nI)}\| = e^{-tn} \left\| \sum_{k=0}^{\infty} \frac{(tn^2R(n,A))^k}{k!} \right\| \leq e^{-tn} \sum_{k=0}^{\infty} \frac{(tn)^k}{k!} = 1.$$

(ii) For $x \in X$, $t > 0$ and $m, n \in \mathbb{N}$ we compute

$$\begin{aligned} e^{tA_m}x - e^{tA_n}x &= \int_0^t \frac{d}{ds} (e^{(t-s)A_n} e^{sA_m} x) ds = \int_0^t e^{(t-s)A_n} (A_m - A_n) e^{sA_m} x ds \\ &= \int_0^t e^{(t-s)A_n} e^{sA_m} (A_m - A_n) x ds \end{aligned}$$

(where in the last equality we have used that A_m, A_n as well as the generated semigroups commute). Recalling part (i) we obtain the estimate

$$\|e^{tA_m}x - e^{tA_n}x\| \leq t \|(A_m - A_n)x\|. \quad (2.3)$$

Let $c > 0$. For $n \in \mathbb{N}$ we define the operator $\mathcal{T}_n^c: X \rightarrow C([0, c]; X)$ (where $C([0, c]; X)$ denotes the Banach space of continuous X -valued functions, equipped with the supremum norm) by

$$\mathcal{T}_n^c x := [t \mapsto e^{tA_n} x] \quad (x \in X).$$

Then part (i) of the proof shows that \mathcal{T}_n^c is a contraction, and inequality (2.3) shows that

$$\|\mathcal{T}_m^c x - \mathcal{T}_n^c x\| \leq c \|A_m x - A_n x\| \quad (m, n \in \mathbb{N}),$$

for all $x \in \text{dom}(A)$, which implies that $(\mathcal{T}_n^c x)_{n \in \mathbb{N}}$ is a Cauchy sequence, because $(A_n x)_{n \in \mathbb{N}}$ is convergent (to Ax). Applying Proposition 1.6 we conclude that there exists $\mathcal{T}^c \in \mathcal{L}(X, C([0, c]; X))$ such that $\mathcal{T}_n^c \rightarrow \mathcal{T}^c$ ($n \rightarrow \infty$) strongly.

Clearly, if $0 < c < c'$, then $\mathcal{T}^{c'} x|_{[0, c]} = \mathcal{T}^c x$ for all $x \in X$, and therefore we can define $T: [0, \infty) \rightarrow \mathcal{L}(X)$ by

$$T(t)x := \mathcal{T}^c x(t) \quad (0 \leq t < c, x \in X).$$

From $T(\cdot)x|_{[0, c]} = \mathcal{T}^c x$ ($c > 0$, $x \in X$) we infer that T is strongly continuous. Since $T(t) = \text{s-lim}_{n \rightarrow \infty} e^{tA_n}$ ($t \geq 0$), the semigroup property carries over from the semigroups $(e^{tA_n})_{t \geq 0}$ to T . As a result, T is a C_0 -semigroup of contractions.

(iii) Let B be the generator of T . Let $x \in \text{dom}(A)$. Using the notation of part (ii), with $c := 1$, we see that $\mathcal{T}_n^1 x \rightarrow \mathcal{T}^1 x$ and $(\mathcal{T}_n^1 x)' = \mathcal{T}_n^1 A_n x \rightarrow \mathcal{T}^1 A x$ ($n \rightarrow \infty$) in $C([0, 1]; X)$ (recall Lemma 1.11). This implies that $T(\cdot)x|_{[0, 1]} = \mathcal{T}^1 x$ is differentiable with continuous derivative $\mathcal{T}^1 A x$, and therefore $x \in \text{dom}(B)$ and $Bx = Ax$.

So far we have shown that $A \subseteq B$. We also know that $(0, \infty) \subseteq \rho(B)$, by Theorem 2.7, and that $(0, \infty) \subseteq \rho(A)$, by hypothesis. Now from $I - A \subseteq I - B$, the injectivity of $I - B$ and $\text{ran}(I - A) = X$ we obtain $I - A = I - B$, and hence $A = B$. \square

2.11 Remarks. (a) The proof of the Hille-Yosida theorem for the general case can be given along the same lines; see Exercise 2.4.

(b) The Yosida approximations for the general case are defined in the same way as for the contractive case as treated in our proof. We point out that the proof given above does not show that the semigroups generated by the Yosida approximations for the non-rescaled semigroup approximate the non-rescaled semigroup (although it is true!). This might seem cumbersome, because sometimes one wants to prove properties of the semigroup using properties of the approximating semigroups. However, in the following section we will show another way of approximation which is similarly (or even more) effective.

(c) As an interesting feature in the proof of Theorem 2.9 we point out that the approximating semigroups are norm continuous, whereas in general the resulting semigroup is only strongly continuous. The norm continuity is lost, obviously, because the convergence of the approximating semigroups is only strong (even though the strong convergence is uniform on bounded intervals).

2.3 An exponential formula

Given $a \in \mathbb{K}$, besides the exponential series there are two well-known ways to approximate e^{ta} , namely

$$e^{ta} = \lim_{n \rightarrow \infty} \left(1 + \frac{t}{n}a\right)^n \quad \text{and} \quad e^{ta} = \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n}a\right)^{-n}.$$

Trying to replace a by an unbounded generator in the first formula leads to hopeless problems with the domains of the powers of the operators involved, whereas the second formula looks more promising because the occurring inverses are just those whose existence is guaranteed by Theorem 2.7. (In fact, the resulting formulas are those known in numerical mathematics as ‘backward Euler method’.)

We mention that the proof of the ‘exponential formula’ presented in this section does not depend on the Hille-Yosida theorem.

2.12 Theorem. *Let X be a Banach space, T a C_0 -semigroup on X , and A its generator. Then*

$$T(t)x = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n}A\right)^{-n} x$$

for all $x \in X$, with uniform convergence on compact intervals.

2.13 Remarks. (a) If M and ω are such that $\|T(t)\| \leq Me^{\omega t}$ ($t \geq 0$), then $(\omega, \infty) \subseteq \rho(A)$, by Theorem 2.7. Assume that $\omega > 0$, and let $c > 0$. If $0 < t \leq c$, then $\frac{n}{t} \geq \frac{n}{c}$, therefore $\frac{n}{t} \in \rho(A)$ if $n > c\omega$, and then

$$\left(I - \frac{t}{n}A\right)^{-1} = \frac{n}{t} \left(\frac{n}{t}I - A\right)^{-1} \in \mathcal{L}(X).$$

This means that the operator $(I - \frac{t}{n}A)^{-n}$ is defined for all $t \in [0, c]$ only for sufficiently large n , depending on c .

This problem does not occur if $\omega \leq 0$.

(b) From elementary analysis we will need the fact that the convergence $(1-t/n)^{-n} \rightarrow e^t$ as $n \rightarrow \infty$ is uniform for t in compact subsets of \mathbb{R} .

(c) As a third remark we note that the expressions $(I - rA)^{-1} = (1/r)((1/r)I - A)^{-1}$, for small $r > 0$ correspond to expressions $\lambda(\lambda I - A)^{-1}$ for large $\lambda > 0$. This implies that the behaviour of $(I - rA)^{-1}$ for $r \rightarrow 0$ is the same as that of $\lambda(\lambda I - A)^{-1}$ for $\lambda \rightarrow \infty$.

Proof of Theorem 2.12. Let $M \geq 1$, $\omega \in \mathbb{R}$, $c > 0$ and $n > c\omega$ be as in Remark 2.13(a). For $0 < t \leq c$ we compute, using Remark 2.4(b),

$$\begin{aligned} \frac{d}{dt} \left(I - \frac{t}{n} A \right)^{-1} &= \frac{d}{dt} \left(\frac{n}{t} \left(\frac{n}{t} I - A \right)^{-1} \right) = -\frac{n}{t^2} \left(\frac{n}{t} I - A \right)^{-1} + \frac{n}{t} \cdot \frac{n}{t^2} \left(\frac{n}{t} I - A \right)^{-2} \\ &= \left(-\frac{n^2}{t^3} I + \frac{n}{t^2} A + \frac{n^2}{t^3} I \right) \left(\frac{n}{t} I - A \right)^{-2} = \frac{1}{n} A \left(I - \frac{t}{n} A \right)^{-2}, \end{aligned}$$

$$\frac{d}{dt} \left(I - \frac{t}{n} A \right)^{-n} = n \left(\left(I - \frac{t}{n} A \right)^{-1} \right)^{n-1} \frac{1}{n} A \left(I - \frac{t}{n} A \right)^{-2} = A \left(I - \frac{t}{n} A \right)^{-n-1}.$$

(In the last computation the product rule for differentiation is used. The result of this computation should not be too surprising; if A is a number, then it is a consequence of the chain rule.)

Let $x \in \text{dom}(A)$. Then the function $[0, t] \ni s \mapsto T(t-s) \left(I - \frac{s}{n} A \right)^{-n} x$ is continuous as well as continuously differentiable on $(0, t)$, with

$$\begin{aligned} \frac{d}{ds} \left(T(t-s) \left(I - \frac{s}{n} A \right)^{-n} x \right) &= T(t-s) \left(-A + A \left(I - \frac{s}{n} A \right)^{-1} \right) \left(I - \frac{s}{n} A \right)^{-n} x \\ &= T(t-s) \left(I - \frac{s}{n} A \right)^{-n} \left(\left(I - \frac{s}{n} A \right)^{-1} - I \right) Ax. \end{aligned}$$

By the fundamental theorem of calculus (see Theorem 1.9) it follows that

$$\begin{aligned} &\left\| \left(I - \frac{t}{n} A \right)^{-n} x - T(t)x \right\| \\ &= \left\| \int_0^t T(t-s) \left(I - \frac{s}{n} A \right)^{-n} \left(\left(I - \frac{s}{n} A \right)^{-1} - I \right) Ax \, ds \right\| \\ &\leq c \sup_{0 < s < t \leq c} \left\| T(t-s) \left(I - \frac{s}{n} A \right)^{-n} \right\| \sup_{0 < s < c} \left\| \left(\left(I - \frac{s}{n} A \right)^{-1} - I \right) Ax \right\|. \end{aligned} \tag{2.4}$$

The first of these suprema is estimated by

$$\begin{aligned} \left\| T(t-s) \left(I - \frac{s}{n} A \right)^{-n} \right\| &\leq M^2 e^{\omega(t-s)} \left(1 - \frac{s}{n} \omega \right)^{-n} \\ &\leq M_0 := M^2 \max\{1, e^{\omega c}\} \sup_{0 < s < c, n > c\omega} \left(1 - \frac{s}{n} \omega \right)^{-n} < \infty, \end{aligned}$$

where the finiteness of the last term is a consequence of Remark 2.13(b). Now (2.4) yields

$$\sup_{0 \leq t \leq c} \left\| \left(I - \frac{t}{n} A \right)^{-n} x - T(t)x \right\| \leq c M_0 \sup_{0 < s < c} \left\| \left(\left(I - \frac{s}{n} A \right)^{-1} - I \right) Ax \right\|,$$

which tends to 0 as $n \rightarrow \infty$, by Lemma 2.10(a) (recall also Remark 2.13(c)).

Now the proof is completed pretty much as in part (ii) of the proof of Theorem 2.9. For $n > c\omega$ we define $\mathcal{T}_n^c: X \rightarrow C([0, c]; X)$,

$$\mathcal{T}_n^c x := [t \mapsto (I - \frac{t}{n}A)^{-n}x] \quad (x \in X).$$

Then $\|\mathcal{T}_n^c\| \leq M_0$ for all $n > c\omega$, and $\mathcal{T}_n^c x \rightarrow T(\cdot)x|_{[0, c]}$ ($n \rightarrow \infty$) for all $x \in \text{dom}(A)$, as shown above. Using that $\text{dom}(A)$ is dense and applying Proposition 1.6 we obtain $\mathcal{T}_n^c x \rightarrow T(\cdot)x|_{[0, c]}$ ($n \rightarrow \infty$) for all $x \in X$. \square

2.14 Remark. An important application of the exponential formula will be the invariance of a closed subset $M \subseteq X$ under a C_0 -semigroup. If M is invariant under $(I - rA)^{-1}$ for small $r > 0$, then Theorem 2.12 implies that M is invariant under T .

Notes

The section on resolvents etc. is pretty standard and can be found in any book on functional analysis treating fundamentals of operator theory. The Hille-Yosida theorem is basic for C_0 -semigroups and can be found (with varying proofs) in any treatment of C_0 -semigroups. Our proof is Yosida's original proof in [Yos48]. The exponential formula, Theorem 2.12, can be found in [HP57; Theorem 11.6.6] or in [Paz83; Theorem 1.8.3], with proofs different from ours. A proof of the Hille-Yosida theorem can also be given using the exponential formula; see [Kat80; IX, Sections 1.2 and 1.3]. It was found by Hille, independently of Yosida, and published in [Hil48]. It is worth mentioning that the exponential formula can also be derived from the Chernoff product formula, for which we refer to [EN00; III, Theorem 5.2]. The proof of the Chernoff product formula requires results from the perturbation theory of C_0 -semigroups.

Exercises

2.1 Show the second equality in (2.2).

2.2 Let T be a C_0 -semigroup on the Banach space X , with generator A . Let $\lambda \in \mathbb{K}$. Show that

$$T_\lambda(t) := e^{-\lambda t}T(t) \quad (t \geq 0),$$

defines a C_0 -semigroup (the rescaled semigroup), and that the generator of T_λ is given by $A - \lambda I$.

2.3 Let T be a bounded C_0 -semigroup on the Banach space X , $M := \sup_{t \geq 0} \|T(t)\|$, and let A be its generator.

(a) Show that

$$\| \|x\| := \sup_{t \geq 0} \|T(t)x\| \quad (x \in X)$$

defines a norm $\| \| \cdot \|$ on X which is equivalent to $\| \cdot \|$, and that T is a C_0 -semigroup of contractions on $(X, \| \| \cdot \|)$.

(b) For any $\alpha_1, \dots, \alpha_n > 0$, show that

$$\|(I - \alpha_1 A)^{-1} \cdots (I - \alpha_n A)^{-1}\| \leq M.$$

2.4 Show the Hille-Yosida theorem for the general case:

Let A be a closed, densely defined operator in the Banach space X . Assume that there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n} \quad (\lambda \in (\omega, \infty), n \in \mathbb{N}).$$

Then A is the generator of a C_0 -semigroup T satisfying the estimate

$$\|T(t)\| \leq Me^{\omega t} \quad (t \geq 0).$$

(Hint: Proceed as in the proof of Theorem 2.9, with adapted estimates.)

2.5 Let T be a C_0 -semigroup on the Banach space X . For $h > 0$ we define $A_h := h^{-1}(T(h) - I)$. Show that $e^{tA_h}x \rightarrow T(t)x$ for all $x \in X$ as $h \rightarrow 0$, uniformly for t in compact subsets of $[0, \infty)$. (Hint: Use a procedure similar to the proof of the exponential formula, Theorem 2.12.)

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Lecture 3

Holomorphic semigroups

The objective of this lecture is to introduce semigroups for which the ‘time parameter’ t can also be chosen in a complex neighbourhood of the positive reals. The foundations of these ‘holomorphic semigroups’ will be given in Section 3.2. The generation of such semigroups will be studied in Section 3.3. And in Section 3.4 the special case of holomorphic semigroups on Hilbert spaces will be treated. We start with an interlude on Banach space valued holomorphy.

3.1 Interlude: vector-valued holomorphic functions

In this section let X, Y be complex Banach spaces.

The first issue of the present section is to show that for Banach space valued functions several notions of holomorphy coincide.

Let $\Omega \subseteq \mathbb{C}$ be open, $f: \Omega \rightarrow X$. The function f is called **holomorphic** if f is (complex) differentiable (at each point of Ω). f is called **scalarly holomorphic** if $x' \circ f$ is holomorphic for all $x' \in X'$ ($= \mathcal{L}(X, \mathbb{C})$, the dual space of X). f is called **analytic** if f can be represented as a power series in a neighbourhood of each point of Ω .

3.1 Remarks. (a) It is evident that holomorphy of a function implies scalar holomorphy.

(b) Let $\Omega \subseteq \mathbb{C}$ be open, and let $f: \Omega \rightarrow X$ be holomorphic. The following facts can be shown in the same way as in the case of \mathbb{C} -valued functions.

If Ω is convex, and γ is a piecewise continuously differentiable closed path in Ω , then $\int_{\gamma} f(z) dz = 0$ (Cauchy’s integral theorem). Note that the path integral is defined by a parametrisation of the path and therefore reduces to integrals over intervals. As a consequence, path integrals enter the context explained in Subsection 1.3.2.

For $z_0 \in \Omega$, $r > 0$ such that $B[z_0, r] \subseteq \Omega$ the function f satisfies Cauchy’s integral formula

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} d\zeta \quad (z \in B(z_0, r)).$$

The function f is analytic, and one has Cauchy’s integral formulas for the derivatives,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad (z \in B(z_0, r), n \in \mathbb{N}),$$

with z_0 and r as before.

A set $E \subseteq X'$ is **separating** (for X) if for all $0 \neq x \in X$ there exists $x' \in E$ such that $x'(x) \neq 0$. The set E is called **almost norming** (for X) if

$$\|x\|_E := \sup\{|x'(x)|; x' \in E, \|x'\| \leq 1\}$$

defines a norm that is equivalent to the norm on X . E is called **norming** if $\|\cdot\|_E = \|\cdot\|$ on X . It is a consequence of the Hahn-Banach theorem that $E = X'$ is norming for X .

3.2 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be open, $f: \Omega \rightarrow X$. Then the following properties are equivalent.*

- (i) f is holomorphic.
- (ii) f is scalarly holomorphic.
- (iii) There exists an almost norming closed subspace $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$.
- (iv) f is locally bounded, and there exists an almost norming subset $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$.
- (v) f is continuous, and there exists a separating set $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$.

Proof. (i) \Rightarrow (ii) is clear (and was already noted above).

(ii) \Rightarrow (iii) is clear, with $E = X'$.

(iii) \Rightarrow (iv). With the canonical embedding $X \subseteq E' = \mathcal{L}(E, \mathbb{C})$ we can consider f as an $\mathcal{L}(E, \mathbb{C})$ -valued function. Then the uniform boundedness theorem implies that f is locally bounded with respect to the norm $\|\cdot\|_E$.

(iv) \Rightarrow (v). Since almost norming subsets are separating we only have to show that f is continuous.

Let $z_0 \in \Omega$, $r > 0$, $B[z_0, r] \subseteq \Omega$, $M := \sup\{\|f(\zeta)\|; |\zeta - z_0| = r\} (< \infty)$. For $x' \in E$, $z \in B(z_0, r/2)$ we then obtain, using Cauchy's integral formula for the derivative,

$$\left| \frac{d}{dz} x'(f(z)) \right| = \left| \frac{1}{2\pi i} \int_{|\zeta - z_0|=r} \frac{x'(f(\zeta))}{(\zeta - z)^2} d\zeta \right| \leq \frac{1}{2\pi} 2\pi r \|x'\| \frac{4M}{r^2} = \frac{4M}{r} \|x'\|.$$

For $z', z'' \in B[z_0, r/2]$ this implies

$$|x'(f(z')) - x'(f(z''))| \leq \frac{4M}{r} |z' - z''| \|x'\| \quad (x' \in E),$$

and therefore

$$\|f(z') - f(z'')\|_E \leq \frac{4M}{r} |z' - z''|.$$

Since the norm $\|\cdot\|_E$ is equivalent to the norm on X the continuity follows on $B(z_0, r/2)$.

(v) \Rightarrow (i). Let $z_0 \in \Omega$. We show that f can be expanded into a power series about z_0 .

Without loss of generality we assume $z_0 = 0$. There exists $r > 0$ such that $B[0, r] \subseteq \Omega$. For $n \in \mathbb{N}_0$ we define

$$a_n := \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta.$$

With $M := \sup\{\|f(\zeta)\|; |\zeta| = r\}$ ($< \infty$) one obtains $\|a_n\| \leq \frac{M}{r^n}$ for all $n \in \mathbb{N}_0$, so $\limsup \|a_n\|^{1/n} \leq 1/r$, and therefore the radius of convergence of the power series $g(z) := \sum_{n=0}^{\infty} z^n a_n$ is greater or equal r . For $x' \in E$, $|z| < r$ one has

$$x'(g(z)) = \sum_{n=0}^{\infty} z^n \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{x'(f(\zeta))}{\zeta^{n+1}} d\zeta = \sum_{n=0}^{\infty} \frac{(x' \circ f)^{(n)}(0)}{n!} z^n = x'(f(z)),$$

where the last equality is just the power series expansion of the holomorphic function $x' \circ f$. Since E is separating we therefore obtain $f(z) = g(z)$. \square

3.3 Remark. If X is a dual Banach space then the predual is a norming closed subspace of its bidual X' . (For instance, c_0 is norming for ℓ_1 .) This illustrates a possible application of condition (iii) of Theorem 3.2.

As a first application of Theorem 3.2 we show the identity theorem for holomorphic functions.

3.4 Corollary. *Let $\Omega \subseteq \mathbb{C}$ be open and connected, let $f: \Omega \rightarrow X$ be holomorphic, and assume that $[f = 0] = \{z \in \Omega; f(z) = 0\}$ has a cluster point in Ω . Then $f = 0$.*

Proof. For each $x' \in X'$ the zeros of the function $x' \circ f$ have a cluster point in Ω , and therefore $x' \circ f = 0$ follows from the identity theorem for \mathbb{C} -valued holomorphic functions.

From $x' \circ f = 0$ for all $x' \in X'$ one obtains $f = 0$. \square

For the experts this proof may not really seem convincing, because the identity theorem can also be concluded in the same way as for \mathbb{C} -valued functions (in the spirit of Remark 3.1(b)).

Finally we come to the characterisation of holomorphy for $\mathcal{L}(X, Y)$ -valued functions. As $\mathcal{L}(X, Y)$ is a Banach space, all previous criteria apply. However, scalar holomorphy is not a useful concept in this case, because the dual of $\mathcal{L}(X, Y)$ mostly is not easily accessible.

3.5 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be open, $F: \Omega \rightarrow \mathcal{L}(X, Y)$. Let B be a dense subset of X , and let $C \subseteq Y'$ be almost norming for Y . Then the following properties are equivalent.*

- (i) F is holomorphic (as an $\mathcal{L}(X, Y)$ -valued function).
- (ii) F is locally bounded, and $F(\cdot)x$ is holomorphic for all $x \in B$.
- (iii) F is locally bounded, and $y'(F(\cdot)x)$ is holomorphic for all $x \in B$, $y' \in C$.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (i). The hypotheses imply that the set

$$E := \{A \mapsto y'(Ax); x \in B, y' \in C\} \subseteq \mathcal{L}(X, Y)'$$

is almost norming for $\mathcal{L}(X, Y)$. Therefore Theorem 3.2, (iv) \Rightarrow (i), implies the assertion. \square

3.6 Remarks. Note that the set E in the proof of Theorem 3.5 is not a subspace of $\mathcal{L}(X, Y)'$. For this application it is convenient to have condition (iv) of Theorem 3.2 at our disposal.

The last issue of this section is the convergence of sequences of holomorphic functions.

3.7 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be open, and let (f_n) be a sequence of holomorphic functions $f_n: \Omega \rightarrow X$. Assume that (f_n) is locally bounded (i.e., for each $z_0 \in \Omega$ there exists $r > 0$ such that $B(z_0, r) \subseteq \Omega$ and $\sup\{\|f_n(z)\|; z \in B(z_0, r), n \in \mathbb{N}\} < \infty$) and that $f(z) := \lim_{n \rightarrow \infty} f_n(z)$ exists for all $z \in \Omega$.*

Then (f_n) converges to f locally uniformly, and f is holomorphic.

Proof. In the first step we show that the sequence (f_n) is locally uniformly equicontinuous. Let $z_0 \in \Omega$, $r > 0$ be such that $B[z_0, r] \subseteq \Omega$. Then $C := \sup_{z \in B[z_0, r], n \in \mathbb{N}} \|f_n(z)\| < \infty$, by hypothesis. From Cauchy's integral formula for the derivative,

$$f'_n(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta \quad (z \in B(z_0, r), n \in \mathbb{N})$$

we conclude that for all $z \in B[z_0, r/2]$ one has

$$\|f'_n(z)\| \leq C \frac{4}{r} \quad (n \in \mathbb{N}).$$

This implies that the sequence (f_n) is uniformly equicontinuous on $B[z_0, r/2]$.

The local uniform equicontinuity of the sequence (f_n) together with the pointwise convergence implies that (f_n) converges to f locally uniformly. In order to show that this implies that f is holomorphic, let again $z_0 \in \Omega$ and $r > 0$ be such that $B[z_0, r] \subseteq \Omega$. Then Cauchy's integral formula

$$f_n(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f_n(\zeta)}{\zeta - z} d\zeta,$$

valid for all $z \in B(z_0, r)$, $n \in \mathbb{N}$, carries over to f ,

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} d\zeta \quad (z \in B(z_0, r)),$$

and this implies that f is holomorphic on $B(z_0, r)$. □

3.8 Corollary. *Let $\Omega \subseteq \mathbb{C}$ be open, and let (F_n) be a sequence of holomorphic functions $F_n: \Omega \rightarrow \mathcal{L}(X, Y)$. Assume that (F_n) is locally bounded and that $F(z) := \text{s-lim}_{n \rightarrow \infty} F_n(z)$ exists for all $z \in \Omega$.*

Then F is holomorphic.

Proof. The hypotheses in combination with Theorem 3.7 imply that $F(\cdot)x$ is holomorphic for all $x \in X$. Therefore Theorem 3.5 implies that F is holomorphic. □

3.2 Holomorphic semigroups

Let X be a complex Banach space. For $\theta \in (0, \pi/2]$ we define the (open) sector

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\}; |\text{Arg } z| < \theta\} = \{re^{i\alpha}; r > 0, |\alpha| < \theta\}.$$

We will also use the notation $\Sigma_{\theta,0} := \Sigma_\theta \cup \{0\}$. A **holomorphic semigroup (of angle θ)**, if we want to make precise the angle) is a function $T: \Sigma_{\theta,0} \rightarrow \mathcal{L}(X)$, holomorphic on Σ_θ , satisfying

(i) $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_{\theta,0}$.

If additionally

(ii) $\lim_{z \rightarrow 0, z \in \Sigma_{\theta'}} T(z)x = x$ for all $x \in X$ and all $\theta' \in (0, \theta)$,

then T will be called a **holomorphic C_0 -semigroup (of angle θ)**.

Saying that T is a holomorphic semigroup we will always mean that T brings along its domain of definition, in particular, *the angle of T* is defined.

3.9 Remark. We note that the definition of a holomorphic C_0 -semigroup implies that for all $\theta' \in (0, \theta)$ there exist $M' \geq 1$, $\omega' \in \mathbb{R}$ such that

$$\|T(z)\| \leq M' e^{\omega' \operatorname{Re} z} \quad (z \in \Sigma_{\theta'});$$

see Exercise 3.3.

The following lemma states that it suffices to check the semigroup property for real times.

3.10 Lemma. *Let $\theta \in (0, \pi/2]$, and let $T: \Sigma_{\theta} \rightarrow \mathcal{L}(X)$ be holomorphic and such that $T(t+s) = T(t)T(s)$ for all $t, s > 0$.*

Then $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_{\theta}$.

Proof. Fixing $t > 0$, we know that the functions $\Sigma_{\theta} \ni z \mapsto T(t+z)$, $\Sigma_{\theta} \ni z \mapsto T(t)T(z)$ and $\Sigma_{\theta} \ni z \mapsto T(z)T(t)$ are holomorphic and coincide on $(0, \infty)$. The identity theorem, Theorem 3.4, implies that they are equal on Σ_{θ} . Repeating the argument with $t \in \Sigma_{\theta}$ we obtain the assertion. \square

3.11 Proposition. *Let T be a C_0 -semigroup on X , and assume that there exist $\theta \in (0, \pi/2]$ and an extension of T to $\Sigma_{\theta,0}$, also called T , holomorphic on Σ_{θ} and satisfying*

$$\sup_{z \in \Sigma_{\theta}, |z| < 1} \|T(z)\| < \infty.$$

Then $\lim_{z \rightarrow 0, z \in \Sigma_{\theta}} T(z)x = x$ for all $x \in X$, and T is a holomorphic C_0 -semigroup.

Proof. First note that Lemma 3.10 implies property (i) from above.

Let $x \in D := \bigcup_{t>0} \operatorname{ran}(T(t))$, i.e., there exist $y \in X$, $t > 0$ such that $x = T(t)y$. Then the continuity of $z \mapsto T(z)x = T(z)T(t)y = T(z+t)y$ at 0 implies $\lim_{z \rightarrow 0, z \in \Sigma_{\theta}} T(z)x = x$.

Note that D is dense, because $T(t) \rightarrow I$ strongly as $t \rightarrow 0$. Therefore the boundedness assumption implies the assertion (recall Proposition 1.6). \square

3.12 Remarks. Let T be a holomorphic C_0 -semigroup of angle $\theta \in (0, \pi/2]$.

(a) Then for each $\alpha \in (-\theta, \theta)$ the mapping $[0, \infty) \ni t \mapsto T_{\alpha}(t) := T(e^{i\alpha t})$ is a C_0 -semigroup.

(b) The set $D := \bigcup_{z \in \Sigma_{\theta}} \operatorname{ran}(T(z))$ is a dense subspace of X . The denseness follows from property (ii). In order to show that D is a vector space we first note that $\operatorname{ran}(T(z_1 + z_2)) \subseteq \operatorname{ran}(T(z_1))$ for all $z_1, z_2 \in \Sigma_{\theta}$. Further, it is not difficult to see that for all $z_1, z_2 \in \Sigma_{\theta}$ there exists $z \in \Sigma_{\theta}$ such that $z_1 - z, z_2 - z \in \Sigma_{\theta}$, and then it follows that $\operatorname{ran}(T(z_j)) \subseteq \operatorname{ran}(T(z))$ for $j = 1, 2$.

3.13 Theorem. *Let T be a holomorphic C_0 -semigroup of angle $\theta \in (0, \pi/2]$, and let A be the generator of T_0 ($= T|_{[0, \infty)}$). Then:*

- (a) *For all $\alpha \in (-\theta, \theta)$ the generator of T_α is given by $e^{i\alpha}A$.*
- (b) *For all $x \in \text{dom}(A)$, $\theta' \in (0, \theta)$ one has*

$$\lim_{z \rightarrow 0, z \in \Sigma_{\theta'}} \frac{1}{z} (T(z)x - x) = Ax.$$

Proof. (a) Let $\alpha \in (-\theta, \theta)$, and denote the generator of the C_0 -semigroup T_α (see Remark 3.12(a)) by A_α . Let $D := \bigcup_{z \in \Sigma_\theta} \text{ran}(T(z))$. Then the differentiability of T on Σ_θ implies that $D \subseteq \text{dom}(A_\alpha)$, and one easily computes that $A_\alpha x = e^{i\alpha}Ax$ for all $x \in D$. Also it follows from the definition of D that D is invariant under T_α . Therefore Proposition 1.14 (together with Remark 3.12(b)) implies that D is a core for A_α . In particular, this holds for $\alpha = 0$, and therefore one obtains $A_\alpha = e^{i\alpha}A$.

- (b) Let $x \in \text{dom}(A)$, $\theta' \in (0, \theta)$. For $z \in \Sigma_{\theta'}$ one then obtains

$$\frac{1}{z} (T(z)x - x) = \frac{1}{z} \int_0^1 \frac{d}{ds} T(sz)x \, ds = \int_0^1 T(sz)Ax \, ds.$$

In view of the continuity at 0 of the restriction of $T(\cdot)Ax$ to $\Sigma_{\theta', 0}$ one therefore obtains the assertion. \square

Let T be a holomorphic C_0 -semigroup. In the light of Theorem 3.13 it is justified to call the generator of the C_0 -semigroup $T|_{[0, \infty)}$ also the **generator** of the holomorphic C_0 -semigroup T . Clearly, the application of Theorem 2.7 yields estimates for the resolvents of the generator of T . We will not pursue this issue but restrict our attention to a special kind of holomorphic semigroups.

3.3 Generation of contractive holomorphic semigroups

As before, X will be a complex Banach space. We will call a holomorphic semigroup of angle θ **contractive** if $\|T(z)\| \leq 1$ for all $z \in \Sigma_{\theta, 0}$.

The following theorem characterises the generation of contractive holomorphic C_0 -semigroups.

3.14 Theorem. *An operator A is the generator of a contractive holomorphic C_0 -semigroup of angle $\theta \in (0, \pi/2]$ if and only if it is closed, densely defined, and $\Sigma_\theta \subseteq \rho(A)$, with*

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{|\lambda|} \quad (\lambda \in \Sigma_\theta).$$

Proof. The necessity of the condition is an almost immediate consequence of Theorem 3.13 and Theorem 2.7. ‘Closed’ and ‘densely defined’ are clear. For $\alpha \in (-\theta, \theta)$, we know from Theorem 3.13(a) that $e^{i\alpha}A$ is the generator of a contractive C_0 -semigroup; therefore $(0, \infty) \subseteq \rho(e^{i\alpha}A)$ and $\|(\lambda I - e^{i\alpha}A)^{-1}\| \leq 1/\lambda$ for all $\lambda > 0$, by Theorem 2.7. The equation $(\lambda I - e^{i\alpha}A) = e^{i\alpha}(e^{-i\alpha}\lambda I - A)$ then shows that $\{e^{-i\alpha}\lambda; \lambda > 0\} \subseteq \rho(A)$ and $\|(e^{-i\alpha}\lambda I - A)^{-1}\| \leq 1/\lambda$ for all $\lambda > 0$. This finishes the proof of the necessity.

In the proof of the sufficiency we will employ the exponential formula, Theorem 2.12. For $n \in \mathbb{N}$ we define the holomorphic function $F_n: \Sigma_\theta \rightarrow \mathcal{L}(X)$,

$$F_n(z) := (I - \frac{z}{n}A)^{-n}.$$

Then the hypotheses imply that $\|F_n(z)\| \leq 1$ for all $z \in \Sigma_\theta$, $n \in \mathbb{N}$. It also follows from the hypotheses and Theorem 2.9 that for each $\alpha \in (-\theta, \theta)$ the operator $e^{i\alpha}A$ generates a contractive C_0 -semigroup; call it T_α . Let $z \in \Sigma_\theta$, $z = e^{i\alpha}t$ with suitable $t > 0$, $\alpha \in (-\theta, \theta)$. Then $F_n(z) = (I - \frac{t}{n}e^{i\alpha}A)^{-n} \rightarrow T_\alpha(t)$ strongly as $n \rightarrow \infty$, by Theorem 2.12, so

$$T(z) := \text{s-lim}_{n \rightarrow \infty} F_n(z)$$

exists for all $z \in \Sigma_{\theta,0}$. From Corollary 3.8 we derive that T is holomorphic on Σ_θ and from Theorem 2.12 that $T|_{[0,\infty)}$ is the C_0 -semigroup generated by A . \square

We add some comments on the generation of holomorphic semigroups which are not necessarily contractive.

3.15 Remarks. (a) The same proof as given for Theorem 3.14 can be used to show the following equivalence.

Let $\theta \in (0, \pi/2]$, $M \geq 1$. Then an operator A is the generator of a **bounded holomorphic C_0 -semigroup** of angle θ and with bound M if and only if for each $\alpha \in (-\theta, \theta)$ the operator $e^{i\alpha}A$ is the generator of a bounded C_0 -semigroup with bound M . (Note that we call the holomorphic semigroup T of angle θ bounded if $\sup_{z \in \Sigma_\theta} \|T(z)\| < \infty$. We point out that the terminology ‘bounded holomorphic semigroup’ is sometimes used differently.)

(b) If A is the generator of a bounded C_0 -semigroup with bound $M \geq 1$, then $[\text{Re} > 0] := \{\lambda \in \mathbb{C}; \text{Re } \lambda > 0\} \subseteq \rho(A)$ and

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{\text{Re } \lambda} \quad (\lambda \in [\text{Re} > 0]),$$

by Theorem 2.7. If $\theta \in (0, \pi/2)$ and $\lambda \in \Sigma_\theta$, then $\frac{\text{Re } \lambda}{|\lambda|} \geq \cos \theta$, and this implies

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{\text{Re } \lambda} \leq \frac{M}{\cos \theta} \frac{1}{|\lambda|}.$$

Applying this observation to the generator A of a bounded holomorphic semigroup of angle $\theta \in (0, \pi/2]$ one obtains

$$\Sigma_{\theta+\pi/2} \subseteq \rho(A) \quad \text{and} \quad \sup_{\lambda \in \Sigma_{\theta'+\pi/2}} \|\lambda(\lambda I - A)^{-1}\| < \infty \quad (\theta' \in (0, \theta)). \quad (3.1)$$

This fact has a kind of converse. Here is the complete information: a closed, densely defined operator A in X is the generator of a holomorphic C_0 -semigroup of angle $\theta \in (0, \pi/2]$ that is bounded on all sectors $\Sigma_{\theta'}$ with $\theta' \in (0, \theta)$ if and only if (3.1) holds (see [Kat80; IX, § 1.6], [EN00; II, Section 4a]).

3.4 The Lumer-Phillips theorem

Let H be a Hilbert space over \mathbb{K} . The scalar product of two elements $x, y \in H$ will be denoted by $(x | y)$, and it is defined to be linear in the first and antilinear in the second argument.

An operator A in H is called **accretive** (or **monotone**) if

$$\operatorname{Re}(Ax | x) \geq 0 \quad (x \in \operatorname{dom}(A)).$$

3.16 Lemma. *Let A be an operator in H . Then A is accretive if and only if*

$$\|(\lambda I + A)x\| \geq \lambda \|x\| \quad (x \in \operatorname{dom}(A)) \quad (3.2)$$

for all $\lambda > 0$.

Proof. Assume that A is accretive, and let $\lambda > 0$, $x \in \operatorname{dom}(A)$. Then

$$\|(\lambda I + A)x\| \|x\| \geq |((\lambda I + A)x | x)| \geq \operatorname{Re}((\lambda I + A)x | x) \geq (\lambda x | x) = \lambda \|x\|^2.$$

This shows the asserted inequality.

On the other hand, assume that the norm inequality holds, and let $x \in \operatorname{dom}(A)$. Then

$$0 \leq \|(\lambda I + A)x\|^2 - \lambda^2 \|x\|^2 = 2\lambda \operatorname{Re}(Ax | x) + \|Ax\|^2$$

for all $\lambda > 0$, and this implies $\operatorname{Re}(Ax | x) \geq 0$. □

3.17 Remark. We point out some simple consequences of the inequality (3.2), in a more general context.

If X, Y are Banach spaces, B is an operator from X to Y , and $\alpha > 0$, then

$$\|Bx\|_Y \geq \alpha \|x\|_X \quad (x \in \operatorname{dom}(B))$$

if and only if B is injective and $\|B^{-1}y\| \leq \alpha^{-1}\|y\|$ for all $y \in \operatorname{ran}(B)$. If these properties are satisfied and additionally $\operatorname{ran}(B) = Y$, then $B^{-1} \in \mathcal{L}(Y, X)$ and $\|B^{-1}\| \leq \alpha^{-1}$.

An accretive operator A satisfying $\operatorname{ran}(I + A) = H$ is called **m-accretive**. Note that A being m-accretive implies that A is closed; this is immediate from Lemma 3.16 and Remark 3.17. A historical note on the prefix ‘m’: it should be remindful of the word ‘maximal’. However, ‘maximal accretive operators’, in the sense that there does not exist a proper accretive extension, need not be m-accretive; see [Phi59; footnote (6)].

3.18 Theorem. (*Lumer-Phillips*) *Let A be an operator in H . Then $-A$ is the generator of a C_0 -semigroup of contractions if and only if A is m-accretive.*

For the sufficiency we need a preparation.

3.19 Lemma. *Let A be an accretive operator in H , and assume that there exists $\lambda_0 > 0$ such that $\operatorname{ran}(\lambda_0 I + A) = H$. Then $(0, \infty) \subseteq \rho(-A)$ (in particular, A is m-accretive), and $\operatorname{dom}(A)$ is dense in H .*

Proof. From Lemma 3.16 and Remark 3.17 we obtain: for $\lambda > 0$ one has $\text{ran}(\lambda I + A) = H$ if and only if $\lambda \in \rho(-A)$, and then $\|(\lambda I + A)^{-1}\| \leq \lambda^{-1}$. In particular $\lambda_0 \in \rho(-A)$; hence $\rho(-A) \cap (0, \infty) \neq \emptyset$. Assuming that $\sigma(-A) \cap (0, \infty) \neq \emptyset$ we find a sequence (λ_n) in $\rho(-A) \cap (0, \infty)$ with $\lambda := \lim_{n \rightarrow \infty} \lambda_n \in \sigma(-A) \cap (0, \infty)$. But then $\|R(\lambda_n, -A)\| \rightarrow \infty$ ($n \rightarrow \infty$), by Remark 2.4(a), which contradicts the bound for the resolvent derived above. So we have shown that $(0, \infty) \subseteq \rho(-A)$.

Let $x \in \text{dom}(A)^\perp$. Then $y := (I + A)^{-1}x \in \text{dom}(A)$, and therefore

$$0 = \text{Re}(x | y) = \text{Re}((I + A)y | y) \geq \|y\|^2.$$

This implies that $y = 0$, $x = (I + A)y = 0$; therefore $\text{dom}(A)^\perp = \{0\}$, i.e., $\text{dom}(A)$ is dense in H . \square

3.20 Remark. We note that Lemma 3.19 implies that every accretive operator $A \in \mathcal{L}(H)$ is automatically m -accretive. Indeed, $\{\lambda \in \mathbb{R}; \lambda > \|A\|\} \subseteq \rho(A)$, by Remark 2.3(b).

Proof of Theorem 3.18. The necessity is an immediate consequence of Theorem 2.7, Remark 3.17 and Lemma 3.16.

The sufficiency follows from Lemma 3.19, Lemma 3.16 (in combination with Remark 3.17) and Theorem 2.9. \square

For the remainder of this section we assume that H is a complex Hilbert space. In order to formulate a conclusion concerning the generation of holomorphic semigroups we state the following definition. For an operator A in H we define the **numerical range** $\text{num}(A) := \{(Ax | x); x \in \text{dom}(A), \|x\| = 1\}$. We call A **sectorial (of angle θ)** if there exists $\theta \in [0, \pi/2)$ such that $\text{num}(A) \subseteq \{z \in \mathbb{C} \setminus \{0\}; |\text{Arg } z| \leq \theta\} \cup \{0\}$, and we call A **m -sectorial** if additionally $\text{ran}(I + A) = H$.

We note that our definition of ‘sectorial’ is slightly more restrictive than the one used in [Kat80; p. V.3.10]. Unhappily, the notation also conflicts with a notion introduced later in [PS93; Section 3] and which meanwhile is an important concept in the functional calculus for operators.

3.21 Remarks. Let A be an operator in H .

(a) Obviously A is accretive if and only if $\text{num}(A) \subseteq [\text{Re} \geq 0]$.

(b) We note that for any angle α one has $\text{num}(e^{i\alpha}A) = e^{i\alpha} \text{num}(A)$. Let $\theta \in (0, \pi/2]$. Then it follows from (a) that $e^{i\alpha}A$ is accretive for all $\alpha \in (-\theta, \theta)$ if and only if $\text{num}(A) \subseteq \{z \in \mathbb{C} \setminus \{0\}; |\text{Arg } z| \leq \pi/2 - \theta\} \cup \{0\}$.

We now draw a conclusion of the Lumer-Phillips theorem for generators of holomorphic semigroups which are contractive on a sector.

3.22 Theorem. *Let A be an operator in the complex Hilbert space H , and let $\theta \in (0, \pi/2]$. Then $-A$ generates a contractive holomorphic C_0 -semigroup of angle θ if and only if A is m -sectorial of angle $\pi/2 - \theta$.*

Proof. For the necessity we note that the hypothesis implies that $-e^{i\alpha}A$ generates a contractive C_0 -semigroup, and therefore by Theorem 3.18, $e^{i\alpha}A$ is accretive, for all $\alpha \in (-\theta, \theta)$. Remark 3.21(b) implies that $\text{num}(A) \subseteq \{z \in \mathbb{C} \setminus \{0\}; |\text{Arg } z| \leq \pi/2 - \theta\} \cup \{0\}$.

As $-A$ generates a contractive C_0 -semigroup one has $\text{ran}(I + A) = H$, by Theorem 2.7. This shows that A is m -sectorial of angle $\pi/2 - \theta$.

For the sufficiency we first note that Theorem 3.18 implies that $-A$ generates a contractive C_0 -semigroup; hence $\text{dom}(A)$ is dense and $\Sigma_\theta \subseteq [\text{Re} > 0] \subseteq \rho(-A)$. Let $\alpha \in (-\theta, \theta)$. From Remark 3.21(b) we know that $e^{i\alpha}A$ is accretive, and therefore

$$\|(e^{-i\alpha}\lambda I + A)^{-1}\| = \|(\lambda I + e^{i\alpha}A)^{-1}\| \leq \frac{1}{\lambda} \quad (\lambda > 0),$$

by Lemma 3.16. This inequality can be rewritten as

$$\|(\lambda I + A)^{-1}\| \leq \frac{1}{|\lambda|} \quad (\lambda \in \Sigma_\theta).$$

Applying Theorem 3.14 we obtain the assertion. □

3.23 Remark. In the context of generators of contractive C_0 -semigroups in Banach spaces, one usually considers *dissipative* instead of *accretive* operators. An operator A is called **dissipative** if $-A$ is accretive. The reason we prefer using the notion of accretive operators is that they will arise naturally in the context of forms.

Notes

The equivalence of (i), (ii), (iii) in Theorem 3.2 is due to Dunford [Dun38; Theorem 76]. Natural as the setup and proof of Theorem 3.2 may seem, it is rather surprising that a further weakening is possible: f is holomorphic if f is locally bounded, and there exist a separating set $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x \in E$. This generalisation is due to Grosse-Erdmann [Gro92] (see also [Gro04]); an elegant short proof, based on the Theorem of Banach-Krein-Šmulian, has been given in [AN00; Theorem 3.1]. Theorem 3.5 is also due to Dunford; see Hille [Hil39; footnote to Theorem 1] (see also [HP57; Theorem 3.10.1]).

The contents of Section 3.2 are standard and can be found in most treatises on C_0 -semigroups. The statement of Theorem 3.14 can be considered as standard. Its proof, however, deviates from the standard proofs. Classically, the generation theorem for holomorphic semigroups is treated by defining the semigroup as a contour integral. We refer to the literature for this kind of proof. Our proof follows the approach presented in [AEH97; Section 4] and [AE12; Section 2]. We note that the characterisation stated at the end of Remark 3.15(b) can also be proved without contour integrals; see [AEH97; Theorem 4.3]. Our treatment of the Lumer-Phillips theorem in Section 3.4 is restricted to Hilbert spaces. This restriction simplifies the treatment significantly in comparison to the treatment in Banach spaces. We refer to the literature for the general treatment; in our lectures we will only need the Hilbert space case.

Exercises

3.1 Define the subspace E of $c'_0 = \ell_1$ by

$$E := \left\{ x = (x_n) \in \ell_1; \sum_{n=1}^{\infty} x_n = 0 \right\}.$$

Show that E is almost norming, but not norming for c_0 .

3.2 Let $\Omega \subseteq \mathbb{C}$ an open set. Let (f_n) be a bounded sequence of bounded holomorphic functions $f_n: \Omega \rightarrow \mathbb{C}$ (i.e., $\sup_{z \in \Omega, n \in \mathbb{N}} |f_n(z)| < \infty$). Define the function $f: \Omega \rightarrow \ell_\infty$ by $f(z) := (f_n(z))_{n \in \mathbb{N}}$.

(a) Show that f is holomorphic. (Hint: Find a suitable norming subset of ℓ_∞ , and use Theorem 3.2.)

(b) Assume additionally that $f_\infty(z) := \lim_{n \rightarrow \infty} f_n(z)$ exists for all $z \in \Omega$. Show that then $f: \Omega \rightarrow c$ is holomorphic (where c denotes the subspace of ℓ_∞ consisting of the convergent sequences), and that f_∞ is holomorphic. (Hint for the last part: The functional $c \ni (x_n) \mapsto \lim x_n \in \mathbb{C}$ belongs to c' .)

(Comment: Continuing this approach one can also show the classical result that the convergence $f_n(z) \rightarrow f_\infty(z)$ is locally uniform. The whole setup could also start with X -valued functions, thereby finally yielding an alternative proof of Theorem 3.7.)

3.3 Let X be a complex Banach space, T a holomorphic C_0 -semigroup of angle $\theta \in (0, \pi/2]$ on X .

(a) Show that, for each $\theta' \in (0, \theta)$, there exists $\delta > 0$ such that

$$\sup_{z \in \Sigma_{\theta'}, \operatorname{Re} z \leq \delta} \|T(z)\| < \infty.$$

(b) Show Remark 3.9.

(c) Show that the estimate in Remark 3.9 can be written equivalently as follows: for each $\theta' \in (0, \theta)$, there exist $M'' \geq 1$, $\omega'' \in \mathbb{R}$ such that

$$\|T(z)\| \leq M'' e^{\omega'' |z|} \quad (z \in \Sigma_{\theta'}).$$

3.4 Let T be a bounded holomorphic C_0 -semigroup of angle $\theta \in (0, \pi/2]$.

(a) Show that there exists a strongly continuous extension (also called T) to the closure of $\Sigma_{\theta,0}$. (Hint: Show first that the extension can be defined on $\bigcup_{z \in \Sigma_\theta} \operatorname{ran}(T(z))$.)

(b) Show that $T_{\pm\theta}$, defined by $T_{\pm\theta}(t) := T(e^{\pm i\theta}t)$ ($t \geq 0$), are C_0 -semigroups (the **boundary semigroups** of T).

(c) If $\theta = \pi/2$, then show that $T_{\pi/2}(t) := T(it)$ ($t \in \mathbb{R}$) defines a C_0 -group $T_{\pi/2}$ (the **boundary group** of T).

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Lecture 4

The Sobolev space H^1 , and applications

In Section 4.1 we present the definition and some basic properties of the Sobolev space H^1 . This treatment is prepared by several important tools from analysis. The main objective of this lecture is the Hilbert space treatment of the Laplace operator in Section 4.2. In particular, the Dirichlet Laplacian will be presented as our first (non-trivial) example of a generator of a contractive holomorphic C_0 -semigroup.

4.1 The Sobolev space H^1

4.1.1 Convolution

We recall the definition of **locally integrable functions** on an open subset Ω of \mathbb{R}^n ,

$$L_{1,\text{loc}}(\Omega) := \left\{ f: \Omega \rightarrow \mathbb{K}; \text{ for all } x \in \Omega \text{ there exists } r > 0 \text{ such that } \right. \\ \left. B(x, r) \subseteq \Omega \text{ and } f|_{B(x,r)} \in L_1(B(x,r)) \right\}.$$

Moreover, $C_c^\infty(\Omega) := C^\infty(\Omega) \cap C_c(\Omega)$ is the space of infinitely differentiable functions with compact support.

4.1 Lemma. *Let $u \in L_{1,\text{loc}}(\mathbb{R}^n)$, $\rho \in C_c^\infty(\mathbb{R}^n)$. We define the **convolution of ρ and u** ,*

$$\rho * u(x) := \int_{\mathbb{R}^n} \rho(x-y)u(y) \, dy = \int_{\mathbb{R}^n} \rho(y)u(x-y) \, dy \quad (x \in \mathbb{R}^n).$$

*Then $\rho * u \in C^\infty(\mathbb{R}^n)$, and for all $\alpha \in \mathbb{N}_0^n$ one has*

$$\partial^\alpha(\rho * u) = (\partial^\alpha \rho) * u.$$

Proof. (i) The integral exists because ρ is bounded and has compact support.

(ii) Continuity of $\rho * u$: There exists $R > 0$ such that $\text{spt } \rho \subseteq B(0, R)$. Let $R' > 0$, $\delta > 0$. For $x, x' \in B(0, R')$, $|x - x'| < \delta$, one obtains

$$\begin{aligned} |\rho * u(x) - \rho * u(x')| &= \left| \int_{B(0, R+R')} (\rho(x-y) - \rho(x'-y))u(y) \, dy \right| \\ &\leq \sup\{|\rho(z) - \rho(z')|; |z - z'| < \delta\} \int_{B(0, R+R')} |u(y)| \, dy. \end{aligned}$$

The second factor in the last expression is finite because u is locally integrable, and the first factor becomes small for small δ because ρ is uniformly continuous.

(iii) Let $1 \leq j \leq n$. The existence of the partial derivative of $\rho * u$ with respect to the j -th variable and the equality $\partial_j(\rho * u) = (\partial_j \rho) * u$ are a consequence of the differentiability of integrals with respect to a parameter. The function $(\partial_j \rho) * u$ is continuous, by step (ii) above, and therefore $\rho * u$ is continuously differentiable with respect to the j -th variable.

(iv) Induction shows the assertion for all $\alpha \in \mathbb{N}_0^n$. \square

A sequence $(\rho_k)_{k \in \mathbb{N}}$ in $C_c(\mathbb{R}^n)$ is called a δ -**sequence** if $\rho_k \geq 0$, $\int \rho_k(x) dx = 1$ and $\text{spt } \rho_k \subseteq B[0, 1/k]$ for all $k \in \mathbb{N}$. (The notation ‘ δ -sequence’ is motivated by the fact that the sequence approximates the ‘ δ -distribution’ $C_c^\infty(\mathbb{R}^n) \ni \varphi \mapsto \varphi(0)$.)

4.2 Remarks. (a) We recall the standard example of a C_c^∞ -function φ . The source of this function is the well-known function $\psi \in C^\infty(\mathbb{R})$,

$$\psi(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ e^{-1/t} & \text{if } t > 0. \end{cases}$$

Then $\varphi(x) := \psi(1 - |x|^2)$ ($x \in \mathbb{R}^n$) defines a function $0 \leq \varphi \in C_c^\infty(\mathbb{R}^n)$, with the property that $\varphi(x) \neq 0$ if and only if $|x| < 1$.

(b) If $0 \leq \rho \in C_c(\mathbb{R}^n)$, $\int \rho(x) dx = 1$, $\text{spt } \rho \subseteq B[0, 1]$, and we define

$$\rho_k(x) := k^n \rho(kx) \quad (x \in \mathbb{R}^n, k \in \mathbb{N}),$$

then (ρ_k) is a δ -sequence.

4.3 Proposition. Let (ρ_k) be a δ -sequence in $C_c(\mathbb{R}^n)$.

(a) Let $f \in C(\mathbb{R}^n)$. Then $\rho_k * f \rightarrow f$ uniformly on compact subsets of \mathbb{R}^n as $k \rightarrow \infty$.

(b) Let $1 \leq p \leq \infty$, $f \in L_p(\mathbb{R}^n)$. Then $\rho_k * f \in L_p(\mathbb{R}^n)$,

$$\|\rho_k * f\|_p \leq \|f\|_p \quad \text{for all } k \in \mathbb{N}.$$

If $1 \leq p < \infty$, then

$$\|\rho_k * f - f\|_p \rightarrow 0 \quad (k \rightarrow \infty).$$

Proof. (a) The assertion is an easy consequence of the uniform continuity of f on compact subsets of \mathbb{R}^n .

(b) (i) For $p = \infty$ the estimate is straightforward.

If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we estimate, using Hölder’s inequality in the second step,

$$\begin{aligned} |\rho_k * f(x)| &= \left| \int \rho_k(x-y)^{\frac{1}{q} + \frac{1}{p}} f(y) dy \right| \\ &\leq \left(\int \rho_k(x-y) dy \right)^{1/q} \left(\int \rho_k(x-y) |f(y)|^p dy \right)^{1/p} \\ &= \left(\int \rho_k(x-y) |f(y)|^p dy \right)^{1/p}. \end{aligned}$$

This estimate also holds (trivially) for $p = 1$. Then, with Fubini's theorem in the second step,

$$\begin{aligned} \int |\rho_k * f(x)|^p dx &\leq \int_x \int_y \rho_k(x-y) |f(y)|^p dy dx \\ &= \int_y \int_x \rho_k(x-y) dx |f(y)|^p dy = \|f\|_p^p. \end{aligned}$$

(ii) Now let $1 \leq p < \infty$. For $k \in \mathbb{N}$ we define $T_k \in \mathcal{L}(L_p(\mathbb{R}^n))$ by $T_k g := \rho_k * g$ for $g \in L_p(\mathbb{R}^n)$; then step (i) shows that $\|T_k\| \leq 1$.

If $g \in C_c(\mathbb{R}^n)$, then $T_k g \rightarrow g$ in $L_p(\mathbb{R}^n)$ as $k \rightarrow \infty$. Indeed, there exists $R > 0$ such that $\text{spt } g \subseteq B[0, R]$. It is easy to check that this implies that $\text{spt}(\rho_k * g) \subseteq B[0, R+1]$ for all $k \in \mathbb{N}$. Also, $\rho_k * g \rightarrow g$ uniformly on $B[0, R+1]$, by part (a). This shows that $\rho_k * g \rightarrow g$ ($k \rightarrow \infty$) in $L_p(\mathbb{R}^n)$.

Now, the denseness of $C_c(\mathbb{R}^n)$ in $L_p(\mathbb{R}^n)$ (which we will accept as a fundamental fact from measure and integration theory) together with Proposition 1.6 implies that $T_k g \rightarrow g$ ($k \rightarrow \infty$) in $L_p(\mathbb{R}^n)$, for all $g \in L_p(\mathbb{R}^n)$. \square

4.4 Corollary. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $1 \leq p < \infty$. Then $C_c^\infty(\Omega)$ is dense in $L_p(\Omega)$.*

Proof. Let (ρ_k) be a δ -sequence in $C_c^\infty(\mathbb{R}^n)$.

Let $g \in C_c(\Omega)$, and extend g by zero to a function in $C_c(\mathbb{R}^n)$. Then $\rho_k * g \in C^\infty(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, by Lemma 4.1. If $\frac{1}{k} < \text{dist}(\text{spt } g, \mathbb{R}^n \setminus \Omega)$, then $\text{spt}(\rho_k * g) \subseteq \text{spt } g + B[0, 1/k] \subseteq \Omega$ (see Exercise 4.1(a)), and therefore $\rho_k * g \in C_c^\infty(\Omega)$. From Lemma 4.3 we know that $\rho_k * g \rightarrow g$ ($k \rightarrow \infty$) in $L_p(\mathbb{R}^n)$. So, we have shown that $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$ with respect to the L_p -norm.

Now the denseness of $C_c(\Omega)$ in $L_p(\Omega)$ yields the assertion. \square

4.1.2 Distributional derivatives

Let $P(\partial) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq k} a_\alpha \partial^\alpha$ be a partial differential operator with constant coefficients $a_\alpha \in \mathbb{K}$ ($|\alpha| \leq k$), with $k \in \mathbb{N}$. Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in C^k(\Omega)$. Then for all “test functions” $\varphi \in C_c^\infty(\Omega)$ one has

$$\int_\Omega (P(\partial)f)\varphi dx = \int_\Omega f \sum_{|\alpha| \leq k} (-1)^{|\alpha|} a_\alpha \partial^\alpha \varphi dx$$

(integration by parts!).

Now, let $f \in L_{1,\text{loc}}(\Omega)$. Then we say that $P(\partial)f \in L_{1,\text{loc}}(\Omega)$ if there exists $g \in L_{1,\text{loc}}(\Omega)$ such that

$$\int g \varphi dx = \int f \sum_{|\alpha| \leq k} (-1)^{|\alpha|} a_\alpha \partial^\alpha \varphi dx$$

for all $\varphi \in C_c^\infty(\Omega)$, and we say that $P(\partial)f = g$ holds **in the distributional sense**. In particular, if $\partial^\alpha f \in L_{1,\text{loc}}(\Omega)$, then we call $\partial^\alpha f$ the **distributional** (or **generalised** or **weak**) **derivative of f** .

In order to justify this definition we have to show that $g = P(\partial)f$ is unique. This is the content of the following “fundamental lemma of the calculus of variations”.

4.5 Lemma. Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in L_{1,\text{loc}}(\Omega)$,

$$\int f\varphi \, dx = 0 \quad (\varphi \in C_c^\infty(\Omega)).$$

Then $f = 0$.

The statement ‘ $f = 0$ ’ means that f is the zero element of $L_{1,\text{loc}}(\Omega)$, i.e., if f is a representative, then $f = 0$ a.e.

Proof of Lemma 4.5. Let $\varphi \in C_c^\infty(\Omega)$. We show that $\varphi f = 0$ a.e. Defining

$$g(x) := \begin{cases} \varphi(x)f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

we have $g \in L_1(\mathbb{R}^n)$.

Let (ρ_k) be a δ -sequence in $C_c^\infty(\mathbb{R}^n)$. From Lemma 4.3 we know that $\rho_k * g \rightarrow g$ in $L_1(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, $k \in \mathbb{N}$ we have

$$\rho_k * g(x) = \int_{\Omega} \rho_k(x-y)\varphi(y)f(y) \, dy = 0,$$

because $\rho_k(x-\cdot)\varphi \in C_c^\infty(\Omega)$. This shows that $\rho_k * g = 0$, and hence $g = 0$ (as an L_1 -function).

From $\varphi f = 0$ a.e. for all $\varphi \in C_c^\infty(\Omega)$ we conclude that $f = 0$ a.e.; see Exercise 4.5. \square

In the remainder of this subsection we give more information on the one-dimensional case. The aim is to show that an $L_{1,\text{loc}}$ -function with $L_{1,\text{loc}}$ -derivative can be written as the integral of its derivative.

4.6 Proposition. Let $-\infty \leq a < x_0 < b \leq \infty$, $f, g \in L_{1,\text{loc}}(a, b)$. Then $f' = g$ in the distributional sense if and only if there exists $c \in \mathbb{K}$ such that

$$f(x) = c + \int_{x_0}^x g(y) \, dy \quad (\text{a.e. } x \in (a, b)).$$

Note that the right hand side of the previous equality is continuous as a function of x , and that therefore f has a continuous representative.

For the proof we need a preparation.

4.7 Lemma. Let $-\infty \leq a < b \leq \infty$, $h \in L_{1,\text{loc}}(a, b)$, and assume that $h' = 0$ in the distributional sense. Then there exists $c \in \mathbb{K}$ such that $h = c$.

Proof. (i) We start with the observation that a function $\psi \in C_c^\infty(a, b)$ is the derivative of a function in $C_c^\infty(a, b)$ if and only if $\int \psi \, dx = 0$.

(ii) Let $\rho \in C_c^\infty(a, b)$, $\int \rho(x) \, dx = 1$, and let $c := \int \rho(x)h(x) \, dx$. For all $\varphi \in C_c^\infty(a, b)$ one obtains

$$\begin{aligned} \int \varphi(x)(h(x) - c) \, dx &= \int \varphi(x)h(x) \, dx - \int \varphi(y) \, dy \int \rho(x)h(x) \, dx \\ &= \int \left(\varphi(x) - \int \varphi(y) \, dy \rho(x) \right) h(x) \, dx = 0 \end{aligned}$$

because of $\int (\varphi(x) - \int \varphi(y) \, dy \rho(x)) \, dx = 0$ and part (i). Now the assertion is a consequence of Lemma 4.5. \square

Proof of Proposition 4.6. (i) We first show the sufficiency. Let $\varphi \in C_c^\infty(a, b)$, and choose $x_1 \in (a, \inf \text{spt } \varphi)$. Then

$$f(x) = c_1 + \int_{x_1}^x g(y) \, dy \quad (\text{a.e. } x \in (a, b)),$$

with $c_1 := c + \int_{x_0}^{x_1} g(y) \, dy$. We obtain

$$\begin{aligned} \int_a^b \varphi' f \, dx &= \int_a^b \varphi'(x) \left(c_1 + \int_{x_1}^x g(y) \, dy \right) dx = c_1 \int_a^b \varphi' \, dx + \iint_{x_1 < y < x < b} \varphi'(x) g(y) \, dy \, dx \\ &= \iint_{x_1 < y < x < b} \varphi'(x) g(y) \, dx \, dy = - \int_a^b \varphi(y) g(y) \, dy. \end{aligned}$$

Thus, $f' = g$.

(ii) For the proof of the necessity we define

$$h(x) := f(x) - \int_{x_0}^x g(y) \, dy \quad (a < x < b).$$

Then part (i) implies that $h' = f' - g = 0$ in the distributional sense, so by Lemma 4.7 there exists $c \in \mathbb{K}$ such that $h = c$. \square

4.1.3 Definition of $H^1(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be open. We define the **Sobolev space**

$$H^1(\Omega) := \left\{ f \in L_2(\Omega); \partial_j f \in L_2(\Omega) \ (j \in \{1, \dots, n\}) \right\},$$

with scalar product

$$(f | g)_1 := (f | g) + \sum_{j=1}^n (\partial_j f | \partial_j g)$$

(where

$$(f | g) := \int_{\Omega} f(x) \overline{g(x)} \, dx$$

denotes the usual scalar product in $L_2(\Omega)$) and associated norm

$$\|f\|_{2,1} := \left(\|f\|_2^2 + \sum_{j=1}^n \|\partial_j f\|_2^2 \right)^{1/2}.$$

4.8 Theorem. *The space $H^1(\Omega)$ is a separable Hilbert space.*

Proof. Clearly, $H^1(\Omega)$ is a pre-Hilbert space.

Let $J: H^1(\Omega) \rightarrow \bigoplus_{j=0}^n L_2(\Omega)$ be defined by $Jf := (f, \partial_1 f, \dots, \partial_n f)$, where $\bigoplus_{j=0}^n L_2(\Omega)$ denotes the orthogonal direct sum. Then evidently J is a norm preserving linear mapping.

Therefore $H^1(\Omega)$ is complete if and only if the range of J is a closed linear subspace of the Hilbert space $\bigoplus_{j=0}^n L_2(\Omega)$. Let (f_k) be a sequence in $H^1(\Omega)$ such that (Jf_k) is convergent in $\bigoplus_{j=0}^n L_2(\Omega)$ to an element (f^0, \dots, f^n) . This means that $f_k \rightarrow f^0$ and $\partial_j f_k \rightarrow f^j$ ($j = 1, \dots, n$) in $L_2(\Omega)$ as $k \rightarrow \infty$. We show that this implies $f^j = \partial_j f^0$ ($j = 1, \dots, n$). Indeed, for all $\varphi \in C_c^\infty(\Omega)$, $k \in \mathbb{N}$ we have

$$\int \partial_j f_k(x) \varphi(x) \, dx = - \int f_k(x) \partial_j \varphi(x) \, dx.$$

Taking $k \rightarrow \infty$ we obtain

$$\int f^j(x) \varphi(x) \, dx = - \int f^0(x) \partial_j \varphi(x) \, dx.$$

So we have shown that $f := f^0 \in H^1(\Omega)$ and that $Jf = (f^0, \dots, f^n)$.

The space $L_2(\Omega)$ is separable, therefore $\bigoplus_{j=0}^n L_2(\Omega)$ is separable, the closed subspace $J(H^1(\Omega))$ of $\bigoplus_{j=0}^n L_2(\Omega)$ is separable, and thus $H^1(\Omega)$ is separable because J is isometric. \square

As in Subsection 4.1.2, we give additional information for the one-dimensional case.

4.9 Theorem. *Let $-\infty < a < b < \infty$. Then every $f \in H^1(a, b)$ possesses a representative in $C[a, b]$, and the inclusion $H^1(a, b) \subseteq C[a, b]$ thus defined is continuous.*

Proof. It is an immediate consequence of Proposition 4.6 that every function $f \in H^1(a, b)$ has a continuous representative, which also can be extended continuously to $[a, b]$.

Let $f \in H^1(a, b)$, with f chosen as the continuous representative. Then

$$\begin{aligned} \|f\|_{C[a,b]} &\leq \inf_{x \in (a,b)} |f(x)| + \int_a^b |f'(x)| \, dx \leq \frac{1}{b-a} \int_a^b |f(x)| \, dx + \int_a^b |f'(x)| \, dx \\ &\leq (b-a)^{-1/2} \|f\|_2 + (b-a)^{1/2} \|f'\|_2 \leq ((b-a)^{-1/2} + (b-a)^{1/2}) \|f\|_{2,1}. \end{aligned}$$

This inequality shows that the inclusion mapping is a bounded operator. \square

4.10 Remarks. (a) It is not difficult to see that Theorem 4.9 implies that, for $c \in \mathbb{R}$, one also obtains continuous embeddings $H^1(-\infty, c) \subseteq C_0(-\infty, c]$, $H^1(c, \infty) \subseteq C_0[c, \infty)$ and $H^1(-\infty, \infty) \subseteq C_0(-\infty, \infty)$. (The important observation is that the norm of the inclusion mapping only depends on the length of the interval (a, b) .)

In fact, the inequality derived in the proof of Theorem 4.9 shows that one obtains

$$\|f\|_\infty \leq (d^{-1/2} + d^{1/2}) \|f\|_{2,1} \quad (f \in H^1(a, b))$$

if (a, b) contains an interval of length d .

(b) Theorem 4.9 is a simple instance of the Sobolev embedding theorems.

4.1.4 Denseness properties

Let $\Omega \subseteq \mathbb{R}^n$ be open. For $f \in L_{1,\text{loc}}(\Omega)$ we define the **support of f** by

$$\text{spt } f := \Omega \setminus \bigcup \{U \subseteq \Omega; U \text{ open, } f|_U = 0\}.$$

(This definition is consistent with the already defined support for continuous functions.) Furthermore we define

$$\begin{aligned} H_c^1(\Omega) &:= \{f \in H^1(\Omega); \text{spt } f \text{ compact in } \Omega\}, \\ H_0^1(\Omega) &:= \overline{H_c^1(\Omega)}^{H^1(\Omega)}. \end{aligned}$$

4.11 Remarks. (a) In a generalised sense, functions in $H_0^1(\Omega)$ ‘vanish at the boundary of Ω ’. This will be made more precise in Lecture 7.

(b) For $f \in H_0^1(\Omega)$ the extension to \mathbb{R}^n by zero belongs to $H^1(\mathbb{R}^n)$. This is easy to see if $f \in H_c^1(\Omega)$ and then carries over to the closure; see Exercise 4.2.

4.12 Theorem. (a) $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

(b) $H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$.

We need the following auxiliary result.

4.13 Lemma. (a) Let $\alpha \in \mathbb{N}_0^n$, $f, \partial^\alpha f \in L_{1,\text{loc}}(\mathbb{R}^n)$, $\rho \in C_c^\infty(\mathbb{R}^n)$. Then

$$\partial^\alpha(\rho * f) = \rho * (\partial^\alpha f).$$

(b) Let $f \in H^1(\mathbb{R}^n)$, and let (ρ_k) be a δ -sequence in $C_c^\infty(\mathbb{R}^n)$. Then $\rho_k * f \rightarrow f$ in $H^1(\mathbb{R}^n)$ as $k \rightarrow \infty$.

Proof. (a) Using Lemma 4.1 in the first equality and the definition of the distributional derivative in the third, one obtains

$$\begin{aligned} \partial^\alpha(\rho * f)(x) &= \int \partial^\alpha \rho(x-y) f(y) \, dy = (-1)^{|\alpha|} \int \partial^\alpha(y \mapsto \rho(x-y)) f(y) \, dy \\ &= \int \rho(x-y) \partial^\alpha f(y) \, dy = \rho * \partial^\alpha f(x). \end{aligned}$$

(b) From Proposition 4.3(b) we know that $\rho_k * f \rightarrow f$ in $L_2(\mathbb{R}^n)$ as $k \rightarrow \infty$. Also, using part (a), for $1 \leq j \leq n$ one obtains

$$\partial_j(\rho_k * f) = \rho_k * \partial_j f \rightarrow \partial_j f \quad (k \rightarrow \infty)$$

in $L_2(\mathbb{R}^n)$, again by Proposition 4.3(b). □

Proof of Theorem 4.12. (a) Let $f \in H^1(\mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$. A straightforward computation shows that $\partial_j(\varphi f) = \partial_j \varphi f + \varphi \partial_j f$, and therefore $\partial_j(\varphi f) \in L_2(\mathbb{R}^n)$ ($1 \leq j \leq n$). Hence $\varphi f \in H_c^1(\mathbb{R}^n)$.

Choose $\psi \in C_c^\infty(\mathbb{R}^n)$, $\psi|_{B(0,1)} = 1$, and define $\psi_k := \psi(\cdot/k)$ ($k \in \mathbb{N}$). Then, with the aid of the previous observation, it is not difficult to show that $\psi_k f \rightarrow f$ in $H^1(\mathbb{R}^n)$ as $k \rightarrow \infty$.

(b) The inclusion ‘ \supseteq ’ is trivial. For the inclusion ‘ \subseteq ’ it is sufficient to show that each $f \in H_c^1(\Omega)$ can be approximated by elements of $C_c^\infty(\Omega)$. This, however, is a consequence of Remark 4.11(b) and Lemma 4.13(b). (Observe that the support of $\rho_k * f$ is a compact subset of Ω , for large $k \in \mathbb{N}$; see Exercise 4.1(a).) \square

4.14 Remark. In general, the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ do not coincide. We show this for the case that $\Omega \neq \emptyset$ is bounded.

As a preparation we note that for all $f \in H_0^1(\Omega)$ one has $\int \partial_1 f(x) dx = 0$. This is clear if $f \in C_c^\infty(\Omega)$ and carries over to $H_0^1(\Omega)$ by continuity. Let $f \in H^1(\Omega)$ be defined by $f(x) := x_1$ ($x \in \Omega$). Then $\int \partial_1 f(x) dx \neq 0$ and therefore $f \in H^1(\Omega) \setminus H_0^1(\Omega)$.

4.15 Example. Right translation semigroups.

We come back to Examples 1.7, but only for the case $p = 2$. We describe the generator A .

(a) On $L_2(\mathbb{R})$: It was mentioned in Remark 1.15 that $D := C_c^1(\mathbb{R})$ is a core for A , i.e., $A = \overline{A|_D}$, and that $Af = -f'$ for all $f \in D$. Since $A|_D$, considered as a subspace of $L_2(\mathbb{R}) \times L_2(\mathbb{R})$, is isomorphic to $C_c^1(\mathbb{R})$, as a subspace of $H^1(\mathbb{R})$, one obtains

$$\text{dom}(A) = \overline{C_c^1(\mathbb{R})} = H_0^1(\mathbb{R}) = H^1(\mathbb{R}), \quad Af = -f' \quad (f \in \text{dom}(A)).$$

(b) On $L_2(0, \infty)$: Similarly to part (a) one obtains

$$\text{dom}(A) = \overline{C_c^1(0, \infty)} = H_0^1(0, \infty), \quad Af = -f' \quad (f \in \text{dom}(A)).$$

(c) We leave it as a homework to show that for $L_2(-\infty, 0)$ the generator is given by

$$\text{dom}(A) = H^1(-\infty, 0), \quad Af = -f' \quad (f \in \text{dom}(A)),$$

whereas for $L_2(0, 1)$ one obtains

$$\text{dom}(A) = \{f \in H^1(0, 1); f(0) = 0\}, \quad Af = -f' \quad (f \in \text{dom}(A)).$$

4.2 The Hilbert space method for the solution of inhomogeneous problems, and the Dirichlet Laplacian

The first aim of this section is to present a variant of the solution of the Poisson equation

$$-\Delta u = f \text{ on } \Omega, \quad u|_{\partial\Omega} = 0.$$

Here, f should be a given function on the open subset $\Omega \subseteq \mathbb{R}^n$, and (for the moment) u should be thought twice differentiable on Ω and continuous on the closure. We will not treat the problem in this form but rather weaken the requirements.

More explicitly, the requirement on the boundary behaviour will be modified to the requirement that u should belong to $H_0^1(\Omega)$, and the equation itself will only be required to hold in the distributional sense.

In the second part of the section we will establish the connection to m -accretivity and holomorphic semigroups.

4.2.1 The equation $u - \Delta u = f$

4.16 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $f \in L_2(\Omega)$. Then there exists a unique function $u \in H_0^1(\Omega)$ such that $u - \Delta u = f$ in the distributional sense.*

We insert a lemma expressing the distributional equality in another form.

4.17 Lemma. *Let $u \in H^1(\Omega)$, $g \in L_2(\Omega)$. Then $\Delta u = g$ in the distributional sense if and only if*

$$(v | g)_{L_2(\Omega)} = - \sum_{j=1}^n (\partial_j v | \partial_j u)_{L_2(\Omega)} \quad (v \in H_0^1(\Omega)). \quad (4.1)$$

Proof. By the definition of distributional derivatives, the equation $\Delta u = g$ is equivalent to

$$(\varphi | g) = (\Delta \varphi | u) = - \sum_{j=1}^n (\partial_j \varphi | \partial_j u) \quad (\varphi \in C_c^\infty(\Omega)),$$

i.e., to the validity of Equation (4.1) for all $v \in C_c^\infty(\Omega)$. As both mappings $H^1(\Omega) \ni v \mapsto (v | g) \in \mathbb{K}$ and $H^1(\Omega) \ni v \mapsto \sum_{j=1}^n (\partial_j v | \partial_j u) \in \mathbb{K}$ are continuous, the equality of the terms in (4.1) extends to the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$, i.e., to $H_0^1(\Omega)$ (recall Theorem 4.12(b)). \square

Proof of Theorem 4.16. We define a linear functional $\eta: H_0^1(\Omega) \rightarrow \mathbb{K}$ by

$$\eta(v) := (v | f)_{L_2(\Omega)} \quad (v \in H_0^1(\Omega)).$$

Then η is continuous; indeed $|\eta(v)| \leq \|f\|_2 \|v\|_2 \leq \|f\|_2 \|v\|_{2,1}$ ($v \in H_0^1(\Omega)$). Applying the representation theorem of Riesz-Fréchet (see, for instance, [Bre83; Théorème V.5]) we obtain $u \in H_0^1(\Omega)$ such that

$$\eta(v) = (v | u)_1 \quad (v \in H_0^1(\Omega)).$$

Putting this equation and the definition of η together we obtain

$$(v | f) = (v | u)_1 = (v | u) + \sum_{j=1}^n (\partial_j v | \partial_j u) \quad (v \in H_0^1(\Omega)).$$

Shifting the first term on the right hand side to the left and applying Lemma 4.17 we conclude that $-\Delta u = f - u$ in the distributional sense.

The uniqueness of u is a consequence of the uniqueness in the Riesz-Fréchet representation theorem. \square

4.2.2 The Dirichlet Laplacian

In this subsection we reformulate the result of Subsection 4.2.1 in operator language. As before, let $\Omega \subseteq \mathbb{R}^n$ be open. In what follows the space $L_2(\Omega)$ will be complex.

We define the **Dirichlet Laplacian** Δ_D in $L_2(\Omega)$,

$$\Delta_D := \{(u, f) \in L_2(\Omega) \times L_2(\Omega); u \in H_0^1(\Omega), \Delta u = f\}.$$

In other words,

$$\begin{aligned} \text{dom}(\Delta_D) &:= \{u \in H_0^1(\Omega); \Delta u \in L_2(\Omega)\}, \\ \Delta_D u &:= \Delta u \quad (u \in \text{dom}(\Delta_D)). \end{aligned}$$

We will show that Δ_D generates a contractive holomorphic C_0 -semigroup of angle $\pi/2$. The name ‘Dirichlet Laplacian’ may be somewhat misleading; so we give a short explanation. In principal, ‘Dirichlet boundary conditions’ are of the form $u|_{\partial\Omega} = \varphi$ for some function φ defined on $\partial\Omega$. We have explained above that the membership of u in H_0^1 is a version of Dirichlet boundary condition zero. So, ‘Dirichlet Laplacian’ should be regarded as an abbreviation of ‘Laplacian with Dirichlet boundary condition zero’.

4.18 Theorem. *The negative Dirichlet Laplacian $-\Delta_D$ is m -sectorial of angle 0. The operator Δ_D is the generator of a contractive holomorphic C_0 -semigroup of angle $\pi/2$ on $L_2(\Omega)$.*

Proof. For $u \in \text{dom}(\Delta_D)$ an application of Lemma 4.17 yields

$$(-\Delta_D u | u) = (-\Delta u | u) = \sum_{j=1}^n \int \partial_j u \overline{\partial_j u} = \sum_{j=1}^n \int |\partial_j u|^2 \in [0, \infty).$$

This means that $\text{num}(-\Delta_D) \subseteq [0, \infty)$. Also, Theorem 4.16 states that $\text{ran}(I - \Delta_D) = L_2(\Omega)$. As a consequence, $-\Delta_D$ is m -sectorial of angle 0.

Now Theorem 3.22 implies that Δ_D generates a contractive holomorphic C_0 -semigroup of angle $\pi/2$. \square

The statement that ‘ $-\Delta_D$ is m -sectorial of angle 0’ is equivalent to saying that $-\Delta_D$ is a positive self-adjoint operator; this will be explained in Lecture 6.

Notes

In Section 4.1 we have collected some basics of Sobolev spaces as far as we will need and use them in the following. For the general definition of Sobolev spaces and more information we refer to [Ada75]. The reader may have noticed that we state (and prove) some properties in more generality than used for the case of the Sobolev space H^1 .

The treatment of the Dirichlet Laplacian as given in the lecture is well-established and can be found in many books on partial differential equations.

Exercises

4.1 (a) Let $f \in L_{1,\text{loc}}(\mathbb{R}^n)$, $\varphi \in C_c(\mathbb{R}^n)$. Show that $\text{spt}(\varphi * f) \subseteq \text{spt } f + \text{spt } \varphi$. (Hint: Show first that $\text{spt } f + \text{spt } \varphi$ is closed.)

(b) Let $K \subseteq U \subseteq \mathbb{R}^n$, K compact, U open. Show that there exists $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\text{spt } \psi \subseteq U$, $\psi|_K = \mathbf{1}$ and $0 \leq \psi \leq 1$. (Hint: Note that $\text{dist}(K, \mathbb{R}^n \setminus U) > 0$. Find ψ as the convolution of a suitable function $\rho \in C_c^\infty(\mathbb{R}^n)$ with a suitable indicator function.)

4.2 Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Let $f \in H_c^1(\Omega)$, and define \tilde{f} as the extension of f to \mathbb{R}^n by zero. Show that $\tilde{f} \in H^1(\mathbb{R}^n)$. (Hint: Using Exercise 4.1, choose a function $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\text{spt } \psi \subseteq \Omega$ and $\psi = 1$ in a neighbourhood of $\text{spt } f$. With the aid of this function show that $\partial_j \tilde{f}$ is the extension of $\partial_j f$ to \mathbb{R}^n by zero.)

(b) Let $f \in H_0^1(\Omega)$, and define \tilde{f} as the extension of f to \mathbb{R}^n by zero. Show that $\tilde{f} \in H^1(\mathbb{R}^n)$ and that $\partial_j \tilde{f}$ is the extension of $\partial_j f$ to \mathbb{R}^n by zero ($1 \leq j \leq n$).

4.3 Let $H \subseteq \mathbb{R}^2$ be the half-plane $H := \{(x_1, x_2); x_1 \geq 0\}$, and let $f \in L_{1,\text{loc}}(\mathbb{R}^2)$ be defined by $f := \mathbf{1}_H$.

(a) Show that $\int \partial_1 \varphi f = \int_{x_2 \in \mathbb{R}} \varphi(0, x_2) dx_2$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ and that there is no $g \in L_{1,\text{loc}}(\mathbb{R}^2)$ such that $\int \partial_1 \varphi f = \int \varphi g$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$.

(b) Decide which of the partial derivatives $\partial_1 f, \partial_2 f, \partial_1 \partial_2 f$ belong to $L_{1,\text{loc}}(\mathbb{R}^2)$.

4.4 Let $n \geq 3$. Show that $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n \setminus \{0\})$. For more ambitious participants: Show this also for $n = 2$.

4.5 Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Show that there exists a **standard exhaustion** $(\Omega_k)_{k \in \mathbb{N}}$ of Ω , i.e., Ω_k is open, relatively compact in Ω_{k+1} ($k \in \mathbb{N}$), and $\bigcup_{k \in \mathbb{N}} \Omega_k = \Omega$. (Hint: For $\Omega \neq \mathbb{R}^n$ use

$$\Omega_k := \left\{ x \in \Omega; |x| < k, \text{dist}(x, \mathbb{R}^n \setminus \Omega) > \frac{1}{k} \right\}.$$

(b) Let $f \in L_{1,\text{loc}}(\Omega)$, and assume that $f = 0$ locally, i.e., for all $x \in \Omega$ there exists $r > 0$ such that $f|_{B(0,r)} = 0$. Then $f = 0$. (All ‘= 0’ should be interpreted as a.e.)

References

- [Ada75] R. A. Adams: *Sobolev spaces*. Academic Press, New York, 1975.
 [Bre83] H. Brezis: *Analyse fonctionnelle – Théorie et applications*. Masson, Paris, 1983.

Lecture 5

Forms and operators

Now we introduce the main object of this course – namely forms in Hilbert spaces. They are so popular in analysis because the Lax-Milgram lemma yields properties of existence and uniqueness which are best adapted for establishing weak solutions of elliptic partial differential equations. What is more, we already have the Lumer-Phillips machinery at our disposal, which allows us to go much further and to associate holomorphic semigroups with forms.

5.1 Forms: algebraic properties

In this section we introduce forms and put together some algebraic properties. As domain we consider a vector space V over \mathbb{K} .

A **sesquilinear form** on V is a mapping $a: V \times V \rightarrow \mathbb{K}$ such that

$$\begin{aligned} a(u + v, w) &= a(u, w) + a(v, w), & a(\lambda u, w) &= \lambda a(u, w), \\ a(u, v + w) &= a(u, v) + a(u, w), & a(u, \lambda v) &= \bar{\lambda} a(u, v) \end{aligned}$$

for all $u, v, w \in V$, $\lambda \in \mathbb{K}$.

If $\mathbb{K} = \mathbb{R}$, then a sesquilinear form is the same as a bilinear form. If $\mathbb{K} = \mathbb{C}$, then a is antilinear in the second variable: it is additive in the second variable but not homogeneous. Thus the form is linear in the first variable, whereas only half of the linearity conditions are fulfilled for the second variable. The form is $1\frac{1}{2}$ -linear; or sesquilinear since the Latin ‘sesqui’ means ‘one and a half’.

For simplicity we will mostly use the terminology **form** instead of sesquilinear form. A form a is called **symmetric** if

$$a(u, v) = \overline{a(v, u)} \quad (u, v \in V),$$

and a is called **accretive** if

$$\operatorname{Re} a(u, u) \geq 0 \quad (u \in V).$$

A symmetric form is also called **positive** if it is accretive.

In the following we will also use the notation

$$a(u) := a(u, u) \quad (u \in V)$$

for the associated quadratic form.

5.1 Remarks. (a) If $\mathbb{K} = \mathbb{C}$, then each form a satisfies the **polarisation identity**

$$a(u, v) = \frac{1}{4}(a(u+v) - a(u-v) + ia(u+iv) - ia(u-iv)) \quad (u, v \in V).$$

In particular, the form is determined by its quadratic terms. The identity also shows that a is symmetric if and only if $a(u) \in \mathbb{R}$ for all $u \in V$. This characterisation is obviously only true if $\mathbb{K} = \mathbb{C}$. So here we have a case where the choice of the field matters.

(b) If $\mathbb{K} = \mathbb{C}$, then a is positive symmetric if and only if $a(u) \in [0, \infty)$ for all $u \in V$.

Now we may again consider $\mathbb{K} = \mathbb{C}$ or \mathbb{R} . Recall that a scalar product is a symmetric form a which is definite, i.e., $a(u) > 0$ for all $u \in V \setminus \{0\}$. For Schwarz's inequality to hold we do not need definiteness. In fact, we may even consider a version which involves two symmetric forms. This will be useful later on.

Note that each form a satisfies the **parallelogram identity**

$$a(u+v) + a(u-v) = 2a(u) + 2a(v) \quad (u, v \in V).$$

5.2 Proposition. (*Schwarz's inequality*) Let $a, b: V \times V \rightarrow \mathbb{K}$ be two symmetric forms. Assume that $|a(u)| \leq b(u)$ for all $u \in V$. Then

$$|a(u, v)| \leq b(u)^{1/2}b(v)^{1/2} \quad (u, v \in V). \quad (5.1)$$

Proof. Let $u, v \in V$. In order to show (5.1) we may assume that $a(u, v) \in \mathbb{R}$ (in the complex case replace u by γu with a suitable $\gamma \in \mathbb{C}$, $|\gamma| = 1$). Then $a(u, v) = a(v, u)$ by the symmetry of a , and therefore

$$a(u, v) = \frac{1}{4}(a(u+v) - a(u-v)).$$

Hence from the hypothesis one obtains

$$|a(u, v)| \leq \frac{1}{4}(b(u+v) + b(u-v)) = \frac{1}{2}(b(u) + b(v)),$$

in virtue of the parallelogram identity.

Let $s > b(u)^{1/2}$, $t > b(v)^{1/2}$. Then

$$|a(u, v)| = st \left| a\left(\frac{1}{s}u, \frac{1}{t}v\right) \right| \leq st \cdot \frac{1}{2} \left(\frac{b(u)}{s^2} + \frac{b(v)}{t^2} \right) \leq st.$$

Taking the infimum over s and t we obtain (5.1). □

Finally, we introduce the adjoint form. Let $a: V \times V \rightarrow \mathbb{K}$ be a form. Then

$$a^*(u, v) := \overline{a(v, u)} \quad (u, v \in V)$$

defines a form $a^*: V \times V \rightarrow \mathbb{K}$. Thus a is symmetric if and only if $a = a^*$. In the case of complex scalars, the forms

$$\operatorname{Re} a := \frac{1}{2}(a + a^*) \quad \text{and} \quad \operatorname{Im} a := \frac{1}{2i}(a - a^*)$$

are symmetric and

$$a = \operatorname{Re} a + i \operatorname{Im} a.$$

We call $\operatorname{Re} a$ the **real part** and $\operatorname{Im} a$ the **imaginary part** of a . Note that $(\operatorname{Re} a)(u) = \operatorname{Re} a(u)$ and $(\operatorname{Im} a)(u) = \operatorname{Im} a(u)$ for all $u \in V$.

There is another algebraic notion – only used for the case $\mathbb{K} = \mathbb{C}$ – that will play a role in this course. A form $a: V \times V \rightarrow \mathbb{C}$ is **sectorial** if there exists $\theta \in [0, \pi/2)$ such that $a(u) \in \{z \in \mathbb{C} \setminus \{0\}; |\operatorname{Arg} z| \leq \theta\} \cup \{0\}$ for all $u \in V$. If we want to specify the angle, we say that a is **sectorial of angle** θ . It is obvious that a form $a: V \times V \rightarrow \mathbb{C}$ is sectorial if and only if there exists a constant $c \geq 0$ such that

$$|\operatorname{Im} a(u)| \leq c \operatorname{Re} a(u) \quad (u \in V).$$

(The angle θ and the constant c are related by $c = \tan \theta$.)

5.2 Representation theorems

Now we consider the case where the underlying form domain is a Hilbert space V over \mathbb{K} . An important result is the classical representation theorem of Riesz-Fréchet: If η is a continuous linear functional on V , then there exists a unique $u \in V$ such that

$$\eta(v) = (v | u)_V \quad (v \in V)$$

(cf. [Bre83; Théorème V.5]).

The purpose of this section is to generalise this result. First of all, in the complex case, it will be natural to consider the antidual V^* of V instead of the dual space V' . More precisely, if $\mathbb{K} = \mathbb{R}$, then $V^* = V'$ is the dual space of V , and if $\mathbb{K} = \mathbb{C}$, then we denote by V^* the space of all continuous antilinear functionals. (We recall that $\eta: V \rightarrow \mathbb{C}$ is called antilinear if $\eta(u + v) = \eta(u) + \eta(v)$ and $\eta(\lambda u) = \bar{\lambda} \eta(u)$ for all $u, v \in V$, $\lambda \in \mathbb{C}$.) Then V^* is a Banach space over \mathbb{C} for the norm $\|\eta\|_{V^*} = \sup_{\|v\|_V \leq 1} |\eta(v)|$. For $\eta \in V^*$ we frequently write

$$\langle \eta, v \rangle := \eta(v) \quad (v \in V).$$

Of course, the theorem of Riesz-Fréchet can be reformulated by saying that for each $\eta \in V^*$ there exists a unique $u \in V$ such that

$$\eta(v) = (u | v)_V \quad (v \in V).$$

We will also need the **Riesz isomorphism** $\Phi: V \rightarrow V^*$, $u \mapsto (u | \cdot)$. It is easy to see that Φ is linear and isometric. The Riesz-Fréchet theorem shows that Φ is surjective.

Next we derive a slight generalisation of the Riesz-Fréchet theorem, the omnipresent Lax-Milgram lemma.

A form $a: V \times V \rightarrow \mathbb{K}$ is called **bounded** if there exists $M \geq 0$ such that

$$|a(u, v)| \leq M \|u\|_V \|v\|_V \quad (u, v \in V). \quad (5.2)$$

It is not difficult to show that boundedness of a form is equivalent to continuity; see Exercise 5.1. The form is **coercive** if there exists $\alpha > 0$ such that

$$\operatorname{Re} a(u) \geq \alpha \|u\|_V^2 \quad (u \in V). \quad (5.3)$$

If $a: V \times V \rightarrow \mathbb{K}$ is a bounded form, then

$$\langle \mathcal{A}u, v \rangle := a(u, v) \quad (u, v \in V)$$

defines a bounded operator $\mathcal{A}: V \rightarrow V^*$ with $\|\mathcal{A}\|_{\mathcal{L}(V, V^*)} \leq M$, where M is the constant from (5.2). Incidentally, each bounded operator from V to V^* is of this form. Coercivity implies that \mathcal{A} is an isomorphism: this is the famous Lax-Milgram lemma.

Before stating and proving the Lax-Milgram lemma we treat the ‘operator version’.

5.3 Remark. Let $A \in \mathcal{L}(V)$ be **coercive**, i.e.,

$$\operatorname{Re}(Au | u) \geq \alpha \|u\|_V^2 \quad (u \in V),$$

with some $\alpha > 0$. Then, obviously, $A - \alpha I$ is accretive, and Remark 3.20 implies that $A - \alpha I$ is m-accretive. Therefore $A = \alpha I + (A - \alpha I)$ is invertible in $\mathcal{L}(V)$, and $\|A^{-1}\| \leq \frac{1}{\alpha}$ (see Lemma 3.16, Remark 3.17 and Lemma 3.19).

5.4 Lemma. (*Lax-Milgram*) Let V be a Hilbert space, $a: V \times V \rightarrow \mathbb{K}$ a bounded and coercive form. Then the operator $\mathcal{A}: V \rightarrow V^*$ defined above is an isomorphism and $\|\mathcal{A}^{-1}\|_{\mathcal{L}(V^*, V)} \leq \frac{1}{\alpha}$, with $\alpha > 0$ from (5.3).

Proof. Composing \mathcal{A} with the inverse of the Riesz isomorphism $\Phi: V \rightarrow V^*$ we obtain an operator $\Phi^{-1}\mathcal{A} \in \mathcal{L}(V)$ satisfying

$$\operatorname{Re}(\Phi^{-1}\mathcal{A}u | u) = \operatorname{Re} \langle \mathcal{A}u, u \rangle = \operatorname{Re} a(u, u) \geq \alpha \|u\|^2 \quad (u \in V).$$

From Remark 5.3 we conclude that $\Phi^{-1}\mathcal{A}$ is invertible in $\mathcal{L}(V)$, and $\|(\Phi^{-1}\mathcal{A})^{-1}\| \leq \frac{1}{\alpha}$. As Φ is an isometric isomorphism we obtain the assertions. \square

If the form is symmetric, then the Lax-Milgram lemma is the same as the theorem of Riesz-Fréchet. In fact, then a is an equivalent scalar product, i.e., $a(u)^{1/2}$ defines an equivalent norm on V .

5.3 Semigroups by forms, the complete case

Here we come to the heart of the course: we prove the first generation theorem. With a coercive form we associate an operator that is m-sectorial and thus yields a contractive, holomorphic semigroup. The Lumer-Phillips theorem in its holomorphic version (Theorem 3.22) characterises generators of such semigroups by sectoriality and a range condition. In concrete cases the range condition leads to a partial differential equation (mostly of elliptic type) which has to be solved. If the operator is associated with a form, then the Lax-Milgram lemma does this job, so the range condition is automatically fulfilled. At first we will explain how we associate an operator with a form.

We use the terminology “complete case” since in this lecture the form domain is a Hilbert space. After having seen a series of examples in diverse further lectures, we will also meet the “non-complete case” where the form domain is just a vector space.

Let V, H be Hilbert spaces over \mathbb{K} and let $a: V \times V \rightarrow \mathbb{K}$ be a bounded form. Let $j \in \mathcal{L}(V, H)$ be an operator with dense range. We consider the condition that

$$u \in V, j(u) = 0, a(u) = 0 \text{ implies } u = 0. \quad (5.4)$$

Let

$$A := \{(x, y) \in H \times H; \exists u \in V: j(u) = x, a(u, v) = (y | j(v)) \ (v \in V)\}.$$

5.5 Proposition. (a) *Assume (5.4). Then the relation A defined above is an operator in H . We call A the **operator associated with** (a, j) and write $A \sim (a, j)$.*

(b) *If a is accretive, then A is accretive.*

(c) *If $\mathbb{K} = \mathbb{C}$ and a is sectorial, then A is sectorial of the same angle as a .*

Proof. (a) It is easy to see that A is a subspace of $H \times H$. Let $(0, y) \in A$. We have to show that $y = 0$. By definition there exists $u \in V$ such that $j(u) = 0$ and $a(u, v) = (y | j(v))_H$ for all $v \in V$. In particular, $a(u) = 0$. Assumption (5.4) implies that $u = 0$. Hence $(y | j(v))_H = 0$ for all $v \in V$. Since j has dense range, it follows that $y = 0$.

(b), (c) If $x \in \text{dom}(A)$, then there exists $u \in V$ such that $j(u) = x$ and such that $a(u, v) = (Aj(u) | j(v))$ for all $v \in V$, and then $a(u, u) = (Aj(u) | j(u)) = (Ax | x)$.

If $\text{Re } a(u, u) \geq 0$ ($u \in V$), then $\text{Re } (Ax | x) \geq 0$ for all $x \in \text{dom}(A)$, and this proves (b). Also, in the complex case, $\text{num}(A)$ is contained in $\{a(v); v \in V\}$, and this proves (c). \square

5.6 Remark. Let V, H, a, j be as above, and let $\omega \in \mathbb{R}$. Then

$$b(u, v) := a(u, v) + \omega (j(u) | j(v)) \quad (u, v \in V)$$

defines a form satisfying (5.4) as well (with a replaced by b). Let B be the operator associated with (b, j) .

Let $x, y \in H$. Then for all $u, v \in V$ with $j(u) = x$ we have

$$a(u, v) = (y | j(v)) \iff b(u, v) = (y + \omega x | j(v)).$$

This shows that

$$(x, y) \in A \iff (x, y + \omega x) \in B.$$

Therefore $B = A + \omega I$.

Now we prove the first generation theorem for forms. Note that coercivity implies (5.4).

5.7 Theorem. (*Generation theorem, complete case, part 1*) *Let $a: V \times V \rightarrow \mathbb{K}$ be bounded and coercive and let $j \in \mathcal{L}(V, H)$ have dense range. Let A be the operator associated with (a, j) . Then A is m -accretive, i.e., $-A$ generates a contractive C_0 -semigroup on H .*

Proof. Clearly, the hypothesis that a is coercive implies that a is accretive. Hence A is accretive by Proposition 5.5(b). In order to show that A is m -accretive we have to show the range condition $\text{ran}(I + A) = H$. Define the form $b: V \times V \rightarrow \mathbb{K}$ by

$$b(u, v) := a(u, v) + (j(u) | j(v)) \quad (u, v \in V).$$

Then b is bounded and coercive; recall from Remark 5.6 that the operator $I + A$ is associated with (b, j) .

Let $y \in H$. Then $\eta(v) := (y | j(v))_H$ defines an element $\eta \in V^*$. By the Lax-Milgram lemma there exists $u \in V$ such that

$$b(u, v) = (y | j(v))_H \quad (v \in V).$$

This implies that $x := j(u) \in \text{dom}(A)$ and $(I + A)x = y$. □

5.8 Remark. For later use we explain that the construction presented in the proof of Theorem 5.7 yields a closed expression for the inverse of $I + A$.

In order to derive this expression we let $\mathcal{B}: V \rightarrow V^*$ denote the ‘Lax-Milgram operator’ associated with the form b used above, i.e.,

$$\langle \mathcal{B}u, v \rangle := b(u, v) \quad (u, v \in V).$$

Further we define $k: H \rightarrow V^*$, $y \mapsto (y | j(\cdot))_H$. Then

$$|\langle k(y), v \rangle| \leq \|y\|_H \|j(v)\|_H \leq \|j\| \|y\|_H \|v\|_V \quad (y \in H, v \in V),$$

and this inequality implies that $k \in \mathcal{L}(H, V^*)$, $\|k\| \leq \|j\|$.

Now, starting with $y \in H$ we obtain (with the notation used in the proof of Theorem 5.7) $\eta = k(y)$, $u = \mathcal{B}^{-1}\eta$, $x = j(u)$. This results in $x = j\mathcal{B}^{-1}k(y)$, and using $(I + A)x = y$ and the invertibility of $I + A$ we obtain $(I + A)^{-1} = j\mathcal{B}^{-1}k$.

In the complex case one also obtains results concerning sectoriality.

5.9 Theorem. (*Generation theorem, complete case, part 2*) Let $\mathbb{K} = \mathbb{C}$, let $a: V \times V \rightarrow \mathbb{C}$ be bounded and coercive and let $j \in \mathcal{L}(V, H)$ have dense range. Let A be the operator associated with (a, j) . Then the form a is sectorial, and the operator A is m -sectorial, i.e., $-A$ generates a contractive holomorphic C_0 -semigroup on H .

Proof. By assumption there exist $M \geq 0$, $\alpha > 0$ such that

$$|a(u, v)| \leq M\|u\|_V \|v\|_V, \quad \text{Re } a(v) \geq \alpha\|v\|_V^2$$

for all $u, v \in V$. Thus

$$\frac{|\text{Im } a(v)|}{\text{Re } a(v)} \leq \frac{M\|v\|_V^2}{\alpha\|v\|_V^2} = \frac{M}{\alpha}$$

for all $v \in V \setminus \{0\}$. This implies that there exists $\theta \in [0, \pi/2)$ such that $|\text{Arg } a(v)| \leq \theta$ for all $v \in V \setminus \{0\}$. Thus a is sectorial. The remaining assertions are immediate consequences of Proposition 5.5(c), Theorem 5.7 and Theorem 3.22. □

We give a first example as an illustration.

5.10 Example. Multiplication operators.

Let (Ω, μ) be a σ -finite measure space and let $m: \Omega \rightarrow \mathbb{C}$ be measurable such that $w(x) := \operatorname{Re} m(x) \geq \delta > 0$ for all $x \in \Omega$. Let $V := L_2(\Omega, w\mu)$. Assume that there exists $c > 0$ such that

$$|\operatorname{Im} m(x)| \leq c \operatorname{Re} m(x) \quad (x \in \Omega).$$

Then $a(u, v) := \int u \bar{v} m \, d\mu$ defines a bounded coercive form $a: V \times V \rightarrow \mathbb{C}$. Let $H := L_2(\Omega, \mu)$, $j(u) = u$ for all $u \in V$. Then $j \in \mathcal{L}(V, H)$, and j has dense range. (For the denseness of $\operatorname{ran}(j)$ note that V is the domain of the maximal multiplication operator by the function \sqrt{w} ; see Exercise 1.3(c).) Let $A \sim (a, j)$. Then one easily sees that

$$\begin{aligned} \operatorname{dom}(A) &= \{u \in V; mu \in L_2(\Omega, \mu)\}, \\ Au &= mu. \end{aligned}$$

From Section 2.2 we recall the concept of rescaling. If $-A$ is the generator of a C_0 -semigroup T and $\omega \in \mathbb{R}$, then $-(A + \omega)$ generates the semigroup $(e^{-\omega t} T(t))_{t \geq 0}$. One frequently uses the word “quasi” as prefix if something is true after rescaling. (The notation ‘ $A + \omega$ ’ is an abbreviation of ‘ $A + \omega I$ ’; the ω stands for multiplication by the scalar ω , which is just the operator ωI .)

Let H be a complex Hilbert space. An operator A in H is **quasi-sectorial** if there exists $\omega \in \mathbb{R}$ such that $A + \omega$ is sectorial. The operator A is **quasi-m-sectorial** if $A + \omega$ is m-sectorial for some $\omega \in \mathbb{R}$. A **quasi-contractive holomorphic semigroup** is a holomorphic semigroup T such that $\|e^{-\omega z} T(z)\| \leq 1$ for all $z \in \Sigma_\theta$, for some $\theta \in (0, \pi/2]$ and some $\omega \in \mathbb{R}$.

Thus A is quasi-m-sectorial if and only if $-A$ generates a quasi-contractive holomorphic C_0 -semigroup.

Let $a: V \times V \rightarrow \mathbb{C}$ be a bounded form and let $j \in \mathcal{L}(V, H)$ have dense range. In analogy to the previous notation we could say that a is *quasi-coercive (with respect to j)* if there exist $\omega \in \mathbb{R}$, $\alpha > 0$ such that

$$\operatorname{Re} a(v) + \omega \|j(v)\|_H^2 \geq \alpha \|v\|_V^2 \quad (v \in V), \quad (5.5)$$

but – for simplicity of notation – we prefer to call a **j -elliptic** in this case. It is obvious that (5.5) implies (5.4). Thus the operator A associated with (a, j) is defined by Proposition 5.5. Consider the form $b: V \times V \rightarrow \mathbb{C}$ given by

$$b(u, v) := a(u, v) + \omega (j(u) | j(v))_H.$$

Then b is bounded and coercive. Thus the operator B associated with (b, j) is m-sectorial. Remark 5.6 implies that $B = A + \omega I$, so we have proved the first part of the following more general generation theorem.

5.11 Corollary. *Let $j \in \mathcal{L}(V, H)$ have dense range, and let $a: V \times V \rightarrow \mathbb{C}$ be bounded and j -elliptic. Then the operator A associated with (a, j) is quasi-m-sectorial. If additionally a is sectorial, then A is m-sectorial.*

For the last statement we observe that Proposition 5.5 implies that A is sectorial. Applying Lemma 3.19 one obtains that A is m -sectorial.

Later we will meet interesting situations where j is not injective. In most applications, however, j is an embedding; then we will usually suppress the letter j . The situation is described as follows. Let H and V be Hilbert spaces such that $V \xrightarrow{d} H$. This is an abbreviation for saying that V is continuously embedded into H (abbreviated by $V \hookrightarrow H$) and that V is dense in H . Of course, that $V \hookrightarrow H$ means that $V \subseteq H$ and that for some constant $c > 0$ one has

$$\|u\|_H \leq c\|u\|_V \quad (u \in V).$$

We call such a constant an **embedding constant**.

Now let $a: V \times V \rightarrow \mathbb{K}$ be a bounded form. We say that a is H -**elliptic** if

$$\operatorname{Re} a(v) + \omega\|v\|_H^2 \geq \alpha\|v\|_V^2$$

for all $v \in V$ and some $\alpha > 0$, $\omega \in \mathbb{R}$. In that case the definition of the operator A associated with a (not mentioning the given embedding of V into H) reads as follows. For $x, y \in H$ one has

$$x \in \operatorname{dom}(A), Ax = y \iff x \in V, a(x, v) = (y | v)_H \quad (v \in V).$$

In the case $\mathbb{K} = \mathbb{C}$, this operator is quasi- m -sectorial by Corollary 5.11.

5.4 The classical Dirichlet form and other examples

Let Ω be an open set in \mathbb{R}^n . The classical Dirichlet form is defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ and given by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx := \sum_{j=1}^n \int_{\Omega} \partial_j u \overline{\partial_j v} \, dx.$$

It is clear that a is bounded; in fact

$$|a(u, v)| \leq \|\nabla u\|_{L_2(\Omega)} \|\nabla v\|_{L_2(\Omega)} \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

Here $\nabla u = (\partial_1 u, \dots, \partial_n u)$ and $\|\nabla u\|_{L_2(\Omega)} := (\sum_{j=1}^n \int_{\Omega} |\partial_j u|^2 \, dx)^{1/2}$. Thus $\|u\|_{H^1(\Omega)}^2 = \|u\|_{L_2(\Omega)}^2 + \|\nabla u\|_{L_2(\Omega)}^2$.

We will prove that the Dirichlet form is coercive if Ω is bounded, or more generally, if Ω **lies in a strip**, i.e., there exist $\delta > 0$ and $j_0 \in \{1, \dots, n\}$ such that $|x_{j_0}| \leq \delta$ for all $x \in \Omega$.

5.12 Theorem. (*Poincaré's inequality*) *Assume that Ω lies in a strip. Then there exists a constant $c_P > 0$ such that*

$$\int_{\Omega} |u|^2 \, dx \leq c_P \int_{\Omega} |\nabla u|^2 \, dx \quad (u \in H_0^1(\Omega)).$$

Proof. Theorem 4.12 implies that it suffices to prove the inequality for all $u \in C_c^\infty(\Omega)$. We may assume that $j_0 = 1$; otherwise we permute the coordinates. Let $\delta > 0$ be such that $|x_1| \leq \delta$ for all $x = (x_1, \dots, x_n) \in \Omega$. Let $h \in C^1[-\delta, \delta]$, $h(-\delta) = 0$. Then by Hölder's inequality we estimate

$$\begin{aligned} \int_{-\delta}^{\delta} |h(x)|^2 dx &= \int_{-\delta}^{\delta} \left| \int_{-\delta}^x h'(y) dy \right|^2 dx \\ &\leq \int_{-\delta}^{\delta} \left(\int_{-\delta}^x |h'(y)|^2 dy \right) \left(\int_{-\delta}^x \mathbf{1} dy \right) dx \\ &\leq (2\delta)^2 \int_{-\delta}^{\delta} |h'(y)|^2 dy. \end{aligned}$$

Let $u \in C_c^\infty(\Omega)$. Applying the above estimate to $h(r) = u(r, x_2, \dots, x_n)$ we obtain

$$\int_{\Omega} |u|^2 dx \leq 4\delta^2 \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \int_{-\delta}^{\delta} |\partial_1 u(x_1, \dots, x_n)|^2 dx_1 \dots dx_n \leq 4\delta^2 \int_{\Omega} |\nabla u|^2 dx. \quad \square$$

In fact, we saw that the constant c_P can be chosen as d^2 where $d := 2\delta$ is an upper estimate for the width of Ω . (For bounded domains the best constant can be determined as $c_P = 1/\lambda_1^D$, where λ_1^D is the first eigenvalue of $-\Delta_D$; we will come back to this later.) At first we revisit the Dirichlet Laplacian.

5.13 Example. The Dirichlet Laplacian.

Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $H := L_2(\Omega)$, $V := H_0^1(\Omega)$ and define $a: V \times V \rightarrow \mathbb{C}$ by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} dx.$$

Then a is bounded and H -elliptic. Observe that $V \xrightarrow{d} H$. Let A be the operator in H associated with a . Then

$$\begin{aligned} \text{dom}(A) &= \{u \in H_0^1(\Omega); \Delta u \in L_2(\Omega)\}, \\ Au &= -\Delta u. \end{aligned}$$

Thus $-A = \Delta_D$, the Dirichlet Laplacian as defined in Subsection 4.2.2.

If Ω lies in a strip, then a is coercive.

Proof. The inequality $a(u) + 1 \|u\|_H^2 \geq \|u\|_V^2$ ($u \in V$) – in fact an equality – shows that a is H -elliptic. Let $A \sim a$. Then for $u, f \in L_2(\Omega)$ one has $u \in \text{dom}(A)$, $Au = f$ if and only if $u \in H_0^1(\Omega)$ and

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} dx = \int_{\Omega} f \overline{v} dx \quad (v \in H_0^1(\Omega)).$$

By Lemma 4.17 the latter is equivalent to $-\Delta u = f$ in the distributional sense (i.e., $-\int_{\Omega} u \Delta \overline{v} dx = \int_{\Omega} f \overline{v} dx$ for all $v \in C_c^\infty(\Omega)$).

Now assume that Ω lies in a strip. Let $u \in H_0^1(\Omega)$. Then $a(u) \geq \frac{1}{c_P} \int_{\Omega} |u|^2 dx$ by Poincaré's inequality. Thus $a(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2c_P} \int_{\Omega} |u|^2 dx \geq \alpha \|u\|_{H^1(\Omega)}^2$ where $\alpha = \min\{\frac{1}{2}, \frac{1}{2c_P}\}$. Thus a is coercive. \square

The semigroup T generated by Δ_D governs the heat equation. In fact, let $u_0 \in L_2(\Omega)$, $u(t) = T(t)u_0$ for $t \geq 0$. Then $u \in C([0, \infty); L_2(\Omega)) \cap C^\infty(0, \infty; L_2(\Omega))$, $u(t) \in \text{dom}(\Delta_D)$ for all $t > 0$, and

$$\begin{aligned} u'(t) &= \Delta u(t), & u(t)|_{\partial\Omega} &= 0 & (t > 0), \\ u(0) &= u_0. \end{aligned}$$

(In which sense ‘ $u(t) \in H_0^1(\Omega)$ ’ can be expressed as ‘ $u(t)|_{\partial\Omega} = 0$ ’ will be explained in Lecture 7.) If we consider a body Ω (a bounded open subset of \mathbb{R}^n) and $u_0(x)$ as the temperature at $x \in \Omega$ at time 0, then $u(t)(x)$ is the temperature at time $t > 0$ at x . The boundary condition means that the temperature is kept at 0 at the boundary. One expects that $\lim_{t \rightarrow \infty} u(t) = 0$. This is the case as we can see in Exercise 5.2.

Finally we give an example where j is not the identity.

5.14 Example. Multiplicative perturbation of Δ_D .

Let $\Omega \subseteq \mathbb{R}^n$ be an open set which lies in a strip. Let $m: \Omega \rightarrow \mathbb{C}$ be measurable such that $|m(x)| \geq \beta > 0$ for all $x \in \Omega$. Define the operator A in $L_2(\Omega)$ by

$$\begin{aligned} \text{dom}(A) &= \{u \in L_2(\Omega); mu \in \text{dom}(\Delta_D), \bar{m}\Delta(mu) \in L_2(\Omega)\}, \\ Au &= -\bar{m}\Delta(mu). \end{aligned}$$

Then A is m -sectorial of angle 0.

Proof. Let $H = L_2(\Omega)$, $V = H_0^1(\Omega)$, $a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} dx$, and let $j \in \mathcal{L}(V, H)$ be given by $j(v) = \frac{1}{m}v$. Then $j(V)$ is dense in $L_2(\Omega)$. In fact, let $g \in j(V)^\perp$. Then $\int_{\Omega} v \frac{1}{m}g dx = 0$ for all $v \in C_c^\infty(\Omega)$. Thus $\frac{1}{m}g = 0$ by Lemma 4.5. Consequently, $g = 0$. We have shown that $j(V)^\perp = \{0\}$, i.e., $\overline{j(V)} = j(V)^{\perp\perp} = L_2(\Omega)$.

Let A be the operator in $L_2(\Omega)$ associated with (a, j) . Then for $u, f \in L_2(\Omega)$ one has $u \in \text{dom}(A)$ and $Au = f$ if and only if there exists $w \in H_0^1(\Omega)$ such that $\frac{w}{m} = u$ and $\int_{\Omega} \nabla w \cdot \overline{\nabla v} dx = \int_{\Omega} f \overline{(\frac{v}{m})} dx$ for all $v \in H_0^1(\Omega)$. This is equivalent to $mu \in \text{dom}(\Delta_D)$ and $-\Delta(mu) = \frac{f}{m}$. \square

Notes

The approach to forms presented here is the ‘French approach’ following Lions [DL92]. However, we have introduced this little j following the article [AE11] (see also [AE12]). It will carry its fruits when we consider the non-complete case and also when we consider the Dirichlet-to-Neumann operator.

The Lax-Milgram lemma was proved in 1954 and is a daily tool for establishing weak solutions since then. It is an interesting part of the history of ideas that Hilbert considered bilinear forms to treat integral equations in his famous papers in the beginning of the 20th century. His ideas led his students to develop the notion of operators in Hilbert spaces. Since then we consider operators as the central objects and formulate physical and other problems with the help of operators. In the 1950’s, form methods were developed to solve equations defined by operators. Forms are most appropriate for numerical treatments.

The reason is that a form $a: V \times V \rightarrow \mathbb{C}$ can easily be restricted to a finite-dimensional subspace $V_m \times V_m$, whereas for operators there might be only few invariant subspaces. The method of finite elements is based on such restrictions.

Exercises

5.1 Let V be a Hilbert space, and let $a: V \times V \rightarrow \mathbb{K}$ be a sesquilinear form. Show that a is bounded if and only if a is continuous.

5.2 (a) Let V, H be Hilbert spaces over \mathbb{C} , $j \in \mathcal{L}(V, H)$ with dense range, $a: V \times V \rightarrow \mathbb{C}$ bounded and coercive. Let $A \sim (a, j)$ and let T be the semigroup generated by $-A$. Show that there exists $\varepsilon > 0$ such that $\|T(t)\| \leq e^{-\varepsilon t}$ for all $t \geq 0$. (Hint: Show that $b(u, v) = a(u, v) - \varepsilon (j(u) | j(v))_H$ defines a coercive form if $\varepsilon > 0$ is small enough.)

(b) Let $\Omega \subseteq \mathbb{R}^n$ be an open set which lies in a strip. Show that

$$\|e^{t\Delta_D}\|_{\mathcal{L}(L_2(\Omega))} \leq e^{-\varepsilon t} \quad (t \geq 0)$$

for some $\varepsilon > 0$. Express $\varepsilon > 0$ in terms of the width of Ω .

5.3 Let $-\infty < a < b < \infty$. In the following we will always use the continuous representative for a function in $H^1(a, b)$; recall Theorem 4.9 for the inclusion $H^1(a, b) \subseteq C[a, b]$.

(a) Show that each $u \in H^1(a, b)$ is Hölder continuous of index $1/2$, i.e., $|u(t) - u(s)| \leq c|t - s|^{1/2}$ for some $c > 0$.

(b) Show that the embedding $H^1(a, b) \hookrightarrow C[a, b]$ is compact, i.e., if $(u_n)_{n \in \mathbb{N}}$ in $H^1(a, b)$, $\|u_n\|_{H^1(a, b)} \leq c$ for all $n \in \mathbb{N}$, then $(u_n)_{n \in \mathbb{N}}$ has a uniformly convergent subsequence.

(c) Let $H^2(a, b) := \{u \in H^1(a, b); u' \in H^1(a, b)\}$. Then $u'' = (u')'$ is defined for all $u \in H^2(a, b)$, and on $H^2(a, b)$ we define the norm by $\|u\|_{H^2} := (\|u\|_{L_2}^2 + \|u'\|_{L_2}^2 + \|u''\|_{L_2}^2)^{1/2}$. Show that $H^2(a, b) \hookrightarrow C^1[a, b]$ if $C^1[a, b]$ carries the norm $\|u\|_{C^1} = \|u\|_{C[a, b]} + \|u'\|_{C[a, b]}$.

5.4 Let $-\infty < a < b < \infty$ and $\alpha, \beta > 0$. Define the operator A in $L_2(a, b)$ by

$$\begin{aligned} \text{dom}(A) &= \{u \in H^2(a, b); -u'(a) + \alpha u(a) = 0, u'(b) + \beta u(b) = 0\}, \\ Au &= -u''. \end{aligned}$$

(See Exercise 5.3(c) for the definition of $H^2(a, b)$ and the existence of $u'(a)$ and $u'(b)$.)

(a) Show that A is m -sectorial. (Hint: Consider the form given by

$$a(u, v) = \int_a^b u' \overline{v'} dx + \alpha u(a) \overline{v(a)} + \beta u(b) \overline{v(b)} \quad (u, v \in H^1(a, b)).$$

(b) Show that $\|e^{-tA}\|_{\mathcal{L}(L_2(a, b))} \leq e^{-\varepsilon t}$ ($t \geq 0$) for some $\varepsilon > 0$.

References

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Lecture 6

Adjoint operators, and compactness

The main objective of this lecture is to show that, for a bounded open set $\Omega \subseteq \mathbb{R}^n$, the Dirichlet Laplacian Δ_D on Ω has an orthonormal basis consisting of eigenfunctions. The main point is that one can show that $(I - \Delta_D)^{-1}$ is a compact self-adjoint operator, and one shows that for such operators there exist sufficiently many eigenfunctions. One of the aims of this lecture is to explain all the notions occurring in the previous sentence.

6.1 Adjoints of operators, and self-adjoint operators

The principal objective of this section is to prove the following result.

6.1 Theorem. *Let H be a Hilbert space, A an operator in H . Then the following properties are equivalent.*

- (a) *A is a positive self-adjoint operator.*
- (b) *A is a symmetric m -accretive operator.*

If H is a complex Hilbert space, then there is the following additional equivalent property.

- (c) *A is m -sectorial of angle 0.*

Evidently, before proceeding to the proof, we first have to explain the notions of self-adjointness and symmetry. The proof will be given at the very end of this section.

Let G, H be Hilbert spaces over \mathbb{K} . Before defining the adjoint of an operator we want to explain the idea behind this notion. If A is an operator from G to H , then the adjoint A^* should be the maximal operator from H to G such that

$$(Ax | y)_H = (x | A^*y)_G \quad (x \in \text{dom}(A), y \in \text{dom}(A^*)).$$

In the following, the product space $G \times H$ will be provided with the scalar product

$$((x, y) | (x_1, y_1)) := (x | x_1)_G + (y | y_1)_H \quad ((x, y), (x_1, y_1) \in G \times H),$$

which makes $G \times H$ a Hilbert space; we will use the notation $G \oplus H$ for this space.

For an operator A from G to H we define the **adjoint**

$$\begin{aligned} A^* &:= \{(y, x) \in H \times G; \forall x_1 \in \text{dom}(A): (Ax_1 | y)_H = (x_1 | x)_G\} \\ &= \{(y, x) \in H \times G; \forall (x_1, y_1) \in A: ((x_1, -y_1) | (x, y)) = 0\} \\ &= ((-A)^\perp)^{-1}. \end{aligned}$$

It is obvious from the last equality that A^* is a closed subspace of $H \times G$. We mention that $-A$, as a linear relation, is given by

$$-A = \{(x, -y); (x, y) \in A\}.$$

(It is somewhat unpleasant that, in principle, A^* is already defined as the antidual space to the subspace A of $G \oplus H$. We will have to live with this ambiguity.)

6.2 Remarks. Let A be an operator from G to H .

(a) If B is an operator from H to G such that

$$(Ax | y) = (x | By) \quad (x \in \text{dom}(A), y \in \text{dom}(B)),$$

then it is immediate that $B \subseteq A^*$ and $A \subseteq B^*$.

(b) If A^* is an operator, then the definition implies that $(Ax | y) = (x | A^*y)$ for all $x \in \text{dom}(A)$, $y \in \text{dom}(A^*)$.

6.3 Theorem. Let A be an operator from G to H . Then:

(a) A^* is an operator if and only if $\text{dom}(A)$ is dense in G .

(b) Assume that $\text{dom}(A)$ is dense. Then $A^{**} := (A^*)^* = \bar{A}$, and $\text{dom}(A^*)$ is dense if and only if A is closable.

Proof. (a) From the definition of A^* it is easy to see that $\{x \in G; (0, x) \in A^*\} = \text{dom}(A)^\perp$. This shows the assertion.

(b) It is easy to see that in the expression $((-A)^\perp)^{-1}$ for A^* the order of the operations $A \mapsto -A$, $A \mapsto A^\perp$ and $A \mapsto A^{-1}$ does not matter. It follows that $A^{**} = A^{\perp\perp} = \bar{A}$.

By part (a), $\text{dom}(A^*)$ is dense if and only if $(A^*)^*$ is an operator, and by $A^{**} = \bar{A}$ this is equivalent to the closability of A . \square

6.4 Example. Define the operator A in $L_2(\mathbb{R})$ by

$$\text{dom}(A) := C_c^\infty(\mathbb{R}), \quad Af := f' \quad (f \in D(A)).$$

For $f, g \in L_2(\mathbb{R})$ we then obtain:

$$\begin{aligned} (g, f) \in A^* &\iff \forall \varphi \in C_c^\infty(\mathbb{R}): (A\varphi | g) = (\varphi | f) \\ &\iff \forall \varphi \in C_c^\infty(\mathbb{R}): \int g\varphi' = \int f\varphi \\ &\iff f = -g' \text{ in the distributional sense.} \end{aligned}$$

This shows that $D(A^*) = H^1(\mathbb{R})$, $A^*g = -g'$ ($g \in \text{dom}(A^*)$).

We claim that $\bar{A} = -A^*$.

The previous considerations show that $A \subseteq -A^*$, and since $-A^*$ is closed we obtain $\bar{A} \subseteq -A^*$. The subset A^* of $L_2(\mathbb{R}) \oplus L_2(\mathbb{R})$ is isometrically isomorphic to $H^1(\mathbb{R})$, under the mapping $A^* \ni (f, -f') \mapsto f \in H^1(\mathbb{R})$. From Theorem 4.12 we know that $\text{dom}(A) = C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$. This shows that $\bar{A} = -A^*$.

An operator A in H is called **symmetric** if $\text{dom}(A)$ is dense, and $A \subseteq A^*$; A is called **self-adjoint** if $\text{dom}(A)$ is dense, and $A = A^*$.

We collect some properties of symmetric operators.

6.5 Remarks. Let A be an operator in H , $\text{dom}(A)$ dense.

(a) Then A is symmetric if and only if $(Ax | y) = (x | Ay)$ for all $x, y \in \text{dom}(A)$. This follows immediately from Remark 6.2.

(b) If A is symmetric, then A is closable, and \bar{A} is symmetric. Indeed from $A \subseteq A^*$ and the fact that A^* is a closed operator one concludes that A is closable and that $\bar{A} \subseteq A^* = \bar{A}^*$ (where the last equality is immediate from the definition).

(c) Let H be complex. Then A is symmetric if and only if $(Ax | x) \in \mathbb{R}$ for all $x \in \text{dom}(A)$. This follows from Remark 5.1(a), applied to the form $\text{dom}(A) \times \text{dom}(A) \ni (x, y) \mapsto (Ax | y)$, and part (a) above.

An operator A in H is called **essentially self-adjoint** if it is symmetric and \bar{A} is self-adjoint. A symmetric operator is called **positive**, $A \geq 0$, if $(Ax | x) \geq 0$ for all $x \in \text{dom}(A)$ (in other words, if A is accretive).

Particularly simple examples of self-adjoint operators are those possessing ‘sufficiently many’ eigenelements, which we will present next. We note that an eigenvalue of a symmetric operator A is always real: If $0 \neq x \in H$, $Ax = \lambda x$, then $\lambda(x | x) = (Ax | x) \in \mathbb{R}$.

6.6 Example. Diagonal self-adjoint operators.

Let A be a self-adjoint operator in an infinite-dimensional separable Hilbert space, and assume that there exists an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ consisting of eigenelements of A , with corresponding eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$.

Then

$$\text{dom}(A) = \left\{ x \in H; \sum_{n \in \mathbb{N}} |\lambda_n|^2 |(x | e_n)|^2 < \infty \right\},$$

$$Ax = \sum_{n \in \mathbb{N}} \lambda_n (x | e_n) e_n \quad (x \in \text{dom}(A)).$$

Define $A_c \subseteq A$ by $\text{dom}(A_c) := \text{lin}\{e_n; n \in \mathbb{N}\}$. Then A_c is essentially self-adjoint (and $\bar{A}_c = A$).

We call A the **diagonal operator** associated with the orthonormal basis $(e_n)_{n \in \mathbb{N}}$ and the sequence $(\lambda_n)_{n \in \mathbb{N}}$.

Proof. Using the unitary operator $J: H \rightarrow \ell_2$, $x \mapsto ((x | e_n))_{n \in \mathbb{N}}$, we transform the situation to the case where A is a self-adjoint operator in the Hilbert space ℓ_2 , possessing the canonical unit vectors as eigenelements.

Now we refer to Exercise 6.2, for the measure space \mathbb{N} with counting measure, with the multiplying function m in Exercise 6.2 given by the sequence $(\lambda_n)_{n \in \mathbb{N}}$. Then $A_c = A_0$, with A_0 defined in Exercise 6.2, and by Exercise 6.2(b) the operator A_c^* is the maximal multiplication operator M_λ by the ‘function’ $(\bar{\lambda}_n)_{n \in \mathbb{N}} = (\lambda_n)_{n \in \mathbb{N}}$. From Exercise 6.2(c) we know that then $\bar{A}_c = M_\lambda$; hence A_c is essentially self-adjoint. Because A is closed, one concludes that $M_\lambda \subseteq A$. Thus $A^* \subseteq M_\lambda \subseteq A = A^*$, and A is the maximal multiplication operator, i.e., A is the operator described above. \square

In order to connect self-adjointness with m-accretivity and m-sectoriality we need further preparations.

6.7 Lemma. *Let G, H be Hilbert spaces, A an operator from G to H .*

Then $\ker(A^) = \text{ran}(A)^\perp$.*

Proof. By the definition of A^* , $y \in \ker(A^*)$ (which by definition means $(y, 0) \in A^*$) is equivalent to $(Ax_1 | y)_H = 0$ for all $x_1 \in \text{dom}(A)$, i.e., to $y \perp \text{ran}(A)$. \square

We point out that in the following result the operator A is not supposed to be densely defined.

6.8 Proposition. *Let A be an operator in H , $(Ax | y) = (x | Ay)$ for all $x, y \in \text{dom}(A)$, and $\text{ran}(A) = H$.*

Then A is self-adjoint.

Proof. Remark 6.2(a) shows that $A \subseteq A^*$, and Lemma 6.7 implies that $\ker(A^*) = \text{ran}(A)^\perp = \{0\}$. These two facts, together with $\text{ran}(A) = H$, imply $A^* = A$. (This comes under the heading that ‘a surjective mapping cannot have a proper injective extension’ – which also holds for relations.) From Theorem 6.3(a) one now infers that $\text{dom}(A)$ is dense, because $A^* = A$ is an operator. \square

6.9 Lemma. *Let A be an operator in H , $\text{dom}(A)$ dense, and let $\lambda \in \mathbb{K}$. Then $(\lambda I + A)^* = \bar{\lambda}I + A^*$.*

Proof. Let $x_1 \in \text{dom}(A)$, $y \in \text{dom}(A^*)$. Then

$$((\lambda + A)x_1 | y) = (x_1 | \bar{\lambda}y) + (x_1 | A^*y) = (x_1 | (\bar{\lambda} + A^*)y).$$

Therefore we obtain $\bar{\lambda} + A^* \subseteq (\lambda + A)^*$. This also implies that

$$A^* = (-\lambda + (\lambda + A))^* \supseteq -\bar{\lambda} + (\lambda + A)^*,$$

i.e., $\bar{\lambda} + A^* \supseteq (\lambda + A)^*$. The two inclusions prove the assertion. \square

After these preparations we will now prove the main result of this section.

Proof of Theorem 6.1. (a) \Rightarrow (b), (c). Clearly, it is sufficient to show that $\text{ran}(I + A) = H$. As A is closed and accretive, the latter is satisfied if $\text{ran}(I + A)$ is dense: the inequality $\|(I + A)x\| \geq \|x\|$ ($x \in \text{dom}(A)$) implies that $\text{ran}(I + A)$ is closed.

By Lemma 6.9 the operator $I + A$ is self-adjoint, and $I + A$ is injective since A is accretive. Now Lemma 6.7 implies $\text{ran}(I + A)^\perp = \ker((I + A)^*) = \ker(I + A) = \{0\}$.

(b) \Rightarrow (a). The hypothesis implies that $I + A$ satisfies the conditions in Proposition 6.8, and therefore $I + A$ is self-adjoint. Then Lemma 6.9 shows that A is self-adjoint.

(c) \Rightarrow (b) Being an m-sectorial operator, A is m-accretive. Sectoriality of angle 0 means that $(Ax | x) \geq 0$ for all $x \in \text{dom}(A)$. Applying Remark 6.5(c) one concludes that A is symmetric. \square

6.2 Adjoints of forms and operators

In this section we assume that V and H are Hilbert spaces over \mathbb{K} and that $j \in \mathcal{L}(V, H)$ has dense range. Let $a: V \times V \rightarrow \mathbb{K}$ be a bounded form. For the reader's convenience we recall the definition of the operator A associated with a and j ,

$$A = \{(x, y) \in H \times H; \exists u \in V: j(u) = x, a(u, v) = (y | j(v)) \ (v \in V)\};$$

see Lecture 5.

The following result shows the close connection between adjoints of forms and operators.

6.10 Theorem. *Let V, H and j be as above, let $a: V \times V \rightarrow \mathbb{K}$ be bounded and coercive, and let $A \sim (a, j)$, $B \sim (a^*, j)$.*

Then $B = A^$. If additionally a is symmetric, then A is self-adjoint and positive.*

Proof. First we note that for $x \in \text{dom}(A)$, $y \in \text{dom}(B)$ there exist $u, v \in V$ such that $j(u) = x$, $j(v) = y$ and

$$(Ax | y) = (Ax | j(v)) = a(u, v) = \overline{a^*(v, u)} = \overline{(By | j(u))} = (x | By).$$

Hence $B \subseteq A^*$, by Remark 6.2(a).

As in the proof of Theorem 5.7 we define the form b by

$$b(u, v) := a(u, v) + (j(u) | j(v)) \quad (u, v \in V).$$

Then $(I+A) \sim (b, j)$, $(I+B) \sim (b^*, j)$, and as just noted this implies that $I+B \subseteq (I+A)^*$. From Theorem 5.7 we know that $\text{ran}(I+B) = \text{ran}(I+A) = H$, and therefore Lemma 6.7 implies that $(I+A)^*$ is injective. These properties imply that $I+B = (I+A)^*$ (recall that 'a surjective mapping cannot have a proper injective extension'). Using Lemma 6.9 we obtain $I+B = I+A^*$, and then $B = A^*$.

If a is symmetric, then $A = B = A^*$, and $a(u) \geq 0$ ($u \in V$) implies $(Ax | x) \geq 0$ ($x \in \text{dom}(A)$). \square

6.3 The spectral theorem for compact self-adjoint operators

For Banach spaces X, Y and an operator A from X to Y we recall that A is called **compact** if $A \in \mathcal{L}(X, Y)$ and $A(B_X(0, 1))$ is a relatively compact subset of Y , where $B_X(0, 1)$ is the open unit ball of X . We recall that the set $\mathcal{K}(X, Y)$ of compact operators is a closed subspace of $\mathcal{L}(X, Y)$, and that the composition of a compact operator with a bounded operator is compact. The latter is called the **ideal property** of compact operators.

In the following let H be a Hilbert space. The next theorem is the spectral theorem for compact self-adjoint operators.

6.11 Theorem. (Hilbert) Let $A \in \mathcal{L}(H)$ be compact and self-adjoint. Then there exist $N \subseteq \mathbb{N}$ and an orthonormal system $(e_j)_{j \in N}$ of eigenelements of A with corresponding eigenvalues $(\lambda_j)_{j \in N}$ in $\mathbb{R} \setminus \{0\}$, $\lambda_j \rightarrow 0$ ($j \rightarrow \infty$), such that

$$Ax = \sum_{j \in N} \lambda_j (x | e_j) e_j \quad (x \in H).$$

For the proof we need two preparations.

6.12 Proposition. Let $A \in \mathcal{L}(H)$ be self-adjoint. Then

$$\|A\| = \sup\{|(Ax | x)|; \|x\| \leq 1\}.$$

Proof. “ \geq ” is obvious from $|(Ax | x)| \leq \|A\| \|x\|^2$.

“ \leq ”. Let $a(x, y) := (Ax | y)$ ($x, y \in H$), $c := \sup\{|a(x)|; \|x\| \leq 1\}$, $b(x, y) := c(x | y)$ ($x, y \in H$). Then Schwarz’s inequality (Proposition 5.2) implies $|(Ax | y)| \leq c \|x\| \|y\|$ ($x, y \in H$); therefore $\|A\| \leq c$. \square

6.13 Proposition. Let $0 \neq A \in \mathcal{L}(H)$ be compact and self-adjoint. Then $\|A\|$ or $-\|A\|$ is an eigenvalue of A .

Proof. By Proposition 6.12 there exist a sequence (x_n) , $\|x_n\| = 1$ ($n \in \mathbb{N}$), and $\lambda \in \mathbb{R}$ with $|\lambda| = \|A\|$ such that $(Ax_n | x_n) \rightarrow \lambda$. Therefore

$$0 \leq \|Ax_n - \lambda x_n\|^2 = \|Ax_n\|^2 - 2\lambda (Ax_n | x_n) + \lambda^2 \leq 2\lambda^2 - 2\lambda (Ax_n | x_n) \rightarrow 0;$$

hence $\|Ax_n - \lambda x_n\| \rightarrow 0$ ($n \rightarrow \infty$).

The compactness of A implies that there exists a subsequence (x_{n_j}) of (x_n) such that (Ax_{n_j}) is convergent, $Ax_{n_j} \rightarrow y$. This implies

$$x_{n_j} = \frac{1}{\lambda} (\lambda x_{n_j} - Ax_{n_j}) + \frac{1}{\lambda} Ax_{n_j} \rightarrow \frac{1}{\lambda} y =: x \neq 0,$$

$Ax = \lambda x$. \square

Proof of Theorem 6.11. We show the assertion with $N = \mathbb{N}$ or $N = \{1, \dots, n\}$ for some $n \in \mathbb{N}_0$. By induction we construct an orthonormal sequence $(e_j)_{j \in N}$ with $\{e_j; j \in N\}^\perp = \ker(A)$ and a corresponding sequence of eigenvalues $(\lambda_j)_{j \in N}$, decreasing in absolute values.

Assume that $n \in \mathbb{N}_0$ and that e_1, \dots, e_n and $\lambda_1, \dots, \lambda_n$ are constructed. Define $H_{n+1} := \{e_1, \dots, e_n\}^\perp$. Then $A(H_{n+1}) \subseteq H_{n+1}$; indeed, $x \perp e_j$ implies $(Ax | e_j) = (x | Ae_j) = \lambda_j (x | e_j) = 0$ ($j = 1, \dots, n$). Then $A_{n+1} := A|_{H_{n+1}}$, considered as an operator in H_{n+1} , is compact. If $A_{n+1} = 0$, then the construction is finished, with $N = \{1, \dots, n\}$. If $A_{n+1} \neq 0$, then Proposition 6.13 implies that there exist $\lambda_{n+1} \in \mathbb{R}$ with $|\lambda_{n+1}| = \|A_{n+1}\|$ and $e_{n+1} \in H_{n+1}$ with $\|e_{n+1}\| = 1$ such that $A_{n+1}e_{n+1} = \lambda_{n+1}e_{n+1}$.

The construction yields $|\lambda_j| = \|A_j\|$ ($j \in N$), and as $(\|A_j\|)_{j \in N}$ is decreasing, so is $(|\lambda_j|)_{j \in N}$.

If $N = \mathbb{N}$, then $e_j \rightarrow 0$ weakly ($j \rightarrow \infty$). This implies that $|\lambda_j| = \|Ae_j\| \rightarrow 0$ ($j \rightarrow \infty$), because A is compact; see Exercise 6.1.

Finally we show the representation of A . For $x \in H$ we have

$$\left\| Ax - \sum_{j=1}^n \lambda_j (x | e_j) e_j \right\| = \left\| A_{n+1} \left(x - \sum_{j=1}^n (x | e_j) e_j \right) \right\| \leq \|A_{n+1}\| \|x\| = |\lambda_{n+1}| \|x\|,$$

and this is 0 if $N = \{1, \dots, n\}$ and converges to 0 as $n \rightarrow \infty$ if $N = \mathbb{N}$. The representation of A shows that $\{e_j; j \in N\}^\perp = \ker(A)$. \square

We are going to draw consequences from Hilbert's theorem, for self-adjoint operators with compact resolvent. We will prepare this by some remarks and an example.

6.14 Remark. Let A be an operator in a Banach space X , and assume that there exists $\lambda \in \rho(A)$ such that $R(\lambda, A)$ is a compact operator. Then it follows from the resolvent equation and the ideal property of compact operators that $R(\mu, A)$ is compact for all $\mu \in \rho(A)$. In this case we will simply call A an **operator with compact resolvent**.

6.15 Proposition. Let V, H be Hilbert spaces, let $j \in \mathcal{L}(V, H)$ have dense range, let $a: V \times V \rightarrow \mathbb{K}$ be a bounded coercive form, and let $A \sim (a, j)$. Assume additionally that j is compact.

Then A has compact resolvent.

Proof. From Remark 5.8 we know that the inverse of $I + A$ can be expressed explicitly as $(I + A)^{-1} = j\mathcal{B}^{-1}k$. As \mathcal{B}^{-1} and k are bounded operators, we see that $(I + A)^{-1}$ is compact. \square

6.16 Example. Diagonal forms.

Let H be an infinite-dimensional separable Hilbert space with orthonormal basis $(e_n)_{n \in \mathbb{N}}$, and let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $[0, \infty)$.

Define the space

$$V := \left\{ u \in H; \sum_{n=1}^{\infty} \lambda_n |(u | e_n)_H|^2 < \infty \right\}.$$

Then V is a Hilbert space for the scalar product

$$(u | v)_V := (u | v)_H + a(u, v),$$

where $a(u, v) := \sum_{n=1}^{\infty} \lambda_n (u | e_n)_H (e_n | v)_H$.

Then the injection j of V into H is continuous, and a is continuous and j -elliptic. The operator A associated with (a, j) is self-adjoint. It is easy to see that A is the diagonal operator associated with $(e_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ as defined in Example 6.6.

Assume additionally that $\lim_{n \rightarrow \infty} \lambda_n = \infty$. Then the injection j is compact. Indeed, it is easy to see that the finite rank operators $P_n j$, where P_n denotes the orthogonal projection onto $\text{lin}\{e_1, \dots, e_n\}$, approximate j in the operator norm. By Proposition 6.15 we see that A has compact resolvent (which is also easily obtained directly).

This example is generic for positive self-adjoint operators with compact resolvent, as the following theorem shows.

6.17 Theorem. *Let A be a positive self-adjoint operator with compact resolvent in an infinite-dimensional Hilbert space H . Then there exist an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ and a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} \lambda_n = \infty$ such that A is the associated diagonal operator.*

Proof. From Theorem 6.1 we know that $(I + A)^{-1}$ exists in $\mathcal{L}(H)$, and it is easy to see that this operator is symmetric, hence self-adjoint. By hypothesis, $(I + A)^{-1}$ is compact. Applying Theorem 6.11 to $(I + A)^{-1}$ one obtains an orthonormal system $(e_n)_{n \in \mathbb{N}}$, with a corresponding sequence $(\mu_n)_{n \in \mathbb{N}}$ of eigenvalues. The representation $(I + A)^{-1}x = \sum_{n \in \mathbb{N}} \mu_n(x | e_n)e_n$ together with the injectivity of $(I + A)^{-1}$ implies that $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis. Hence N is countably infinite, without loss of generality $N = \mathbb{N}$.

It follows that e_n is an eigenelement of A with eigenvalue $\lambda_n := \mu_n^{-1} - 1$, for all $n \in \mathbb{N}$. The positivity of A implies

$$\lambda_n = (\lambda_n e_n | e_n) = (Ae_n | e_n) \geq 0 \quad (n \in \mathbb{N}).$$

Since $(\mu_n)_{n \in \mathbb{N}}$ is a null sequence, it follows that $\lim_{n \rightarrow \infty} \lambda_n = \infty$. \square

A sequence (λ_n) in \mathbb{R} with $\lim_{n \rightarrow \infty} \lambda_n = \infty$ can always be rearranged to an increasing sequence, and in applications of Theorem 6.17 one usually assumes the sequence of eigenvalues to be increasing.

Note that the hypotheses of Theorem 6.17 imply that H is separable. We formulate the consequence of this theorem for operators associated with forms.

6.18 Corollary. *Let V, H be infinite-dimensional Hilbert spaces, and let $a: V \times V \rightarrow \mathbb{K}$ be a bounded coercive symmetric form. Let $j \in \mathcal{L}(V, H)$ have dense range, and assume that j is compact. Let A be the operator associated with (a, j) .*

Then A is a positive self-adjoint operator with compact resolvent, and therefore all the conclusions of Theorem 6.17 hold. Additionally, all eigenvalues of A are positive.

Proof. We know from Theorem 6.10 that A is a positive self-adjoint operator, and it was shown in Proposition 6.15 that A has compact resolvent. In order to show that all the eigenvalues are positive, we note that, if λ is an eigenvalue of A with eigenelement x , then $\lambda \|x\|^2 = (Ax | x) = a(u) > 0$, where $u \in V$ with $j(u) = x$. \square

We finally come to the property announced in the introduction of this lecture. In order to draw the conclusion for the Dirichlet Laplacian Δ_D in $L_2(\Omega)$, for bounded open $\Omega \subseteq \mathbb{R}^n$ we will use the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$, which will be proved in the next section.

6.19 Example. Spectral decomposition of the Dirichlet Laplacian.

If $\Omega \subseteq \mathbb{R}^n$ is open and bounded, then Δ_D has compact resolvent. There exists an orthonormal basis $(\varphi_k)_{k \in \mathbb{N}}$ of $L_2(\Omega)$ and an increasing sequence $(\lambda_k)_{k \in \mathbb{N}}$ in $(0, \infty)$, with $\lim_{k \rightarrow \infty} \lambda_k = \infty$, such that $-\Delta_D$ is the associated diagonal operator. In particular, $\varphi_k \in \text{dom}(\Delta_D)$ and

$$-\Delta_D \varphi_k = \lambda_k \varphi_k$$

for all $k \in \mathbb{N}$.

Proof. From Example 5.13 we know that $-\Delta_D$ is associated with the classical Dirichlet form and $j: H_0^1(\Omega) \hookrightarrow L_2(\Omega)$. As the embedding j is compact, by Theorem 6.21, the assertions follow from Corollary 6.18. \square

6.20 Remark. The existence of an orthonormal basis of eigenelements of $-\Delta_D$ is a highlight and triumph of Hilbert space theory as applied to partial differential equations. There is no way to obtain this kind of result by computing eigenfunctions, even if the boundary is nice.

In the case of an interval in one dimension there is no problem (see Exercise 6.4), and this can be generalised to n -dimensional rectangles. For a ball one already has to use Bessel functions.

6.4 Compactness of the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$

The main result of this section is the following special case of the Rellich-Kondrachov theorem.

6.21 Theorem. (*Rellich-Kondrachov*) *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Then the inclusion map $j: H_0^1(\Omega) \rightarrow L_2(\Omega)$ is compact.*

The proof will be given at the end of the section. As a preparation we start with explaining the convolution of functions for a context different from Subsection 4.1.1.

6.22 Proposition. *Let $\rho \in L_1(\mathbb{R}^n)$, and let $1 \leq p \leq \infty$. Then, for all $u \in L_p(\mathbb{R}^n)$,*

$$\rho * u(x) := \int_{\mathbb{R}^n} \rho(x-y)u(y) \, dy = \int_{\mathbb{R}^n} \rho(y)u(x-y) \, dy$$

exists for a.e. $x \in \mathbb{R}^n$, and

$$\|\rho * u\|_p \leq \|\rho\|_1 \|u\|_p.$$

Proof. Let $u \in L_p(\mathbb{R}^n)$. Then the function $\mathbb{R}^{2n} \ni (x, y) \mapsto \rho(x-y)u(y)$ is measurable because the mapping $(x, y) \mapsto (x-y, y)$ is measurable. Let $1 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L_q(\mathbb{R}^n)$ we estimate, using Fubini-Tonelli and Hölder,

$$\begin{aligned} \iint |\rho(y)u(x-y)| \, dy |g(x)| \, dx &= \int \rho(y) \int |u(x-y)||g(x)| \, dx \, dy \\ &\leq \int |\rho(y)| \|u(\cdot - y)\|_p \|g\|_q \, dy \leq \|\rho\|_1 \|u\|_p \|g\|_q. \end{aligned}$$

This inequality shows the assertions. \square

The fundamental result in this context is the following theorem.

6.23 Theorem. *Let $\rho \in L_1(\mathbb{R}^n)$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $1 \leq p \leq \infty$. Then the mapping*

$$L_p(\mathbb{R}^n) \ni u \mapsto (\rho * u)|_{\Omega} \in L_p(\Omega)$$

is compact.

Proof. If $\rho \in C_c(\mathbb{R}^n)$, then we show that

$$F := \{(\rho * u)|_{\bar{\Omega}}; u \in L_p(\mathbb{R}^n), \|u\|_p \leq 1\}$$

is relatively compact in $C(\bar{\Omega})$. Clearly, the set is bounded because $|\rho * u(x)| \leq \|\rho\|_q \|u\|_p$ (with $\frac{1}{q} + \frac{1}{p} = 1$, as usual). Also, the estimate

$$|\rho * u(x) - \rho * u(y)| \leq \int |(\rho(x-z) - \rho(y-z))u(z)| dz \leq \|\rho(x-\cdot) - \rho(y-\cdot)\|_q \|u\|_p$$

shows that F is equicontinuous. By the Arzelà-Ascoli theorem it follows that F is relatively compact in $C(\bar{\Omega})$. The embedding $C(\bar{\Omega}) \hookrightarrow L_p(\Omega)$ is continuous, and therefore F is also relatively compact in $L_p(\Omega)$.

Now let $\rho \in L_1(\mathbb{R}^n)$. Then there exists a sequence (ρ_k) in $C_c(\mathbb{R}^n)$ such that $\rho_k \rightarrow \rho$ in $L_1(\mathbb{R}^n)$. For $k \in \mathbb{N}$ let $J_k \in \mathcal{L}(L_p(\mathbb{R}^n), L_p(\Omega))$ be defined by $J_k u = (\rho_k * u)|_{\Omega}$ ($u \in L_p(\mathbb{R}^n)$), and let J be associated with ρ in the same way. As shown above, J_k is compact for all $k \in \mathbb{N}$. Proposition 6.22 implies that $\|J - J_k\| \leq \|\rho - \rho_k\|_1 \rightarrow 0$ ($k \rightarrow \infty$), and as $\mathcal{K}(L_p(\mathbb{R}^n), L_p(\Omega))$ is closed in $\mathcal{L}(L_p(\mathbb{R}^n), L_p(\Omega))$ we conclude that J is compact. \square

In order to use Theorem 6.23 for the compactness of the embedding $H_0^1 \hookrightarrow L_2$, we derive a representation of functions in $C_c^1(\mathbb{R}^n)$ in terms of their derivatives.

6.24 Lemma. *Let $\varphi \in C_c^1(\mathbb{R}^n)$. Then*

$$\varphi(0) = -\frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla \varphi(y) \cdot \frac{y}{|y|^n} dy,$$

where σ_{n-1} is the $(n-1)$ -dimensional volume of the unit sphere S_{n-1} of \mathbb{R}^n .

Proof. Let $z \in S_{n-1}$. Then

$$\varphi(0) = -\int_0^\infty \frac{d}{dt} \varphi(tz) dt = -\int_0^\infty \nabla \varphi(tz) \cdot z dt.$$

Integrating this formula over S_{n-1} , provided with the surface measure σ , we obtain

$$\begin{aligned} \sigma_{n-1} \varphi(0) &= -\int_{z \in S_{n-1}} \int_{t=0}^\infty \nabla \varphi(tz) \cdot z dt d\sigma(z) \\ &= -\int_{t=0}^\infty \int_{z \in S_{n-1}} \nabla \varphi(tz) \cdot \frac{tz}{t^n} d\sigma(z) t^{n-1} dt = -\int_{\mathbb{R}^n} \nabla \varphi(y) \cdot \frac{y}{|y|^n} dy, \end{aligned}$$

where in the last equality we have used generalised polar coordinates. \square

Proof of Theorem 6.21. In this proof we extend all functions defined on Ω to \mathbb{R}^n , by setting them 0 on $\mathbb{R}^n \setminus \Omega$.

There exists $R > 0$ such that $\Omega \subseteq B(0, R)$. For $\varphi \in C_c^1(\Omega)$, $x \in \Omega$ we obtain, applying Lemma 6.24 to $\varphi(x - \cdot)$,

$$\begin{aligned}\varphi(x) &= \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla \varphi(x - y) \cdot \frac{y}{|y|^n} dy \\ &= \frac{1}{\sigma_{n-1}} \int_{B(0, 2R)} \nabla \varphi(x - y) \cdot \frac{y}{|y|^n} dy = \int_{\mathbb{R}^n} \nabla \varphi(x - y) \cdot \rho(y) dy\end{aligned}$$

with $\rho(y) := \frac{1}{\sigma_{n-1}} \mathbf{1}_{B(0, 2R)}(y) \frac{y}{|y|^n}$.

Due to the cut-off at $2R$, the function ρ belongs to the vector valued $L_1(\mathbb{R}^n; \mathbb{R}^n)$. Therefore Theorem 6.23 implies that the mapping

$$J_2: L_2(\mathbb{R}^n; \mathbb{K}^n) \rightarrow L_2(\Omega), u \mapsto (\rho * u)|_\Omega$$

is compact. Obviously, the mapping

$$J_1: H_0^1(\Omega) \rightarrow L_2(\Omega; \mathbb{K}^n), f \mapsto \nabla f$$

is continuous. This implies that $J_2 J_1: H_0^1(\Omega) \rightarrow L_2(\Omega)$ is compact.

The formula presented above shows that for $\varphi \in C_c^1(\Omega)$ one has $j(\varphi) = J_2 J_1 \varphi$. As both operators j and $J_2 J_1$ are continuous and $C_c^1(\Omega)$ is dense in $H_0^1(\Omega)$ one concludes that $j = J_2 J_1$ is compact. \square

Notes

It is difficult to attribute the development of adjoint operators to a source. Maybe one of the first more systematic treatments is given in [Neu32]. The idea to include also linear relations is contained in [Are61], where the adjoint is also defined for linear relations (whereas we define the adjoint only for operators).

The first version of the spectral theorem of compact self-adjoint operators is contained in [Hil06]. (This paper is also contained in the collection [Hil12].) The diagonal structure of the Dirichlet Laplacian seems to be classical and difficult to attribute; however, the use of compactness methods for this purpose can be attributed to Rellich [Rel30]. In this paper the first version of the Rellich-Kondrachev theorem appeared as well as the application to the Dirichlet Laplacian. The proof we give for the Rellich-Kondrachev theorem is not the one usually found in text books. In particular, the virtual lecturer could not find a version of Theorem 6.23 in the literature.

Exercises

6.1 Let G, H be Hilbert spaces, $A \in \mathcal{L}(G, H)$. Show that A is a compact operator if and only if A maps weakly convergent sequences in G to convergent sequences in H .

Hints: 1. Recall that (x_n) in G is called weakly convergent to $x \in G$ if $(x_n | y) \rightarrow (x | y)$ ($y \in G$), and that every bounded sequence in G contains a weakly convergent subsequence.

2. Recall that $A \in \mathcal{L}(G, H)$ is also continuous with respect to the weak topologies in G and H ; in particular, if (x_n) in G is weakly convergent to $x \in G$, then (Ax_n) is weakly convergent to Ax in H .

6.2 Let (Ω, μ) be a σ -finite measure space, and let $m: \Omega \rightarrow \mathbb{C}$ be measurable. Let A_0 be defined by

$$\text{dom}(A_0) := \text{lin}\{\mathbf{1}_C; C \in \mathcal{C}\},$$

with

$$\mathcal{C} := \left\{ C \subseteq \Omega; C \text{ measurable, } \mu(C) < \infty, \sup_{x \in C} |m(x)| < \infty \right\},$$

$$A_0 f := m f \quad (f \in \text{dom}(A_0)).$$

(a) Show that $\text{dom}(A_0)$ is dense in $L_2(\Omega, \mu)$, and that $A_0^* = M_{\bar{m}}$, where $M_{\bar{m}}$ denotes the maximal multiplication operator by \bar{m} ,

$$\text{dom}(M_{\bar{m}}) := \{f \in L_2(\Omega, \mu); \bar{m}f \in L_2(\Omega, \mu)\},$$

$$M_{\bar{m}} f := \bar{m} f \quad (f \in \text{dom}(M_{\bar{m}})).$$

(b) Show further that $M_{\bar{m}}^* = M_m$ (maximal multiplication operator by m).

(c) Assume that m is real-valued. Show that A_0 is essentially self-adjoint, and $\overline{A_0} = M_m$.

6.3 (a) Let X be a vector space, Y an n -dimensional subspace, Z an $(n-1)$ -codimensional subspace (i.e., there exists an $(n-1)$ -dimensional subspace Z_0 of X such that $Z \cap Z_0 = \{0\}$, $Z + Z_0 = X$). Show that $Y \cap Z \neq \{0\}$.

(b) Let H be an infinite-dimensional separable Hilbert space, and let A be a self-adjoint diagonal operator in H associated with the form a , described as in Examples 6.6 and 6.16. Assume that the sequence of eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ is increasing, with $\lim_{n \rightarrow \infty} \lambda_n = \infty$.

For $x_1, \dots, x_{n-1} \in H$ define

$$I(x_1, \dots, x_{n-1}) := \inf \{a(x); x \in \{x_1, \dots, x_{n-1}\}^\perp \cap V, \|x\| = 1\}.$$

Show the **min-max principle**:

$$\lambda_n = \max \{I(x_1, \dots, x_{n-1}); x_1, \dots, x_{n-1} \in H\} \quad (n \in \mathbb{N}).$$

(c) Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n$ be bounded open sets, and let Δ_j be the Dirichlet-Laplacian on Ω_j and $(\lambda_k^j)_{k \in \mathbb{N}}$ the corresponding increasing sequence of eigenvalues, for $j = 1, 2$.

Show the **domain monotonicity of eigenvalues**: $\lambda_k^1 \geq \lambda_k^2$ for all $k \in \mathbb{N}$.

6.4 (a) Let $f \in C(-1, 1)$, and assume that $g_1 := (f|_{(-1,0)})' \in L_1(-1, 0)$, $g_2 := (f|_{(0,1)})' \in L_1(0, 1)$. Define $g \in L_1(-1, 1)$ by $g|_{(-1,0)} := g_1$, $g|_{(0,1)} := g_2$. Show that $f' = g$ (all derivatives in the distributional sense).

(b) Let $a, b \in \mathbb{R}$, $a < b$. Show that $H_0^1(a, b) = \{f \in H^1(a, b); f(a) = f(b) = 0\}$.

Hints: 1. For $f \in H^1(a, b)$ with $f(a) = f(b) = 0$, show that the extension of f to \mathbb{R} by 0 belongs to $H^1(\mathbb{R})$.

2. Let $c \in (a, b)$, and let $f \in H^1(a, b)$, $f(a) = 0$, $f|_{(c,b)} = 0$. Show that $f \in H_0^1(a, b)$ (by using suitable translates of f).

(c) Compute the orthonormal basis of eigenfunctions and the eigenvalues of $-\Delta_D$ for $\Omega = (0, \pi)$.

(d) Determine the optimal value of the Poincaré constant for the open set $(0, \pi)$ (see Section 5.4).

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Lecture 7

Robin boundary conditions

So far we have only studied the Laplacian with Dirichlet boundary conditions. Our aim of this lecture is to investigate Neumann boundary conditions

$$\partial_\nu u = 0 \quad \text{on } \partial\Omega$$

and more generally Robin boundary conditions

$$\partial_\nu u + \beta u = 0 \quad \text{on } \partial\Omega.$$

If we think of heat conduction in a body Ω , then the Neumann boundary condition describes an isolated body, whereas Robin boundary conditions describe when part of the heat is absorbed at the boundary.

We start with the description of properties of the boundary for an open subset of \mathbb{R}^n . The main issue of Section 7.1 will be the statement of Gauss' theorem and some discussion of its consequences. In particular we point out that it can be considered as an n -dimensional version of the fundamental theorem of calculus.

In an interlude in Section 7.2 we present properties of $H^1(\Omega)$ that will be needed in order to formulate Neumann and Robin boundary conditions and to derive properties of the associated operators.

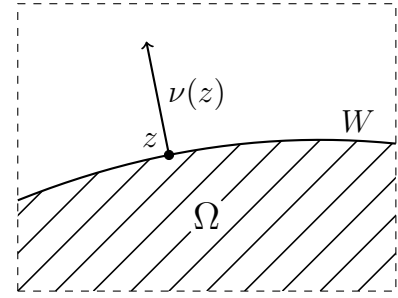
7.1 Gauss' theorem

The aim of this section is to generalise the fundamental theorem of calculus to higher dimensions. For this we need to define the outer normal. We will mainly consider open sets with C^1 -boundary, but in order to be complete we occasionally mention domains with more general boundaries. For $x, y \in \mathbb{K}^n$ we use the notation $x \cdot y := \sum_{j=1}^n x_j y_j$ (which for \mathbb{R}^n is the natural scalar product) and $|x| := \sqrt{x \cdot \bar{x}}$ (which for \mathbb{R}^n is the Euclidean norm). All over this section, Ω is a bounded open subset of \mathbb{R}^n .

Let $W \subseteq \partial\Omega$ be an open subset (of the metric space $\partial\Omega$). Then we say that W is a **normal C^1 -graph** (with respect to Ω) if $W' := \{(z_1, \dots, z_{n-1}); z = (z_1, \dots, z_n) \in W\} \subseteq \mathbb{R}^{n-1}$ is open and there exist an open interval $(a, b) \subseteq \mathbb{R}$ and a C^1 -function $g: W' \rightarrow (a, b)$ such that $W = \{(y, g(y)); y \in W'\}$, i.e., W is the graph of g , and for a point $(y, t) \in W' \times (a, b)$ one has

$$(y, t) \in \Omega \quad \text{if and only if} \quad t < g(y).$$

It is easy to see that then $(y, t) \notin \bar{\Omega}$ if and only if $t > g(y)$. The set W is a C^1 -**graph** (with respect to Ω), if there exists an orthogonal matrix $B \in \mathbb{R}^{n \times n}$ such that $\Phi(W)$ is a normal C^1 -graph with respect to $\Phi(\Omega)$, where $\Phi(x) = Bx$ ($x \in \mathbb{R}^n$). This means of course that W is a normal C^1 -graph with respect to another cartesian coordinate system. We say that Ω has C^1 -**boundary** if for each $z \in \partial\Omega$ there exists an open neighbourhood $W \subseteq \partial\Omega$ of z such that W is a C^1 -graph.



We define right away other possible regularity properties of the boundary.

7.1 Remarks. (C^k -, Lipschitz, continuous boundary)

(a) For $k \in \mathbb{N} \cup \{\infty\}$ we call W a C^k -**graph** if the function g in the definition given above is a C^k -function.

We talk of a **Lipschitz graph** if $g: W' \rightarrow (a, b)$ satisfies $|g(x) - g(y)| \leq L|x - y|$ for all $x, y \in W'$ and some $L > 0$. We call V a **continuous graph** if we merely require that $g: W' \rightarrow (a, b)$ is continuous.

(b) We say that Ω has C^k -**boundary** (**Lipschitz boundary**, **continuous boundary**), if for each $z \in \partial\Omega$ there exists an open neighbourhood $W \subseteq \partial\Omega$ of z such that W is a C^k -graph (or Lipschitz graph, or continuous graph).

In this way we have defined a hierarchy of regularity properties. If Ω has C^1 -boundary, then it also has Lipschitz boundary. Continuous boundary is the weakest property we consider and C^∞ -boundary is the strongest. Each polygon in \mathbb{R}^2 and each convex polyhedron in \mathbb{R}^3 has Lipschitz boundary, but not C^1 -boundary. So there are good reasons to consider Lipschitz boundary. But things become much easier in C^1 -domains, on which we will focus.

At first we introduce the outer (or exterior) normal of a C^1 -domain. This can be done intrinsically without mentioning the graph.

7.2 Proposition. *Assume that Ω has C^1 -boundary. Then for each $z \in \partial\Omega$ there is a unique vector $\nu(z) \in \mathbb{R}^n$ satisfying*

- (i) $|\nu(z)| = 1$;
- (ii) if $\psi \in C^1(-1, 1; \mathbb{R}^n)$ is such that $\psi(0) = z$ and $\psi(t) \in \partial\Omega$ for all $t \in (-1, 1)$, then $\nu(z) \cdot \psi'(0) = 0$;
- (iii) there exists $\varepsilon > 0$ such that $z + t\nu(z) \notin \bar{\Omega}$ (and $z - t\nu(z) \in \Omega$) for all $0 < t < \varepsilon$.

We call $\nu(z)$ the **outer normal at z** . It is a continuous function on $\partial\Omega$ with values in \mathbb{R}^n .

Condition (ii) says that $\nu(z)$ is orthogonal to the boundary and (iii) that $\nu(z)$ points out of Ω .

7.3 Remark. We do not give a proof of Proposition 7.2 but we mention that

$$\nu(z) = \frac{(-\nabla g(z'), 1)}{\sqrt{|\nabla g(z')|^2 + 1}}$$

if $z = (z', z_n) \in W$, with W as in the description of a normal C^1 -graph.

Now we can formulate Gauss' theorem. By $C^1(\bar{\Omega})$ we denote the space of all functions $u \in C^1(\Omega) \cap C(\bar{\Omega})$ for which $\partial_j u$ has a continuous extension to $\bar{\Omega}$ for each $j \in \{1, \dots, n\}$. We keep the notation $\partial_j u$ for this extension.

7.4 Theorem. (*Gauss*) Assume that $\Omega \subseteq \mathbb{R}^n$ is open and bounded and has C^1 -boundary. There exists a unique Borel measure σ on $\partial\Omega$, the **surface measure** on $\partial\Omega$, such that

$$\int_{\Omega} \partial_j u(x) \, dx = \int_{\partial\Omega} u(z) \nu_j(z) \, d\sigma(z)$$

for each $j \in \{1, \dots, n\}$ and all $u \in C^1(\bar{\Omega})$. Here $\nu \in C(\partial\Omega; \mathbb{R}^n)$ is the outer normal with coordinates $\nu(z) = (\nu_1(z), \dots, \nu_n(z))$.

7.5 Remarks. (a) If $W \subseteq \partial\Omega$ is a normal graph and $u \in C(\partial\Omega)$ has support in W , then

$$\int_{\partial\Omega} u(z) \, d\sigma(z) = \int_{W'} u(z', g(z')) \sqrt{|\nabla g(z')|^2 + 1} \, dz',$$

where g is as in the definition for the normal graph. This formula is one of the ingredients for the construction of the measure σ . The weight factor in the integral on the right hand side is such that the $(n-1)$ -dimensional Lebesgue measure on W' is transferred to the appropriate $(n-1)$ -dimensional measure on the $(n-1)$ -dimensional manifold $\partial\Omega$.

(b) For the proof of Theorem 7.4 and Proposition 7.2 we refer to [AU10; Sections 7.1, 7.2] or to calculus books.

Gauss' theorem can be considered as the n -dimensional version of the fundamental theorem of calculus. In fact, for $\Omega = (a, b)$ we have $\partial\Omega = \{a, b\}$, $\nu(a) = -1$, $\nu(b) = 1$. Then for $u \in C^1[a, b]$ we can write

$$\int_a^b u'(x) \, dx = u(b) - u(a) = \int_{\{a, b\}} u(z) \nu(z) \, d\sigma(z),$$

with the counting measure σ .

Next we derive an important consequence of Gauss' theorem. We define $C^2(\bar{\Omega}) := \{u \in C^1(\bar{\Omega}); \partial_j u \in C^1(\bar{\Omega}) \ (j = 1, \dots, n)\}$. Then for $u \in C^2(\bar{\Omega})$ the functions $\partial_i \partial_j u$ are in $C(\bar{\Omega})$ for all $i, j = 1, \dots, n$. For $u \in C^1(\bar{\Omega})$ the function $\partial_\nu u: \partial\Omega \rightarrow \mathbb{K}$, given by

$$\partial_\nu u(z) := \nabla u(z) \cdot \nu(z) = \sum_{j=1}^n \partial_j u(z) \nu_j(z),$$

is called the **normal derivative** of u . Note that $\partial_\nu u \in C(\partial\Omega)$.

7.6 Corollary. (*Green's formulas*) Let $u \in C^2(\bar{\Omega})$. Then

$$\int_{\Omega} (\Delta u)v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} (\partial_\nu u)v \, d\sigma \quad (v \in C^1(\bar{\Omega})), \quad (7.1)$$

$$\int_{\Omega} (v\Delta u - u\Delta v) \, dx = \int_{\partial\Omega} (v\partial_\nu u - u\partial_\nu v) \, d\sigma \quad (v \in C^2(\bar{\Omega})). \quad (7.2)$$

Proof. By Gauss' theorem one has

$$\int_{\Omega} \partial_j u \partial_j v = - \int_{\Omega} (\partial_j^2 u) v + \int_{\Omega} \partial_j ((\partial_j u) v) = - \int_{\Omega} (\partial_j^2 u) v + \int_{\partial\Omega} (\partial_j u) v \nu_j \, d\sigma.$$

Summation over $j = 1, \dots, n$ yields (7.1).

Exchanging u and v in (7.1) and subtracting the result from (7.1) one obtains (7.2). \square

7.2 Interlude: more on $H^1(\Omega)$; denseness, trace and compactness

In this section we give some information on $H^1(\Omega)$ that will be needed in the following. We could have simply quoted these results, because our main interest is to present form methods. However, we felt that just quoting the results would have the effect that the reader will not be aware of the analysis facts behind the treated situations. So we give the proofs, but we will be somewhat sketchy in the presentation. In a first reading there should be no problem if you just take notice of the results and first look at the further development.

The first issue is a denseness theorem for $H^1(\Omega)$. To recall, denseness theorems are needed to transfer properties one can show classically to more general functions if suitable estimates are provided. An example for this procedure is the proof of Poincaré's inequality in Lecture 5; further examples follow in this section.

7.7 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with continuous boundary. Then the set*

$$\check{C}^\infty(\Omega) := \{\varphi|_{\Omega}; \varphi \in C_c^\infty(\mathbb{R}^n)\}$$

is dense in $H^1(\Omega)$. As a consequence, $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$.

Proof. (i) Let $W \subseteq \partial\Omega$ be a normal continuous graph, with (a, b) and g as described at the beginning of Section 7.1. Let $u \in H^1(\Omega)$ be such that $\text{spt } u$ is a relatively compact subset of $W' \times (a, b)$, and extend u by 0 to \mathbb{R}^n . For $\tau > 0$ we define

$$u_\tau(x) := u(x', x_n - \tau) \quad (x = (x', x_n) \in \mathbb{R}^n).$$

Then $u_\tau|_{\Omega} \in H^1(\Omega)$ for small τ , and $u_\tau|_{\Omega} \rightarrow u$ in $H^1(\Omega)$ as $\tau \rightarrow 0$. Moreover, for $\tau > 0$ one has that $u_\tau \in H^1(\Omega + \tau e_n)$ (with the n -th unit vector $e_n \in \mathbb{R}^n$). Let $(\rho_k)_{k \in \mathbb{N}}$ be a δ -sequence in $C_c^\infty(\mathbb{R}^n)$. Then $\rho_k * u_\tau \in C_c^\infty(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, and it is not too difficult to see that $(\rho_k * u_\tau)|_{\Omega} \rightarrow u_\tau|_{\Omega}$ in $H^1(\Omega)$ as $k \rightarrow \infty$. (In the last step one has to use a 'local version' of Lemma 4.13.)

From the above we conclude: for each continuous graph $W \subseteq \partial\Omega$ there exists an open set $U \subseteq \mathbb{R}^n$ such that $W = U \cap \partial\Omega$, and such that each $u \in H^1(\Omega)$ with relatively compact support in U can be approximated by functions in $\check{C}^\infty(\Omega)$.

(ii) A compactness argument shows that $\partial\Omega$ can be covered by open sets W_1, \dots, W_m such that each W_k is a continuous graph, with a corresponding open set $U_k \subseteq \mathbb{R}^n$ as indicated at the end of part (i) above. Then, defining $U_0 := \Omega$, the family $(U_k)_{k \in \{0, \dots, m\}}$

is an open covering of $\bar{\Omega}$, and there exists a partition of unity $(\varphi_k)_{k \in \{0, \dots, m\}}$ in $C_c^\infty(\mathbb{R}^n)$ on $\bar{\Omega}$, subordinate to $(U_k)_{k \in \{0, \dots, m\}}$. ('On $\bar{\Omega}$ ' means that $\sum_{k=0}^m \varphi_k|_{\bar{\Omega}} = 1$, and 'subordinate' means that $\text{spt } \varphi_k \subseteq U_k$ ($k = 0, \dots, m$). We refer to [AU10; Satz 7.12] for the existence of a partition of unity as above.)

Let $u \in H^1(\Omega)$. Then $\varphi_0 u \in H_0^1(\Omega)$ can be approximated by $C_c^\infty(\Omega)$ -functions, by Theorem 4.12(b), and $\varphi_1 u, \dots, \varphi_m u$ can be approximated by $\check{C}^\infty(\Omega)$ -functions, by part (i). In consequence, u can be approximated by $\check{C}^\infty(\Omega)$ -functions. \square

7.8 Remarks. (a) The proof we have given uses that Ω having continuous boundary implies that Ω satisfies the 'segment property'. We refer to [Ada75; 3.17] for this property and to [Ada75; Theorem 3.18] for Theorem 7.7. It is not difficult to show that the segment property is in fact equivalent to continuous boundary.

(b) The following important observation will be used in the proofs of Theorems 7.9 and 7.10. The procedure used in the proof of Theorem 7.7 can be adapted to yield simultaneous approximation with respect to other properties. For instance, if $u \in H^1(\Omega) \cap C(\bar{\Omega})$, then the approximations can be chosen to additionally approximate u in the sup-norm. And also: if $u \in H^1(\Omega)$ is such that $\Delta u \in L_2(\Omega)$, then the approximations u_k can be chosen such that additionally $\Delta u_k \rightarrow \Delta u$ in $L_2(\Omega)$.

In order to explain this in somewhat more detail we first mention that multiplying u by a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ does not effect the additional properties mentioned in the previous paragraph. This is clear for the case that u is continuous. But also, if $\Delta u \in L_2(\Omega)$, then $\Delta(\varphi u) = (\Delta\varphi)u + 2\nabla\varphi \cdot \nabla u + \varphi\Delta u \in L_2(\Omega)$. In view of part (ii) of the proof of Theorem 7.7, this means that for the approximation we only have to treat the 'local' case considered in part (i).

Concerning part (i) of the proof of Theorem 7.7, the case of continuous u is done by invoking a local version of Proposition 4.3(a). The case $\Delta u \in L_2(\Omega)$ is slightly more involved: one has to use a local version of Lemma 4.13(a), with ∂^α replaced by Δ , and for the convergence a local version of Proposition 4.3(b).

Next we show that for Ω with C^1 -boundary one can define a trace mapping $\text{tr}: H^1(\Omega) \rightarrow L_2(\partial\Omega)$ such that for $u \in C^1(\bar{\Omega})$ one has $\text{tr } u = u|_{\partial\Omega}$.

7.9 Theorem. *Assume that $\Omega \subseteq \mathbb{R}^n$ is open, bounded, and with C^1 -boundary. Then there exists $c \geq 0$ such that*

$$\|u|_{\partial\Omega}\|_{L_2(\partial\Omega)}^2 \leq c \|u\|_{L_2(\Omega)} \|u\|_{H^1(\Omega)} \quad (7.3)$$

for all $u \in C^1(\bar{\Omega})$.

There is a unique bounded operator $\text{tr}: H^1(\Omega) \rightarrow L_2(\partial\Omega)$, called the **trace operator**, such that $\text{tr } u = u|_{\partial\Omega}$ for all $u \in C(\bar{\Omega}) \cap H^1(\Omega)$, and then (7.3) holds for all $u \in H^1(\Omega)$ (with $u|_{\partial\Omega}$ replaced by $\text{tr } u$ on the left hand side).

Proof. (i) Let $W \subseteq \partial\Omega$ be a normal C^1 -graph, with (a, b) and g as in the definition. Let $x \in W$, and let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\text{spt } \varphi \subseteq W' \times (a, b)$, $\varphi \geq 0$, and $\varphi = 1$ on a neighbourhood of x . Then there exists an open neighbourhood $W_x \subseteq W$ of x such that $\varphi = 1$ on W_x . As $W \ni z \mapsto \nu(z) \in \mathbb{R}^n$ is continuous and $\nu_n(z) > 0$ for all $z \in W$, one concludes that $\delta := \inf_{z \in W_x} \nu_n(z) > 0$.

Let $u \in C^1(\bar{\Omega})$. Then, by Gauss' theorem,

$$\begin{aligned} \delta \int_{W_x} |u|^2 d\sigma &\leq \int_{\partial\Omega} (\varphi u) \bar{u} \nu_n d\sigma = \int_{\Omega} (\partial_n(\varphi u) \bar{u} + (\varphi u) \partial_n \bar{u}) \\ &\leq \|\varphi u\|_{H^1} \|u\|_{L_2} + \|\varphi u\|_{L_2} \|u\|_{H^1} \leq c_\varphi \|u\|_{L_2} \|u\|_{H^1}, \end{aligned}$$

with c_φ only depending on φ .

We have shown that for each $x \in \partial\Omega$ there exist an open neighbourhood $W_x \subseteq \partial\Omega$ and a constant $c_x \geq 0$ such that

$$\int_{W_x} |u|^2 d\sigma \leq c_x \|u\|_{L_2} \|u\|_{H^1}$$

for all $u \in C^1(\bar{\Omega})$. A standard compactness argument finishes the proof of (7.3) for $u \in C^1(\bar{\Omega})$.

(ii) The inequality (7.3) together with the denseness of $C^1(\bar{\Omega})$ in $H^1(\Omega)$ implies that the mapping $u \mapsto u|_{\partial\Omega}$ has a continuous extension $\text{tr}: H^1(\Omega) \rightarrow L_2(\partial\Omega)$.

(iii) So far, we only have shown that $\text{tr} u = u|_{\partial\Omega}$ holds for $u \in C^1(\bar{\Omega})$. In order to show it for $u \in C(\bar{\Omega}) \cap H^1(\Omega)$, we use the remarkable feature of the proof of Theorem 7.7 mentioned in Remark 7.8(b). As explained there, for $u \in C(\bar{\Omega}) \cap H^1(\Omega)$ an approximating sequence (u_k) in $C^1(\bar{\Omega})$ can be chosen converging to u in $C(\bar{\Omega})$ as well as in $H^1(\Omega)$. For this sequence, $(u_k|_{\partial\Omega})$ converges to $\text{tr} u$ in $L_2(\partial\Omega)$ and uniformly to $u|_{\partial\Omega}$, and this implies $\text{tr} u = u|_{\partial\Omega}$. \square

By abuse of notation, we still write $u|_{\partial\Omega} := \text{tr} u$ for $u \in H^1(\Omega)$. In integrals we will frequently omit the trace notation to make things more readable. Here and in the following we always understand $L_2(\partial\Omega)$ with respect to the surface measure.

The trace is compatible with our definition of $H_0^1(\Omega)$ as the following result shows.

7.10 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with C^1 -boundary. Then*

$$H_0^1(\Omega) = \{u \in H^1(\Omega); \text{tr} u = 0\}.$$

Proof. The inclusion ' \subseteq ' follows from the continuity of the trace operator since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$.

Let $u \in H^1(\Omega)$ be such that $\text{tr} u = 0$. Our aim is to show that $u \perp H_0^1(\Omega)^\perp$ (where throughout this proof the orthogonality symbol ' \perp ' refers to the scalar product $(\cdot | \cdot)_1$ in $H^1(\Omega)$). If this is proved we are done, because $H_0^1(\Omega)^{\perp\perp} = H_0^1(\Omega)$.

As a first step we note that one easily shows – using the definition of distributional derivatives – that $H_0^1(\Omega)^\perp = \{v \in H^1(\Omega); \Delta v = v\}$.

Now, let $v \in H_0^1(\Omega)^\perp$; then $\Delta v = v$. Thus, as explained in Remark 7.8(b), there exists a sequence (v_k) in $C^2(\bar{\Omega})$ such that $v_k \rightarrow v$ in $H^1(\Omega)$ and $\Delta v_k \rightarrow \Delta v = v$ in $L_2(\Omega)$. Then (7.1) in combination with Theorem 7.9 implies

$$\int_{\Omega} (\Delta v_k) \bar{u} + \int_{\Omega} \nabla v_k \cdot \bar{\nabla} u = \int_{\partial\Omega} (\partial_\nu v_k) \text{tr} \bar{u} d\sigma = 0$$

for all $k \in \mathbb{N}$, and for $k \rightarrow \infty$ one obtains

$$(v | u)_1 = \int_{\Omega} v \bar{u} + \int_{\Omega} \nabla v \cdot \bar{\nabla} u = 0. \quad \square$$

For another proof of Theorem 7.10 we refer to [Eva10; Section 5.5, Theorem 2].

We close the section by transferring the compactness of the embedding $H_0^1(\Omega) \subseteq L_2(\Omega)$ to $H^1(\Omega)$.

7.11 Theorem. (*Rellich-Kondrachov*) *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with continuous boundary. Then the embedding $j: H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact.*

Proof. The hypothesis implies that there exist a covering of $\partial\Omega$ by continuous graphs W_1, \dots, W_m and vectors $y^1, \dots, y^m \in \mathbb{R}^n$ with the following properties: $\Omega \setminus \bigcup_{k=1}^m U_{k,\varepsilon}$ is a compact subset of Ω for all $\varepsilon > 0$, where

$$U_{k,\varepsilon} := \bigcup_{0 < s < \varepsilon} (W_k + sy^k) \quad (k = 1, \dots, m, \varepsilon > 0),$$

and $U_{k,2} \subseteq \Omega$ for $k = 1, \dots, m$.

Let $u \in C^1(\bar{\Omega})$. Let $k \in \{1, \dots, m\}$, $0 < \varepsilon < 1$. Then for $x \in U_{k,\varepsilon}$ we have

$$u(x) = - \int_0^1 \frac{d}{dt} ((1-t)u(x + ty^k)) dt.$$

It follows that

$$\int_0^\varepsilon |u(z + sy^k)|^2 ds \leq \varepsilon \int_0^2 (|u(z + ty^k)|^2 + |\nabla u(z + ty^k)|^2 |y^k|^2) dt$$

for all $z \in W_k$. Therefore, $\int_{U_{k,\varepsilon}} |u|^2 dx \leq 2\varepsilon (\|u\|_{L_2(U_{k,2})}^2 + |y^k|^2 \|\nabla u\|_{L_2(U_{k,2}; \mathbb{R}^n)}^2)$. Summing over $k = 1, \dots, m$ we obtain

$$\int_{U_\varepsilon} |u|^2 dx \leq \varepsilon C \|u\|_{H^1(\Omega)}^2, \quad (7.4)$$

with the ‘boundary layer’ $U_\varepsilon := \bigcup_{k=1}^m U_{k,\varepsilon}$ and $C := \sum_{k=1}^m 2(1 + |y^k|^2)$. From Theorem 7.7 we conclude that (7.4) holds for all $u \in H^1(\Omega)$.

There exists a function $\psi_\varepsilon \in C_c^\infty(\Omega)$ with $\mathbf{1}_{\Omega \setminus U_\varepsilon} \leq \psi_\varepsilon \leq 1$. Define $j_\varepsilon: H^1(\Omega) \rightarrow L_2(\Omega)$ by $j_\varepsilon(u) := \psi_\varepsilon u$. Then Theorem 6.21 implies that j_ε is compact. Further, (7.4) implies that $\|j - j_\varepsilon\|_{\mathcal{L}(H^1(\Omega), L_2(\Omega))} \leq (\varepsilon C)^{1/2}$.

Summing up, we have shown that the embedding j can be approximated in $\mathcal{L}(H^1(\Omega), L_2(\Omega))$ by compact operators, and this shows that j is compact. \square

We also refer to [EE87; Theorem 4.17] for Theorem 7.11.

7.12 Remark. We note that (7.3) in combination with Theorem 7.11 also shows that the trace mapping in Theorem 7.9 is compact. Indeed, if (u_k) is a bounded sequence in $H^1(\Omega)$, then Theorem 7.11 implies that there exists a subsequence (u_{k_i}) converging in $L_2(\Omega)$, and then (7.3) implies that $(\text{tr } u_{k_i})$ is a Cauchy sequence, hence convergent.

7.3 Weak normal derivative

Now we want to define the normal derivative for certain functions in $H^1(\Omega)$. At first we recall the weak definition of the Laplace operator.

Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $u, f \in L_2(\Omega)$. Then $\Delta u = f$ if

$$\int_{\Omega} u \Delta \varphi = \int_{\Omega} f \varphi \quad (\varphi \in C_c^\infty(\Omega)).$$

For $u \in L_2(\Omega)$ we say that $\Delta u \in L_2(\Omega)$ if there exists $f \in L_2(\Omega)$ such that $\Delta u = f$.

Now we define the normal derivative in a weak sense by asking Green's formula (7.1) to be valid.

Let Ω be bounded, with C^1 -boundary, and let $u \in H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$. We say that $\partial_\nu u \in L_2(\partial\Omega)$ if there exists $h \in L_2(\partial\Omega)$ such that

$$\int_{\Omega} (\Delta u)v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} h v \, d\sigma \quad (v \in H^1(\Omega)).$$

In this case we let $\partial_\nu u := h$. (In order to show the uniqueness of h we note that $C^1(\bar{\Omega}) \subseteq H^1(\Omega)$. The Stone-Weierstrass theorem ([Yos68; Section 0.2]) implies that the set $\{\varphi|_{\partial\Omega}; \varphi \in C^1(\bar{\Omega})\}$ is dense in $C(\partial\Omega)$. As $C(\partial\Omega)$ is dense in $L_2(\partial\Omega)$ we obtain the uniqueness.) In the integral over $\partial\Omega$ we did omit the trace sign. The integral over Ω is always with respect to the Lebesgue measure and that over $\partial\Omega$ always with respect to the surface measure σ .

7.4 The Neumann Laplacian

Now we consider Neumann boundary conditions. Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with C^1 -boundary, and let $u \in H^1(\Omega)$ be such that $\Delta u \in L_2(\Omega)$. Then, by the definition given in the previous section, $\partial_\nu u = 0$ if and only if

$$\int_{\Omega} (\Delta u)v + \int_{\Omega} \nabla u \cdot \nabla v = 0 \quad (v \in H^1(\Omega)). \quad (7.5)$$

It is remarkable that (7.5) makes sense for arbitrary open sets. Therefore, for open sets $\Omega \subseteq \mathbb{R}^n$ and $u \in H^1(\Omega)$ we will write ' $\partial_\nu u = 0$ ' (including the quotes!) if (7.5) holds. This leads us to the following definition.

Let $\Omega \subseteq \mathbb{R}^n$ be open (not necessarily bounded). We define the operator Δ_N in $L_2(\Omega)$ by

$$\begin{aligned} \text{dom}(\Delta_N) &:= \{u \in H^1(\Omega); \Delta u \in L_2(\Omega), \text{ '}\partial_\nu u = 0\text{'}\}, \\ \Delta_N u &:= \Delta u \quad (u \in \text{dom}(\Delta_N)). \end{aligned}$$

We call Δ_N the **Laplacian with Neumann boundary condition** or simply the **Neumann Laplacian**.

7.13 Theorem. *The negative Neumann Laplacian $-\Delta_N$ is self-adjoint and positive. It is associated with the classical Dirichlet form on $H^1(\Omega) \times H^1(\Omega)$.*

Proof. Define $a: H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{K}$ by $a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx$. Then a is continuous. We consider $H^1(\Omega)$ as a subspace of $H := L_2(\Omega)$. Since $a(u) + \|u\|_{L_2(\Omega)}^2 = \|u\|_{H^1(\Omega)}^2$, the form a is H -elliptic. Moreover, a is accretive. Let $A \sim a$. We show that $A = -\Delta_N$. Let $u \in \text{dom}(A)$, $Au = f$. Then by definition $\int_{\Omega} \nabla u \cdot \overline{\nabla v} = \int_{\Omega} f \overline{v}$ for all $v \in H^1(\Omega)$. Inserting test functions $v \in C_c^\infty(\Omega)$ one obtains $-\Delta u = f$. Thus $\int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\Omega} (\Delta u) \overline{v} = 0$ for all $v \in H^1(\Omega)$, i.e., u satisfies (7.5). We have shown that $A \subseteq -\Delta_N$. Conversely, if $u \in \text{dom}(\Delta_N)$ and $-\Delta_N u = f$, then

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} = \int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\Omega} (\Delta u) \overline{v} + \int_{\Omega} f \overline{v} = \int_{\Omega} f \overline{v} \quad (v \in H^1(\Omega)).$$

Thus $u \in \text{dom}(A)$ and $Au = f$. \square

Applying Theorem 7.11 one concludes that A has compact resolvent if Ω satisfies our weakest regularity property.

7.14 Theorem. (*Spectral decomposition of the Neumann Laplacian*) *If $\Omega \subseteq \mathbb{R}^n$ is open, bounded and has continuous boundary, then Δ_N has compact resolvent. There exist an orthonormal basis $(\varphi_k)_{k \in \mathbb{N}}$ of $L_2(\Omega)$ and an increasing sequence $(\lambda_k)_{k \in \mathbb{N}}$ in $[0, \infty)$, with $\lambda_1 = 0$ and $\lim_{k \rightarrow \infty} \lambda_k = \infty$, such that $-\Delta_N$ is the associated diagonal operator. In particular, $\varphi_k \in \text{dom}(\Delta_N)$ and*

$$-\Delta_N \varphi_k = \lambda_k \varphi_k$$

for all $k \in \mathbb{N}$.

Proof. From Theorem 7.11 we know that the embedding $j: H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact. Therefore Proposition 6.15 – in combination with Theorem 7.13 – implies that Δ_N has compact resolvent.

The statement concerning the eigenfunctions and eigenvalues now follows from Theorem 6.17, except for the property that $\lambda_1 = 0$. However, it is immediate that $\varphi_1 = \text{vol}(\Omega)^{-1/2} \mathbf{1}_{\Omega}$ is an eigenfunction of $-\Delta_N$ with eigenvalue 0. \square

As a consequence, we deduce the following:

$$\begin{aligned} \text{dom}(\Delta_N) &= \left\{ u \in L_2(\Omega); \sum_{j=1}^{\infty} \lambda_j^2 |(u | \varphi_j)_{L_2(\Omega)}|^2 < \infty \right\}, \\ -\Delta_N u &= \sum_{j=1}^{\infty} \lambda_j (u | \varphi_j)_{L_2(\Omega)} \varphi_j, \\ T(t)u &= \sum_{j=1}^{\infty} e^{-\lambda_j t} (u | \varphi_j)_{L_2(\Omega)} \varphi_j, \end{aligned}$$

where T denotes the C_0 -semigroup generated by Δ_N .

We will see later that $\lambda_2 > 0$ if Ω is connected. This gives important information concerning the asymptotic behaviour of $T(t)$ as $t \rightarrow \infty$. We have to wait until we have discussed positivity.

7.5 The Robin Laplacian

Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with C^1 -boundary. Here we consider Robin boundary conditions. Given $\beta \in L_\infty(\partial\Omega)$ we define the operator Δ_β in $L_2(\Omega)$ by

$$\begin{aligned} \text{dom}(\Delta_\beta) &:= \{u \in H^1(\Omega); \Delta u \in L_2(\Omega), \partial_\nu u + \beta u|_{\partial\Omega} = 0\}, \\ \Delta_\beta u &:= \Delta u. \end{aligned}$$

Note that the condition ' $\partial_\nu u + \beta u|_{\partial\Omega} = 0$ ' should be read as ' $\partial_\nu u = -\beta u|_{\partial\Omega}$ ', in the sense of the definition in Section 7.3. We call Δ_β the **Laplacian with Robin boundary conditions** or briefly **Robin Laplacian**. We now state and prove properties about the Robin Laplacian, announced in the title of the lecture.

7.15 Theorem. *Let β be real-valued. Then the operator $-\Delta_\beta$ is self-adjoint and quasi-accretive, with compact resolvent. In particular, Δ_β generates a quasi-contractive C_0 -semigroup T_β on $L_2(\Omega)$. If $\beta \geq 0$, then $-\Delta_\beta$ is accretive and $\|T_\beta(t)\| \leq 1$ for all $t \geq 0$.*

Proof. Consider the form $a: H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{K}$ given by $a(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_{\partial\Omega} \beta u \bar{v}$. Then $|a(u, v)| \leq \|\nabla u\|_2 \|\nabla v\|_2 + \|\beta\|_{L_\infty(\partial\Omega)} \|\text{tr } u\|_{L_2(\partial\Omega)} \|\text{tr } v\|_{L_2(\partial\Omega)}$. Since the trace is continuous, it follows that a is continuous.

We consider $H^1(\Omega)$ as a subspace of $H := L_2(\Omega)$ and claim that a is H -elliptic, i.e., that

$$\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \beta |u|^2 + \omega \int_\Omega |u|^2 \geq \alpha \|u\|_{H^1(\Omega)}^2$$

for some $\omega \geq 0$, $\alpha > 0$ and all $u \in H^1(\Omega)$. By Theorem 7.9 and Euclid's inequality (i.e., $ab \leq \frac{1}{2}(\gamma a^2 + \frac{1}{\gamma} b^2)$ for all $a, b \geq 0$, $\gamma > 0$) there exists $c > 0$ such that

$$\|\beta\|_{L_\infty(\partial\Omega)} \int_{\partial\Omega} |u|^2 \leq \frac{1}{2} \|u\|_{H^1(\Omega)}^2 + c \|u\|_{L_2(\Omega)}^2$$

for all $u \in H^1(\Omega)$. Hence

$$\int_{\partial\Omega} \beta |u|^2 \geq -\frac{1}{2} \int_\Omega |\nabla u|^2 - \left(\frac{1}{2} + c\right) \int_\Omega |u|^2,$$

and therefore

$$\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \beta |u|^2 \geq \frac{1}{2} \int_\Omega |\nabla u|^2 - \left(\frac{1}{2} + c\right) \int_\Omega |u|^2$$

for all $u \in H^1(\Omega)$. This proves the claim.

Let A be the operator associated with a . We show that $A = -\Delta_\beta$. Let $(u, f) \in A$. Then

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_{\partial\Omega} \beta u \bar{v} = \int_\Omega f \bar{v} \quad (v \in H^1(\Omega)). \quad (7.6)$$

Taking $v \in C_c^\infty(\Omega)$ we see that $-\Delta u = f$. Replacing f by $-\Delta u$ in (7.6) we find

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (\Delta u) \bar{v} = - \int_{\partial\Omega} \beta u \bar{v} \quad (v \in H^1(\Omega)). \quad (7.7)$$

This is equivalent to $\partial_\nu u = -\beta u|_{\partial\Omega}$. Thus $(u, f) \in -\Delta_\beta$. Conversely, if $u \in \text{dom}(\Delta_\beta)$, then (7.7) holds. Letting $f = -\Delta u$ we obtain (7.6) and thus $(u, f) \in A$.

Finally, since by Theorem 7.11 the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, the operator A has compact resolvent. Since a is symmetric, A is self-adjoint, and since a is H -elliptic, A is quasi-accretive. \square

Notes

In Section 7.1 we partially follow [AU10]. It is possible to extend Gauss' theorem to Lipschitz domains, for which a suitable surface measure on $\partial\Omega$ is defined analogously. Most of the properties of $H^1(\Omega)$ presented in Section 7.2 can be found in the standard literature on Sobolev spaces. The proofs of (7.3) and Theorem 7.10 have been contributed by H. Vogt.

The theorem of Gauss is due to Lagrange in 1792 but has been rediscovered by Carl Friedrich Gauss in 1813, by George Green in 1825, and by Mikhail V. Ostrogradsky in 1831. For this reason one finds it in the literature under these different names. We could call it 'fundamental theorem of calculus' but this is not usual. Obviously, it can also be written as

$$\int_{\Omega} \operatorname{div} u = \int_{\partial\Omega} u \cdot \nu \, d\sigma,$$

for each vector field $u \in C^1(\bar{\Omega}; \mathbb{R}^n)$. In this form it is frequently called the **divergence theorem**. Physicists and engineers love this theorem because of immediate interpretation.

Victor Gustave Robin (1855–1897) was a French mathematician and the reader should correctly pronounce the nasal. He was teaching mathematical physics at the Sorbonne in Paris. Not much is known about him since he burnt his manuscripts. But he worked on thermodynamics, and the Russian school introduced the name Robin boundary conditions. These boundary conditions had already been introduced by Isaac Newton (1643–1727). Neumann boundary conditions carry their name to honour Carl G. Neumann (1832–1925) who was professor at Halle, Basel, Tübingen and Leipzig. He introduced the Neumann series for matrices.

None of these 'forefathers' used forms it seems. Time was not yet ripe and Hilbert had to come into play first.

Exercises

7.1 Let $\Omega \subseteq \mathbb{R}^n$ be open, connected, bounded, and with C^1 -boundary. Let $0 \leq \beta \in L_{\infty}(\partial\Omega)$ such that $\int_{\partial\Omega} \beta \, d\sigma > 0$ (i.e., β is not 0 in $L_{\infty}(\partial\Omega)$). Denote by T_{β} the C_0 -semigroup generated by the Robin Laplacian Δ_{β} . Show that

$$\|T_{\beta}(t)\| \leq e^{-\varepsilon t} \quad (t \geq 0)$$

for some $\varepsilon > 0$. (Hint: If $u \in H^1(\Omega)$ is such that $\nabla u = 0$, then u is constant. This can be used without proof; it holds because Ω is connected.)

7.2 Let $\Omega \subseteq \mathbb{R}^n$ be a C^1 -domain with a hole, i.e., we assume that there exist bounded open sets $\tilde{\Omega}$ and ω with C^1 -boundary such that $\bar{\omega} \subseteq \tilde{\Omega}$ and $\Omega = \tilde{\Omega} \setminus \bar{\omega}$. Let $\Gamma_1 = \partial\tilde{\Omega}$,

$\Gamma_2 = \partial\omega$ so that $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Let $\beta \in L_\infty(\Gamma_2)$ be real-valued. Define the Laplacian with Robin boundary condition $\partial_\nu u + \beta u|_{\Gamma_2} = 0$ on Γ_2 and Dirichlet boundary condition zero on Γ_1 . Show that it is a self-adjoint operator.

7.3 Let $a: V \times V \rightarrow \mathbb{C}$ be a symmetric continuous form that is j -elliptic, where V, H are complex Hilbert spaces and $j \in \mathcal{L}(V, H)$ has dense range. Let $b: V \times V \rightarrow \mathbb{C}$ be sesquilinear and assume that there exists $c \geq 0$ such that

$$|b(u, v)| \leq c \|u\|_V \|j(v)\|_H \quad (u, v \in V).$$

(a) Show that $a + b: V \times V \rightarrow \mathbb{C}$ is continuous and j -elliptic.

(b) Denote by A the operator associated with $(a + b, j)$. Show that the numerical range $\text{num}(A)$ lies in a parabola with vertex on the real axis and opened in the direction of the positive real axis.

(c) Assume that $a = 0$ and that b is j -elliptic. Denote by B the operator associated with (b, j) . Show that j is an isomorphism and that B is bounded.

7.4 In this problem we let $\mathbb{K} = \mathbb{C}$. Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with C^1 -boundary. Let $\beta \in L_\infty(\partial\Omega)$ (not necessarily real-valued), and let Δ_β be the Robin Laplacian.

(a) Show that $-\Delta_\beta$ is H -elliptic (where $H := L_2(\Omega)$), and that $-\Delta_\beta$ is m -accretive if $\text{Re } \beta \geq 0$.

(b) Show that $\text{num}(-\Delta_\beta)$ is contained in the region ‘surrounded’ by a parabola with vertex on the real axis and opened in the direction of the positive real axis.

(c) Show that $-\Delta_\beta$ is quasi- m -sectorial of any angle $\varphi < \pi/2$ and that Δ_β generates a holomorphic semigroup of angle $\pi/2$.

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Lecture 8

The Dirichlet-to-Neumann operator

The Dirichlet-to-Neumann operator plays an important role in the theory of inverse problems. In fact, from measurements of electrical currents at the surface of the human body one wishes to determine conductivity inside the body. But the Dirichlet-to-Neumann operator also plays a big role in many parts of analysis. Here we prove by form methods that it is a self-adjoint operator in $L_2(\partial\Omega)$. Our form methods come to fruition; in particular, to allow general mappings j is very useful here: throughout this lecture j will be the trace operator.

8.1 The Dirichlet-to-Neumann operator for the Laplacian

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1 -boundary. We use the classical Dirichlet form

$$a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} \quad (u, v \in H^1(\Omega)). \quad (8.1)$$

If we choose the canonical injection of $H^1(\Omega)$ into $L_2(\Omega)$, the associated operator is the Neumann Laplacian. Here we will choose as j the trace operator from $H^1(\Omega)$ to $L_2(\partial\Omega)$, introduced in Theorem 7.9. We will show that the form a is j -elliptic. Thus we obtain as associated operator a self-adjoint operator in $L_2(\partial\Omega)$. It turns out that this is the Dirichlet-to-Neumann operator D_0 which maps $g \in \text{dom}(D_0)$ to $\partial_\nu u \in L_2(\partial\Omega)$ where $u \in H^1(\Omega)$ is the harmonic function with $u|_{\partial\Omega} = g$. This will be made more precise in the following. (We recall that a function u defined on an open set is called **harmonic** if it is twice continuously differentiable, and $\Delta u = 0$.)

Observe that the trace operator $\text{tr}: H^1(\Omega) \rightarrow L_2(\partial\Omega)$ has dense image. In fact, by the Stone-Weierstrass theorem ([Yos68; Section 0.2]) the set $\{\varphi|_{\partial\Omega}; \varphi \in C_c^\infty(\mathbb{R}^n)\}$ is dense in $C(\partial\Omega)$, and $C(\partial\Omega)$ is dense in $L_2(\partial\Omega)$. Next we prove j -ellipticity.

8.1 Proposition. *With $j = \text{tr}$, the classical Dirichlet form is j -elliptic.*

For the proof we need an auxiliary result. We state it for the general case of Banach spaces; in our context it will only be needed for Hilbert spaces.

8.2 Lemma. *Let X, Y, Z be Banach spaces, X reflexive, $K \in \mathcal{L}(X, Y)$ compact and $S \in \mathcal{L}(X, Z)$ injective. Then for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that*

$$\|Kx\|_Y \leq \varepsilon \|x\|_X + c_\varepsilon \|Sx\|_Z.$$

Proof. If not, there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$\|Kx_n\|_Y > \varepsilon\|x_n\|_X + n\|Sx_n\|_Z$$

and $\|x_n\|_X = 1$. Passing to a subsequence we may assume that $x_n \rightarrow x$ weakly. Hence $Sx_n \rightarrow Sx$ weakly. The inequality implies that $\|Sx_n\|_Z \rightarrow 0$, hence $Sx = 0$. Since S is injective, it follows that $x = 0$. Since K is compact it follows that $Kx_n \rightarrow 0$ in norm. But $\|Kx_n\|_Y \geq \varepsilon$ for all $n \in \mathbb{N}$, a contradiction. \square

Proof of Proposition 8.1. (i) Define $S: H^1(\Omega) \rightarrow L_2(\Omega; \mathbb{K}^n) \oplus L_2(\partial\Omega)$ by

$$Su := (\nabla u, u|_{\partial\Omega}).$$

We will show below that S is injective. Because the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, by Theorem 7.11, the application of Lemma 8.2 yields a constant $c \geq 0$ such that

$$\begin{aligned} \int_{\Omega} |u|^2 dx &\leq \frac{1}{2} \|u\|_{H^1(\Omega)}^2 + c \|Su\|^2 \\ &= \frac{1}{2} \int_{\Omega} |u|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + c \int_{\Omega} |\nabla u|^2 dx + c \int_{\partial\Omega} |u|^2 d\sigma. \end{aligned}$$

Adding $\frac{1}{2}(\int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |u|^2 dx)$ to this inequality we obtain

$$\frac{1}{2} \|u\|_{H^1(\Omega)}^2 \leq c \|u\|_{L_2(\partial\Omega)}^2 + (c+1)a(u) \quad (u \in H^1(\Omega)),$$

and this shows that a is j -elliptic.

(ii) For the proof that S is injective let $u \in H^1(\Omega)$ be such that $Su = 0$. Then $\nabla u = 0$, and therefore u is constant on each connected component ω of Ω . As $\text{tr } u = 0$, this constant is zero. Therefore $u = 0$. \square

8.3 Remark. In the proof given above (as well as in Exercise 7.1) it was used that a function u on a connected open set Ω with $\nabla u = 0$ (distributionally) is constant.

This is shown as follows. If one takes the convolution of u with a function $\rho \in C_c^\infty(\mathbb{R}^n)$ with support in a ‘small’ neighbourhood of 0, then $\rho * u$ is infinitely differentiable and satisfies $\nabla(\rho * u) = 0$ far enough away from $\partial\Omega$, and therefore is locally constant far enough away from $\partial\Omega$. Convoluting u with a δ -sequence in $C_c^\infty(\mathbb{R}^n)$ one therefore infers that, locally on Ω , u is the limit of constant functions. Thus u has a locally constant representative, which is constant since Ω is connected.

8.4 Theorem. Let j be the trace operator, and let a be the classical Dirichlet form (8.1). Then the operator D_0 in $L_2(\partial\Omega)$ associated with (a, j) is described by

$$D_0 = \{(g, h) \in L_2(\partial\Omega) \times L_2(\partial\Omega); \exists u \in H^1(\Omega): \Delta u = 0, u|_{\partial\Omega} = g, \partial_\nu u = h\}.$$

The operator D_0 is self-adjoint and positive and has compact resolvent. We call D_0 the **Dirichlet-to-Neumann operator** (with respect to Δ).

Proof. Let $(g, h) \in D_0$. Then there exists $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = g$ and

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} = a(u, v) = \int_{\partial\Omega} h \bar{v} \quad (8.2)$$

for all $v \in H^1(\Omega)$. Employing this equality with $v \in C_c^\infty(\Omega)$ we obtain $-\Delta u = 0$. Thus adding $\Delta u = 0$ to (8.2) we have

$$\int_{\Omega} (\Delta u) \bar{v} + \int_{\Omega} \nabla u \cdot \overline{\nabla v} = \int_{\partial\Omega} h \bar{v} \quad (v \in H^1(\Omega)).$$

Hence $\partial_\nu u = h$ by our definition of the normal derivative. This shows “ \subseteq ” in the asserted equality for D_0 .

In order to get the converse inclusion let $u \in H^1(\Omega)$ such that $\Delta u = 0$, $g := u|_{\partial\Omega}$, $h := \partial_\nu u \in L_2(\partial\Omega)$. Then

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} = \int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\Omega} (\Delta u) \bar{v} = \int_{\partial\Omega} h \bar{v} \quad (v \in H^1(\Omega)).$$

Thus $a(u, v) = (h | j(v))_{L_2(\partial\Omega)}$ for all $v \in H^1(\Omega)$. Consequently $(g, h) \in D_0$.

The symmetry of a implies that D_0 is self-adjoint. Finally, it was shown in Remark 7.12 that $\text{tr}: H^1(\Omega) \rightarrow L_2(\partial\Omega)$ is compact; hence D_0 has compact resolvent, by Proposition 6.15. \square

Our next aim is to define Dirichlet-to-Neumann operators with respect to more general Dirichlet problems. The following interlude is a preparation for this treatment.

8.2 Interlude: the Fredholm alternative in Hilbert space

We need a detail from the spectral theory of compact operators which we formulate and prove only for operators in Hilbert spaces. It will be used in the proof of Proposition 8.10.

8.5 Proposition. *Let H be a Hilbert space, $K \in \mathcal{L}(H)$ compact. Then $I + K$ is injective if and only if $\text{ran}(I + K) = H$, and in this case $I + K$ is invertible in $\mathcal{L}(H)$.*

Proof. (i) First we note that $K(B_H[0, 1])$ is a relatively compact set in a metric space, and as such is separable; therefore $\text{ran}(K)$ is separable.

(ii) In this step we treat the case that $\dim \text{ran}(K) < \infty$. We define $H_1 := \ker(K)$, $H_2 := \ker(K)^\perp$ and denote by P_1, P_2 the orthogonal projections onto H_1, H_2 , respectively. Note that $\dim H_2 < \infty$. (Indeed, from $K = \sum_{j=1}^n (\cdot | x_j) y_j$ one obtains $\ker(K) \supseteq \{x_1, \dots, x_n\}^\perp$, and therefore $\ker(K)^\perp \subseteq \text{lin}\{x_1, \dots, x_n\}$.) On the orthogonal sum $H_1 \oplus H_2$ the operator $I + K$ can be written as the operator matrix

$$I + K = \begin{pmatrix} I_1 & P_1 K|_{H_2} \\ 0 & I_2 + P_2 K|_{H_2} \end{pmatrix},$$

where I_1, I_2 are the identity operators in H_1, H_2 , respectively. In the matrix representation it is easy to see that $I + K$ is injective if and only if $I_2 + P_2 K|_{H_2}$ is injective. From

(finite-dimensional) linear algebra it is known that the operator $I_2 + P_2K|_{H_2}$ is injective if and only if it is surjective. Again, using the matrix representation one easily can see that $I_2 + P_2K|_{H_2}$ is surjective if and only if $I + K$ is surjective.

If $I + K$ is injective, then in the (upper diagonal) block matrix version of $I + K$ the diagonal entries are invertible, and it is straightforward to show that then $I + K$ is invertible in $\mathcal{L}(H)$.

(iii) If $\text{ran}(K)$ is not finite-dimensional, let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $\overline{\text{ran}(K)}$, and let P_n be the orthogonal projection onto $\text{lin}\{e_1, \dots, e_n\}$. Then $P_n x \rightarrow x$ ($n \rightarrow \infty$) for all $x \in \overline{\text{ran}(K)}$, and the compactness of K implies that $P_n K \rightarrow K$ in the operator norm.

Therefore K can be written as a sum $K = K_1 + K_2$, where $\|K_1\| < 1$ and $\text{ran}(K_2)$ is finite-dimensional. Then $I + K_1$ is invertible in $\mathcal{L}(H)$ (recall Remark 2.3(a)), and therefore $(I + K_1)^{-1}(I + K) = I + F$, with the finite rank operator $F = (I + K_1)^{-1}K_2$. Hence $I + K$ is injective if and only if $I + F$ is injective if and only if $I + F$ is surjective (by step (ii)) if and only if $I + K$ is surjective.

If $I + K$ is injective, then the continuity of the inverse follows from part (ii) and $(I + K)^{-1} = (I + F)^{-1}(I + K_1)^{-1}$. \square

8.6 Remark. If, in the situation of Proposition 8.5, $B \in \mathcal{L}(H)$ is invertible in $\mathcal{L}(H)$, then $B + K$ is injective if and only if $\text{ran}(B + K) = H$. This is immediate from $B + K = B(I + B^{-1}K)$ and Proposition 8.5, applied to $I + B^{-1}K$.

Proposition 8.5 carries the name ‘alternative’ because the only *alternative* to $I + K$ being invertible is that $I + K$ is not injective *and* $I + K$ is not surjective.

8.3 Quasi-m-accretive and self-adjoint operators via compactly elliptic forms

Let V, H be Hilbert spaces, and let $a: V \times V \rightarrow \mathbb{K}$ be a continuous sesquilinear form. Let $j \in \mathcal{L}(V, H)$ have dense range. We assume that

$$u \in \ker(j), a(u, v) = 0 \text{ for all } v \in \ker(j) \text{ implies } u = 0. \quad (8.3)$$

This is slightly more general than (5.4).

As in Section 5.3, let

$$A := \{(x, y) \in H \times H; \exists u \in V: j(u) = x, a(u, v) = (y | j(v)) \ (v \in V)\}.$$

8.7 Proposition. *Assume (8.3). Then the relation A defined above is an operator. As before, we call A the operator associated with (a, j) . If (8.3) is also satisfied for a^* , and B is the operator associated with (a^*, j) , then $B \subseteq A^*$ and $A \subseteq B^*$. In particular, if a is symmetric and $\text{dom}(A)$ is dense, then A is symmetric.*

The proof is delegated to Exercise 8.1. For the following we define

$$V_j(a) := \{u \in V; a(u, v) = 0 \ (v \in \ker(j))\}.$$

8.8 Remarks. (a) Condition (8.3) is equivalent to $V_j(a) \cap \ker(j) = \{0\}$.

(b) $V_j(a)$ is a closed subspace of V . One can think of $V_j(a)$ as the ‘orthogonal complement’ of $\ker(j)$ with respect to a .

We call the form a **compactly elliptic** if there exist a Hilbert space \tilde{H} and a compact operator $\tilde{j}: V \rightarrow \tilde{H}$ such that a is ‘ \tilde{j} -elliptic’, i.e.,

$$\operatorname{Re} a(u) + \|\tilde{j}(u)\|_{\tilde{H}}^2 \geq \tilde{\alpha} \|u\|_V^2 \quad (8.4)$$

for all $u \in V$ and some $\tilde{\alpha} > 0$. We are going to show that the operator associated with (a, j) is quasi- m -accretive if in addition (8.3) is satisfied.

8.9 Lemma. *If (8.3) is satisfied and a is compactly elliptic, then there exist $\omega \geq 0$, $\alpha > 0$ such that*

$$\operatorname{Re} a(u) + \omega \|j(u)\|_H^2 \geq \alpha \|u\|_V^2 \quad (u \in V_j(a)). \quad (8.5)$$

Proof. According to Remark 8.8(a), j is injective on $V_j(a)$. Therefore Lemma 8.2 implies

$$\|\tilde{j}(u)\|_{\tilde{H}}^2 \leq \frac{\tilde{\alpha}}{2} \|u\|_V^2 + c \|j(u)\|_H^2 \quad (u \in V_j(a)),$$

with $\tilde{\alpha} > 0$ from (8.4) and some $c \geq 0$. From (8.4) we then obtain

$$\operatorname{Re} a(u) + c \|j(u)\|_H^2 \geq \frac{\tilde{\alpha}}{2} \|u\|_V^2. \quad \square$$

8.10 Proposition. *Assume that (8.3) is satisfied and that a is compactly elliptic. Then $V = V_j(a) \oplus \ker(j)$ is a (not necessarily orthogonal) topological direct sum.*

Proof. There exists $R \in \mathcal{L}(V)$ such that

$$(Ru | v)_V = a(u, v) + \omega (j(u) | j(v))_H \quad (u, v \in V),$$

with ω from (8.5). (For fixed u , the right hand side, as a function of v , belongs to V^* , and Ru results from the Riesz-Fréchet theorem.) Then R is injective: If $Ru = 0$, then $a(u, v) = 0$ for all $v \in \ker(j)$, i.e., $u \in V_j(a)$, and then $\alpha \|u\|_V^2 \leq \operatorname{Re} (Ru | u) = 0$ by (8.5).

Define $B \in \mathcal{L}(V)$ by

$$(Bu | v)_V = a(u, v) + \omega (j(u) | j(v))_H + (\tilde{j}(u) | \tilde{j}(v))_{\tilde{H}} \quad (u, v \in V).$$

Then the ‘operator version’ of the Lax-Milgram lemma (Remark 5.3) implies that B is invertible in $\mathcal{L}(V)$, by (8.4). The operator $K := \tilde{j}^* \tilde{j} \in \mathcal{L}(V)$ is compact, and from the definitions we see that $R = B - K$. As R is injective, the ‘Fredholm alternative’, Proposition 8.5 in the guise of Remark 8.6, shows that R is invertible in $\mathcal{L}(V)$. Note that then also R^* is invertible in $\mathcal{L}(V)$; see Exercise 8.2(c).

Let $J: V_j(a) \hookrightarrow V$ be the embedding and $S := J^* R^* J$. Then $S \in \mathcal{L}(V_j(a))$, and

$$\operatorname{Re} (Su | u)_{V_j(a)} = \operatorname{Re} (R^* u | u)_V = \operatorname{Re} (u | Ru)_V \geq \alpha \|u\|_V^2 \quad (u \in V_j(a)).$$

This shows that S is coercive, hence invertible in $\mathcal{L}(V_j(a))$, by the operator version of the Lax-Milgram lemma, Remark 5.3. Note that J^* is the orthogonal projection onto $V_j(a)$; see Exercise 8.2(d).

Now, let $P := JS^{-1}J^*R^* \in \mathcal{L}(V)$. Then $P^2 = JS^{-1}SS^{-1}J^*R^* = P$, and clearly $\text{ran}(P) = V_j(a)$. This shows that P is a projection onto $V_j(a)$. We note that

$$\text{ran}(RJ) = R(V_j(a)) = \ker(j)^\perp.$$

Indeed, $u \in V_j(a)$ if and only if $(Ru | v) = a(u, v) = 0$ for all $v \in \ker(j)$, i.e., $Ru \in \ker(j)^\perp$. As R is invertible, this shows the equality. This implies

$$\ker P = \ker(J^*R^*) = \text{ran}(RJ)^\perp = \ker(j).$$

(For the second of these equalities we refer to Exercise 8.2(b) and Lemma 6.7.) Therefore P is the (continuous) projection onto $V_j(a)$ along $\ker(j)$, and this shows the assertion. \square

We insert the definition that a self-adjoint operator A in a Hilbert space H is called **bounded below** or **bounded above** if the set $\{(Ax | x); x \in \text{dom}(A), \|x\| = 1\}$ is bounded below or bounded above, respectively. Now we – finally – can show that the operator associated with (a, j) is as one should hope for.

8.11 Theorem. *Let $a: V \times V \rightarrow \mathbb{K}$ be a bounded form, and let $j \in \mathcal{L}(V, H)$ have dense range. Assume that (8.3) is satisfied and that a is compactly elliptic.*

Let A be the operator associated with (a, j) . Then A is quasi- m -accretive. If a is symmetric, then A is self-adjoint and bounded below. If j is compact, then A has compact resolvent.

Moreover, let $\check{a} := a|_{V_j(a) \times V_j(a)}$ and $\check{j} := j|_{V_j(a)}$. Then $\text{ran}(\check{j})$ is dense, and A is also associated with (\check{a}, \check{j}) .

Proof. From Proposition 8.10 one obtains $\check{j}(V_j(a)) = j(V)$, in particular $\text{ran}(\check{j})$ is dense. By Lemma 8.9, \check{a} is \check{j} -elliptic. Hence the operator \check{A} associated with (\check{a}, \check{j}) is quasi- m -accretive. If a is symmetric, then \check{a} is symmetric, and \check{A} is self-adjoint. If j is compact, then \check{j} is compact, and \check{A} has compact resolvent, by Proposition 6.15. It remains to show that $A = \check{A}$.

Let $(x, y) \in A$. Then there exists $u \in V$ such that $j(u) = x$ and $a(u, v) = (y | j(v))$ for all $v \in V$. In particular, for $v \in \ker(j)$ one has $a(u, v) = (y | j(v)) = 0$. Hence $u \in V_j(a)$ and $a(u, v) = (y | j(v))$ for all $v \in V_j(a)$. This implies that $(x, y) \in \check{A}$.

Conversely, let $(x, y) \in \check{A}$. Then there exists $u \in V_j(a)$ such that $j(u) = x$ and $a(u, v) = (y | j(v))$ for all $v \in V_j(a)$. For $v \in \ker(j)$ one has $a(u, v) = 0 = (y | j(v))$, because $u \in V_j(a)$. Now Proposition 8.10 implies that $a(u, v) = (y | j(v))$ for all $v \in V$, and therefore $(x, y) \in A$. \square

8.12 Remark. The last statement of Theorem 8.11 says that it is possible to specialise to the case of *embedded forms*, i.e., to the case that $j: V \rightarrow H$ is injective. This also applies to the situation that a is j -elliptic, even if j is not compact, which can be seen as follows.

There exists $\omega \in \mathbb{R}$ such that the form b defined by $b(u, v) := a(u, v) + \omega(j(u) | j(v))$ is coercive. Recall from Remark 5.6 that (b, j) is associated with $\omega + A$. One easily sees that $V_j(b) = V_j(a)$. Moreover, b is compactly elliptic, with $\tilde{j} = 0$, so Proposition 8.10 yields $V = V_j(b) \oplus \ker(j)$, and Theorem 8.11 applies to b . Let $\check{a}, \check{b}, \check{j}$ be the restrictions as in the last assertion of the theorem. Then $\omega + A$ is associated with (\check{b}, \check{j}) , and from Remark 5.6 we conclude that A is associated with (\check{a}, \check{j}) .

8.4 The Dirichlet-to-Neumann operator with respect to

$$\Delta + m$$

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1 -boundary. Let $m \in L_\infty(\Omega)$ be real-valued. Our aim is to consider the Dirichlet-to-Neumann operator D_m with respect to $(\Delta + m)$ -harmonic functions. This means that we define D_m in $L_2(\partial\Omega)$ by requiring that for $g, h \in L_2(\partial\Omega)$ one has $g \in \text{dom}(D_m)$ and $D_m g = h$ if there is a solution $u \in H^1(\Omega)$ of $\Delta u + mu = 0$, $u|_{\partial\Omega} = g$ such that $\partial_\nu u = h$. We will show that D_m is a self-adjoint operator if

$$0 \notin \sigma(\Delta_D + m) \quad (8.6)$$

which we want to suppose throughout. Here $\Delta_D + m$ is the Dirichlet Laplacian perturbed by the bounded multiplication operator by the function m . We observe that this operator is self-adjoint, bounded above and has compact resolvent; see Exercise 8.3.

As in Section 8.1 we consider $H = L_2(\partial\Omega)$, $V = H^1(\Omega)$ and as $j \in \mathcal{L}(H^1(\Omega), L_2(\partial\Omega))$ the trace operator. According to the intended setup we now define the form $a: V \times V \rightarrow \mathbb{C}$ by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} - \int_{\Omega} mu\bar{v} \quad (u, v \in H^1(\Omega)).$$

8.13 Remark. The form a is in general not j -elliptic. In fact, let $\lambda > \lambda_1$, where λ_1 is the first Dirichlet eigenvalue, and $m := \lambda$. Choose $u \in H_0^1(\Omega)$ as an eigenfunction of $-\Delta_D$ belonging to λ_1 . Then $\int_{\Omega} |\nabla u|^2 = \lambda_1 \int_{\Omega} |u|^2$, and

$$\int_{\Omega} |\nabla u|^2 - \lambda \int_{\Omega} |u|^2 + \omega \int_{\partial\Omega} |u|^2 = (\lambda_1 - \lambda) \int_{\Omega} |u|^2 < 0$$

for all $\omega \geq 0$. Thus the form is not j -elliptic.

Since the form a is not j -elliptic the theory developed in the previous lectures is not applicable. However, we can apply the results of the previous section.

8.14 Theorem. *Suppose that (8.6) holds. Then*

$$D_m := \{(g, h) \in L_2(\partial\Omega) \times L_2(\partial\Omega); \exists u \in H^1(\Omega): \Delta u + mu = 0, u|_{\partial\Omega} = g, \partial_\nu u = h\}$$

defines a self-adjoint operator with compact resolvent, and D_m is bounded below.

Proof. Let a be the form defined above, and let $j \in \mathcal{L}(H^1(\Omega), L_2(\partial\Omega))$ be the trace. We first show that condition (8.3) is satisfied. We recall from Theorem 7.10 that $\ker(j) = H_0^1(\Omega)$. Let $u \in \ker(j)$ such that $a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} - \int_{\Omega} mu\bar{v} = 0$ for all $v \in \ker(j) = H_0^1(\Omega)$. Then $u \in \text{dom}(\Delta_D + m)$ and $\Delta_D u + mu = 0$, by the definition of Δ_D . This implies $u = 0$ since $0 \notin \sigma(\Delta_D + m)$ by our assumption.

In order to show that a is compactly elliptic we choose $\tilde{H} := L_2(\Omega)$ and as \tilde{j} the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$, multiplied by $c := (\|m\|_\infty + 1)^{1/2}$. Then

$$a(u) + \|\tilde{j}(u)\|_2^2 = \int_{\Omega} |\nabla u|^2 - \int_{\Omega} m|u|^2 + (\|m\|_\infty + 1)\|u\|_2^2 \geq \int_{\Omega} |\nabla u|^2 + \|u\|_2^2 = \|u\|_{H^1}^2$$

for all $u \in H^1(\Omega)$, and from Theorem 7.11 we know that \tilde{j} is compact.

Let A be the operator associated with (a, j) . By Theorem 8.11, A is self-adjoint and bounded below. We show that $A = D_m$. In fact, for $g, h \in L_2(\partial\Omega)$ we have $(g, h) \in A$ if and only if there exists $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = g$ and

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} - \int_{\Omega} mu\bar{v} = \int_{\partial\Omega} h\bar{v} \quad (v \in H^1(\Omega)). \quad (8.7)$$

Inserting test functions $v \in C_c^\infty(\Omega)$ one concludes that $-\Delta u - mu = 0$. Plugging $mu = -\Delta u$ into (8.7) we deduce that $\partial_\nu u = h$. Thus $(g, h) \in D_m$. Conversely, if $(g, h) \in D_m$, then there exists $u \in H^1(\Omega)$ such that $\Delta u + mu = 0$, $u|_{\partial\Omega} = g$ and $\partial_\nu u = h$. Thus, by the definition of the normal derivative,

$$\int_{\partial\Omega} h\bar{v} = \int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\Omega} (\Delta u)\bar{v} = \int_{\Omega} \nabla u \cdot \overline{\nabla v} - \int_{\Omega} mu\bar{v} = a(u, v) \quad (v \in H^1(\Omega)),$$

hence $(g, h) \in A$.

As j is compact, by Remark 7.12, Theorem 8.11 implies that A has compact resolvent. \square

Notes

A large part of the material of this lecture is an extract from [AEKS14]. The main result of Section 8.3, Theorem 8.11, goes beyond this paper and is due to H. Vogt. Our proof of Theorem 8.11 for the general non-symmetric case is based on the decomposition in Proposition 8.10. A different proof can be given based on results in [Sau13].

We may add that assumption (8.6) is not really needed. However, if $0 \in \sigma(\Delta_D + m)$, then D_m is a self-adjoint relation and no longer an operator. We did not want to introduce the notion of self-adjoint relations here. However, it is well motivated. In the complex case, the resolvent $(is - D_m)^{-1}$ is a bounded operator on $L_2(\partial\Omega)$, and the mapping $L_\infty(\Omega) \ni m \mapsto (is - D_m)^{-1} \in \mathcal{L}(L_2(\partial\Omega))$ is continuous. This gives valuable information on the stability of the inverse problem which interests engineers and doctors at the same time.

Exercises

8.1 Prove Proposition 8.7.

8.2 Let F, G, H be Hilbert spaces.

(a) Let A be a densely defined operator from G to H , $B \in \mathcal{L}(G, H)$. Show that $(A + B)^* = A^* + B^*$.

(b) Let $A \in \mathcal{L}(F, G)$, $B \in \mathcal{L}(G, H)$. Show that $(BA)^* = A^*B^*$.

(c) Let $A \in \mathcal{L}(H)$ be invertible in $\mathcal{L}(H)$. Show that A^* is invertible in $\mathcal{L}(H)$.

(d) Let $H_0 \subseteq H$ be a closed subspace, $J: H_0 \rightarrow H$ the embedding. Show that J^* is the orthogonal projection from H onto H_0 .

8.3 Let $\Omega \subseteq \mathbb{R}^n$ be open, $m \in L_\infty(\Omega)$ real-valued. Show that $\Delta_D + m$ is self-adjoint, bounded above and has compact resolvent. (Hint: Use Exercise 8.2(a).)

8.4 Let $-\infty < a < b < \infty$. (a) Compute the Dirichlet-to-Neumann operator D_0 for $\Omega = (a, b)$, and compute the C_0 -semigroup generated by $-D_0$.

(b) For $a = -1, b = 1$, interpret the result in the light of Exercise 8.5.

8.5 Let $U_n := B_{\mathbb{R}^n}(0, 1)$ be the open unit ball in \mathbb{R}^n , $S_{n-1} := \partial U_n$ the unit sphere. The following facts can be used for the solution of this exercise: For each $\varphi \in C(S_{n-1})$ there exists a unique solution $u \in C(\overline{U_n})$ of the Dirichlet problem,

$$u|_{S_{n-1}} = \varphi, \quad u|_{U_n} \text{ harmonic.}$$

(We mention that the solution can be written down explicitly with the aid of the Poisson kernel, but this will not be needed for solving the exercise.) The solution satisfies $u|_{U_n} \in C^\infty(U_n)$ and $\|u\|_\infty \leq \|\varphi\|_\infty$. We will use the notation $G\varphi := u$; thus $G \in \mathcal{L}(C(S_{n-1}), C(\overline{U_n}))$.

Define $T(t) \in \mathcal{L}(C(S_{n-1}))$ by

$$T(t)\varphi(z) := u(e^{-t}z) \quad (z \in S_{n-1}, t \geq 0)$$

(with φ and $u = G\varphi$ as above).

(a) Show that T is a C_0 -semigroup of contractions on $C(S_{n-1})$.

(b) Let A be the generator of T . Show that $D := \bigcup_{t>0} \text{ran}(T(t))$ is a core for A , and that $A\varphi = -\partial_\nu(G\varphi)$ for all $\varphi \in D$.

(c) Define $A_{\min} := A|_D$. Show that $-A_{\min}$ is a restriction of the Dirichlet-to-Neumann operator in $L_2(S_{n-1})$, and that $D_0 = -\overline{A_{\min}}$ (where A_{\min} is interpreted as an operator in $L_2(S_{n-1})$). Conclude that T extends to a C_0 -semigroup T_2 of contractions on $L_2(S_{n-1})$, and that $-D_0$ is the generator of T_2 .

(We refer to [Lax02; Section 36.2] for this exercise.)

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Lecture 9

Invariance of closed convex sets

In this lecture we investigate criteria for a closed convex set to be invariant under a semigroup. To begin with, we present criteria involving properties of the generator. Applying these criteria to the Dirichlet and Neumann Laplacian one realises that further properties of H^1 -functions are needed. These will be provided in an interlude on lattice properties of H^1 . In the last section we present criteria involving properties of forms. Their applicability to a wide range of problems will be presented in a later lecture on elliptic operators.

9.1 Invariance for semigroups

Let T be a C_0 -semigroup on a Banach space X over \mathbb{K} , with generator A . Our aim is to characterise when a closed convex subset C of X is **invariant under the semigroup T** , i.e., $T(t)(C) \subseteq C$ for all $t \geq 0$. At first we note that invariance under T is equivalent to invariance under the resolvent.

9.1 Proposition. *Let $C \subseteq X$ be closed and convex. Then the following assertions are equivalent.*

- (i) C is invariant under T .
- (ii) There exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and $\lambda R(\lambda, A)(C) \subseteq C$ for all $\lambda > \omega$.

Noting that $\lambda R(\lambda, A) = (I - \frac{1}{\lambda}A)^{-1}$ we see that condition (ii) can be expressed equivalently by requiring that there exists $r_0 > 0$ such that $\{1/r; 0 < r < r_0\} \subseteq \rho(A)$ and $(I - rA)^{-1}(C) \subseteq C$ for all $0 < r < r_0$. It is this version of condition (ii) that will mostly be used in the following.

For the proof of the implication '(i) \Rightarrow (ii)' we need a fact concerning integration. It should be understood as a statement on generalised convex combinations.

9.2 Lemma. *Let C be a closed convex subset of a Banach space X , let $-\infty < a < b < \infty$, $u \in C([a, b]; X)$ with $u(t) \in C$ for all $t \in [a, b]$, and $\varphi \in C[a, b]$, $\varphi \geq 0$, $\int_a^b \varphi(t) dt = 1$.*

Then $\int_a^b \varphi(t)u(t) dt \in C$.

Proof. For simplicity of notation (and without loss of generality) we assume that $[a, b] = [0, 1]$.

For $n \in \mathbb{N}$ define

$$u_n := u(0)\mathbf{1}_{\{0\}} + \sum_{k=1}^n u(k/n)\mathbf{1}_{((k-1)/n, k/n]}.$$

Then $\|\varphi u_n - \varphi u\|_\infty \rightarrow 0$ ($n \rightarrow \infty$); hence $\int_0^1 \varphi u_n dt \rightarrow \int_0^1 \varphi u dt$. Moreover $\int_0^1 \varphi u_n dt = \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \varphi(t) dt u(k/n) \in C$, as a convex combination of elements of C . As C is closed we obtain the assertion. \square

Proof of Proposition 9.1. (i) \Rightarrow (ii). Let $\omega \in \mathbb{R}$, $M \geq 0$ be such $\|T(t)\| \leq Me^{\omega t}$ ($t \geq 0$), and let $\lambda > \omega$. Then $\lambda R(\lambda, A) = \int_0^\infty \lambda e^{-\lambda t} T(t) dt$ (strong improper integral). Let $x \in C$. For $r > 0$ we obtain $(1 - e^{-\lambda r})^{-1} \int_0^r \lambda e^{-\lambda t} T(t)x dt \in C$, by Lemma 9.2. Letting $r \rightarrow \infty$ we conclude that $\lambda R(\lambda, A)x \in C$.

(ii) \Rightarrow (i). This follows from ‘Euler’s formula’ (Theorem 2.12):

$$T(t)x = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n} A \right)^{-n} x \in C \quad (x \in C). \quad \square$$

In order to motivate why one may be interested in the invariance of closed convex sets, we indicate several examples.

9.3 Remarks. Let (Ω, μ) be a measure space, $H := L_2(\mu; \mathbb{K})$.

(a) Let $C \subseteq L_2(\mu)$ be the **positive cone**, $C := L_2(\mu)_+ := \{u \in L_2(\mu); u \geq 0\}$. Clearly, C is a closed convex subset of $L_2(\mu)$. An operator $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is **positive**, or **positivity preserving**, i.e., $Su \geq 0$ for all $u \geq 0$.

(b) Let $\mathbb{K} = \mathbb{C}$, and let $C := L_2(\mu; \mathbb{R})$ be the subset of real-valued functions. An operator $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is ‘real’, i.e., Su is real-valued for all real-valued u .

(c) Let $C := \{u \in L_2(\mu); \|u\|_\infty \leq 1\}$. Then C is convex and closed, and $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is **L_∞ -contractive**, i.e., $\|Su\|_\infty \leq \|u\|_\infty$ for all $u \in L_2(\mu) \cap L_\infty(\mu)$.

(d) Let $C := \{u \in L_2(\mu); u \leq 1\}$. Then C is convex and closed, and $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is **sub-Markovian**, i.e., S is positive (in particular, real) and L_∞ -contractive.

Indeed, let $u \in L_2(\mu)_+$. Then $-\alpha u \leq 1$ and therefore $-\alpha Su \leq 1$, for all $\alpha \geq 0$. This implies that $Su \geq 0$. This shows that S is a positive operator.

Let $u \in L_2(\mu) \cap L_\infty(\mu)$, $\|u\|_\infty \leq 1$. Then, for any $\gamma \in \mathbb{K}$ with $|\gamma| = 1$ one obtains

$$\operatorname{Re}(\gamma Su) = \operatorname{Re}(S(\gamma u)) = S(\operatorname{Re}(\gamma u)) \leq 1.$$

It is not difficult to show that this implies that $\|Su\|_\infty \leq 1$.

Conversely, if S is positive and L_∞ -contractive, then $u \leq 1$ implies $Su \leq Su^+ \leq 1$.

We are looking for another characterisation involving more directly the generator A and not just its resolvent. This is possible in Hilbert spaces. Let H be a Hilbert space over \mathbb{K} and let $\emptyset \neq C \subseteq H$ be closed and convex. We denote by $P_C: H \rightarrow C$ the **minimising projection** of H onto C , i.e., for $x \in H$ the element $P_C x \in C$ is the unique element satisfying

$$\|x - P_C x\| = \inf\{\|x - y\|; y \in C\}.$$

In other words, $P_C x$ is the best approximation to x in C . The mapping P_C is a contraction; in particular, P_C is continuous. It will be important for us that $P_C x$ can also be characterised as the unique element of C such that

$$\operatorname{Re}(y - P_C x | x - P_C x) \leq 0 \quad (y \in C); \quad (9.1)$$

see [Bou07; V, §1, N° 5, Théorème 1], [Bre83; Théorème V.2]. Geometrically, this means that $x - P_C x$ is ‘orthogonal’ to the boundary of C . Clearly, the mapping P_C satisfies $P_C \circ P_C = P_C$; so it deserves the name ‘projection’. We could not find a commonly accepted name for this mapping in the literature. One should keep in mind that in general P_C is not a linear operator.

The following result has a geometric appeal. It tells that C is invariant under the motion if the ‘driving term’ $Au(t)$ in the equation $u'(t) = Au(t)$ always points ‘sufficiently’ from $u(t)$ towards C . (For $\omega \leq 0$ this is quite intuitive. If $\omega > 0$, then one can interpret that it is more and more true, the closer $u(t)$ is to C .)

9.4 Proposition. *Let H, T, A be as before, $\emptyset \neq C \subseteq H$ a closed convex set, and denote by $P := P_C$ the minimising projection. Assume that there exists $\omega \in \mathbb{R}$ such that*

$$\operatorname{Re}(Ax \mid x - Px) \leq \omega \|x - Px\|^2 \quad (9.2)$$

for all $x \in \operatorname{dom}(A)$.

Then C is invariant under T .

Proof. Because of Proposition 9.1 we only have to show that $(I - rA)^{-1}(C) \subseteq C$ for small $r > 0$. (Observe that $(I - rA)^{-1} \in \mathcal{L}(H)$ for small $r > 0$.) Without loss of generality we assume $\omega > 0$. Let $0 < r < 1/\omega$ and let $x \in \operatorname{dom}(A)$ such that $(I - rA)x \in C$. We have to show that $x \in C$. Applying (9.1) with $y = (I - rA)x \in C$ we obtain

$$\operatorname{Re}((I - rA)x - Px \mid x - Px) \leq 0.$$

Thus

$$\begin{aligned} \|x - Px\|^2 &= \operatorname{Re}(rAx + (I - rA)x - Px \mid x - Px) \\ &\leq r \operatorname{Re}(Ax \mid x - Px) \leq r\omega \|x - Px\|^2. \end{aligned}$$

Using $r\omega < 1$ we conclude that $\|x - Px\| = 0$, $x = Px \in C$. \square

The converse of Proposition 9.4 holds for quasi-contractive semigroups.

9.5 Proposition. *Let H, T, A be as before, and assume that T is quasi-contractive, i.e., there exists $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$. Let $\emptyset \neq C \subseteq H$ be a closed and convex set, and assume that C is invariant under T .*

Then (9.2) holds for all $x \in \operatorname{dom}(A)$, with the minimising projection $P := P_C$.

Proof. Let $x \in \operatorname{dom}(A)$. Then (9.1) implies $\operatorname{Re}(T(t)Px - Px \mid x - Px) \leq 0$, and this inequality can be rewritten as $0 \leq \operatorname{Re}(-T(t)Px + Px \mid x - Px)$. One then obtains

$$\begin{aligned} \operatorname{Re}(T(t)x - x \mid x - Px) &\leq \operatorname{Re}(T(t)(x - Px) - (x - Px) \mid x - Px) \\ &\leq (e^{\omega t} - 1)\|x - Px\|^2. \end{aligned}$$

Dividing by t and taking the limit $t \rightarrow 0+$ we conclude that

$$\operatorname{Re}(Ax \mid x - Px) \leq \omega \|x - Px\|^2. \quad \square$$

For the case of contractive C_0 -semigroups we summarise the results of Propositions 9.4 and 9.5 as an equivalence.

9.6 Corollary. *Let T be a contractive C_0 -semigroup on a Hilbert space H , let A be the generator of T , and let $C \neq \emptyset$ be a closed convex subset of H . Then C is invariant under T if and only if*

$$\operatorname{Re}(Ax \mid x - Px) \leq 0 \quad (x \in \operatorname{dom}(A)), \quad (9.3)$$

where $P = P_C$ is the minimising projection.

For an illustration we expand the example where $H = L_2(\mu; \mathbb{R})$ and $C = L_2(\mu)_+$ is the positive cone; see Remark 9.3(a).

9.7 Corollary. *Let T be a contractive C_0 -semigroup on $L_2(\mu; \mathbb{R})$. Then T is **positive**, i.e., $T(t)$ is positive for all $t \geq 0$, if and only if*

$$(Au \mid u^+) \leq 0 \quad (u \in \operatorname{dom}(A)). \quad (9.4)$$

Here, $u^+ := u \vee 0$ is the **positive part of u** . If A satisfies the condition (9.4), then A is sometimes called **dispersive**.

Proof. Clearly, the projection P_C is given by $P_C u = u^+$. With $u^- := (-u)^+ = u^+ - u$, the condition (9.3) translates to $(Au \mid -u^-) \leq 0$ ($u \in \operatorname{dom}(A)$). Replacing u by $-u$ one obtains

$$0 \geq (A(-u) \mid -(-u)^-) = (Au \mid u^+) \quad (u \in \operatorname{dom}(A)). \quad \square$$

9.2 Application to Laplacians

We recall that the Dirichlet Laplacian Δ_D is associated with the classical Dirichlet form on $V = H_0^1(\Omega)$, embedded into $L_2(\Omega)$. We also recall that the Neumann Laplacian Δ_N is associated with the classical Dirichlet form on $V = H^1(\Omega)$, embedded into $L_2(\Omega)$. (See Example 5.13 and Theorem 7.13.)

9.8 Example. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then the C_0 -semigroup generated by the Dirichlet Laplacian Δ_D in $L_2(\Omega; \mathbb{C})$ leaves $L_2(\Omega; \mathbb{R})$ invariant. Moreover, $e^{t\Delta_D}$ is sub-Markovian for all $t \geq 0$, i.e., $e^{t\Delta_D}$ is positivity preserving and $\|e^{t\Delta_D} u\|_\infty \leq \|u\|_\infty$ for all $0 \leq u \in L_2 \cap L_\infty(\Omega)$.

Proof. (i) For the proof of the first property we use Corollary 9.6. As $-\Delta_D$ is accretive, the semigroup $(e^{t\Delta_D})_{t \geq 0}$ is contractive. The minimising projection $P: L_2(\Omega; \mathbb{C}) \rightarrow C := L_2(\Omega; \mathbb{R})$ is given by $Pu := \operatorname{Re} u$. Let $u \in \operatorname{dom}(\Delta_D)$. Then

$$\operatorname{Re}(\Delta_D u \mid u - Pu) = \operatorname{Re}(\Delta_D(\operatorname{Re} u) + i\Delta_D(\operatorname{Im} u) \mid i\operatorname{Im} u) = (\Delta_D(\operatorname{Im} u) \mid \operatorname{Im} u) \leq 0.$$

Therefore Corollary 9.6 implies that $L_2(\Omega; \mathbb{R})$ is invariant under $(e^{t\Delta_D})_{t \geq 0}$.

(ii) We now restrict $e^{t\Delta_D}$ a priori to $L_2(\Omega; \mathbb{R})$, and we show that the (closed convex) set $C := \{u \in L_2(\Omega; \mathbb{R}); u \leq 1\}$ is invariant under $(e^{t\Delta_D})_{t \geq 0}$. The minimising projection onto C is given by $Pu = u \wedge 1$. We have to show that

$$\operatorname{Re}(\Delta_D u \mid u - u \wedge 1) \leq 0$$

for all $u \in \text{dom}(\Delta_D)$.

We are going to use that $u \wedge 1 \in H_0^1(\Omega)$, and that $\nabla(u \wedge 1) = \mathbf{1}_{[u < 1]} \nabla u$. These properties will be shown in the following section (and the present example should serve as a motivation for this treatment); see Theorem 9.14. Accepting these properties we obtain

$$(\Delta_D u | u - u \wedge 1) = -(\nabla u | \mathbf{1}_{[u \geq 1]} \nabla u) = - \int_{[u \geq 1]} |\nabla u|^2 \leq 0.$$

Now the application of Corollary 9.6 yields the invariance of C . It was shown in Remark 9.3(d) that then $e^{t\Delta_D}$ is sub-Markovian for all $t \geq 0$. \square

We note that identically the same arguments show the same properties for the Neumann Laplacian. We refer to Exercise 9.3 for the discussion of invariance properties for the Robin Laplacian.

9.3 Interlude: lattice properties of $H^1(\Omega)$

We start this section with a warm-up.

9.9 Lemma. *Let $-\infty \leq a < b \leq \infty$, and let $u: (a, b) \rightarrow \mathbb{R}$ be continuously differentiable. Then $\partial|u| = (\text{sgn } u)u'$ in the distributional sense, where $\text{sgn}: \mathbb{C} \rightarrow \mathbb{C}$ is the signum function, $\text{sgn } \alpha := \frac{\alpha}{|\alpha|}$ if $0 \neq \alpha \in \mathbb{C}$, and $\text{sgn } 0 := 0$.*

For the proof we use a sequence (F_k) of functions $F_k \in C^\infty(\mathbb{R}; \mathbb{R})$ with $F_k(t) = |t| - \frac{1}{k}$ for $|t| \geq \frac{2}{k}$, $F_k(t) = 0$ for $|t| \leq \frac{1}{2k}$ and $|F_k'(t)| \leq 1$ for all $t \in \mathbb{R}$. (F_1 can be obtained as a convolution of $t \mapsto (|t| - 1)^+$ with a suitable C_c^∞ -function; and then $F_k(t) := \frac{1}{k} F_1(kt)$ ($t \in \mathbb{R}$).

Proof of Lemma 9.9. By the chain rule we have $(F_k \circ u)' = (F_k' \circ u)u'$. Then $F_k \circ u(x) \rightarrow |u(x)|$, uniformly for x in compact subsets of (a, b) , and $(F_k \circ u)'(x) \rightarrow (\text{sgn } u(x))u'(x)$ for all $x \in (a, b)$, with $|(F_k \circ u)'(x)| \leq |u'(x)|$ for all $x \in (a, b)$, $k \in \mathbb{N}$. Therefore $(F_k \circ u)' \rightarrow (\text{sgn } u)u'$ locally in L_1 on (a, b) . This implies $\partial|u| = (\text{sgn } u)u'$. \square

9.10 Remark. It is an interesting observation that, in the situation of Lemma 9.9, one also obtains that $u' = 0$ on $[u = 0]$ ($= \{x \in (a, b); u(x) = 0\}$).

Indeed, using a sequence (F_k) of functions in $C^\infty(\mathbb{R})$ satisfying $F_k(0) = 0$, $F_k'(0) = 1$, $0 \leq F_k' \leq 1$, $F_k(t) = -1/k$ ($t \leq -2/k$), $F_k(t) = 1/k$ ($t \geq 2/k$), we obtain $F_k \circ u \rightarrow 0$ uniformly, $(F_k \circ u)' = (F_k' \circ u)u' \rightarrow \mathbf{1}_{[u=0]}u'$ locally in L_1 on (a, b) . This shows that $\mathbf{1}_{[u=0]}u'$ is the distributional derivative of the zero-function; hence $u' = 0$ a.e. on $[u = 0]$.

Our aim is to show similar properties in more general situations. The first point is that the chain rule holds also for distributional derivatives.

In the following let $\Omega \subseteq \mathbb{R}^n$ be an open set. Until further notice all the function spaces will consist of real-valued functions.

9.11 Proposition. *Let $F \in C^1(\mathbb{R}; \mathbb{R})$, $|F'(t)| \leq 1$ for all $t \in \mathbb{R}$, $u \in L_{1,\text{loc}}(\Omega)$, $j \in \{1, \dots, n\}$, $\partial_j u \in L_{1,\text{loc}}(\Omega)$.*

Then $\partial_j(F \circ u) = (F' \circ u)\partial_j u$.

Proof. Without loss of generality we assume that $F(0) = 0$; then $|F(t)| \leq |t|$ for all $t \in \mathbb{R}$. Being the distributional derivative of a function is a local property, and therefore (after suitable multiplication by a C_c^∞ -function) it is sufficient to treat the case that $\Omega = \mathbb{R}^n$ and $u, \partial_j u \in L_1(\mathbb{R}^n)$.

Let $(\rho_k)_{k \in \mathbb{N}}$ be a δ -sequence in $C_c^\infty(\mathbb{R}^n)$. Then $u_k := \rho_k * u \rightarrow u$, $\partial_j u_k = \rho_k * \partial_j u \rightarrow \partial_j u$ in $L_1(\mathbb{R}^n)$, and for a suitable subsequence (ρ_{k_i}) these convergences also hold a.e. as well as “boundedly”, in the sense that there exists $h \in L_1(\mathbb{R}^n)$ such that $|u_{k_i}|, |\partial_j u_{k_i}| \leq h$. As $u_k \in C^\infty(\mathbb{R}^n)$, one has $F \circ u_{k_i} \in C^1(\mathbb{R}^n)$, and

$$F \circ u_{k_i} \rightarrow F \circ u, \quad \partial_j(F \circ u_{k_i}) = (F' \circ u_{k_i})\partial_j u_{k_i} \rightarrow (F' \circ u)\partial_j u \quad \text{a.e.},$$

since F, F' are continuous. Furthermore, $|F \circ u_{k_i}| \leq |u_{k_i}| \leq h$, $|\partial_j(F \circ u_{k_i})| \leq h$, by the hypotheses on F and the subsequence, and therefore $F \circ u_{k_i} \rightarrow F \circ u$, $\partial_j(F \circ u_{k_i}) \rightarrow (F' \circ u)\partial_j u$ in $L_1(\mathbb{R}^n)$, and this implies that $\partial_j(F \circ u) = (F' \circ u)\partial_j u$. \square

Next, we extend the chain rule of Proposition 9.11 to more general composition functions F .

9.12 Proposition. *Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be continuous, and assume that there exist a function $G: \mathbb{R} \rightarrow \mathbb{R}$ and a sequence (F_k) in $C^1(\mathbb{R}; \mathbb{R})$ with $\|F'_k\|_\infty \leq 1$ ($k \in \mathbb{N}$), $F_k \rightarrow F$ pointwise, and $F'_k \rightarrow G$ pointwise ($k \rightarrow \infty$).*

Let $u \in L_{1,\text{loc}}(\Omega)$, $j \in \{1, \dots, n\}$, $\partial_j u \in L_{1,\text{loc}}(\Omega)$. Then $F \circ u \in L_{1,\text{loc}}(\Omega)$, $\partial_j(F \circ u) = (G \circ u)\partial_j u$.

Proof. From Proposition 9.11 we know that $F_k \circ u \in L_{1,\text{loc}}(\Omega)$, $\partial_j(F_k \circ u) = (F'_k \circ u)\partial_j u$. Applying the dominated convergence theorem on relatively compact subsets of Ω one obtains the assertions. \square

9.13 Corollary. *Let $u \in L_{1,\text{loc}}(\Omega)$, $j \in \{1, \dots, n\}$, $\partial_j u \in L_{1,\text{loc}}(\Omega)$. Then $u^+, u \wedge 1 \in L_{1,\text{loc}}(\Omega)$, $\partial_j(u^+) = \mathbf{1}_{[u>0]}\partial_j u$, $\partial_j(u \wedge 1) = \mathbf{1}_{[u<1]}\partial_j u$.*

Proof. Similarly to the construction of the sequence (F_k) at the beginning of the section one can construct a sequence (F_k) converging to $F(t) := t^+$, with the properties mentioned in Proposition 9.12 and such that $F'_k \rightarrow \mathbf{1}_{(0,\infty)}$ pointwise. Then Proposition 9.12 implies the assertion for u^+ . The reasoning for $u \wedge 1$ is analogous. \square

The following result is the conclusion for the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. It implies that they are Stonian sublattices of $L_2(\mu)$.

9.14 Theorem. *Let $u \in H^1(\Omega)$. Then $u^+, u \wedge 1 \in H^1(\Omega)$, $\nabla u^+ = \mathbf{1}_{[u>0]}\nabla u$, $\nabla(u \wedge 1) = \mathbf{1}_{[u<1]}\nabla u$.*

If $u \in H_0^1(\Omega)$, then $u^+, u \wedge 1 \in H_0^1(\Omega)$.

Proof. It was shown in Corollary 9.13 that the indicated derivatives for u^+ and $u \wedge 1$ are the distributional derivatives. As they belong to $L_2(\Omega; \mathbb{R}^n)$, the first part of the theorem is proved.

Now let $u \in H_0^1(\Omega)$. There exists a sequence (u_k) in $H_c^1(\Omega)$ such that $u_k \rightarrow u$ in $H^1(\Omega)$. Then (u_k^+) is a bounded sequence in $H_c^1(\Omega)$, and $u_k^+ \rightarrow u^+$ in $L_2(\Omega)$. The fact formulated in Remark 9.15 below shows that u belongs to the closure of $H_c^1(\Omega)$ in $H^1(\Omega)$, i.e., to $H_0^1(\Omega)$. The argument for $u \wedge 1$ is similar. \square

9.15 Remark. Let V, H be Hilbert spaces, $V \subseteq H$ with continuous embedding. Let (v_n) be a bounded sequence in V that is weakly convergent in H to $u \in H$. Then $u \in V$, and $v_n \rightarrow u$ weakly in V .

Indeed, there exist $v \in V$ and a subsequence $(v_{n_k}), v_{n_k} \rightarrow v$ weakly in V . Then also $v_{n_k} \rightarrow v$ weakly in H ; hence $v = u$. A standard sub-sub-sequence argument shows $v_n \rightarrow u$ weakly in V .

9.4 Invariance described by forms

In this section we transform the invariance criteria obtained in Propositions 9.4 and 9.5 to conditions on forms instead of operators. This means that we only treat C_0 -semigroups associated with forms. In this case the C_0 -semigroup is quasi-contractive, and the condition (9.2) is equivalent to the invariance of C under the semigroup.

We restrict our treatment to the case of embedded forms, i.e., we assume that V is a Hilbert space that is densely embedded into H and that $a: V \times V \rightarrow \mathbb{K}$ is a bounded H -elliptic form. We recall that this means that there exist $\omega \in \mathbb{R}, \alpha > 0$ such that

$$\operatorname{Re} a(u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad (u \in V). \quad (9.5)$$

In the following the quantity

$$\omega_0(a) := \inf \{ \omega \in \mathbb{R}; \operatorname{Re} a(u) + \omega \|u\|_H^2 \geq 0 \ (u \in V) \}$$

will be needed. Then $-\omega_0(a)$ is the ‘lower bound’ of a , in particular $\operatorname{Re} a(u) \geq -\omega_0(a) \|u\|_H^2$ for all $u \in V$. It follows from Proposition 5.5(b) that $\|T(t)\| \leq e^{\omega_0(a)t}$ for all $t \geq 0$, where T is the C_0 -semigroup associated with a .

The notation used above will be fixed throughout this section. Coming back to invariance, let $\emptyset \neq C \subseteq H$ be convex and closed, and let $P := P_C$ be the minimising projection.

9.16 Proposition. *Let C be invariant under T . Then $P(V) \subseteq V$.*

9.17 Remark. At the first glance, this property might look rather unexpected, because the elements of V have some quality (or ‘regularity’), and it is surprising that this quality is preserved under P . To make the point, the elements of the domain of the generator will not be mapped to the domain of the generator, in general.

For the proof we single out a technical detail, which will be useful in several of the subsequent proofs.

9.18 Lemma. *Let $(u_n), (v_n)$ be sequences in V , $u_n \rightarrow u$ in H , (v_n) bounded in V , and*

$$\operatorname{Re} a(u_n, u_n - v_n) \leq 0 \quad (n \in \mathbb{N}).$$

Then $u \in V$, and $u_n \rightarrow u$ weakly in V .

Proof. Using (9.5) we estimate

$$\begin{aligned} \alpha \|u_n\|_V^2 &\leq \operatorname{Re} a(u_n, u_n) + \omega \|u_n\|_H^2 \leq \operatorname{Re} a(u_n, v_n) + \omega \|u_n\|_H^2 \\ &\leq M \|u_n\|_V \|v_n\|_V + \omega \|u_n\|_H \|u_n\|_V \end{aligned}$$

(where M denotes the bound of a , and without loss of generality we have assumed $\|\cdot\|_H \leq \|\cdot\|_V$). This implies that (u_n) is bounded in V , and therefore Remark 9.15 implies that $u_n \rightarrow u$ weakly in V . \square

Proof of Proposition 9.16. Adding the inequality $(y - Px | Px - y) \leq 0$ to (9.1) one obtains

$$\operatorname{Re}(Px - y | y - x) = \operatorname{Re}(y - Px | x - y) \leq 0 \quad (x \in H, y \in C). \quad (9.6)$$

For $r > 0$ small enough (say, $0 < r < r_0 < \infty$) we define $R_r := (I + rA)^{-1}$. Then $AR_r = \frac{1}{r}(I + rA - I)R_r = \frac{1}{r}(I - R_r)$, so

$$a(R_r u, v) = (AR_r u | v) = \frac{1}{r}(u - R_r u | v) \quad (u \in H, v \in V). \quad (9.7)$$

We recall from Proposition 9.1 that the invariance of C under T implies that $R_r(C) \subseteq C$. Let $u \in V$. Using (9.7) and applying (9.6) with $y = R_r Pu \in C$, we obtain

$$\operatorname{Re} a(R_r Pu, R_r Pu - u) = \frac{1}{r} \operatorname{Re}(Pu - R_r Pu | R_r Pu - u) \leq 0.$$

Since $R_r Pu \rightarrow Pu$ in H as $r \rightarrow 0$ (by Lemma 2.10(a)), Lemma 9.18 implies that $Pu \in V$. \square

We insert an auxiliary result that will be used in the proof of the next theorem.

9.19 Lemma. (a) *As in the proof of Proposition 9.16 we define $R_r := (I + rA)^{-1}$ for $0 < r < r_0$ (suitable). Then $R_r u \rightarrow u$ ($r \rightarrow 0$) weakly in V for all $u \in V$.*

(b) *Assume that (u_n) is a sequence converging weakly in V to u . Then $a(u_n, v) \rightarrow a(u, v)$ for all $v \in V$.*

Proof. (a) Let $u \in V$. By (9.7) we obtain

$$a(R_r u, R_r u - u) = \frac{1}{r}(u - R_r u | R_r u - u) \leq 0.$$

Since $R_r u \rightarrow u$ ($r \rightarrow 0$) in H , Lemma 9.18 implies that $R_r u \rightarrow u$ weakly in V as $r \rightarrow 0$.

(b) For $v \in V$, the functional $V \ni u \mapsto a(u, v) \in \mathbb{K}$ is continuous, by the boundedness of a . This implies the assertion. \square

Now we come to the fundamental result concerning invariance characterised by conditions on the form. The inequality (9.9) appearing below has already been commented upon before Proposition 9.4. In order to give a geometrical interpretation to (9.8) we note that, loosely speaking, $a(Pu, u - Pu)$ can be understood as $(A(Pu) | u - Pu)$ (only Pu is not necessarily in $\operatorname{dom}(A)$). So, the condition gives information on the driving term $-Au(t)$, whenever $u(t)$ is the image Pu of some $u \in H \setminus C$: in these points, the driving term ‘points towards C ’.

9.20 Theorem. *Under the previous assumptions the following properties are equivalent.*

- (i) C is invariant under T ;
- (ii) $P(V) \subseteq V$, and

$$\operatorname{Re} a(Pu, u - Pu) \geq 0 \quad (9.8)$$

for all $u \in V$;

(iii) there exists a dense subset D of V such that $P(D) \subseteq V$, and (9.8) holds for all $u \in D$;

- (iv) $P(V) \subseteq V$, and

$$\operatorname{Re} a(u, u - Pu) \geq -\omega \|u - Pu\|^2 \quad (u \in V), \quad (9.9)$$

for some $\omega \in \mathbb{R}$ / for $\omega = \omega_0(a)$.

Proof. (i) \Rightarrow (ii). $P(V) \subseteq V$ was shown in Proposition 9.16. Let $u \in V$. Then for $0 < r < r_0$ we have

$$\operatorname{Re} a(R_r Pu, u - Pu) = \frac{1}{r} \operatorname{Re} (Pu - R_r Pu | u - Pu) \geq 0$$

by (9.7) and (9.1), and from Lemma 9.19 we obtain

$$\operatorname{Re} a(Pu, u - Pu) \geq 0.$$

(ii) \Leftrightarrow (iii). ‘(ii) \Rightarrow (iii)’ is trivial. For the proof of ‘(iii) \Rightarrow (ii)’ let $u \in V$. There exists a sequence (u_n) in D such that $u_n \rightarrow u$ in V as $n \rightarrow \infty$. By the hypothesis we have

$$\operatorname{Re} a(Pu_n, Pu_n - u_n) \leq 0 \quad (n \in \mathbb{N}). \quad (9.10)$$

From the continuity of P we obtain $Pu_n \rightarrow Pu$ in H , and therefore Lemma 9.18 implies that $Pu \in V$, and $Pu_n \rightarrow Pu$ weakly in V .

In order to show (9.8) we use that on V an equivalent norm is given by

$$\|v\|_a := (\operatorname{Re} a(v) + \omega \|v\|_H^2)^{1/2} \quad (v \in V),$$

associated with the scalar product $\frac{1}{2}(a + a^*) + \omega (\cdot | \cdot)_H$ (with ω from (9.5)). It follows that for the weakly convergent sequence (Pu_n) in V one has $\|Pu\|_a \leq \liminf_{n \rightarrow \infty} \|Pu_n\|_a$. Using (9.10) we obtain

$$\begin{aligned} \operatorname{Re} a(Pu) + \omega \|Pu\|_H^2 &\leq \liminf_{n \rightarrow \infty} (\operatorname{Re} a(Pu_n) + \omega \|Pu_n\|_H^2) \\ &\leq \liminf_{n \rightarrow \infty} \operatorname{Re} a(Pu_n, u_n) + \omega \|Pu\|_H^2. \end{aligned}$$

Since $a(Pu_n, u_n) = a(Pu_n, u_n - u) + a(Pu_n, u) \rightarrow 0 + a(Pu, u)$ as $n \rightarrow \infty$, we conclude that $\operatorname{Re} a(Pu, u - Pu) = \operatorname{Re} a(Pu, u) - \operatorname{Re} a(Pu) \geq 0$.

(ii) \Rightarrow (iv) with ‘ $\omega = \omega_0(a)$ ’ follows from the identity

$$a(u, u - Pu) = a(Pu, u - Pu) + a(u - Pu)$$

and the definition of $\omega_0(a)$.

(iv) with ‘some $\omega \in \mathbb{R}$ ’ \Rightarrow (i). Because of $a(u, u - Pu) = (Au | x - Pu)$ ($u \in \operatorname{dom}(A)$), condition (9.9) implies (9.2) for the generator $-A$ of T . Then the assertion follows from Proposition 9.4. \square

9.21 Example. We come back to Example 9.8 and show again part (ii), i.e., that the C_0 -semigroup generated by the Dirichlet or Neumann Laplacian is sub-Markovian. Using $C = \{u \in L_2(\Omega; \mathbb{R}); u \leq 1\}$ and the minimising projection $Pu = u \wedge 1$, we check property (ii) of Theorem 9.20: Theorem 9.14 implies that P leaves $V = H_0^1(\Omega)$ (and also $V = H^1(\Omega)$) invariant, and

$$a(Pu, u - Pu) = \int \nabla(u \wedge 1) \cdot (\nabla u - \nabla(u \wedge 1)) = \int \mathbf{1}_{[u < 1]} \nabla u \cdot \mathbf{1}_{[u \geq 1]} \nabla u = 0$$

shows that property (ii) is satisfied. Hence C is invariant by Theorem 9.20.

9.22 Remark. We emphasise that in Theorem 9.20 the associated semigroup is not assumed to be contractive. If the convex set C is not a cone, then the quasi-contractive case cannot be reduced to the contractive case by scaling.

An example for an application to a non-contractive semigroup can be found in Exercise 9.5.

Notes

Clearly, it is of fundamental interest to ask for criteria describing when certain sets are invariant under the time evolution of a system, and questions of this kind have a long history, in particular in the finite-dimensional case, for linear and non-linear problems.

The seminal papers for investigating such questions in infinite-dimensional spaces are probably the papers by Beurling and Deny, [BD58], [BD59]. Another fundamental paper on positive contraction semigroups is by Phillips [Phi62]. Skipping a lot of history we refer to Kunita [Kun70] for the idea to include non-contractive non-symmetric semigroups in the treatment, and we mention the more recent papers by Ouhabaz, [Ouh96] and [MVV05] as well as Ouhabaz' book [Ouh05] and refer to the literature mentioned in these sources. The invariance criterion in Theorem 9.20 is taken from [MVV05; Theorem 2.1]. An investigation on invariance for non-linear evolution equations can be found in [Bar96].

The lattice properties of the Sobolev spaces $H_0^1(\Omega)$ and $H^1(\Omega)$ have been developed in the 70's (at the latest). We refer to a paper of Marcus and Mizel [MM79] where also earlier references can be found. Meanwhile, these properties can be found in several books on Sobolev spaces or partial differential equations. We refer to [EE87; p. VI.2] for a more general chain rule than presented in the lecture.

Exercises

9.1 Let (Ω, μ) be a measure space, $1 \leq p \leq \infty$, and let $A \in \mathcal{L}(L_p(\mu))$ be positive, i.e., $Au \geq 0$ for all $u \in L_p(\mu)$ with $u \geq 0$.

(a) Show that $|Au| \leq A|u|$ for all $u \in L_p(\mu)$.

(b) Show that

$$\|A\| = \sup\{\|Au\|_p; u \in L_p(\mu), u \geq 0, \|u\|_p \leq 1\}.$$

9.2 Let (Ω, μ) be a measure space, $\check{C} \subseteq \mathbb{K}$ convex and closed, $0 \in \check{C}$; let $\check{P}: \mathbb{K} \rightarrow \check{C}$ be the minimising projection. Then clearly

$$C := \{u \in L_2(\mu); u(x) \in \check{C} \text{ for } \mu\text{-a.e. } x\} \neq \emptyset$$

is convex and closed.

Show that the minimising projection $P: L_2(\mu) \rightarrow C$ is given by $(Pu)(x) = \check{P}(u(x))$ ($x \in \Omega$).

9.3 Let Δ_β be the Robin Laplacian from Section 7.5.

(a) Let β be real-valued. Show that the C_0 -semigroup generated by Δ_β is positive.

(b) Let $\beta \geq 0$. Show that the C_0 -semigroup generated by Δ_β is sub-Markovian.

9.4 (a) Assume that H, V, a, j are as in Proposition 5.5, and such that minus the operator associated with (a, j) is a generator. Let $\emptyset \neq C \subseteq H$ be convex and closed, P the minimising projection onto C . Let $\hat{P}: V \rightarrow V$ be a mapping satisfying $Pj = j\hat{P}$. Further assume

$$a(u, u - \hat{P}u) \geq 0 \quad (v \in V). \quad (9.11)$$

Show that C is invariant under the C_0 -semigroup associated with (a, j) . (Hint: Use Proposition 9.4.)

(For the more ambitious: Show the assertion if (9.11) is replaced by

$$a(\hat{P}u, u - \hat{P}u) \geq 0 \quad (v \in V),$$

additionally assuming that a is j -elliptic.)

(b) Show that the C_0 -semigroup generated by the Dirichlet-to-Neumann operator of Section 8.1 is sub-Markovian. (Hint: Show that the operation $u \mapsto u \wedge 1$ for an H^1 -function is consistent with the trace operator.)

9.5 Let $\Omega \subseteq \mathbb{R}^n$ be open, let $b \in L_\infty(\Omega; \mathbb{R}^n)$, and define the operator A in $L_2(\Omega)$ by

$$\begin{aligned} \text{dom}(A) &:= \{u \in H_0^1(\Omega); -\Delta u + b \cdot \nabla u \in L_2(\Omega)\}, \\ Au &:= -\Delta u + b \cdot \nabla u \quad (u \in \text{dom}(A)). \end{aligned}$$

(a) Show that A is associated with an H -elliptic form on $V \times V$, with $V := H_0^1(\Omega) \subseteq H := L_2(\Omega)$ (and therefore $-A$ generates a quasi-contractive C_0 -semigroup). Show that the semigroup generated by $-A$ is holomorphic of angle $\pi/2$, if $\mathbb{K} = \mathbb{C}$ (cf. Exercises 7.3 and 7.4).

(b) Show that the C_0 -semigroup generated by $-A$ is sub-Markovian.

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Lecture 10

Interpolation of holomorphic semigroups

In the first section of this lecture we will present an extremely powerful tool of Functional Analysis, important in many areas: complex interpolation. It should be looked upon as the surprising fact that (elementary) complex methods are a useful tool for deriving inequalities. The main result is the Stein interpolation theorem; as a particular case one also obtains the famous and important Riesz-Thorin interpolation theorem.

For us, the important consequence will be that a holomorphic semigroup on some L_{p_1} that is bounded on some L_{p_0} for real times can be ‘interpolated’ holomorphically to other L_p ’s. In the last section we demonstrate the interplay of invariance, interpolation and duality in applications to C_0 -semigroups on L_2 .

10.1 Interlude: the Stein interpolation theorem

Throughout this section the scalar field will be $\mathbb{K} = \mathbb{C}$.

10.1.1 The three lines theorem

The content of this section is a version of the maximum principle for holomorphic functions on an unbounded set. First we recall the maximum principle.

If $\Omega \subseteq \mathbb{C}$ is a bounded open set, and $h: \bar{\Omega} \rightarrow \mathbb{C}$ is continuous and holomorphic on Ω , then $\|h\|_{\bar{\Omega}} \leq \|h\|_{\partial\Omega}$. This is an easy consequence of Cauchy’s integral formula. Here and in the following we denote by $\|\cdot\|_M$ the supremum norm taken over the set M .

In this and the following section the set $S \subseteq \mathbb{C}$ will be the open strip

$$S := \{z \in \mathbb{C}; 0 < \operatorname{Re} z < 1\}.$$

10.1 Theorem. *Let $h: \bar{S} \rightarrow \mathbb{C}$ be continuous and bounded, and $h|_S$ holomorphic. Then*

$$\|h\|_{\bar{S}} \leq \|h\|_{\partial S}.$$

Proof. For $n \in \mathbb{N}$, the function $\psi_n(z) := \frac{n}{z+n}$ is continuous on \bar{S} and holomorphic on S . From the maximum principle one obtains

$$\|\psi_n h\|_{\bar{S}_k} \leq \|\psi_n h\|_{\partial S_k} \quad (k \in \mathbb{N}),$$

where $S_k := \{x \in S; |\operatorname{Im} z| < k\}$. As $\|\psi_n h\|_{\{z \in \partial S_k; |\operatorname{Im} z|=k\}} \rightarrow 0$ ($k \rightarrow \infty$), we conclude that

$$\|\psi_n h\|_{\bar{S}} \leq \|\psi_n h\|_{\partial S} \leq \|h\|_{\partial S} \quad (n \in \mathbb{N}).$$

Letting $n \rightarrow \infty$ we obtain the assertion. \square

10.2 Remark. Taking more astute functions ψ_n one may weaken the assumption that h is bounded; it suffices that $|h(z)| \leq c e^{\alpha |\operatorname{Im} z|}$ with $c \geq 0$ and $\alpha < \pi$ (see Exercise 10.1).

It is of interest to distinguish between the suprema of h at $[\operatorname{Re} = 0]$ and $[\operatorname{Re} = 1]$. This is expressed in the following statement.

10.3 Corollary. (*Three lines theorem*) *Let h be as in Theorem 10.1. Then*

$$\|h\|_{[\operatorname{Re}=\tau]} \leq \|h\|_{[\operatorname{Re}=0]}^{1-\tau} \|h\|_{[\operatorname{Re}=1]}^{\tau}$$

for all $0 \leq \tau \leq 1$.

Proof. Let $b_j > \|h\|_{[\operatorname{Re}=j]}$ for $j = 0, 1$. Then we apply Theorem 10.1 to the function $z \mapsto \left(\frac{b_0}{b_1}\right)^z h(z)$ and obtain

$$\left(\frac{b_0}{b_1}\right)^{\tau} \|h\|_{[\operatorname{Re}=\tau]} \leq b_0 = \left(\frac{b_0}{b_1}\right)^1 b_1 \quad (0 \leq \tau \leq 1);$$

hence

$$\|h\|_{[\operatorname{Re}=\tau]} = \left(\frac{b_1}{b_0}\right)^{\tau} b_0 = b_0^{1-\tau} b_1^{\tau}.$$

Taking the infima over the b_j 's one obtains the assertion. \square

10.4 Remark. It follows that $\tau \mapsto \|h\|_{[\operatorname{Re}=\tau]}$ is a log-convex function, i.e., $\tau \mapsto \ln \|h\|_{[\operatorname{Re}=\tau]}$ is convex.

10.1.2 The Stein interpolation theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{A}_c \subseteq \{A \in \mathcal{A}; \mu(A) < \infty\}$ be a ring of subsets of Ω , with the property that the space of simple functions over \mathcal{A}_c ,

$$S(\mathcal{A}_c) := \operatorname{lin}\{\mathbf{1}_A; A \in \mathcal{A}_c\}$$

is dense in $L_1(\mu)$. We will also use the notation

$$L_1(\mathcal{A}_c) := \left\{ u: \Omega \rightarrow \mathbb{C}; u \text{ measurable, } \mathbf{1}_A u \in L_1(\mu) \text{ for all } A \in \mathcal{A}_c, \right. \\ \left. \exists (A_n) \text{ in } \mathcal{A}_c: [u \neq 0] \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\},$$

where we understand the elements of $L_1(\mathcal{A}_c)$ as equivalence classes of a.e. equal functions (with the equivalence classes formed in the set of measurable functions). With these conventions one has $uv \in L_1(\mu)$ for all $u \in S(\mathcal{A}_c), v \in L_1(\mathcal{A}_c)$. We note that for σ -finite

measure spaces the requirement on $[u \neq 0]$ in the definition of $L_1(\mathcal{A}_c)$ can be dispensed with. (The index ‘c’ should be remindful of ‘compact’: if $\Omega \subseteq \mathbb{R}^n$ is an open set, then one can choose \mathcal{A}_c as the system of relatively compact measurable subsets of Ω .)

Let the strip $S \subseteq \mathbb{C}$ be defined as in Subsection 10.1.1, and let $p_0, p_1, q_0, q_1 \in [1, \infty]$, $M_0, M_1 \geq 0$. For $\tau \in (0, 1)$ we denote

$$\frac{1}{p_\tau} := \frac{1-\tau}{p_0} + \frac{\tau}{p_1}, \quad \frac{1}{q_\tau} := \frac{1-\tau}{q_0} + \frac{\tau}{q_1}, \quad M_\tau := M_0^{1-\tau} M_1^\tau.$$

Finally, let $L(S(\mathcal{A}_c), L_1(\mathcal{A}_c))$ denote the linear operators from $S(\mathcal{A}_c)$ to $L_1(\mathcal{A}_c)$ (without continuity requirement), and let $\Phi: \bar{S} \rightarrow L(S(\mathcal{A}_c), L_1(\mathcal{A}_c))$ be a mapping satisfying the following two conditions.

(i) $\|\Phi(j+is)u\|_{q_j} \leq M_j \|u\|_{p_j}$ for all $u \in S(\mathcal{A}_c)$, all $s \in \mathbb{R}$ and $j = 0, 1$. (The estimate means, in particular, that $\Phi(j+is)u \in L_{q_j}(\mu)$ for all the indicated terms.)

(ii) For all $A, B \in \mathcal{A}_c$ the function $\bar{S} \ni z \mapsto \int (\Phi(z)\mathbf{1}_A)\mathbf{1}_B d\mu$ is continuous and bounded, and its restriction to S is holomorphic.

After these preparations we can state the Stein interpolation theorem, the main result of this section.

10.5 Theorem. (Stein) *In the context described above it follows that*

$$\|\Phi(\tau+is)u\|_{q_\tau} \leq M_\tau \|u\|_{p_\tau}$$

for all $u \in S(\mathcal{A}_c)$, $s \in \mathbb{R}$ and $\tau \in (0, 1)$.

Before proceeding to the proof we mention the important application to the situation where the function Φ is constant.

10.6 Corollary. (Riesz-Thorin) *Let $(\Omega, \mu), p_0, p_1, q_0, q_1, M_0, M_1$ be as before, and let $B \in L(S(\mathcal{A}_c), L_1(\mathcal{A}_c))$ be such that $\|Bu\|_{q_j} \leq M_j \|u\|_{p_j}$ ($u \in S(\mathcal{A}_c)$, $j = 0, 1$).*

Then for all $\tau \in (0, 1)$ one has $\|Bu\|_{q_\tau} \leq M_\tau \|u\|_{p_\tau}$ ($u \in S(\mathcal{A}_c)$).

As an application we recall Example 9.8, where it was shown that the operators $e^{t\Delta_D}$ are sub-Markovian. Since we know that $e^{t\Delta_D}$ is contractive in $L_2(\Omega)$, we conclude from Corollary 10.6 that $e^{t\Delta_D}$ is contractive in $L_p(\Omega)$ for all $p \in [2, \infty]$.

The following fact will be needed in the proof of Theorem 10.5.

10.7 Lemma. *Let the notation be as above. Let $p, p' \in [1, \infty]$, $\frac{1}{p} + \frac{1}{p'} = 1$, $u \in L_1(\mathcal{A}_c)$, and assume that there exists $c \geq 0$ such that*

$$\left| \int uv d\mu \right| \leq c \|v\|_{p'} \tag{10.1}$$

for all $v \in S(\mathcal{A}_c)$. Then $u \in L_p(\mu)$, $\|u\|_p \leq c$.

Proof. (i) In the first step we show that (10.1) carries over to all

$$v \in L_{\infty, c}(\mathcal{A}_c) := \{w: \Omega \rightarrow \mathbb{C}; w \text{ measurable, bounded, } \exists A \in \mathcal{A}_c: [w \neq 0] \subseteq A\}.$$

Let $v \in L_{\infty,c}(\mathcal{A}_c)$, $A \in \mathcal{A}_c$ such that $[v \neq 0] \subseteq A$. The hypotheses imply that there exists a sequence (v_n) in $S(\mathcal{A}_c)$, such that $v_n \rightarrow v$ a.e. Since \mathcal{A}_c is a ring, we can choose (v_n) such that $[v_n \neq 0] \subseteq A$ and $\|v_n\|_{\infty} \leq \|v\|_{\infty}$ for all $n \in \mathbb{N}$. Then $\int uv_n d\mu \rightarrow \int uv d\mu$ as well as $v_n \rightarrow v$ in $L_{p'}(\mu)$ ($n \rightarrow \infty$), and this implies (10.1).

(ii) Recall that $[u \neq 0] \subseteq \bigcup_{n \in \mathbb{N}} A_n$ for a suitable sequence (A_n) in \mathcal{A}_c .

If $p = 1$, then $\int_A |u| d\mu = \int u(\overline{\text{sgn } u} \mathbf{1}_A) d\mu \leq c$ for all $A \in \mathcal{A}_c$, and therefore $\|u\|_1 \leq c$.

In the case $1 < p < \infty$ we use that $|u|$ can be approximated pointwise by an increasing sequence $(v_k)_{k \in \mathbb{N}}$ in $L_{\infty,c}(\mathcal{A}_c)_+$. We estimate

$$\|v_k\|_p^p = \int v_k^p d\mu \leq \int u(\overline{\text{sgn } u} v_k^{p-1}) d\mu \leq c \|v_k^{p-1}\|_{p'} = c \|v_k\|_p^{p-1};$$

hence $\|v_k\|_p \leq c$. Now the monotone convergence theorem implies $u \in L_p(\mu)$, $\|u\|_p \leq c$.

If $p = \infty$ and $[|u| > c]$ is not a null set, then there exists $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ and $A \subseteq [|u| > c]$. Then

$$\int u(\overline{\text{sgn } u} \mathbf{1}_A) d\mu = \int |u| \mathbf{1}_A d\mu > c\mu(A) = c\|\mathbf{1}_A\|_1$$

leads to a contradiction. □

Proof of Theorem 10.5. Let $\tau \in (0, 1)$, and let $u, v \in S(\mathcal{A}_c)$, $\|u\|_{p_\tau} = 1$, $\|v\|_{q'_\tau} = 1$ (where $\frac{1}{q_\tau} + \frac{1}{q'_\tau} = 1$). We are going to show that then $|\int (\Phi(\tau)u)v d\mu| \leq M_\tau$.

For $z \in \bar{S}$ define $\alpha(z) := \frac{1-z}{p_0} + \frac{z}{p_1}$, $\beta(z) := \frac{1-z}{q'_0} + \frac{z}{q'_1}$,

$$F(z) := \begin{cases} |u|^{\alpha(z)p_\tau} \text{sgn } u & \text{if } p_\tau \neq \infty, \\ u & \text{if } p_\tau = \infty, \end{cases}$$

$$G(z) := \begin{cases} |v|^{\beta(z)q'_\tau} \text{sgn } v & \text{if } q'_\tau \neq \infty, \\ v & \text{if } q'_\tau = \infty. \end{cases}$$

Then $F(\tau) = u$, $G(\tau) = v$. Note that $F(z), G(z) \in S(\mathcal{A}_c)$ for all $z \in \bar{S}$. Indeed, u can be written as $u = \sum_{j=1}^n c_j \mathbf{1}_{A_j}$, with $A_1, \dots, A_n \in \mathcal{A}_c$ pairwise disjoint, and then $F(z) = \sum_{j=1}^n |c_j|^{\alpha(z)p_\tau} (\text{sgn } c_j) \mathbf{1}_{A_j}$, if $p_\tau \neq \infty$, and similarly for $G(z)$.

Finally we define

$$h(z) := \int (\Phi(z)F(z))G(z) d\mu \quad (z \in \bar{S}).$$

Then h is continuous, bounded, and holomorphic on S . Indeed, it is sufficient to show this for the case $u = c\mathbf{1}_A$, $v = d\mathbf{1}_B$, with $c, d \in \mathbb{C}$, $A, B \in \mathcal{A}_c$. If $p_\tau, q'_\tau < \infty$, then

$$\int (\Phi(z)F(z))G(z) d\mu = |c|^{\alpha(z)p_\tau} (\text{sgn } c) |d|^{\beta(z)q'_\tau} (\text{sgn } d) \int (\Phi(z)\mathbf{1}_A)\mathbf{1}_B d\mu \quad (z \in \bar{S}),$$

and this function has the required properties, by condition (ii). An analogous – easier – computation shows that this also holds if one or both of p_τ, q'_τ are equal to ∞ .

The definition of F is such that $\|F(\sigma + is)\|_{p_\sigma} = 1$ for all $\sigma \in [0, 1]$, $s \in \mathbb{R}$. For the proof recall that $\alpha(\sigma) = 1/p_\sigma$. If $p_\sigma < \infty$, then $p_\tau < \infty$, and therefore

$$\|F(\sigma + is)\|_{p_\sigma}^{p_\sigma} = \int (|u|^{\alpha(\sigma)p_\tau})^{p_\sigma} d\mu = \|u\|_{p_\tau}^{p_\tau} = 1.$$

If $p_\sigma = \infty$, then $\alpha(\sigma) = 0$, and therefore

$$\|F(\sigma + is)\|_{p_\sigma} = \||u|^0 \mathbf{1}_{[u \neq 0]}\|_\infty = 1.$$

An analogous observation applies to G . This shows that

$$\begin{aligned} \|\Phi(is)F(is)\|_{q_0} &\leq M_0 \|F(is)\|_{p_0} = M_0, \\ |h(is)| &= \left| \int (\Phi(is)F(is))G(is) d\mu \right| \leq \|\Phi(is)F(is)\|_{q_0} \|G(is)\|_{q'_0} \leq M_0 \end{aligned}$$

for all $s \in \mathbb{R}$. In the same way one obtains $|h(1 + is)| \leq M_1$ for all $s \in \mathbb{R}$.

At this point we can apply Corollary 10.3 and obtain $|\int (\Phi(\tau)u)v d\mu| = |h(\tau)| \leq M_\tau$. So, we have shown that

$$\left| \int (\Phi(\tau)u)v d\mu \right| \leq M_\tau \|u\|_{p_\tau} \|v\|_{q'_\tau} \quad (u, v \in S(\mathcal{A}_c)).$$

In view of Lemma 10.7 this implies the assertion for $s = 0$.

If $s \in \mathbb{R}$, then the result proved so far can be applied to the function $z \mapsto \Phi(z + is)$, and this yields the asserted inequality for general s . \square

10.2 Interpolation of semigroups

As in Section 10.1, the scalar field in this section will be $\mathbb{K} = \mathbb{C}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $p_1 \in [1, \infty)$, $\theta \in (0, \pi/2]$, and let T be a bounded holomorphic C_0 -semigroup on $L_{p_1}(\mu)$ of angle θ , $M_1 := \sup_{z \in \Sigma_\theta} \|T(z)\|_{\mathcal{L}(L_{p_1}(\mu))} < \infty$. Let $p_0 \in [1, \infty]$, $p_0 \neq p_1$, and assume that $T|_{[0, \infty)}$ is L_{p_0} -bounded; by this we mean that there exists $M_0 \geq 0$ such that

$$\|T(t)u\|_{p_0} \leq M_0 \|u\|_{p_0} \quad (u \in L_{p_1} \cap L_{p_0}(\mu), t \geq 0).$$

10.8 Theorem. *Let the hypotheses be as above, and let $\tau \in (0, 1)$, $\theta_\tau := \tau\theta$, $\frac{1}{p_\tau} := \frac{1-\tau}{p_0} + \frac{\tau}{p_1}$, $M_\tau := M_0^{1-\tau} M_1^\tau$.*

Then for all $z \in \Sigma_{\theta_\tau}$ the operator $T(z)|_{L_{p_1} \cap L_{p_\tau}(\mu)}$ extends (uniquely) to an operator $T_\tau(z) \in \mathcal{L}(L_{p_\tau}(\mu))$, and T_τ is a bounded holomorphic C_0 -semigroup of angle θ_τ , $\|T_\tau(z)\| \leq M_\tau$ for all $z \in \Sigma_{\theta_\tau}$.

Proof. We define $\mathcal{A}_c := \{A \in \mathcal{A}; \mu(A) < \infty\}$ and use the notation $S(\mathcal{A}_c), L_1(\mathcal{A}_c)$ from the beginning of Subsection 10.1.2.

(i) The essential part of the theorem is the boundedness statement for $T(z)|_{S(\mathcal{A}_c)}$; its proof will require the Stein interpolation theorem.

Let $0 < \theta' < \theta$. The function $\psi(z) := e^{i\theta'z}$ maps the strip \bar{S} continuously onto the ‘semi-sector’ $\Sigma'_{\theta'} := \{z \in \mathbb{C} \setminus \{0\}; 0 \leq \text{Arg } z \leq \theta'\}$, and ψ is holomorphic on S . Now one can see that the function $\Phi := T \circ \psi: \bar{S} \rightarrow L(S(\mathcal{A}_c), L_1(\mathcal{A}_c))$ satisfies the hypotheses of Theorem 10.5 with $q_0 = p_0$, $q_1 = p_1$. Let us just comment on the holomorphy hypothesis: for $A \in \mathcal{A}_c$ the function $S \ni z \mapsto T(\psi(z))\mathbf{1}_A \in L_{p_1}(\mu)$ is holomorphic, and this implies that for all $B \in \mathcal{A}_c$ the function $S \ni z \mapsto \int (T(\psi(z))\mathbf{1}_A)\mathbf{1}_B d\mu$ is holomorphic. The other properties are checked similarly.

For every $s \in \mathbb{R}$, Theorem 10.5 implies that $T(\psi(\tau + is))|_{S(\mathcal{A}_c)}$ is bounded with respect to the L_{p_τ} -norm, with norm $\leq M_\tau$. The points $\psi(\tau + is) = e^{i\theta'(\tau + is)} = e^{-\theta's}e^{i\theta'\tau}$ are contained in the open semi-sector $\overset{\circ}{\Sigma}'_{\theta'}$, and in fact all points of this open semi-sector can be obtained by a suitable choice of s and θ' .

For the complementary open semi-sector $\{\bar{z}; z \in \overset{\circ}{\Sigma}'_{\theta'}\}$ the reasoning is analogous, and for $z \geq 0$ the boundedness statement follows from Corollary 10.6.

(ii) As $S(\mathcal{A}_c)$ is a dense subspace of $L_{p_\tau}(\mu)$, the operator $T(z)|_{S(\mathcal{A}_c)}$ has a unique extension $T_\tau(z) \in \mathcal{L}(L_{p_\tau}(\mu))$. We show that $T(z)|_{L_{p_1} \cap L_{p_\tau}(\mu)} = T_\tau(z)|_{L_{p_1} \cap L_{p_\tau}(\mu)}$: Let $u \in L_{p_1} \cap L_{p_\tau}(\mu)$. Then there exists a sequence (u_n) in $S(\mathcal{A}_c)$ with $u_n \rightarrow u$ in $L_{p_1}(\mu)$ as well as in $L_{p_\tau}(\mu)$. This implies $T(z)u = T_\tau(z)u$.

In order to show that $\Sigma_{\theta_\tau} \ni z \mapsto T_\tau(z)$ is holomorphic we use the results of Section 3.1. For $u, v \in S(\mathcal{A}_c)$ the function $\Sigma_{\theta_\tau} \ni z \mapsto \int (T(z)u)v d\mu$ is holomorphic. As $S(\mathcal{A}_c)$ is dense in $L_{p_\tau}(\mu)$ and in $L_{p'_\tau}(\mu)$ and $T_\tau: \Sigma_{\theta_\tau} \rightarrow \mathcal{L}(L_{p_\tau}(\mu))$ is bounded, Theorem 3.5 implies that $z \mapsto T_\tau(z) \in \mathcal{L}(L_{p_\tau}(\mu))$ is holomorphic on Σ_{θ_τ} .

(iii) In order to show the strong continuity of T_τ at 0 we use Hölder’s inequality

$$\|u\|_{p_\tau} \leq \|u\|_{p_0}^{1-\tau} \|u\|_{p_1}^\tau \quad (u \in S(\mathcal{A}_c)).$$

Let $u \in S(\mathcal{A}_c)$. Then the boundedness of $\{\|T(t)u\|_{p_0}; t \geq 0\}$ together with the continuity of $T(\cdot)u$ at 0 in $L_{p_1}(\mu)$ implies that $\|T(t)u - u\|_{p_\tau} \rightarrow 0$ as $t \rightarrow 0$. Then the combination of Lemma 1.5 and Proposition 3.11 implies that T_τ is strongly continuous at 0. \square

10.3 Adjoint semigroups

In this section we insert some information on adjoint semigroups. Let H be a Hilbert space, T a C_0 -semigroup on H . Then clearly $T^* := (T(t)^*)_{t \geq 0}$ is a one-parameter semigroup on H , but it is not obvious that T^* is strongly continuous. We will show this by looking at the generator of T . However, we will restrict our treatment to the case of quasi-contractive semigroups.

10.9 Theorem. *Let H be a Hilbert space, let T be a quasi-contractive C_0 -semigroup on H , and let A be its generator. Then A^* is the generator of a quasi-contractive C_0 -semigroup, and the generated C_0 -semigroup is T^* as defined above.*

Proof. By rescaling we can reduce the situation to the case that T is contractive. Then $-A$ is an m -accretive operator.

As A is closed, A^* is densely defined; see Theorem 6.3(b). We know from Theorem 3.18 that $(0, \infty) \subseteq \rho(A)$, and that $\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$. Similarly as in the proof of

Theorem 6.3(b) one obtains $(\lambda - A^*)^{-1} = ((\lambda - A)^{-1})^*$. It follows that $\lambda \in \rho(A^*)$ and $\|(\lambda - A^*)^{-1}\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$. Therefore the Hille-Yosida theorem (Theorem 2.9) implies that A^* generates a contractive C_0 -semigroup.

From the exponential formula, Theorem 2.12, we conclude that the C_0 -semigroup generated by A^* is the adjoint semigroup T^* . \square

10.10 Remark. If the semigroup T in Theorem 10.9 is holomorphic of some angle $\theta \in (0, \pi/2]$, then T^* (defined as the adjoint of $T|_{[0, \infty)}$) has a holomorphic extension to the sector Σ_θ . This extension is given by

$$T^*(z) := T(\bar{z})^* \quad (z \in \Sigma_\theta). \quad (10.2)$$

Indeed, it is not difficult to show that T^* , defined by (10.2), is holomorphic. Hence T^* is a holomorphic C_0 -semigroup.

10.11 Remarks. (a) Using the general Hille-Yosida generation theorem (see Exercise 2.4) one also obtains Theorem 10.9 for general C_0 -semigroups on H .

(b) If X is a Banach space and T is a C_0 -semigroup on X , then it is not generally true that $T'(t) := T(t)'$ ($t \geq 0$) defines a C_0 -semigroup on X' , where $T(t)' \in \mathcal{L}(X')$ is the dual operator. It is true, however, if X is reflexive. More generally, if T is a one-parameter semigroup on X that is weakly continuous, then T is a C_0 -semigroup.

10.4 Applications of invariance criteria and interpolation

Throughout this section let (Ω, μ) be a measure space.

An operator $S \in \mathcal{L}(L_1(\mu))$ is called **substochastic** if S is positive and contractive. An operator $S \in \mathcal{L}(L_2(\mu))$ is called **L_1 -contractive** if $\|Su\|_1 \leq \|u\|_1$ for all $u \in L_2 \cap L_1(\mu)$, and S is **substochastic** if S is positive and L_1 -contractive. The same notation will be used for semigroups if all the semigroup operators satisfy the corresponding property.

10.12 Theorem. *Let $V \subseteq H := L_2(\mu)$ be continuously and densely embedded, and let $a: V \times V \rightarrow \mathbb{K}$ be a bounded H -elliptic form. Let A be the operator associated with a , and let T be the C_0 -semigroup generated by $-A$. Then one has the following properties.*

- (a) T is real if and only if $\operatorname{Re} u \in V$ for all $u \in V$ and $a(u, v) \in \mathbb{R}$ for all real $u, v \in V$.
- (b) T is positive if and only if T is real, and $u^+ \in V$, $a(u^+, u^-) \leq 0$ for all real $u \in V$.
- (c) T is sub-Markovian if and only if T is real, and $u \wedge 1 \in V$, $a(u \wedge 1, (u - 1)^+) \geq 0$ for all real $u \in V$.

- (d) T is substochastic if and only if T is real, and $u \wedge 1 \in V$, $a((u - 1)^+, u \wedge 1) \geq 0$ for all real $u \in V$.

For the proof of (d) we need an auxiliary result.

10.13 Lemma. *Let $S \in \mathcal{L}(L_2(\mu))$. Then S is L_∞ -contractive if and only if S^* is L_1 -contractive, and S is sub-Markovian if and only if S^* is substochastic.*

Proof. Let S be L_∞ -contractive. Let $v \in L_2 \cap L_1(\mu)$, $\|v\|_1 \leq 1$. Then

$$\left| \int u \overline{S^*v} \, d\mu \right| = \left| \int (Su) \bar{v} \, d\mu \right| \leq 1 \quad (u \in L_2 \cap L_\infty(\mu), \|u\|_\infty \leq 1),$$

and from Lemma 10.7 we conclude that $\|S^*v\|_1 \leq 1$.

The converse statement is proved in the same way.

For the second statement it is now sufficient to notice that S is positive if and only if S^* is positive. \square

Proof of Theorem 10.12. In view of Remarks 9.3 the statements (a), (b), (c) are easy consequences of '(i) \Leftrightarrow (ii)' in Theorem 9.20. For the necessity in (a) we note that only the case $\mathbb{K} = \mathbb{C}$ is of interest and that for real $u, v \in V$ one obtains

$$0 \leq \operatorname{Re} a(\operatorname{Re}(u \pm iv), u \pm iv - \operatorname{Re}(u \pm iv)) = \operatorname{Re} a(u, \pm iv) = \pm \operatorname{Im} a(u, v),$$

which implies that $a(u, v) \in \mathbb{R}$. For (b) we note that $u - u^+ = -u^-$, and for (c) we note that $u - u \wedge 1 = (u - 1)^+$.

For (d) we note that $-A^*$ is the generator of the C_0 -semigroup T^* , by Theorem 10.9, and that A^* is associated with the form a^* , by Theorem 6.10 (which by Remark 5.6 and Lemma 6.9 also holds for H -elliptic forms). It is easy to see that T is real if and only if T^* is real, and therefore Lemma 10.13 implies that T is substochastic if and only if T is real and T^* is sub-Markovian, or equivalently, $u \wedge 1 \in V$ and $a((u - 1)^+, u \wedge 1) = a^*(u \wedge 1, (u - 1)^+) \geq 0$ for all real $u \in V$, by part (c). \square

10.14 Remarks. (a) A form a satisfying conditions (c) and (d) of Theorem 10.12 is called a (**non-symmetric**) **Dirichlet form**. The conditions formulated in (b), (c) and (d) are the **Beurling-Deny criteria**.

(b) Using the condition (iii) of Theorem 9.20 one could also formulate the conditions in Theorem 10.12 with a dense subset of V .

10.15 Theorem. *Let T be a C_0 -semigroup on $L_2(\mu)$.*

(a) *Assume that T is sub-Markovian and substochastic. Then for all $p \in [1, \infty)$ the operators $T(t)|_{L_2 \cap L_p(\mu)}$ extend to operators $T_p(t) \in \mathcal{L}(L_p(\mu))$, and T_p thus defined is a contractive C_0 -semigroup on $L_p(\mu)$. For $1 \leq p, q < \infty$ the semigroups T_p, T_q are **consistent**, i.e., $T_p(t)|_{L_p \cap L_q(\mu)} = T_q(t)|_{L_p \cap L_q(\mu)}$ for all $t \geq 0$.*

(b) *Assume that $T(t)$ is self-adjoint for all $t \geq 0$ and that T is sub-Markovian. Then the assertions of (a) hold. If $\mathbb{K} = \mathbb{C}$, then for all $p \in (1, \infty)$ the semigroup T_p extends to a contractive holomorphic C_0 -semigroup of angle*

$$\theta_p = \begin{cases} (1 - \frac{1}{p})\pi & \text{if } 1 < p < 2, \\ \frac{1}{p}\pi & \text{if } 2 \leq p < \infty. \end{cases}$$

Proof. (a) Let $1 \leq p < \infty$. For every $t > 0$, Exercise 10.3 (or Corollary 10.6, if $\mathbb{K} = \mathbb{C}$) implies that $T(t)|_{L_1 \cap L_\infty(\mu)}$ extends to a contractive operator $T_p(t)$ on $L_p(\mu)$. It is standard to show that T_p is a one-parameter semigroup. The strong continuity of T_p at 0 is obtained as follows. If $u \in L_p \cap L_2(\mu)$ and (t_n) is a null sequence in $(0, \infty)$, then $T(t_n)u \rightarrow u$ in

$L_2(\mu)$ implies that for a subsequence one has $T(t_{n_k})u \rightarrow u$ a.e., and the contractivity of T_p in combination with Lemma 10.16, proved subsequently, implies that $T_p(t_{n_k})u \rightarrow u$ in $L_p(\mu)$. Applying a standard sub-sub-sequence argument one obtains $T_p(t)u \rightarrow u$ in $L_p(\mu)$ as $t \rightarrow 0+$. Lemma 1.5 concludes the argument.

The consistency is shown as in (ii) of the proof of Theorem 10.8.

(b) From $T(t)^* = T(t)$ for all $t \geq 0$ and Lemma 10.13 it follows that T is also substochastic. Thus (a) is applicable. Also, Theorem 10.9 implies that the generator A of T is self-adjoint, and as T is contractive, $-A$ is accretive.

Now let $\mathbb{K} = \mathbb{C}$. Then it follows that $-A$ is sectorial of angle 0. Hence $-A$ is the generator of a contractive holomorphic C_0 -semigroup of angle $\pi/2$; see Theorem 3.22. In view of Theorem 10.8 this implies the remaining assertions. \square

10.16 Lemma. *Let $1 \leq p < \infty$, and let (f_n) be a sequence in $L_p(\mu)$, $f \in L_p(\mu)$ such that $f_n \rightarrow f$ a.e. and $\limsup_{n \rightarrow \infty} \|f_n\|_p \leq \|f\|_p$. Then $f_n \rightarrow f$ in $L_p(\mu)$.*

Proof. For $n \in \mathbb{N}$ let $\check{f}_n := \operatorname{sgn} f_n (|f| \wedge |f_n|)$. Then $\check{f}_n \rightarrow f$ in $L_p(\mu)$ by the dominated convergence theorem. Moreover $|f_n| = |\check{f}_n| + |f_n - \check{f}_n|$, hence $|f_n|^p \geq |\check{f}_n|^p + |f_n - \check{f}_n|^p$ for all $n \in \mathbb{N}$. Therefore $\|f_n - \check{f}_n\|_p^p \leq \|f_n\|_p^p - \|\check{f}_n\|_p^p \rightarrow 0$, and this implies that $f_n = (f_n - \check{f}_n) + \check{f}_n \rightarrow f$ in $L_p(\mu)$. \square

Notes

The three lines theorem is generally attributed to J. Hadamard. The Stein interpolation theorem, essentially in the form presented here, is contained in [Ste56]. We refer to this paper for some history of the Riesz-Thorin convexity theorem, finally proved by Thorin by the complex variable method, which initiated a whole new branch of Functional Analysis. In fact, the paper [Ste56] can be considered as the start of interpolation theory, for which we refer to the seminal paper of Calderón [Cal64] as well as to the monographs [BL76], [Lun09].

The application of invariance and interpolation as described in Section 10.4 is well-established in the theory of semigroups for diffusion equations, Schrödinger semigroups and related. Interestingly enough, nice and elegant as the proof of Theorem 10.15 may seem, the angle of holomorphy for the L_p -semigroup is not optimal, in this case, neither for holomorphy nor for contractivity. There exist other methods providing more sophisticated estimates.

Exercises

10.1 For this exercise let S be the strip

$$S := \{z \in \mathbb{C}; -1/2 < \operatorname{Re} z < 1/2\}$$

(of width 1). Let $h: \bar{S} \rightarrow \mathbb{C}$ be continuous and holomorphic on S . Assume that h is bounded on ∂S , and that there exists $\alpha < \pi$ such that

$$|h(z)| \leq e^{\alpha |\operatorname{Im} z|} \quad (z \in S).$$

Show that h is bounded by $\|h\|_{\partial S}$. (Hint: Use $\psi_n(z) := e^{-\frac{1}{n}(e^{i\beta z} + e^{-i\beta z})}$, with $\alpha < \beta < \pi$.)

10.2 Let (Ω, μ) be a measure space. Show that

$$\left(\left\|\frac{1}{2}(f+g)\right\|_p^p + \left\|\frac{1}{2}(f-g)\right\|_p^p\right)^{1/p} \leq 2^{-1/p}(\|f\|_p^p + \|g\|_p^p)^{1/p} \quad (10.3)$$

for all $f, g \in L_p(\mu)$, $2 \leq p \leq \infty$.

Hint: Use the mapping

$$T: L_p(\mu) \times L_p(\mu) \rightarrow L_p(\mu) \times L_p(\mu), (f, g) \mapsto \left(\frac{1}{2}(f+g), \frac{1}{2}(f-g)\right).$$

Compute the norm of T for $p = 2$ and for $p = \infty$ and use the Riesz-Thorin interpolation theorem. (The inequality (10.3) is one of Clarkson's inequalities. The other inequalities of Clarkson involve p and the conjugate exponent and can also be obtained by interpolation, but this is more complicated.)

10.3 (a) Let $p \in (1, \infty)$, $r \in [0, \infty)$. Show that

$$r = \inf_{\alpha \in (0, \infty) \cap \mathbb{Q}} \left(\frac{1}{p}\alpha^{1-p}r^p + \left(1 - \frac{1}{p}\right)\alpha\right).$$

(b) Let (Ω, μ) be a measure space, and let $S \in \mathcal{L}(L_2(\mu))$ be sub-Markovian and substochastic. Show that S is L_p -contractive for all $p \in (1, \infty)$. (The case $\mathbb{K} = \mathbb{C}$ is already covered by Corollary 10.6, but not the case $\mathbb{K} = \mathbb{R}$!)

Hint: Using (a) twice show first that $S|u| \leq (S|u|^p)^{1/p}$ for simple functions.

(c) Let (Ω, μ) be a measure space, let $S \in \mathcal{L}(L_2(\mu))$ be sub-Markovian, and assume that there exists $c > 0$ such that $\frac{1}{c}S$ is substochastic. Show that S interpolates to an operator $S_p \in \mathcal{L}(L_p(\mu))$ with $\|S_p\| \leq c^{1/p}$, for all $1 < p < \infty$.

10.4 (Continuation of Exercise 9.5) Let the hypotheses be as in Exercise 9.5, and additionally $b \in C^1(\Omega)$. Assume that $\omega \in \mathbb{R}$ is such that $\operatorname{div} b(x) \leq \omega$ for all $x \in \Omega$.

(a) Show that $\|T(t)u\|_1 \leq e^{\omega t}\|u\|_1$ ($u \in L_2 \cap L_1(\Omega)$, $t \geq 0$), where T is the C_0 -semigroup generated by the operator $-A$.

Hint: Use $C_c^1(\Omega)$ as the dense subset of $V = H_0^1(\Omega)$ for the application of the invariance criterion to the semigroup $(e^{-\omega t}T(t))_{t \geq 0}$. Observe that on $C_c^1(\Omega)$ one can transform $(b \cdot \nabla u | v)$ – using integration by parts – to an expression where b only appears in the second argument of the scalar product, and u in the first argument appears without derivative.

(b) Compute estimates for $\|T_p(t)\|$ in terms of $\omega := \sup \operatorname{div} b$ for $t \geq 0$, $1 \leq p < \infty$, where T_p is the interpolated semigroup on $L_p(\Omega)$, analogous to Theorem 10.15(b).

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Lecture 11

Elliptic operators

Elliptic operators with measurable coefficients are a classical topic in partial differential equations. Their realisations under diverse boundary conditions generate semigroups and thus lead to solutions of parabolic initial boundary value problems. Form methods are most efficient to treat these problems and to derive properties of the solutions of the equations. It will become apparent that a large amount of the topics presented so far enter the treatment of these equations. In particular the properties of the Sobolev space $H^1(\Omega)$ and the invariance criteria will play a decisive role in the treatment. The latter lead to positivity and to sub-Markovian and substochastic behaviour of the generated semigroups.

In order to achieve these goals it turns out that again additional order properties of $H^1(\Omega)$ – treated in an interlude – are needed.

11.1 Perturbation of continuous forms

We start with a perturbation result. We refer to the proof of Theorem 7.15 and to Exercise 7.3 for related methods.

11.1 Lemma. *Let V, H be Hilbert spaces with $V \xhookrightarrow{d} H$. Let $a: V \times V \rightarrow \mathbb{K}$ be an H -elliptic continuous form. Let $b: V \times V \rightarrow \mathbb{K}$ be a continuous form such that*

$$|b(u)| \leq M \|u\|_V \|u\|_H \quad (u \in V).$$

Then $a + b: V \times V \rightarrow \mathbb{K}$ is H -elliptic.

Proof. By the H -ellipticity of a there exist $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\operatorname{Re} a(u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2$$

for all $u \in V$.

By the “Peter-Paul inequality” (i.e., Young’s inequality, $ab \leq \frac{1}{2}(\gamma a^2 + \frac{1}{\gamma} b^2)$ for all $a, b \geq 0$, $\gamma > 0$) one has

$$\begin{aligned} \operatorname{Re} a(u) + \operatorname{Re} b(u) + \omega \|u\|_H^2 &\geq \alpha \|u\|_V^2 - M \|u\|_V \|u\|_H \\ &\geq \alpha \|u\|_V^2 - \frac{1}{2} \left(\alpha \|u\|_V^2 + \frac{1}{\alpha} M^2 \|u\|_H^2 \right). \end{aligned}$$

This implies

$$\operatorname{Re}(a(u) + b(u)) + \left(\omega + \frac{M^2}{2\alpha} \right) \|u\|_H^2 \geq \frac{\alpha}{2} \|u\|_V^2 \quad (u \in V). \quad \square$$

11.2 Elliptic operators

Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $a_{jk} \in L_\infty(\Omega)$ ($j, k = 1, \dots, n$) be coefficient functions satisfying the **ellipticity condition**

$$\operatorname{Re} \sum_{j,k=1}^n a_{jk}(x) \xi_k \bar{\xi}_j \geq \alpha |\xi|^2 \quad (\xi \in \mathbb{K}^n) \quad (11.1)$$

for a.e. $x \in \Omega$, with $\alpha > 0$, and let $b_j, c_j \in L_\infty(\Omega)$ ($j = 1, \dots, n$), $d \in L_\infty(\Omega)$. Our aim is to define operators in $L_2(\Omega)$ corresponding to the “elliptic operator in divergence form” \mathcal{A} written formally as

$$\begin{aligned} \mathcal{A}u &= \sum_{j,k=1}^n -\partial_j(a_{jk}\partial_k u) + \sum_{j=1}^n (b_j\partial_j u - \partial_j(c_j u)) + du \\ &= -\operatorname{div}((a_{jk})\nabla u) + b \cdot \nabla u - \operatorname{div}(cu) + du. \end{aligned} \quad (11.2)$$

We consider the form

$$a: H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{K}$$

given by

$$a(u, v) = \int_{\Omega} \left(\sum_{j,k=1}^n a_{jk} \partial_k u \bar{\partial}_j v + \sum_{j=1}^n (b_j \partial_j u \bar{v} + c_j u \bar{\partial}_j v) + du \bar{v} \right) dx.$$

11.2 Proposition. *The form a is continuous and $L_2(\Omega)$ -elliptic.*

Proof. Let the form $a_0: H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{K}$ be given by

$$a_0(u, v) = \int_{\Omega} \sum_{j,k=1}^n a_{jk} \partial_k u \bar{\partial}_j v dx.$$

Then the boundedness of the coefficients a_{jk} implies that a_0 is continuous. The ellipticity condition (11.1) implies that $\operatorname{Re} a_0(u) + \alpha \|u\|_{L_2}^2 \geq \alpha \|\nabla u\|_{L_2}^2 + \alpha \|u\|_{L_2}^2 = \alpha \|u\|_{H^1}^2$ for all $u \in H^1(\Omega)$. Thus a_0 is $L_2(\Omega)$ -elliptic.

Define $a_1: H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{K}$ by

$$a_1(u, v) = \int_{\Omega} \left(\sum_{j=1}^n (b_j \partial_j u \bar{v} + c_j u \bar{\partial}_j v) + du \bar{v} \right) dx.$$

Then $a = a_0 + a_1$.

The boundedness of the coefficient functions b_j, c_j and d implies that a_1 is continuous. It also implies that there exists $M \geq 0$ such that

$$\begin{aligned} |a_1(u)| &\leq M \int_{\Omega} |\nabla u| |u| dx + \|d\|_{L_\infty} \|u\|_{L_2}^2 \\ &\leq M \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}} + \|d\|_{L_\infty} \|u\|_{L_2}^2 \\ &\leq (M + \|d\|_{L_\infty}) \|u\|_{H^1} \|u\|_{L_2} \end{aligned}$$

for all $u \in H^1(\Omega)$. Now the claim follows from Lemma 11.1. \square

Now let V be a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Then the restriction of a to $V \times V$ is also continuous and $L_2(\Omega)$ -elliptic. Denote by A_V the operator associated with $a|_{V \times V}$. Then $-A_V$ generates a C_0 -semigroup T_V on $L_2(\Omega)$, and if $\mathbb{K} = \mathbb{C}$, then $-A_V$ generates a holomorphic C_0 -semigroup T_V on $L_2(\Omega)$; see Section 5.3.

Coming back to (11.2), let $u \in H^1(\Omega)$ and multiply (11.2) by a test function $v \in C_c^\infty(\Omega)$. A formal integration by parts gives

$$\int_{\Omega} (\mathcal{A}u)\bar{v} \, dx = a(u, v).$$

So we might say that for $u \in H^1(\Omega)$ one has $\mathcal{A}u = f \in L_2(\Omega)$ if

$$\int_{\Omega} f\bar{v} \, dx = a(u, v) \quad (v \in C_c^\infty(\Omega)).$$

This leads to the definition of the maximal operator

$$A_{\max} := \left\{ (u, f) \in H^1(\Omega) \times L_2(\Omega); a(u, v) = \int_{\Omega} f\bar{v} \, dx \ (v \in C_c^\infty(\Omega)) \right\}.$$

11.3 Remark. For each closed subspace $V \subseteq H^1(\Omega)$ with $H_0^1(\Omega) \subseteq V$ the operator A_V is a restriction of A_{\max} . For $(u, f) \in A_{\max}$ to be in A_V it is needed that

- (i) $u \in V$ (and not merely for $u \in H^1(\Omega)$),
- (ii) $a(u, v) = (f | v)_{L_2}$ for all $v \in V$ (and not merely for $v \in C_c^\infty(\Omega)$).

These conditions can be interpreted as a boundary condition.

11.4 Examples. (a) Let us first consider the case $V = H_0^1(\Omega)$. Then $A_{H_0^1}$ is just the restriction of A_{\max} to $\text{dom}(A_{\max}) \cap H_0^1(\Omega)$. We write $A_D := A_{H_0^1}$ and call A_D the realisation of the elliptic operator \mathcal{A} with **Dirichlet boundary conditions**. We define $T_D := T_{H_0^1}$.

(b) Next we consider $V = H^1(\Omega)$. We define $T_N := T_{H^1}$ and call $A_N := A_{H^1}$ the realisation of the elliptic operator \mathcal{A} with **Neumann boundary conditions**. However, it is not the normal derivative of u which is 0 at the boundary, but rather the **conormal derivative**

$$\sum_{j,k=1}^n (a_{jk} \partial_k u) \nu_j + \sum_{j=1}^n c_j u \nu_j$$

on $\partial\Omega$, which is defined in terms of the coefficients of \mathcal{A} .

Clearly, the conormal derivative has no meaning for general domains, but even for nice domains there is the problem that the partial derivatives of u occur, and there is no reason why they should have a trace on $\partial\Omega$. We will motivate the Neumann boundary condition involving the conormal derivative under suitable “smooth hypotheses”.

Assume that Ω is bounded and has C^1 -boundary. Assume that $a_{jk} \in C^1(\bar{\Omega})$, $c_j \in C^1(\bar{\Omega})$, $b_j \in C(\bar{\Omega})$, $d \in C(\bar{\Omega})$. Let $u \in \text{dom}(A_N)$, $A_N u = f$, and assume additionally that $u \in C^2(\bar{\Omega})$. Since $A_{\max} u = f$ by Remark 11.3 and since $a_{jk} \partial_k u \in C^1(\bar{\Omega})$, $c_j u \in C^1(\bar{\Omega})$, partial integration yields

$$f = - \sum_{j,k=1}^n \partial_j (a_{jk} \partial_k u) + \sum_{j=1}^n b_j \partial_j u - \sum_{j=1}^n \partial_j (c_j u) + du.$$

By the definition of the associated operator A_N we have

$$\begin{aligned} \int_{\Omega} f \bar{v} \, dx &= a(u, v) \\ &= \int_{\Omega} \left(\sum_{j,k=1}^n a_{jk} \partial_k u \bar{\partial}_j v + \sum_{j=1}^n b_j \partial_j u \bar{v} + \sum_{j=1}^n c_j u \bar{\partial}_j v + d u \bar{v} \right) dx \end{aligned}$$

for all $v \in C^1(\bar{\Omega}) \subseteq H^1(\Omega)$. Applying Gauss' theorem (Theorem 7.4) we deduce that

$$\int_{\Omega} f \bar{v} \, dx = \int_{\Omega} f \bar{v} \, dx + \int_{\partial\Omega} \left(\sum_{j,k=1}^n (a_{jk} \partial_k u) \nu_j \right) \bar{v} \, d\sigma + \int_{\partial\Omega} \left(\sum_{j=1}^n c_j u \nu_j \right) \bar{v} \, d\sigma$$

for all $v \in C^1(\bar{\Omega}) \subseteq H^1(\Omega)$, where $\nu = (\nu_1, \dots, \nu_n)$ is the outer normal. (These computations correspond to the argument given at the beginning of Section 7.4 concerning the Neumann boundary condition for the Laplacian.) This implies that

$$\sum_{j=1}^n \left(\sum_{k=1}^n a_{jk} \partial_k u + c_j u \right) \nu_j = ((a_{jk}) \nabla u + c u) \cdot \nu = 0 \quad \text{on } \partial\Omega. \quad (11.3)$$

Thus (11.3) is the Neumann boundary condition we are realising by the form a on $H^1(\Omega) \times H^1(\Omega)$.

(c) There are other possible choices of V . For example, assume that Ω has C^1 -boundary, and let $\Gamma \subseteq \partial\Omega$ be a Borel set. Then we may consider

$$V = \{u \in H^1(\Omega); \operatorname{tr} u = 0 \text{ on } \Gamma\}.$$

We call A_V the realisation of the elliptic operator \mathcal{A} with **mixed boundary conditions** (i.e., Dirichlet on Γ , Neumann on $\partial\Omega \setminus \Gamma$).

11.5 Remarks. Assume that the coefficient matrix $(a_{jk}(x))$ is self-adjoint, i.e., $a_{jk}(x) = \overline{a_{kj}(x)}$, for all $x \in \Omega$.

(a) Then the ellipticity condition (11.1) says that the smallest eigenvalue of each matrix $(a_{jk}(x))$ is $\geq \alpha$.

(b) If $b_j = \bar{c}_j$ ($j = 1, \dots, n$) and d is real-valued, then a is symmetric and the operator A_V is self-adjoint.

We conclude this section with comments on the interplay between the real and complex cases. Obviously, in order to obtain a holomorphic C_0 -semigroup one has to work with the field \mathbb{C} .

11.6 Remarks. Assume that all the coefficient functions a_{jk}, b_j, c_j, d are real-valued. Assume that the ellipticity condition (11.1) is satisfied. (Take notice of Exercise 11.1!)

(a) Let V be a closed subspace of $H^1(\Omega; \mathbb{C})$ containing $H_0^1(\Omega; \mathbb{C})$ with the property that $u \in V$ implies $\operatorname{Re} u \in V$. Then $T_V(t)$ leaves $L_2(\Omega; \mathbb{R})$ invariant, for $t \geq 0$; this follows from Theorem 10.12(a).

Let $V_{\mathbb{R}} = V \cap L_2(\Omega; \mathbb{R})$. It is clear that the restriction of a to $V_{\mathbb{R}} \times V_{\mathbb{R}}$ is again continuous and elliptic and that minus the generator of T_V restricted to $L_2(\Omega; \mathbb{R})$ is the operator associated with $a|_{V_{\mathbb{R}} \times V_{\mathbb{R}}}$.

(b) Conversely, if V is a closed subspace of $H^1(\Omega; \mathbb{R})$ containing $H_0^1(\Omega; \mathbb{R})$, then $V_{\mathbb{C}} = V \oplus iV$ is a space as we considered in (a).

In order to study further properties of the semigroup on $L_2(\Omega; \mathbb{R})$ generated by an elliptic operator with real coefficients we need additional properties of the Sobolev space $H^1(\Omega)$.

11.3 Interlude: Further order properties of $H^1(\Omega)$

In this section let $\mathbb{K} = \mathbb{R}$. Let $\Omega \subseteq \mathbb{R}^n$ be open. In Section 9.3 we have seen that $H^1(\Omega)$ is a sublattice of $L_2(\Omega)$. More precisely, if $u \in H^1(\Omega)$, then Theorem 9.14 implies that $u^+, u^- = (-u)^+, |u| = u^+ + u^- \in H^1(\Omega)$ and $\partial_j u^+ = \mathbf{1}_{[u>0]} \partial_j u$, $\partial_j u^- = -\mathbf{1}_{[u<0]} \partial_j u$, $\partial_j |u| = \partial_j u^+ + \partial_j u^- = (\text{sgn } u) \partial_j u$. It follows that

$$\| |u| \|_{H^1(\Omega)} = \|u\|_{H^1(\Omega)} \quad (u \in H^1(\Omega)) \quad (11.4)$$

since $\partial_j u = \partial_j u^+ - \partial_j u^- = \mathbf{1}_{[u \neq 0]} \partial_j u$ ($j = 1, \dots, n$). Incidentally, this last equality implies that $\mathbf{1}_{[u=0]} \partial_j u = 0$, i.e., $\partial_j u = 0$ a.e. on $[u = 0]$ (“Stampacchia-Lemma”); see also Remark 9.10.

Next we show that the lattice operations are continuous.

11.7 Proposition. (a) *The mapping $H^1(\Omega) \ni u \mapsto |u| \in H^1(\Omega)$ is continuous.*

(b) *The mappings $(u, v) \mapsto u \wedge v$ and $(u, v) \mapsto u \vee v$ are continuous from $H^1(\Omega) \times H^1(\Omega)$ to $H^1(\Omega)$. In particular, the mappings $H^1(\Omega) \ni u \mapsto u^+, u^- \in H^1(\Omega)$ are continuous.*

Proof. (a) Let (u_k) be a sequence in $H^1(\Omega)$, $u_k \rightarrow u$ in $H^1(\Omega)$. Then $|u_k| \rightarrow |u|$ in $L_2(\Omega)$, and (11.4) implies that $(|u_k|)$ is bounded in $H^1(\Omega)$. Therefore Remark 9.15 implies that $|u_k| \rightarrow |u|$ weakly in $H^1(\Omega)$.

From (11.4) we also obtain

$$\| |u_k| \|_{H^1} = \|u_k\|_{H^1} \rightarrow \|u\|_{H^1} = \| |u| \|_{H^1} \quad (k \rightarrow \infty),$$

and this implies that $|u_k| \rightarrow |u|$ in $H^1(\Omega)$; see the subsequent Remark 11.8.

(b) follows from $u \wedge v = \frac{1}{2}(u + v - |u - v|)$, $u \vee v = \frac{1}{2}(u + v + |u - v|)$ and part (a). \square

11.8 Remark. Let H be a Hilbert space, (u_k) a sequence in H , $u_k \rightarrow u$ weakly in H , and $\|u_k\| \rightarrow \|u\|$ ($k \rightarrow \infty$). Then

$$\|u_k - u\|^2 = \|u_k\|^2 + \|u\|^2 - 2 \operatorname{Re}(u_k | u) \rightarrow 0;$$

so $u_k \rightarrow u$ in H ($k \rightarrow \infty$).

For a function $u: \Omega \rightarrow \mathbb{R}$ we denote by $\tilde{u}: \mathbb{R}^n \rightarrow \mathbb{R}$ the extension of u by 0,

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

From Subsection 4.1.4 we recall the definitions

$$\begin{aligned} H_c^1(\Omega) &:= \{f \in H^1(\Omega); \text{spt } f \text{ compact in } \Omega\}, \\ H_0^1(\Omega) &:= \overline{H_c^1(\Omega)}^{H^1(\Omega)} \end{aligned}$$

and the denseness property $H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$. We also recall that for $u \in H_0^1(\Omega)$ one has $\tilde{u} \in H^1(\mathbb{R}^n)$, $\partial_j \tilde{u} = \widetilde{\partial_j u}$ ($j = 1, \dots, n$); see Remark 4.11(b) and Exercise 4.2(b). We already know that $H_0^1(\Omega)$ is a sublattice of $H^1(\Omega)$ (see Theorem 9.14). Now we show that it is even an ideal (in the sense of vector lattices).

11.9 Proposition. *Let $u \in H_0^1(\Omega)$, $v \in H^1(\Omega)$, $0 \leq v \leq u$. Then $v \in H_0^1(\Omega)$.*

Proof. There exists a sequence (u_k) in $H_c^1(\Omega)$ such that $u_k \rightarrow u$ in $H^1(\Omega)$. It follows from Proposition 11.7 that $u_k \wedge v \rightarrow u \wedge v = v$ in $H^1(\Omega)$ as $k \rightarrow \infty$. Since $u_k \wedge v \in H_c^1(\Omega)$ we conclude that $v \in H_0^1(\Omega)$. \square

11.10 Proposition. *Let Ω be bounded and with C^1 -boundary. Let $\Gamma \subseteq \partial\Omega$ be a Borel subset, and let V be as in Example 11.4(c). Then V is a sublattice of $H^1(\Omega)$, and $u \in V$, $v \in H^1(\Omega)$, $0 \leq v \leq u$ implies $v \in V$.*

Proof. The fundamental observation is that the mapping $\text{tr}: H^1(\Omega) \rightarrow L_2(\partial\Omega)$ is a continuous lattice homomorphism. This means that additionally to continuity and linearity one also has $\text{tr}(u \vee v) = \text{tr } u \vee \text{tr } v$ ($u, v \in H^1(\Omega)$). This last property is clear for $u, v \in H^1(\Omega) \cap C(\bar{\Omega})$ and carries over to $H^1(\Omega)$ by denseness and continuity; see Theorems 7.7 and 7.9.

Therefore, if $u \in V$, then $\text{tr}(u^+)|_\Gamma = (\text{tr } u)^+|_\Gamma = 0$, i.e., $u^+ \in V$. This implies that V is a sublattice.

Let u, v be as in the statement of the proposition. Then $0 \leq \text{tr } v|_\Gamma \leq \text{tr } u|_\Gamma = 0$. This shows that $v \in V$. \square

We will also need the following denseness properties.

11.11 Lemma. *The set $C_c^\infty(\Omega)_+ := \{\varphi \in C_c^\infty(\Omega); \varphi \geq 0\}$ is dense in $H_0^1(\Omega)_+ := \{u \in H_0^1(\Omega); u \geq 0\}$.*

Proof. (i) In this step we show that $H_c^1(\Omega)_+ := \{u \in H_c^1(\Omega); u \geq 0\}$ is dense in $H_0^1(\Omega)_+$.

Let $u \in H_0^1(\Omega)_+$. There exists a sequence (u_k) in $H_c^1(\Omega)$ such that $u_k \rightarrow u$. Then clearly $u_k^+ \in H_c^1(\Omega)_+$ for all $k \in \mathbb{N}$, and Proposition 11.7 implies that $u_k^+ \rightarrow u^+ = u$.

(ii) Let $u \in H_c^1(\Omega)_+$. Let (ρ_k) be a δ -sequence in $C_c^\infty(\mathbb{R}^n)$. It was shown in the proof of Theorem 4.12(b) that then $(\rho_k * \tilde{u})|_\Omega \in C_c^\infty(\Omega)$ for large k , and $(\rho_k * \tilde{u})|_\Omega \rightarrow u$ in $H^1(\Omega)$. Clearly $\rho_k * \tilde{u} \geq 0$ for all $k \in \mathbb{N}$, and this shows that $u \in \overline{C_c^\infty(\Omega)_+}^{H^1(\Omega)}$. \square

11.12 Proposition. *Let Ω be bounded and with continuous boundary. Then $\check{C}^\infty(\Omega)_+ := \{u \in \check{C}^\infty(\Omega); u \geq 0\}$ is dense in $H^1(\Omega)_+ := \{u \in H^1(\Omega); u \geq 0\}$.*

Proof. We refer to the proof of Theorem 7.7. Following the lines of this proof one can see that, starting with a function $u \in H^1(\Omega)_+$, in all the steps one stays in the realm of positive functions. As a consequence, the approximating function constructed in the proof of Theorem 7.7 will be positive. \square

11.4 Elliptic operators with real coefficients

In this section we assume throughout that $\mathbb{K} = \mathbb{R}$. Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $a_{jk}, b_j, c_j, d \in L_\infty(\Omega)$ (all real-valued) be as in Section 11.2; in particular, we assume that the ellipticity condition (11.1) is satisfied.

A **vector sublattice** of $H^1(\Omega)$ is a subspace with the property that $u \in V$ implies $u^+ \in V$.

11.13 Proposition. *Let V be a closed vector sublattice of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Then the semigroup T_V generated by $-A_V$ is positive.*

Proof. Let $u \in H^1(\Omega)$. Then Theorem 9.14 implies $\partial_j u^+ = \mathbf{1}_{[u>0]} \partial_j u$, $\partial_j u^- = -\mathbf{1}_{[u<0]} \partial_j u$; therefore $\partial_k u^+ \partial_j u^- = 0$, $\partial_j u^+ u^- = 0$, $u^+ \partial_j u^- = 0$, $u^+ u^- = 0$. Thus $a(u^+, u^-) = 0$. Now it follows from Theorem 10.12(b) that T_V is positive. \square

Proposition 11.13 implies that the semigroup is positive for Dirichlet, Neumann and mixed boundary conditions; see Theorem 9.14 and Proposition 11.10.

Next we show additional properties for Dirichlet boundary conditions.

11.14 Proposition. (a) *Assume that $c = (c_1, \dots, c_n) \in C^1(\Omega; \mathbb{R}^n)$ and $\operatorname{div} c \leq d$. Then T_D is sub-Markovian.*

(b) *Let $b = (b_1, \dots, b_n) \in C^1(\Omega; \mathbb{R}^n)$, $\operatorname{div} b \leq d$. Then T_D is substochastic.*

Proof. (a) Let $u \in H_0^1(\Omega)$. Then $u \wedge 1 \in H_0^1(\Omega)$ and $\partial_j(u \wedge 1) = \mathbf{1}_{[u<1]} \partial_j u$, by Theorem 9.14. Since $u = u \wedge 1 + (u - 1)^+$, it follows that $(u - 1)^+ \in H_0^1(\Omega)$ and $\partial_j(u - 1)^+ = \mathbf{1}_{[u \geq 1]} \partial_j u$. Thus $\partial_k(u \wedge 1) \partial_j(u - 1)^+ = 0$ and $\partial_j(u \wedge 1)(u - 1)^+ = 0$. It follows that

$$\begin{aligned} a(u \wedge 1, (u - 1)^+) &= \int_{\Omega} \left(\sum_{j=1}^n c_j (u \wedge 1) \partial_j (u - 1)^+ + d (u \wedge 1) (u - 1)^+ \right) dx \\ &= \int_{\Omega} \left(\sum_{j=1}^n c_j \partial_j (u - 1)^+ + d (u - 1)^+ \right) dx \end{aligned}$$

(where the last equality holds because $u \wedge 1 = 1$ on $[u \geq 1]$). From the hypotheses we obtain

$$\int_{\Omega} \left(\sum_{j=1}^n c_j \partial_j \varphi + d \varphi \right) dx = \int_{\Omega} \left(- \sum_{j=1}^n \partial_j c_j + d \right) \varphi dx \geq 0$$

for all $0 \leq \varphi \in C_c^\infty(\Omega)$. Since $(u - 1)^+$ can be approximated by positive test functions, by Lemma 11.11, it follows that $a(u \wedge 1, (u - 1)^+) \geq 0$. Now it follows from Theorem 10.12(c) that T_D is sub-Markovian.

(b) The proof is analogous to (a) and uses Theorem 10.12(d). \square

11.15 Remark. In Proposition 11.14(a) it would be sufficient to require the validity of $\operatorname{div} c \leq d$ in the distributional sense, without requiring c to be differentiable, and similarly for b in Proposition 11.14(b). (A corresponding statement for the subsequent result would be somewhat more subtle, because of the occurrence of the boundary terms.)

Next we consider boundary conditions which are defined by more general spaces V .

11.16 Proposition. *Assume that Ω is bounded and has C^1 -boundary. Assume that $u \in V$ implies $u \wedge 1 \in V$.*

(a) *If $c \in C^1(\bar{\Omega}; \mathbb{R}^n)$, $\operatorname{div} c \leq d$ on Ω and $c \cdot \nu \geq 0$ on $\partial\Omega$, then T_V is sub-Markovian.*

(b) *If $b \in C^1(\bar{\Omega}; \mathbb{R}^n)$, $\operatorname{div} b \leq d$ on Ω and $b \cdot \nu \geq 0$ on $\partial\Omega$, then T_V is substochastic. (As before, $\nu(z) = (\nu_1(z), \dots, \nu_n(z))$ denotes the exterior normal at $z \in \partial\Omega$.)*

Proof. (a) As in the proof of Proposition 11.14 one has

$$a(u \wedge 1, (u - 1)^+) = \int_{\Omega} \left(\sum_{j=1}^n c_j \partial_j (u - 1)^+ + d(u - 1)^+ \right) dx \quad (u \in V).$$

For $0 \leq \varphi \in C^1(\bar{\Omega})$ we now have by Gauss' theorem (Theorem 7.4) that

$$\int_{\Omega} \left(\sum_{j=1}^n c_j \partial_j \varphi + d\varphi \right) dx = \int_{\Omega} \left(- \sum_{j=1}^n \partial_j c_j + d \right) \varphi dx + \int_{\partial\Omega} \sum_{j=1}^n \nu_j c_j \varphi d\sigma \geq 0.$$

By approximation – applying Proposition 11.12 – we deduce that $a(u \wedge 1, (u - 1)^+) \geq 0$.

The proof of (b) is analogous. \square

11.17 Remarks. Let Ω and V be as in Proposition 11.16, and assume that $\mathbf{1}_{\Omega} \in V$.

(a) If in Proposition 11.16(a) one has the equalities

$$\operatorname{div} c = d \quad \text{on } \Omega, \quad c \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

then $T_V(t)\mathbf{1}_{\Omega} = \mathbf{1}_{\Omega}$ for all $t \geq 0$. This means that T_V is not only sub-Markovian but **Markovian**.

Indeed, the proof of Proposition 11.16 shows that $a(\mathbf{1}_{\Omega}, v) = 0$ for all $v \in C^1(\bar{\Omega})$ and hence for all $v \in H^1(\Omega)$, by Proposition 11.12. This implies that $\mathbf{1}_{\Omega} \in \operatorname{dom}(A_V)$ and $A_V \mathbf{1}_{\Omega} = 0$. Now Theorem 1.12(a) implies the assertion.

(b) Similarly, if in Proposition 11.16(b) one has the equalities

$$\operatorname{div} b = d \quad \text{on } \Omega, \quad b \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

then T_V is not only substochastic but **stochastic**, i.e., $\|T_V(t)u\|_1 = \|u\|_1$ for all $0 \leq u \in L_2 \cap L_1(\Omega)$, $t \geq 0$. See Exercise 11.4.

11.18 Remark. Deviating from the initial announcement of this section that only $\mathbb{K} = \mathbb{R}$ is treated, we include a comment on the complex case. It should be noticed that then Theorems 10.8 and 10.15 are applicable to the situations treated in Propositions 11.14 and 11.16 and that they yield L_p -properties and holomorphic extensions for the generated semigroups.

11.5 Domination

We assume $\mathbb{K} = \mathbb{R}$ throughout this section. Let $\Omega \subseteq \mathbb{R}^n$ be open, V a closed vector sublattice of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Recall from Theorem 11.13 that then the semigroup T_V generated by $-A_V$ is positive. We recall the notation A_D, T_D, A_N, T_N from Examples 11.4.

11.19 Theorem. *For all $t \geq 0$ one has $0 \leq T_D(t) \leq T_V(t)$, i.e., $0 \leq T_D(t)f \leq T_V(t)f$ for all $0 \leq f \in L_2(\Omega)$.*

Proof. Because of the exponential formula, Theorem 2.12, it suffices to show the domination property for the resolvents; i.e., for large λ we have to show that

$$(\lambda + A_D)^{-1}f \leq (\lambda + A_V)^{-1}f$$

for all $0 \leq f \in L_2(\Omega)$. Adding λ to the coefficient d we may assume $\lambda = 0$ and also that the form a is coercive on $H^1(\Omega) \times H^1(\Omega)$.

Let $0 \leq f \in L_2(\Omega)$, $u_1 = A_D^{-1}f$, $u_2 = A_V^{-1}f$. We know that $u_1 \in H_0^1(\Omega)_+$, $u_2 \in V_+ := V \cap L_2(\Omega)_+$ and

$$\begin{aligned} a(u_1, v) &= (f | v)_{L_2} \quad \text{for all } v \in H_0^1(\Omega), \\ a(u_2, v) &= (f | v)_{L_2} \quad \text{for all } v \in V. \end{aligned}$$

Thus $a(u_1 - u_2, v) = 0$ for all $v \in H_0^1(\Omega)$. Observe that $0 \leq (u_1 - u_2)^+ \leq u_1$. Thus $(u_1 - u_2)^+ \in H_0^1(\Omega)$ by the ideal property Proposition 11.9. Taking $v := (u_1 - u_2)^+$ we obtain $a(u_1 - u_2, (u_1 - u_2)^+) = 0$. But $a(w^-, w^+) = 0$ for all $w \in H^1(\Omega)$ as we had seen in the proof of Proposition 11.13. Thus $a((u_1 - u_2)^+) = 0$, which implies $(u_1 - u_2)^+ = 0$ by our coerciveness assumption. Consequently, $u_1 \leq u_2$. We have shown that $A_D^{-1} \leq A_V^{-1}$. \square

11.20 Theorem. *Assume additionally that V is an ideal in $H^1(\Omega)$, i.e., $u \in V, v \in H^1(\Omega)$, $0 \leq v \leq u$ implies $v \in V$. Then $T_V(t) \leq T_N(t)$ for all $t \geq 0$.*

The proof is analogous to the proof of Theorem 11.19; see Exercise 11.5.

If A_V is the elliptic operator with mixed boundary conditions, then V is an ideal in $H^1(\Omega)$, by Proposition 11.10, and so $T_V(t) \leq T_N(t)$ for all $t \geq 0$.

Finally we want to prove domain monotonicity for Dirichlet boundary conditions. Let $\Omega_1 \subseteq \Omega$ be open. We consider the semigroup T_D on $L_2(\Omega)$. But we may also restrict the coefficients to Ω_1 and consider the semigroup T_D^1 on $L_2(\Omega_1)$. We identify $L_2(\Omega_1)$ with a subspace of $L_2(\Omega)$ by extending functions in $L_2(\Omega_1)$ by 0 on $\Omega \setminus \Omega_1$.

11.21 Theorem. *One has $T_D^1(t)f \leq T_D(t)f$ for all $0 \leq f \in L_2(\Omega_1)$, $t \geq 0$.*

Proof. By the exponential formula it suffices to show that $(\lambda + A_D^1)^{-1}f \leq (\lambda + A_D)^{-1}f$ on Ω_1 for large enough λ and $0 \leq f \in L_2(\Omega_1)$. As in the proof of Theorem 11.19 we may assume that the form a is coercive and $\lambda = 0$.

Let $0 \leq f \in L_2(\Omega_1)$, $u_1 = (A_D^1)^{-1}f$, $u_2 = (A_D)^{-1}f$. Then $u_1 \in H_0^1(\Omega_1)_+$, $u_2 \in H_0^1(\Omega)_+$ and

$$\begin{aligned} a(u_1, v) &= \int_{\Omega_1} f v \, dx \quad \text{for all } v \in H_0^1(\Omega_1), \\ a(u_2, v) &= \int_{\Omega} f v \, dx \quad \text{for all } v \in H_0^1(\Omega). \end{aligned}$$

Observe that for $v \in H_0^1(\Omega_1)$ the extension \tilde{v} is in $H_0^1(\Omega)$ and $\partial_j \tilde{v} = \widetilde{\partial_j v}$ for all $j = 1, \dots, n$ (see the paragraph before Proposition 11.9). Thus $a(u_1 - u_2, v) = 0$ for all $v \in H_0^1(\Omega_1)$. Since $u_1, u_2 \geq 0$ one has $0 \leq (u_1 - u_2)^+ \leq u_1$. It follows from Proposition 11.9 that $v := (u_1 - u_2)^+ \in H_0^1(\Omega_1)$. Hence $a(u_1 - u_2, (u_1 - u_2)^+) = 0$. Since $a((u_1 - u_2)^-, (u_1 - u_2)^+) = 0$ it follows that $a((u_1 - u_2)^+) = 0$ and hence $(u_1 - u_2)^+ = 0$ by the coerciveness assumption. Thus $u_1 \leq u_2$. \square

Notes

The application of invariance criteria – in the form of the Beurling-Deny criteria – to symmetric elliptic operators, in particular in connection with the heat equation with potential (“Schrödinger semigroups”), has a longer history; see for instance [RS78], [Dav89]. The application to non-symmetric operators seems to start with [MR92], [Ouh92], [Ouh96]; see [Ouh05] and [Are06] for more recent presentations. Domination is also considered in the above references. It is interesting that the conditions for domination given in Exercise 11.5 are also necessary; see [MVV05; Corollary 4.3]. We refer to [MVV05] and the literature quoted there for the treatment of more general domination results, which are then treated in the context of invariance criteria.

Perturbations as in Lemma 11.1 play an important role for the second order equation (cosine functions), see [ABHN11; Chapter 7 and Section 3.14] as well as the corresponding notes.

We note that the treatment of second order elliptic operators by forms is particularly effective for operators in divergence form, as written in (11.2). This nomenclature concerns the second order part of the operator. Transforming an expression $\sum_{j,k=1}^n a_{jk} \partial_j \partial_k u$ into divergence form would require differentiability properties of a_{jk} and produce first order terms (called drift terms).

Exercises

11.1 Let $(a_{jk}) \in \mathbb{R}^{n \times n}$, $\alpha > 0$ such that

$$\sum_{j,k=1}^n a_{jk} \xi_k \xi_j \geq \alpha |\xi|^2$$

for all $\xi \in \mathbb{R}^n$. Show that

$$\operatorname{Re} \sum_{j,k=1}^n a_{jk} \xi_k \bar{\xi}_j \geq \alpha |\xi|^2$$

for all $\xi \in \mathbb{C}^n$.

11.2 Let $\Omega \subseteq \mathbb{R}^2$ be open, $(a_{jk}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $b = c = 0$, $d = 0$.

(a) Show that $A_D = -\Delta_D$.

(b) Assume that Ω is bounded with C^1 -boundary. Find the conormal derivative corresponding to A_N ; cf. Example 11.4(b). Find Ω with $A_N \neq -\Delta_N$. Can one see that for all of these $\Omega \neq \emptyset$ one has $A_N \neq -\Delta_N$?

11.3 Let a_{jk}, b_j, c_j, d be as in Section 11.2.

(a) Assume additionally that $b \in C_b^1(\Omega; \mathbb{K}^n)$ (bounded derivatives!). Let the formal elliptic operators $\mathcal{A}_1, \mathcal{A}_2$, in the sense of (11.2), be defined by

$$\mathcal{A}_1 u := - \sum_{j,k=1}^n \partial_j (a_{jk} \partial_k u) + b \cdot (\nabla u), \quad \mathcal{A}_2 u := - \sum_{j,k=1}^n \partial_j (a_{jk} \partial_k u) + \operatorname{div}(bu) - (\operatorname{div} b)u.$$

Show that $A_{1,D} = A_{2,D}$.

(b) Assume additionally that $b, c \in C_b^1(\Omega; \mathbb{K}^n)$, $c = b$, and let \mathcal{A} be defined by

$$\mathcal{A} u := - \sum_{j,k=1}^n \partial_j (a_{jk} \partial_k u) + b \cdot (\nabla u) - \operatorname{div}(cu).$$

Show that A_D is associated with a formal elliptic operator without drift terms.

11.4 Prove Remark 11.17(b).

11.5 (a) Let $\mathbb{K} = \mathbb{R}$, (Ω, μ) a measure space, $H := L_2(\mu)$, V, W Hilbert spaces, $V \xrightarrow{d} H$, $W \xrightarrow{d} H$ and let $a: V \times V \rightarrow \mathbb{R}$, $b: W \times W \rightarrow \mathbb{R}$ be continuous bilinear forms, both H -elliptic. Denote by A the operator associated with a and by B the operator associated with b . Assume that the semigroups T generated by $-A$ and S generated by $-B$ are both positive. Assume

(i) $V \subseteq W$, and if $v \in V$, $w \in W$, $0 \leq w \leq v$, then $w \in V$;

(ii) $a(u, v) \geq b(u, v)$ for all $0 \leq u, v \in V$.

Show that $T(t) \leq S(t)$ ($t \geq 0$).

(b) Prove Theorem 11.20.

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Lecture 12

Sectorial forms

In this lecture we study sectorial forms. If the form is closed, then the approach presented here is equivalent to the approach via elliptic forms. We point out that even if the form is not closed, we can always associate an m -sectorial operator. Two examples are given that illustrate very well the theory: the Robin Laplacian and the Dirichlet-to-Neumann operator are revisited, but now on rough domains.

12.1 Closed sectorial forms

In this section we want to establish a slightly different formulation of the generation theorems of Section 5.3. As defined in Lecture 5, a form a is a sesquilinear mapping $a: V \times V \rightarrow \mathbb{K}$, for some \mathbb{K} -vector space V . In contrast to previous sections, V will carry no additional structure. For our purposes it will be convenient to simply call V the **domain of a** and set $\text{dom}(a) := V$. Evidently, this is a misuse of the symbol ‘dom’, since actually the domain of a , in the usual sense, is the cartesian product $V \times V$. However, this notation is useful, has a long history, and should not lead to (too much) confusion.

Let $a: \text{dom}(a) \times \text{dom}(a) \rightarrow \mathbb{C}$ be a form. We recall that, by definition, a is sectorial if there exists $\theta \in [0, \pi/2)$ such that $a(u) \in \{z \in \mathbb{C} \setminus \{0\}; |\text{Arg } z| \leq \theta\} \cup \{0\}$ for all $u \in \text{dom}(a)$. This is the same as saying that there exists $c \geq 0$ such that

$$|\text{Im } a(u)| \leq c \text{Re } a(u) \quad (12.1)$$

for all $u \in \text{dom}(a)$. We further recall that $\text{Re } a, \text{Im } a: \text{dom}(a) \times \text{dom}(a) \rightarrow \mathbb{C}$ are symmetric forms such that $a = \text{Re } a + i \text{Im } a$ and that $\text{Re } a(u) = (\text{Re } a)(u)$ and $\text{Im } a(u) = (\text{Im } a)(u)$ for all $u \in \text{dom}(a)$. Using Proposition 5.2 we deduce from (12.1) the inequality

$$|a(u, v)| \leq (1 + c)(\text{Re } a(u))^{1/2}(\text{Re } a(v))^{1/2}. \quad (12.2)$$

This is a sort of intrinsic continuity of a which we will use throughout.

Let H be a complex Hilbert space. A **sectorial form in H** is a couple (a, j) where $a: \text{dom}(a) \times \text{dom}(a) \rightarrow \mathbb{C}$ is a sectorial form and $j: \text{dom}(a) \rightarrow H$ is linear. We say that (a, j) is **densely defined** if j has dense range.

12.1 Remark. Later we want to use the results of this section also for the case $\mathbb{K} = \mathbb{R}$, if a is an accretive symmetric form $a: \text{dom}(a) \times \text{dom}(a) \rightarrow \mathbb{R}$. Note that then the inequality (12.2) holds with $c = 0$; in fact the validity of (12.2) is a key point for the results of this section. We ask the reader to keep this in mind and to read the section under this aspect, observing that the results also hold in this case.

12.2 Proposition. *Let (a, j) be a densely defined sectorial form in H . Then*

$$A_0 := \{(x, y) \in H \times H; \exists u \in \text{dom}(a): j(u) = x, a(u, v) = (y | j(v))_H \ (v \in \text{dom}(a))\} \quad (12.3)$$

defines a sectorial operator in H .

Proof. (i) Let $(0, y) \in A_0$. We show that $y = 0$; this implies that A_0 is an operator. Since $(0, y) \in A_0$ there exists $u \in \text{dom}(a)$ such that $j(u) = 0$ and $a(u, v) = (y | j(v))_H$ for all $v \in \text{dom}(a)$. In particular, $a(u) = 0$. By (12.2) this implies that $a(u, v) = 0$ for all $v \in \text{dom}(a)$. Consequently $(y | j(v))_H = 0$ for all $v \in \text{dom}(a)$. Since j has dense range it follows that $y = 0$.

(ii) Let $x \in \text{dom}(A_0)$. There exists $u \in \text{dom}(a)$ such that $j(u) = x$ and $a(u, v) = (A_0 x | j(v))_H$ for all $v \in \text{dom}(a)$. In particular, $(A_0 x | x) = a(u)$. Thus A_0 is sectorial. \square

In general, A_0 is not m-sectorial, but later we will construct an m-sectorial extension. We will give a condition which implies that A_0 is m-sectorial.

Let (a, j) be a densely defined sectorial form in H . Then

$$(u | v)_a := (\text{Re } a)(u, v) + (j(u) | j(v))_H \quad (12.4)$$

defines a semi-inner product, i.e., a symmetric, accretive sesquilinear form on $\text{dom}(a)$. Thus

$$\|u\|_a := \sqrt{(u | u)_a} = (\text{Re } a(u) + \|j(u)\|_H^2)^{1/2}$$

defines a seminorm on $\text{dom}(a)$.

12.3 Remark. (Completion of a semi-inner product space) Let E be a vector space over \mathbb{K} and $(\cdot | \cdot)$ a **semi-inner product** on E (i.e, the form $(\cdot | \cdot)$ is symmetric and accretive), with associated semi-norm $\|\cdot\|$.

(a) There exist a Hilbert space \tilde{E} and a linear mapping $q: E \rightarrow \tilde{E}$ which

- (i) is **isometric**, i.e., $(q(u) | q(v))_{\tilde{E}} = (u | v)$ for all $u, v \in E$, and
- (ii) has dense range.

Indeed, note that $F := \{u \in E; \|u\| = 0\}$ is a subspace of E , and let $q: E \rightarrow E/F =: G$ denote the quotient map. Since $|(u | v)| \leq \|u\| \|v\|$, by Proposition 5.2, $\|u\| = 0$ implies $(u | v) = 0$ for all $v \in E$. Thus, setting $(q(u) | q(v))_G = (u | v)$ one obtains a well-defined scalar product on G such that q is isometric. We define \tilde{E} as the completion of G .

(b) The space \tilde{E} is unique up to unitary equivalence. We call the pair (\tilde{E}, q) the **completion** of E .

(c) The mapping q is injective if and only if $(\cdot | \cdot)$ is positive definite, and q is surjective if and only if $(E, \|\cdot\|)$ is complete. If $(E, \|\cdot\|)$ is complete, but $(\cdot | \cdot)$ is not positive definite, then its completion $(\tilde{E}, q) = (E/F, q)$ is different from E . (Recall that a semi-normed space is called complete if every Cauchy sequence is convergent.)

(d) If H is a Hilbert space, and $j: E \rightarrow H$ is a continuous linear mapping, then there exists a unique continuous linear mapping $\tilde{j}: \tilde{E} \rightarrow H$ such that $\tilde{j} \circ q = j$. Similarly, if $a: E \times E \rightarrow \mathbb{K}$ is a bounded form (i.e., $|a(u, v)| \leq M \|u\| \|v\|$ for all $u, v \in E$), then there exists a uniquely determined bounded form $\tilde{a}: \tilde{E} \times \tilde{E} \rightarrow \mathbb{K}$ such that $\tilde{a}(q(u), q(v)) = a(u, v)$ for all $u, v \in E$.

The form (a, j) is called **closed** if the semi-inner product space $(\text{dom}(a), (\cdot | \cdot)_a)$ is complete.

12.4 Theorem. *Let (a, j) be a densely defined closed sectorial form in H . Then the operator A_0 defined in Proposition 12.2 is m -sectorial. We call $A := A_0$ **the operator associated with** (a, j) and write $A \sim (a, j)$.*

Proof. Denote the completion of $(\text{dom}(a), (\cdot | \cdot)_a)$ by (V, q) . Since $\text{dom}(a)$ is complete, q is surjective. From Remark 12.3(d) we recall the existence and continuity of $\tilde{j}: V \rightarrow H$ and $\tilde{a}: V \times V \rightarrow \mathbb{C}$. Observe that \tilde{a} is sectorial. Moreover,

$$\text{Re } \tilde{a}(q(u)) + \|\tilde{j}(q(u))\|_H^2 = \text{Re } a(u) + \|j(u)\|_H^2 = \|u\|_a^2 = \|q(u)\|_V^2$$

for all $u \in \text{dom}(a)$. Thus \tilde{a} is \tilde{j} -elliptic. By Corollary 5.11 the operator \tilde{A} associated with (\tilde{a}, \tilde{j}) is m -sectorial.

We show that $\tilde{A} = A_0$. Indeed, due to the surjectivity of q one has $(x, y) \in \tilde{A}$ if and only if there exists $u \in \text{dom}(a)$ such that $\tilde{j}(q(u)) = x$ and $\tilde{a}(q(u), q(v)) = (y | \tilde{j}(q(v)))_H$ for all $v \in \text{dom}(a)$. This is the same as saying that there exists $u \in \text{dom}(a)$ such that $j(u) = x$ and $a(u, v) = (y | j(v))_H$ for all $v \in \text{dom}(a)$, which is equivalent to $(x, y) \in A_0$. \square

12.2 The Friedrichs extension

Let H be a complex Hilbert space. In this section we consider the situation that a is a form whose domain is a subspace of H and that $j: \text{dom}(a) \hookrightarrow H$ is the embedding. In that case we drop the j in our notation and call a a **form in H** . If we want to emphasise that we are in this situation we will sometimes call a an **embedded** form. If a is a sectorial form in H and $\text{dom}(a)$ is dense, the operator A_0 of Proposition 12.2 is described by

$$A_0 = \{(u, y) \in \text{dom}(a) \times H; a(u, v) = (y | v)_H \text{ (} v \in \text{dom}(a)\text{)}\}.$$

The form a is closed if and only if $\text{dom}(a)$ is complete for the norm $\|\cdot\|_a$ given by

$$\|u\|_a^2 = \text{Re } a(u) + \|u\|_H^2.$$

In that case the associated operator $A = A_0$ is m -sectorial, by Theorem 12.4.

12.5 Theorem. (*Friedrichs extension*) *Let B be a densely defined sectorial operator in H . Then there exists a unique densely defined embedded closed sectorial form a in H such that $\text{dom}(B) \subseteq \text{dom}(a)$, $\text{dom}(B)$ dense in $(\text{dom}(a), \|\cdot\|_a)$, and*

$$a(u, v) = (Bu | v)_H$$

for all $u \in \text{dom}(B)$, $v \in \text{dom}(a)$.

Let $A \sim a$. Then $B \subseteq A$.

The operator A is called the **Friedrichs extension** of B . Note that the theorem also holds for accretive symmetric operators in real Hilbert spaces; see Remark 12.1.

Proof of Theorem 12.5. Define $b: \text{dom}(B) \times \text{dom}(B) \rightarrow \mathbb{C}$ by $b(u, v) := (Bu | v)_H$. Then b is densely defined and sectorial. We use the embedding $j: \text{dom}(b) \hookrightarrow H$ and the scalar product $(\cdot | \cdot)_b$ analogous to (12.4) on $\text{dom}(b) = \text{dom}(B)$. Let (V, q) be the completion of $(\text{dom}(b), \|\cdot\|_b)$; then q is injective by Remark 12.3(c). We consider $\text{dom}(b)$ as a subset of V and drop the notation q . We show that $\tilde{j} \in \mathcal{L}(V, H)$ – from Remark 12.3(d) – is injective. Let $u \in V$ such that $\tilde{j}(u) = 0$. There exists a sequence (u_n) in $\text{dom}(b)$ such that $u_n \rightarrow u$ in V . Then $u_n = \tilde{j}(u_n) \rightarrow \tilde{j}(u) = 0$ in H . Using (12.2) one obtains

$$\begin{aligned} \text{Re } b(u_n) &= \text{Re } b(u_n, u_n - u_k) + \text{Re } b(u_n, u_k) \\ &\leq (1 + c)\|u_n\|_b\|u_n - u_k\|_b + |(Bu_n | u_k)_H| \quad (k, n \in \mathbb{N}). \end{aligned}$$

Sending $k \rightarrow \infty$ we deduce that $\text{Re } b(u_n) \leq (1 + c)\|u_n\|_V\|u_n - u\|_V$ for all $n \in \mathbb{N}$. It follows that $\|u_n\|_V^2 = \text{Re } b(u_n) + \|u_n\|_H^2 \rightarrow 0$ as $n \rightarrow \infty$, $\|u\|_V = \lim_{n \rightarrow \infty} \|u_n\|_V = 0$.

Because \tilde{j} is injective we now can consider V as a subspace of H . Then $a := \tilde{b}$ with $\text{dom}(a) := V$ (with the notation of Remark 12.3(d)) is a closed sectorial form in H with the required properties, and the associated operator A is an extension of B .

In order to show uniqueness let a and b be two forms in H with the required properties. Then $\text{dom}(B) \subseteq \text{dom}(a) \cap \text{dom}(b)$, $a(u, v) = (Bu | v) = b(u, v)$ for all $u, v \in \text{dom}(B)$, and $\text{dom}(B)$ is dense in $(\text{dom}(a), \|\cdot\|_a)$ and in $(\text{dom}(b), \|\cdot\|_b)$. Since $\|\cdot\|_a = \|\cdot\|_b$ on $\text{dom}(B)$ it follows that $a = b$. \square

We now prove a converse version of Theorem 12.4.

12.6 Corollary. *Let A be an m -sectorial operator in H . Then there exists a unique densely defined embedded closed sectorial form a in H such that A is associated with a .*

Proof. The existence follows from Theorem 12.5 since m -sectorial operators do not have proper m -sectorial extensions.

Let a be as asserted. Then $\text{dom}(A) \subseteq \text{dom}(a)$ and $a(u, v) = (Au | v)$ for all $u, v \in \text{dom}(A)$. Moreover it follows from Lemma 9.19(a) that $\text{dom}(A)$ is dense in $(\text{dom}(a), \|\cdot\|_a)$. Therefore the uniqueness follows from the uniqueness in Theorem 12.5. \square

12.3 Sectorial versus elliptic

We now have two different generation results: on the one hand based on the notion of sectoriality in Section 12.1, on the other hand based on the notion of ellipticity in Lecture 5. We will show that they are basically the same. Before making this precise we first apply the usual rescaling procedure.

Let H be a complex Hilbert space. A **quasi-sectorial** form is a couple (a, j) where $a: \text{dom}(a) \times \text{dom}(a) \rightarrow \mathbb{C}$ is a sesquilinear form whose domain is a complex vector space and $j: \text{dom}(a) \rightarrow H$ is linear, with the property that there exists $\omega \in \mathbb{R}$ such that the form

$$a_\omega(u, v) = a(u, v) + \omega(j(u) | j(v))_H$$

is sectorial. If a is densely defined (i.e., j has dense range) we define the operator A_0 corresponding to (a, j) as in Proposition 12.2. One easily sees that $A_0 + \omega$ corresponds

to a_ω . Now assume that a is **closed**, i.e., a_ω is closed. Then $A_\omega \sim (a_\omega, j)$ is m -sectorial, $A_\omega = A_0 + \omega$, and thus $A := A_0$ is quasi- m -sectorial.

In the following remark we will make it clear that the notions of ‘densely defined closed quasi-sectorial form’ and ‘ j -elliptic form’ are equivalent.

12.7 Remarks. (a) ‘Closed sectorial’ implies ‘continuous j -elliptic’.

Let (a, j) be a densely defined closed sectorial form in H . Then the completion (V, q) of $(\text{dom}(a), \|\cdot\|_a)$ is a Hilbert space for the norm given by $\|u\|_V := \|u\|_{\tilde{a}} = (\text{Re } \tilde{a}(u) + \|\tilde{j}(u)\|_H^2)^{1/2}$. The form \tilde{a} is continuous for this norm and \tilde{j} -elliptic.

(b) ‘Continuous j -elliptic’ implies ‘closed quasi-sectorial, $\|\cdot\|_V \sim \|\cdot\|_{a_\omega}$ ’.

Let V be a Hilbert space, and assume that $j \in \mathcal{L}(V, H)$ has dense range. Let $a: V \times V \rightarrow \mathbb{C}$ be continuous and j -elliptic, say

$$\begin{aligned} |a(u, v)| &\leq M \|u\|_V \|v\|_V, \\ \text{Re } a(u) + \omega \|j(u)\|_H^2 &\geq \alpha \|u\|_V^2. \end{aligned}$$

Then a_ω is sectorial (see Theorem 5.9) and the norm $\|\cdot\|_{a_\omega}$ is equivalent to the given norm $\|\cdot\|_V$ on V . The latter shows that a_ω and hence a is closed.

(c) Sectorial on a complete domain.

Let V, H be Hilbert spaces, $V \subseteq H$ dense, with continuous embedding, and let $a: V \times V \rightarrow \mathbb{C}$ be a closed sectorial form in H . Observe that the norm of V is not used for these properties. They imply that $\|u\|_a = (\text{Re } a(u) + \|u\|_H^2)^{1/2}$ defines a complete norm on V . Since also $(V, \|\cdot\|_a) \hookrightarrow H$, it follows from the closed graph theorem that both norms, $\|\cdot\|_a$ and $\|\cdot\|_V$, are equivalent. Thus the form a is H -elliptic also with respect to the given norm on V .

12.4 The non-complete case

Here we consider densely defined sectorial forms that are not closed. Let H be a Hilbert space over \mathbb{C} . Let (a, j) be a densely defined sectorial form in H . Let (V, q) be the completion of $(\text{dom}(a), \|\cdot\|_a)$, and let \tilde{j}, \tilde{a} be as explained in Remark 12.3(d). Then (\tilde{a}, \tilde{j}) is a closed sectorial form in H . Let A be the m -sectorial operator associated with (\tilde{a}, \tilde{j}) according to Theorem 12.4. We call A the **operator associated with** (a, j) . Recall that

$$A = \{(x, y) \in H \times H; \exists u \in V: \tilde{j}(u) = x, \tilde{a}(u, v) = (y | \tilde{j}(v))_H \ (v \in V)\}.$$

It follows from the proof of Theorem 12.4 that this definition is consistent with the definition given in Theorem 12.4.

12.8 Theorem. *For the operator A defined above one has the description*

$$\begin{aligned} A = \{(x, y) \in H \times H; \text{ there exists } (u_k) \text{ in } \text{dom}(a) \text{ such that} \\ \text{(a) } j(u_k) \rightarrow x \text{ as } k \rightarrow \infty, \\ \text{(b) } \sup_{k \in \mathbb{N}} \text{Re } a(u_k) < \infty, \\ \text{(c) } a(u_k, v) \rightarrow (y | j(v))_H \ (v \in \text{dom}(a))\}. \end{aligned}$$

In this description the condition (b) can be replaced by

$$(b') \lim_{k, \ell \rightarrow \infty} \operatorname{Re} a(u_k - u_\ell) = 0.$$

The operator A is an extension of A_0 defined in Proposition 12.2.

Proof. Let $(x, y) \in A$. Then there exists $w \in V$ such that $\tilde{j}(w) = x$ and $\tilde{a}(w, v) = (y | \tilde{j}(v))_H$ for all $v \in V$. Since q has dense range there exists a sequence (u_k) in $\operatorname{dom}(a)$ such that $q(u_k) \rightarrow w$ in V . By the continuity of \tilde{j} it follows that $j(u_k) = \tilde{j}(q(u_k)) \rightarrow \tilde{j}(w) = x$. Since q is isometric, we obtain

$$\operatorname{Re} a(u_k - u_\ell) + \|j(u_k) - j(u_\ell)\|_H^2 = \|u_k - u_\ell\|_a^2 = \|q(u_k) - q(u_\ell)\|_V^2 \rightarrow 0$$

as $k, \ell \rightarrow \infty$. Finally, for $v \in \operatorname{dom}(a)$ we have

$$a(u_k, v) = \tilde{a}(q(u_k), q(v)) \rightarrow \tilde{a}(w, q(v)) = (y | \tilde{j}(q(v)))_H = (y | j(v))_H.$$

This shows the existence of a sequence (u_k) as asserted, and for this sequence one even has the stronger property (b').

Conversely, let $(x, y) \in H \times H$ such that there exists a sequence (u_k) in $\operatorname{dom}(a)$ satisfying (a), (b), (c). It follows from (a) and (b) that $\sup_{k \in \mathbb{N}} \|u_k\|_a < \infty$. Thus, taking a subsequence if necessary, we can assume that $(q(u_k))$ converges weakly to some $w \in V$. Hence $j(u_k) = \tilde{j}(q(u_k)) \rightarrow \tilde{j}(w)$ weakly in H , and $\tilde{j}(w) = x$ by (a). Property (c) implies

$$\tilde{a}(w, q(v)) = \lim_{k \rightarrow \infty} \tilde{a}(q(u_k), q(v)) = \lim_{k \rightarrow \infty} a(u_k, v) = (y | j(v))_H$$

for all $v \in \operatorname{dom}(a)$. Since q has dense range and \tilde{a}, \tilde{j} are continuous, it follows that $\tilde{a}(w, \tilde{v}) = (y | \tilde{j}(\tilde{v}))_H$ for all $\tilde{v} \in V$. This implies that $(x, y) \in A$.

If $(x, y) \in A_0$ and $u \in \operatorname{dom}(a)$ is as in (12.3), then the constant sequence $(u_k) = (u)$ satisfies the conditions required in the description of A , so $(x, y) \in A$. \square

12.9 Remarks. In this remark we assume that a is an embedded densely defined sectorial form in H . Then the description of the operator associated with a is as in Theorem 12.8, but without 'j' in conditions (a) and (c).

(a) Let (\tilde{a}, \tilde{j}) be the 'completion' of (a, j) as described at the beginning of the section. We point out that in general \tilde{j} is not injective. We call the form a **closable** if \tilde{j} is injective. Then we may identify $\operatorname{dom}(\tilde{a})$ with a subspace of H .

An example of a closable form has been given in Section 12.2, leading to the Friedrichs extension.

(b) We have seen that we may associate an m-sectorial operator A with the form a , no matter whether a is closable or not. This means that in our context we can simply forget about the notion of closability.

From Corollary 12.6 we know that there exists a unique closed form \bar{a} in H that is associated with A . In general there is no simple relation between a and \bar{a} .

(c) We add a comment on the terminology. The notation 'closed form' (in the situation of embedded forms) was forged by Kato – see, for instance, [Kat80; VI, §1.3] – in a vague analogy to 'closed operators'. However, whereas closed operators *are* closed in the product topology, for forms there is no visible closed set. Note that the definition of 'closable' – see [Kat80; VI, §1.4] – requires the associated closed form to be an embedded form.

12.10 Remark. Coming back to accretive symmetric forms in a real Hilbert space – recall Remark 12.1 – we note that in this case one obtains an associated accretive self-adjoint operator A and the same description as in Theorem 12.8.

12.5 The Robin Laplacian for rough domains

We choose $\mathbb{K} = \mathbb{R}$ throughout this section. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. On $\partial\Omega$ we consider the $(n - 1)$ -dimensional Hausdorff measure σ ; we refer to the end of this section for the definition. If Ω has C^1 -boundary, then σ coincides with the surface measure. We assume that $\sigma(\partial\Omega) < \infty$. Let $0 < c \leq \beta \in L_\infty(\partial\Omega)$.

We define the **trace** tr as the closure of the operator $u \mapsto u|_{\partial\Omega}: C(\bar{\Omega}) \cap H^1(\Omega) \rightarrow L_2(\partial\Omega)$ in $H^1(\Omega) \times L_2(\partial\Omega)$. For $u \in H^1(\Omega)$ we denote $\text{tr } u := \{g \in L_2(\partial\Omega); (u, g) \in \text{tr}\}$; this means that

$$\text{tr } u = \{g \in L_2(\partial\Omega); \text{ there exists } (u_k) \text{ in } C(\bar{\Omega}) \cap H^1(\Omega) \text{ such that} \\ u_k \rightarrow u \text{ in } H^1(\Omega), u_k|_{\partial\Omega} \rightarrow g \text{ in } L_2(\partial\Omega)\}.$$

In general the set $\text{tr } u$ may consist of more than one element; see Exercise 12.4. But if Ω has Lipschitz boundary, then $\text{tr } u$ is a singleton for each $u \in H^1(\Omega)$, i.e., tr is an operator. For the case of C^1 -boundary this has been shown in Theorem 7.9, and for the case of Lipschitz boundary we refer to [Alt85; A 5.7].

Let $u \in H^1(\Omega)$ such that $\Delta u \in L_2(\Omega)$. Let $h \in L_2(\partial\Omega)$. We say that $\partial_\nu u = h$ if

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} (\Delta u)v \, dx = \int_{\partial\Omega} hv \, d\sigma$$

for all $v \in C(\bar{\Omega}) \cap H^1(\Omega)$. We say that $\partial_\nu u \in L_2(\partial\Omega)$ if there exists $h \in L_2(\partial\Omega)$ such that $\partial_\nu u = h$. Note that this is analogous to the definition of the weak normal derivative in Section 7.3. The uniqueness of $\partial_\nu u$ is obtained from the denseness of $\{v|_{\partial\Omega}; v \in C^1(\bar{\Omega})\}$ in $L_2(\partial\Omega)$.

Now we can define the **Robin Laplacian**

$$\Delta_\beta := \{(u, f) \in H^1(\Omega) \times L_2(\Omega); \Delta u = f, \exists g \in \text{tr } u: \partial_\nu u = -\beta g\}$$

under our present general hypotheses.

12.11 Theorem. *The operator $-\Delta_\beta$ is an accretive self-adjoint operator.*

Proof. Let $\text{dom}(a) := C(\bar{\Omega}) \cap H^1(\Omega)$ and

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta uv \, d\sigma.$$

Then a is symmetric, densely defined and accretive. Let A be the operator associated with a ; see Section 12.4. Then A is self-adjoint and accretive by Remark 12.10. Using Theorem 12.8 and Remark 12.10 we are going to show that $-\Delta_\beta = A$.

‘ $A \subseteq -\Delta_\beta$ ’. Let $(u, f) \in A$. Then there exists a sequence (u_k) in $\text{dom}(a)$ such that

(a) $u_k \rightarrow u$ in $L_2(\Omega)$,

(b') $\lim_{k,\ell \rightarrow \infty} a(u_k - u_\ell) = 0$,

(c) $\int_{\Omega} \nabla u_k \cdot \nabla v \, dx + \int_{\partial\Omega} \beta u_k v \, d\sigma \rightarrow \int_{\Omega} f v \, dx$ for all $v \in \text{dom}(a)$.

It follows from (a) and (b') that $u \in H^1(\Omega)$ and $u_k \rightarrow u$ in $H^1(\Omega)$ and that $g := \lim_{k \rightarrow \infty} u_k|_{\partial\Omega}$ exists in $L_2(\partial\Omega)$. Now (c) implies that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta g v \, d\sigma = \int_{\Omega} f v \, dx$$

for all $v \in \text{dom}(a)$. Taking $v \in C_c^\infty(\Omega)$ we conclude that $-\Delta u = f$. Thus

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} (\Delta u) v \, dx = - \int_{\partial\Omega} \beta g v \, d\sigma$$

for all $v \in \text{dom}(a)$. This means by our definition that $\partial_\nu u = -\beta g$.

' $-\Delta_\beta \subseteq A$ '. Let $(u, f) \in -\Delta_\beta$. Then $u \in H^1(\Omega)$, $f = -\Delta u$, and there exists $g \in \text{tr } u$ such that $\partial_\nu u = -\beta g$. Thus, there exists a sequence (u_k) in $\text{dom}(a)$ such that $u_k \rightarrow u$ in $H^1(\Omega)$ and $u_k|_{\partial\Omega} \rightarrow g$ in $L_2(\partial\Omega)$ as $k \rightarrow \infty$. Thus (a) and (b') hold, and

$$a(u_k, v) = \int_{\Omega} \nabla u_k \cdot \nabla v \, dx + \int_{\partial\Omega} \beta u_k v \, d\sigma \rightarrow \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta g v \, d\sigma = \int_{\Omega} f v \, dx$$

for all $v \in \text{dom}(a)$ since $-\Delta u = f$ and $\partial_\nu u = -\beta g$. It follows that $(u, f) \in A$, by Theorem 12.8. \square

We close the section by a short introduction to the d -dimensional Hausdorff measure on \mathbb{R}^n , for $d \geq 0$. Let $A \subseteq \mathbb{R}^n$. For $\varepsilon > 0$ and coverings $(C_j)_{j \in \mathbb{N}}$ of A by sets $C_j \subseteq \mathbb{R}^n$ satisfying $\sup_{j \in \mathbb{N}} \text{diam } C_j \leq \varepsilon$ we let

$$\sigma_{d,\varepsilon}(A) := \omega_d \inf \sum_{j=1}^{\infty} \left(\frac{\text{diam } C_j}{2} \right)^d,$$

where the infimum is taken over all coverings as above, and

$$\omega_d := \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)},$$

which is the volume of the unit ball in \mathbb{R}^d if $d \in \mathbb{N}$. Then

$$\sigma_d^*(A) := \lim_{\varepsilon \rightarrow 0} \sigma_{d,\varepsilon}(A)$$

is the **outer d -dimensional Hausdorff measure** of A , where the limit exists because $\sigma_{d,\varepsilon}(A)$ is decreasing in ε . Carathéodory's construction of measurable sets yields a measure σ_d , the **d -dimensional Hausdorff measure**, and it turns out that Borel sets are measurable. If $d \in \mathbb{N}$, and $E = \mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^n$, then σ_d is the Lebesgue measure on $\mathbb{R}^d \cong E$. For all of these properties (and more) we refer to [EG92; Chapter 2].

12.6 The Dirichlet-to-Neumann operator for rough domains

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. We keep the definitions concerning σ and ∂_ν from Section 12.5. Again we use $\mathbb{K} = \mathbb{R}$ and assume that $\sigma(\partial\Omega) < \infty$. Now we define the **Dirichlet-to-Neumann operator** D_0 .

12.12 Theorem. *The relation*

$$D_0 = \{(g, h) \in L_2(\partial\Omega) \times L_2(\partial\Omega); \exists u \in H^1(\Omega): \Delta u = 0, g \in \text{tr } u, h = \partial_\nu u\}.$$

is an accretive self-adjoint operator in $L_2(\partial\Omega)$.

It is part of the assertion in Theorem 12.12 that the relation D_0 defines an operator even though Ω may have rough boundary. For the proof we need a remarkable inequality due to Maz'ja. Let $q := \frac{2n}{n-1}$. There exists a constant $c_M > 0$ such that

$$\|u\|_{L_q(\Omega)}^2 \leq c_M \left(\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} |u|^2 d\sigma \right) \quad (12.5)$$

for all $u \in C(\bar{\Omega}) \cap H^1(\Omega)$ (see [Maz85; Example 3.6.2/1 and Theorem 3.6.3]).

Proof of Theorem 12.12. Let $\text{dom}(a) = C(\bar{\Omega}) \cap H^1(\Omega)$, $a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$. Then a is an accretive symmetric form. Let $j: \text{dom}(a) \rightarrow L_2(\partial\Omega)$ be given by $j(u) := u|_{\partial\Omega}$. Then j has dense range because $C^1(\bar{\Omega}) \subseteq \text{dom}(a)$. Let A be the operator associated with (a, j) ; see Section 12.4. Then A is self-adjoint and accretive by Remark 12.10. We show that $A = D_0$.

Let $(g, h) \in A$. Then there exists a sequence (u_k) in $C(\bar{\Omega}) \cap H^1(\Omega)$ such that $u_k|_{\partial\Omega} \rightarrow g$ in $L_2(\partial\Omega)$, $\lim_{k, \ell \rightarrow \infty} \int_\Omega |\nabla(u_k - u_\ell)|^2 dx = 0$ and $\lim_{k \rightarrow \infty} a(u_k, v) = \int_{\partial\Omega} hv d\sigma$ for all $v \in C(\bar{\Omega}) \cap H^1(\Omega)$. Now Maz'ja's inequality implies that (u_k) is a Cauchy sequence in $H^1(\Omega)$. (Here we use that $q > 2$ and Ω has finite measure, and therefore $L_q(\Omega) \subseteq L_2(\Omega)$, with continuous embedding.) Let $u := \lim_{k \rightarrow \infty} u_k$ in $H^1(\Omega)$. Then $g \in \text{tr } u$. Moreover,

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} hv d\sigma \quad (v \in C(\bar{\Omega}) \cap H^1(\Omega)).$$

Taking test functions $v \in C_c^\infty(\Omega)$ one obtains $\Delta u = 0$. Thus

$$\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega (\Delta u)v dx = \int_{\partial\Omega} hv d\sigma$$

for all $v \in C(\bar{\Omega}) \cap H^1(\Omega)$. Hence $\partial_\nu u = h$ by the definition in Section 12.5. We have shown that $A \subseteq D_0$.

Conversely, let $(g, h) \in D_0$. Then there exists $u \in H^1(\Omega)$ such that $g \in \text{tr } u$, $\Delta u = 0$, $\partial_\nu u = h$. Hence there exists a sequence (u_k) in $\text{dom}(a)$ such that $u_k|_{\partial\Omega} \rightarrow g$ in $L_2(\partial\Omega)$ and $u_k \rightarrow u$ in $H^1(\Omega)$. Thus $\sup_{k \in \mathbb{N}} a(u_k) < \infty$ and

$$a(u_k, v) = \int_\Omega \nabla u_k \cdot \nabla v dx \rightarrow \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega (\Delta u)v dx = \int_{\partial\Omega} hv d\sigma$$

for all $v \in \text{dom}(a)$. This shows that $(g, h) \in A$. \square

12.13 Remark. Regrettably, it is completely beyond the scope of the Internet Seminar to provide a proof of Maz'ja's inequality. We refer to Maz'ja's book [Maz85] as well as to a self-contained presentation in a manuscript of Daners [Daned].

We mention that for our purpose it would have been sufficient to have (12.5) with $q = 2$. In this case, and for Ω with C^1 -boundary, we can show (12.5). Indeed, in the proof of Proposition 8.1 we have shown the inequality

$$\int_{\Omega} |u|^2 dx \leq \frac{1}{2} \int_{\Omega} |u|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + c \int_{\Omega} |\nabla u|^2 dx + c \int_{\partial\Omega} |u|^2 d\sigma,$$

and this inequality implies

$$\|u\|_2^2 \leq (2c + 1) \int_{\Omega} |\nabla u|^2 dx + 2c \int_{\partial\Omega} |u|^2 d\sigma.$$

However, this inequality is deficient in comparison to (12.5) with respect to several features: Maz'ja's inequality holds for arbitrary open sets – this is the salient point in our application – and on the left hand side one has the stronger q -norm.

Notes

The material of this lecture comes from [AE11a] and [AE11b]. The trace may be multi-valued if Ω does not have Lipschitz boundary; see Exercise 12.4. Much more information on the Dirichlet-to-Neumann operator is contained in [AE11b].

For the case of embedded non-closable accretive *symmetric* forms, a procedure to define an associated closed form has been presented in [Sim78].

Exercises

12.1 Let H be a complex Hilbert space, a an embedded sectorial form in H . Show the following criteria for a being closed or closable:

(a) a is closed if and only if for any Cauchy sequence (u_n) in $(\text{dom}(a), \|\cdot\|_a)$ with $u_n \rightarrow u$ in H one has $\|u_n - u\|_a \rightarrow 0$.

(b) a is closable if and only if for any Cauchy sequence (u_n) in $(\text{dom}(a), \|\cdot\|_a)$ with $u_n \rightarrow 0$ in H one has $\|u_n\|_a \rightarrow 0$.

12.2 Let $H := L_2(-1, 1)$, a_1, a_2 in H defined by $\text{dom}(a_j) = C_c^\infty(-1, 1)$,

$$\begin{aligned} a_1(u, v) &:= u(0)\overline{v(0)}, \\ a_2(u, v) &:= \int_{-1}^1 u'(x)\overline{v'(x)} dx + u(0)\overline{v(0)} \end{aligned}$$

for all $u, v \in C_c^\infty(-1, 1)$.

For $j = 1, 2$ determine whether a_j is closable. Find the completion of $(\text{dom}(a_j), (\cdot | \cdot)_{a_j})$ and the operator associated with a_j .

12.3 Let $\Omega := (-1, 0) \cup (0, 1)$.

- (a) Determine the relation tr of Section 12.5, and show that $\text{dom}(\text{tr})$ is not dense.
- (b) Find $\partial_\nu u$ for those $u \in H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$ for which $\partial_\nu u \in L_2(\partial\Omega)$.
- (c) Determine the Robin-Laplacian for $\beta = 1$.

12.4 Let $S := [0, 1] \times \{0\} \subseteq \mathbb{R}^2$, and let (x_n) be a bounded sequence in $\mathbb{R}^2 \setminus S$ having the set S as its cluster values. Let (r_n) be a sequence in $(0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$, $B[x_n, r_n] \cap S = \emptyset$ ($n \in \mathbb{N}$) and $B[x_n, r_n] \cap B[x_m, r_m] = \emptyset$ ($m, n \in \mathbb{N}$, $m \neq n$). Let $\Omega := \bigcup_{n \in \mathbb{N}} B(x_n, r_n)$.

- (a) Determine $\partial\Omega$ and $\sigma_1(\partial\Omega)$ (1-dimensional Hausdorff measure).
- (b) Show that $\text{dom}(\text{tr})$ is dense in $H^1(\Omega)$ and that $\text{tr} 0 = L_2(S)$ (where $L_2(S) \subseteq L_2(\partial\Omega)$ is the natural embedding).
- (c) Let D_0 be the Dirichlet-to-Neumann operator for Ω . Show that $L_2(S) \subseteq \text{dom}(D_0)$ and that $D_0|_{L_2(S)} = 0$.

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Lecture 13

The Stokes operator

The Stokes operator arises in the context of the (non-linear!) Navier-Stokes equation and acts in a subspace of a \mathbb{K}^n -valued L_2 -space. In our context we define it using a variant of the classical Dirichlet form. One of the features appearing in the description of the Stokes operator is the use of a Sobolev space of negative order, which is introduced at the beginning. Another important feature is the appearance of divergence free vector fields. This extra condition of vanishing divergence has interesting implications for the theory of the related Sobolev spaces, and a large part of the lecture is devoted to the investigation of these properties.

13.1 Interlude: the Sobolev space $H^{-1}(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be open. The space $H_0^1(\Omega)$ is a Hilbert space, and by the Riesz-Fréchet theorem, each continuous antilinear functional on $H_0^1(\Omega)$ is represented by an element of $H_0^1(\Omega)$. For some purposes, however, it is more convenient to work with antilinear functionals directly, i.e., to consider the antidual $H_0^1(\Omega)^*$ without this identification. The basic idea is to use that an element $f \in L_2(\Omega)$ acts on $H_0^1(\Omega)$ in a natural way as a continuous antilinear functional by

$$H_0^1(\Omega) \ni u \mapsto (f | u)_{L_2(\Omega)} =: \langle f, u \rangle_{H^{-1}, H_0^1}.$$

The operator $L_2(\Omega) \ni f \mapsto \langle f, \cdot \rangle_{H^{-1}, H_0^1} \in H_0^1(\Omega)^*$ is injective, and because the subspace $\{\langle f, \cdot \rangle_{H^{-1}, H_0^1} ; f \in L_2(\Omega)\}$ of $H_0^1(\Omega)^*$ separates the points of $H_0^1(\Omega)$, it is dense in $H_0^1(\Omega)^*$. In this context the antidual $H_0^1(\Omega)^*$ is denoted by $H^{-1}(\Omega)$.

13.1 Remark. We note that in the situation described above one has

$$H_0^1(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow H^{-1}(\Omega) = H_0^1(\Omega)^*,$$

with dense embeddings, and the embeddings are dual to each other. In such a situation one calls $(H_0^1(\Omega), L_2(\Omega), H^{-1}(\Omega))$ a **Gelfand triple**; this will be treated in more generality in Section 14.1.

If $f \in L_2(\Omega)$ and $j \in \{1, \dots, n\}$, then the mapping

$$H_0^1(\Omega) \ni u \mapsto \langle \partial_j f, u \rangle_{H^{-1}, H_0^1} := - (f | \partial_j u)_{L_2(\Omega)}$$

belongs to $H_0^1(\Omega)^*$. This definition of $\partial_j f$ is consistent with the definition of the distributional derivative in Section 4.1 because $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, and the mapping $\partial_j: L_2(\Omega) \rightarrow H^{-1}(\Omega)$ is linear and continuous. With these definitions the differential operator $\Delta = \sum_{j=1}^n \partial_j \partial_j$ acts as a continuous operator $\Delta: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$. This interpretation of the Laplace operator implies that

$$\langle \Delta u, \varphi \rangle_{H^{-1}, H_0^1} = (u | \Delta \varphi)_{L_2(\Omega)}$$

for all $u \in H_0^1(\Omega)$, $\varphi \in C_c^\infty(\Omega)$.

It turns out that the mapping $I - \Delta: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is an isometric isomorphism. If Ω is bounded and one provides $H_0^1(\Omega)$ with the scalar product $(u, v) \mapsto \int \nabla u \cdot \overline{\nabla v}$, then $-\Delta: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is an isometric isomorphism. See Exercise 13.1 for these properties. (Strictly speaking, the notation ‘ $I - \Delta$ ’ is not correct: the identity is meant to be the embedding $H_0^1(\Omega) \hookrightarrow H^{-1}(\Omega)$.)

13.2 The Stokes operator

Let $\Omega \subseteq \mathbb{R}^n$ be open. The Stokes operator is an operator in a subspace H of $L_2(\Omega; \mathbb{K}^n)$. It is defined as the operator associated with the classical Dirichlet form on

$$V := H_{0,\sigma}^1(\Omega; \mathbb{K}^n) := \{u \in H_0^1(\Omega; \mathbb{K}^n) = H_0^1(\Omega)^n; \operatorname{div} u = 0\}.$$

Note that V is a closed subspace of $H_0^1(\Omega; \mathbb{K}^n)$. The Hilbert space H is defined as the closure of V in $L_2(\Omega; \mathbb{K}^n)$,

$$H := L_{2,\sigma,0}(\Omega; \mathbb{K}^n) := \overline{V}^{L_2(\Omega; \mathbb{K}^n)}.$$

13.2 Remarks. (a) The space $L_{2,\sigma}(\Omega; \mathbb{K}^n) := \{f \in L_2(\Omega; \mathbb{K}^n); \operatorname{div} f = 0\}$ is a closed subspace of $L_2(\Omega; \mathbb{K}^n)$; this is because $\operatorname{div}: L_2(\Omega; \mathbb{K}^n) \rightarrow H^{-1}(\Omega)$ is continuous. Therefore H is a (closed) subspace of $L_{2,\sigma}(\Omega; \mathbb{K}^n)$.

(b) In previous instances when working with forms, we always had the Hilbert space H given beforehand, and then had to make sure that the domain of the form is dense (or that $j: V \rightarrow H$ has dense range). In contrast to this procedure, here we have a situation where V is given and H is adapted to V . We point out that $C_c^\infty(\Omega; \mathbb{K}^n) \cap H$ need not be dense in H , so that the notation $L_{2,\sigma,0}$ is not entirely consistent with previous notation. (There is no such reservation for sets Ω that are bounded and have Lipschitz boundary; cf. Theorem 13.13.)

(c) A comment on the notation: the index ‘ σ ’ should be remindful of ‘solenoidal’, which is the classical term for divergence free vectors fields.

We define the form $a: V \times V \rightarrow \mathbb{R}$ by

$$a(u, v) := \sum_{j=1}^n \int_{\Omega} \nabla u_j \cdot \overline{\nabla v_j}.$$

In each component of $H_0^1(\Omega)^n$, the form a is the classical Dirichlet form. Therefore we conclude from Section 5.4 that a is symmetric, accretive and H -elliptic; for bounded Ω it is coercive. The (accretive self-adjoint) operator associated with a is therefore given by

$$A = \left\{ (u, f) \in V \times H; \sum_{j=1}^n \int \nabla u_j \cdot \overline{\nabla v_j} = (f | v)_H \quad (v \in V) \right\}. \quad (13.1)$$

In view of Section 13.1, the equality appearing in the description of A can be rewritten as

$$0 = - \sum_{j=1}^n \int \nabla u_j \cdot \overline{\nabla v_j} + (f | v)_H = \sum_{j=1}^n \langle \Delta u_j + f_j, v_j \rangle_{H^{-1}, H_0^1} \quad (v \in V),$$

where f_j is considered as an element of $H^{-1}(\Omega)$ via the injection $L_2(\Omega) \hookrightarrow H^{-1}(\Omega)$, and Δ is the operator $\Delta: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ from Section 13.1.

Note that the antidual of $H_0^1(\Omega)^n$ is given by $H^{-1}(\Omega)^n$, with the dual pairing

$$\langle \eta, u \rangle_{H^{-1}, H_0^1} := \sum_{j=1}^n \langle \eta_j, u_j \rangle_{H^{-1}, H_0^1} \quad (\eta \in H^{-1}(\Omega)^n, u \in H_0^1(\Omega)^n).$$

It will be convenient to have a notation for the subset of $H^{-1}(\Omega)^n$ “orthogonal” to V . In order to avoid confusion with the orthogonal complement in Hilbert spaces we will use the notation as the polar,

$$V^\circ := \{ \eta \in H^{-1}(\Omega)^n; \langle \eta, v \rangle_{H^{-1}, H_0^1} = 0 \quad (v \in V) \}.$$

It is immediate from the definition of the derivative that for all $f \in L_2(\Omega)$ the gradient $\nabla f \in H^{-1}(\Omega)^n$ belongs to V° .

After these preliminaries we get a description of the Stokes operator. Abbreviating, we will use the notation $\Delta u = (\Delta u_1, \dots, \Delta u_n)$ for $u \in H_0^1(\Omega)^n$.

13.3 Theorem. *The operator A in H associated with the form a is given by*

$$A = \{ (u, f) \in V \times H; \exists \eta \in V^\circ: -\Delta u + \eta = f \}$$

(where the equality ‘ $-\Delta u + \eta = f$ ’ is an equality in $H^{-1}(\Omega)^n$, with $f \in H \hookrightarrow H^{-1}(\Omega)^n$).
Written differently,

$$\begin{aligned} \text{dom}(A) &= \{ u \in V; \exists \eta \in V^\circ: -\Delta u + \eta \in H \}, \\ Au &= -\Delta u + \eta \quad (\text{with } \eta \text{ as in } \text{dom}(A)). \end{aligned}$$

Proof. According the previous discussion, the condition appearing in (13.1) can be rephrased as

$$\langle \Delta u + f, v \rangle_{H^{-1}, H_0^1} = 0 \quad (v \in V),$$

which means that $\eta = \Delta u + f \in V^\circ$. □

As mentioned above, the **Stokes operator** A in Theorem 13.3 is self-adjoint and accretive.

For the physical interpretation one wants to express the element $\eta \in V^\circ$ appearing in the description of A in a more explicit way. In order to do this we need an additional property:

$$\text{for all } \eta \in V^\circ \text{ there exists } p \in L_2(\Omega) \text{ with } \eta = \nabla p. \quad (\text{H})$$

We will comment on this hypothesis in Remarks 13.5. In Section 13.4 it will be shown that (H) holds if Ω is bounded with Lipschitz boundary. With this hypothesis we get another description of the Stokes operator A .

13.4 Theorem. *Assume that Ω satisfies hypothesis (H). Then*

$$A = \{(u, f) \in V \times H; \exists p \in L_2(\Omega): -\Delta u + \nabla p = f\}$$

is the operator associated with the form a. Expressed differently,

$$\begin{aligned} \text{dom}(A) &= \{u \in V; \exists p \in L_2(\Omega): -\Delta u + \nabla p \in H\}, \\ Au &= -\Delta u + \nabla p \quad (\text{with } p \text{ as in } \text{dom}(A)). \end{aligned}$$

In the application in fluid dynamics one would like to interpret p in the statement of Theorem 13.4 as a pressure – the reason for the notation. However, we will not endeavour to enter the physical interpretation of the Stokes operator.

13.5 Remarks. Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $u \in L_2(\Omega; \mathbb{K}^n)$ satisfy

$$\int u \cdot \varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^1(\Omega; \mathbb{K}^n) \text{ with } \text{div } \varphi = 0. \quad (13.2)$$

(a) Then $\partial_j u_k - \partial_k u_j = 0$ in $H^{-1}(\Omega)$, for all $j, k \in \{1, \dots, n\}$.

Indeed, let $\psi \in C_c^\infty(\Omega)$, and define $\varphi_j := \partial_k \psi$, $\varphi_k := -\partial_j \psi$, $\varphi_\ell := 0$ for all other components. Then $\text{div } \varphi = 0$, and therefore

$$\int (\partial_j u_k - \partial_k u_j) \psi = \int (u_j \partial_k \psi - u_k \partial_j \psi) = \int u \cdot \varphi = 0.$$

(b) Assume additionally that u is a continuous vector field. In view of (a), the condition (13.2) implies that u satisfies the ‘compatibility conditions’ for a vector field to be locally the gradient of a potential.

However, the condition (13.2) is not only local, and in fact one can show that it implies that the potential exists also globally. For the idea of the proof we mention that it is sufficient to treat the case that Ω is connected. Then, fixing an “initial point” $x^0 \in \Omega$, one defines $p(x) := \int_0^1 u(\gamma(t)) \cdot \gamma'(t) \, dt$, where $\gamma: [0, 1] \rightarrow \Omega$ is a continuously differentiable path connecting x^0 with x . Using (13.2) it can be shown that this is well-defined – this is the main issue – and that then $u = \nabla p$.

We will not carry out this proof but refer to the proof of Theorem 13.13, where an analogous problem is treated.

13.3 Interlude: the Bogovskiĭ formula

This section could also run under the heading “some functions are divergences”. For $u \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$ one has $\int \operatorname{div} u \, dx = 0$, and it is well-known that the converse holds as well: for each $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\int \varphi \, dx = 0$ there exists $u \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$ with $\operatorname{div} u = \varphi$. In this section we will show that this holds in a more general and more precise version: for suitable bounded Ω one has that for any $f \in L_2(\Omega)$ with $\int f \, dx = 0$ there exists $u \in H_0^1(\Omega)^n$ such that $\operatorname{div} u = f$; see Theorem 13.9.

Let $v \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$ be a vector field satisfying the compatibility conditions $\partial_j v_k = \partial_k v_j$ for all $j, k = 1, \dots, n$. Then it is well-known and easy to show that, for any $y \in \mathbb{R}^n$, a potential for v is given by

$$p(x) := \int_0^1 v(y + t(x - y)) \cdot (x - y) \, dt,$$

i.e., $v = \nabla p$. We smooth this formula out with the help of a function $\rho \in C_c^\infty(\mathbb{R}^n)_+$ satisfying $\int \rho \, dx = 1$: we define

$$Av(x) := \int \rho(y) \int_0^1 v(ty + (1 - t)x) \cdot (x - y) \, dt \, dy \quad (13.3)$$

and obtain $Av \in C^\infty(\mathbb{R}^n)$, $\nabla(Av) = v$. In order to write A as an integral operator, we substitute $z = ty + (1 - t)x$ and $r = \frac{1}{t}$ to obtain

$$\begin{aligned} Av(x) &= \int_0^1 \int \rho\left(\frac{1}{t}(z - (1 - t)x)\right) v(z) \cdot \frac{x - z}{t} t^{-n} \, dz \, dt \\ &= \int_1^\infty \int \rho(x + r(z - x)) v(z) \cdot (x - z) \, dz \, r^{n-1} \, dr. \end{aligned}$$

This means that one can write $Av(x) = \int k(x, y) \cdot v(y) \, dy$, with

$$k(x, y) = \int_1^\infty \rho(x + r(y - x)) r^{n-1} \, dr (x - y).$$

Let ℓ be the negative transposed kernel of k , i.e.,

$$\ell(x, y) := -k(y, x) = (x - y) \int_1^\infty \rho(y + r(x - y)) r^{n-1} \, dr \quad (x, y \in \mathbb{R}^n).$$

It will be shown in the following theorem that then the definition

$$\begin{aligned} Bf(x) &:= \int \ell(x, y) f(y) \, dy \\ &= \int f(y) (x - y) \int_1^\infty \rho(y + r(x - y)) r^{n-1} \, dr \, dy, \end{aligned} \quad (13.4)$$

for $x \in \mathbb{R}^n$, $f \in C_c^\infty(\mathbb{R}^n)$, yields a mapping $B: C_c^\infty(\mathbb{R}^n) \rightarrow C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$. This definition is such that for all $v \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$, $f \in C_c^\infty(\mathbb{R}^n)$ one has $\int (Av) f \, dx = -\int v \cdot Bf \, dx$.

13.6 Theorem. For all $f \in C_c^\infty(\mathbb{R}^n)$ one has $Bf \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$,

$$\text{spt}(Bf) \subseteq \{\lambda z_1 + (1 - \lambda)z_2; z_1 \in \text{spt } f, z_2 \in \text{spt } \rho, 0 \leq \lambda \leq 1\} =: E. \quad (13.5)$$

If $\int f \, dx = 0$, then $\text{div } Bf = f$.

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$.

(i) First we show that $Bf = 0$ on $\mathbb{R}^n \setminus E$. Note that E is a compact set (in general a proper subset of the convex hull of $\text{spt } f \cup \text{spt } \rho$). Let $x \in \mathbb{R}^n \setminus E$. If $y \in \text{spt } f$ and $r \geq 1$ then $y + r(x - y) \notin \text{spt } \rho$ (because $z = y + r(x - y) \in \text{spt } \rho$ would lead to $x = \frac{1}{r}z + (1 - \frac{1}{r})y \in E$ - a contradiction), and therefore $\rho(y + r(x - y)) = 0$. Hence (13.4) implies $Bf(x) = 0$.

(ii) By the variable transformation $z = x - y$ and then $r = 1 + \frac{s}{|z|}$ in the inner integral we obtain Bf in the form

$$Bf(x) = \int f(x - z) \frac{z}{|z|^n} \int_0^\infty \rho\left(x + s \frac{z}{|z|}\right) (s + |z|)^{n-1} \, ds \, dz.$$

In this form one can differentiate under the integral to obtain $Bf \in C^\infty(\mathbb{R}^n; \mathbb{K}^n)$. (If $R > 0$ is such that ρ, f have their supports in $B(0, R)$, then we know from (i) that $Bf = 0$ on $\mathbb{R}^n \setminus B(0, R)$, and for $x \in B(0, R)$ we can use a multiple of $g(s, z) := \mathbf{1}_{(0, 2R)}(s) \mathbf{1}_{B(0, 2R)}(z) |z|^{1-n}$ as a dominating function.)

(iii) Let $f, \varphi \in C_c^\infty(\mathbb{R}^n)$, $\int f \, dx = 0$. Then

$$\int (\text{div } Bf) \varphi \, dx = - \int Bf \cdot \nabla \varphi \, dx = \int f A(\nabla \varphi) \, dx,$$

and this implies

$$0 = \int (\text{div } Bf) \varphi \, dx - \int f A(\nabla \varphi) \, dx = \int (\text{div } Bf - f) \varphi \, dx + \int f(\varphi - A(\nabla \varphi)) \, dx. \quad (13.6)$$

Note that in the above computation $\nabla \varphi$ satisfies the compatibility conditions, therefore $\nabla A(\nabla \varphi) = \nabla \varphi$, and as a consequence $\nabla(\varphi - A(\nabla \varphi)) = 0$, i.e., $\varphi - A(\nabla \varphi)$ is constant. From $\int f \, dx = 0$ we therefore conclude that the last term in (13.6) vanishes, which implies that $\int (\text{div } Bf - f) \varphi \, dx = 0$. As this holds for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ we obtain $\text{div } Bf = f$. \square

13.7 Remarks. (a) The formula (13.4) is the **Bogovskiĭ formula**.

(b) If $n \geq 2$, then there exist vector fields $0 \neq v \in C_c^\infty(\mathbb{R}^n; \mathbb{K}^n)$ satisfying $\text{div } v = 0$. Therefore the vector field obtained by the Bogovskiĭ formula is not the unique solution of $\text{div } v = f$.

Next we give reasons why, for suitable bounded Ω , the Bogovskiĭ formula provides a continuous linear operator $B: L_2^0(\Omega) \rightarrow H_0^1(\Omega)$, where $L_2^0(\Omega) := \{f \in L_2(\Omega); \int f \, dx = 0\}$. Let us note immediately that $L_2^0(\Omega)$ is a closed subspace of $L_2(\Omega)$ and that $C_c^\infty \cap L_2^0(\Omega)$ is dense in $L_2^0(\Omega)$; see Exercise 13.2(a).

In the following considerations all functions on Ω should be considered as extended by zero to \mathbb{R}^n .

13.8 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set containing a ball $B(x_0, r)$ such that Ω is star-shaped with respect to every point of $B(x_0, r)$. Let $\rho \in C_c^\infty(\mathbb{R}^n)_+$ with $\text{spt } \rho \subseteq B(x_0, r)$, $\int \rho \, dx = 1$.*

Then $B: C_c^\infty \cap L_2^0(\Omega) \rightarrow C_c^\infty(\Omega; \mathbb{K}^n)$, defined by (13.4), has a continuous (linear) extension $B: L_2^0(\Omega) \rightarrow H_0^1(\Omega)$. (The formula (13.5) implies that indeed $Bf \in C_c^\infty(\Omega; \mathbb{K}^n)$ for all $f \in C_c^\infty \cap L_2^0(\Omega)$.) For all $f \in L_2^0(\Omega)$ one has $\text{div } Bf = f$.

Unfortunately we cannot give the proof of this theorem because it relies on a piece of Analysis and Operator Theory that is beyond the scope of the Internet Seminar. In fact, the problem consists in two parts: one needs the Calderón-Zygmund theory of singular integral operators, and then one has to check the applicability to the operator at hand.

Showing that there exists $c > 0$ such that $\|Bf\|_2 \leq c\|f\|_2$ is not the problem; see Exercise 13.5. The problem is to find the corresponding estimate for the derivatives of Bf . For the Calderón-Zygmund theorem needed for this purpose we refer to the original paper [CZ56; Theorem 2] and to [Gal11; Theorem II.11.4]. The application to the Bogovskiĭ operator is treated in [Gal11; proof of Lemma III.3.1, pp. 164, 165].

We emphasise that the main interest in the mapping B in Theorem 13.8 consists in the circumstance that it implies the surjectivity of the map $\text{div}: H_0^1(\Omega) \rightarrow L_2^0(\Omega)$. (Note that $\text{div}(C_c^\infty(\Omega; \mathbb{K}^n)) \subseteq L_2^0(\Omega)$, hence $\text{div}(H_0^1(\Omega)) \subseteq L_2^0(\Omega)$, for *any* bounded open $\Omega \subseteq \mathbb{R}^n$.)

We show that Theorem 13.8 implies the surjectivity of $\text{div}: H_0^1(\Omega) \rightarrow L_2^0(\Omega)$ for more general Ω .

13.9 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, connected and with Lipschitz boundary. Let $f \in L_2^0(\Omega)$. Then there exists $v \in H_0^1(\Omega; \mathbb{K}^n)$ such that $\text{div } v = f$.*

Proof. (i) It is not difficult to see that for all $x \in \bar{\Omega}$ there exists an open neighbourhood U_x such that $U_x \cap \Omega$ is star-shaped with respect to the points of a ball in $U_x \cap \Omega$. This is obvious for $x \in \Omega$, and for $x \in \partial\Omega$ it results from the Lipschitz property of $\partial\Omega$. Compactness of $\bar{\Omega}$ implies that there exists a finite open covering $(\Omega_j)_{j=1, \dots, m}$ of Ω by sets to which Theorem 13.8 can be applied.

(ii) It is not too difficult to show that there exist functions $f_1, \dots, f_m \in L_2^0(\Omega)$ such that $[f_j \neq 0] \subseteq \Omega_j$ for all $j \in \{1, \dots, m\}$ and $f = \sum_{j=1}^m f_j$. (Let us illustrate this for $m = 2$. In this case choose $g \in L_2(\Omega)$ with $[g \neq 0] \subseteq \Omega_1 \cap \Omega_2$, $\int g \, dx = \int \mathbf{1}_{\Omega_1} f \, dx$, and define $f_1 := \mathbf{1}_{\Omega_1} f - g$, $f_2 := \mathbf{1}_{\Omega_2 \setminus \Omega_1} f + g$. See Exercise 13.2(b) for the general case.) Then for all $j \in \{1, \dots, m\}$ there exists $v^j \in H_0^1(\Omega_j; \mathbb{K}^n)$ with $\text{div } v^j = f_j$, and $v := \sum_{j=1}^m v^j$ has the required properties. \square

13.10 Remark. We mention that in Theorem 13.9 one can also construct a continuous linear operator $B: L_2^0(\Omega) \rightarrow H_0^1(\Omega; \mathbb{K}^n)$ such that $\text{div}(Bf) = f$ for all $f \in L_2^0(\Omega)$.

13.4 The hypothesis (H) and the Bogovskiĭ formula

In this section we show that the hypothesis (H) is satisfied if $\Omega \subseteq \mathbb{R}^n$ is open, bounded and has Lipschitz boundary.

13.11 Theorem. Let Ω be as stated above. Then for any $\eta \in H_{0,\sigma}^1(\Omega; \mathbb{K}^n)^\circ$, i.e., $\eta \in H^{-1}(\Omega)^n$ with $\langle \eta, v \rangle_{H^{-1}, H_0^1} = 0$ for all $v \in H_{0,\sigma}^1(\Omega; \mathbb{K}^n)$, there exists $p \in L_2(\Omega)$ such that $\eta = \nabla p$. In other words, Ω satisfies hypothesis (H).

For the proof we need the following special case of the ‘closed range theorem’.

13.12 Theorem. Let G, H be Hilbert spaces, $A \in \mathcal{L}(G, H)$ with the property that $\text{ran}(A)$ is closed.

Then $\text{ran}(A^*)$ is closed.

Proof. (i) First we treat the case that A is bijective. Then the closed graph theorem implies that A^{-1} is continuous, $(A^*)^{-1} = (A^{-1})^*$ is in $\mathcal{L}(G, H)$, and therefore $\text{ran}(A^*) = \text{dom}((A^*)^{-1}) = G$ is closed.

(ii) We can apply part (i) to $A_1 := A|_{\ker(A)^\perp} : \ker(A)^\perp \rightarrow \text{ran}(A)$ and obtain that $\text{ran}(A_1^*)$ is closed. Let $J : \ker(A)^\perp \hookrightarrow G$, and let P be the orthogonal projection from H onto $\text{ran}(A) = \ker(A^*)^\perp$. Then J^* is the orthogonal projection from G onto $\ker(A)^\perp = \text{ran}(A^*)$, and $P^* : \ker(A^*)^\perp \hookrightarrow H$; see Exercise 8.2(d). Then $A_1^* = (PAJ)^* = J^*A^*P^*$ shows that $\text{ran}(A^*) = \text{ran}(A_1^*)$ is closed. \square

Proof of Theorem 13.11. Without loss of generality we assume that Ω is connected.

The fundamental observation for the proof is that the continuous linear operators $\text{div} : H_0^1(\Omega; \mathbb{K}^n) \rightarrow L_2(\Omega)$ and $\nabla : L_2(\Omega) \rightarrow H^{-1}(\Omega)^n$ are negative adjoints of each other. Indeed, for $v \in H_0^1(\Omega; \mathbb{K}^n)$, $f \in L_2(\Omega)$ one has

$$(f | \text{div } v) = \sum_{j=1}^n \int f \overline{\partial_j v_j} \, dx = - \sum_{j=1}^n \langle \partial_j f, v_j \rangle_{H^{-1}, H_0^1} = - \langle \nabla f, v \rangle_{H^{-1}, H_0^1}.$$

We know from Lemma 6.7 that $\text{ran}(\nabla)^\circ = \ker(\text{div})$, and this implies $\overline{\text{ran}(\nabla)} = \ker(\text{div})^\circ$. (Note that for this argument no special properties of Ω are required.) It is shown in Theorem 13.9 that $\text{ran}(\text{div}) = L_2^0(\Omega)$, and clearly $L_2^0(\Omega)$ is a closed subspace of $L_2(\Omega)$. Therefore Theorem 13.12 implies that $\text{ran}(\nabla)$ is closed, $\text{ran}(\nabla) = \overline{\text{ran}(\nabla)} = \ker(\text{div})^\circ$. As $\ker(\text{div}) = H_{0,\sigma}^1(\Omega; \mathbb{K}^n)$ by definition, we obtain the assertion of the theorem. \square

13.5 Supplement: the space $H_{\sigma,0}^1(\Omega; \mathbb{K}^n)$ and the hypothesis (H')

The final issue of the lecture will be to investigate a slightly stronger version of hypothesis (H). Using the space $H_{\sigma,0}^1(\Omega; \mathbb{K}^n) := \overline{C_c^\infty(\Omega)^n \cap H_{0,\sigma}^1(\Omega; \mathbb{K}^n)}^{H_0^1}$ we can formulate it as follows:

$$\text{for all } \eta \in H_{\sigma,0}^1(\Omega; \mathbb{K}^n)^\circ \text{ there exists } p \in L_2(\Omega) \text{ with } \eta = \nabla p. \quad (\text{H}')$$

The space $H_{\sigma,0}^1(\Omega; \mathbb{K}^n)$ is a closed subspace of $H_{0,\sigma}^1(\Omega; \mathbb{K}^n)$, and (H') can be rephrased as the property that $H_{\sigma,0}^1(\Omega; \mathbb{K}^n)^\circ = \text{ran}(\nabla)$, where $\nabla : L_2(\Omega) \rightarrow H^{-1}(\Omega)^n$, $f \mapsto \nabla f$.

We insert the important observation that for any open set $\Omega \subseteq \mathbb{R}^n$ one has $\text{ran}(\nabla) \subseteq H_{0,\sigma}^1(\Omega; \mathbb{K}^n)^\circ \subseteq H_{\sigma,0}^1(\Omega; \mathbb{K}^n)^\circ (= C_{c,\sigma}^\infty(\Omega; \mathbb{K}^n)^\circ)$. Recall that (H) is equivalent to $H_{0,\sigma}^1(\Omega; \mathbb{K}^n)^\circ = \text{ran}(\nabla)$.

13.13 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and with Lipschitz boundary. Then (H') is satisfied, and $H_{\sigma,0}^1(\Omega; \mathbb{K}^n) = H_{0,\sigma}^1(\Omega; \mathbb{K}^n)$, i.e., $C_{c,\sigma}^\infty(\Omega; \mathbb{K}^n)$ is dense in $H_{0,\sigma}^1(\Omega; \mathbb{K}^n)$.*

Proof. (i) Assume additionally that Ω is star-shaped with respect to every point of some ball $B(x^0, r)$ contained in Ω . Without loss of generality assume that $x^0 = 0$. Then one can easily see that $\lambda\bar{\Omega} \subseteq \Omega$ for all $\lambda \in (0, 1)$. Let $u \in H_{0,\sigma}^1(\Omega; \mathbb{K}^n)$. Extend u to \mathbb{R}^n by 0. Then it follows from Exercise 4.2 that $\operatorname{div} u = 0$ also for the extended function. For $\lambda \in (0, 1)$ define $u_\lambda := u(\lambda^{-1}\cdot)$; then $\operatorname{spt} u_\lambda \subseteq \lambda\bar{\Omega}$ is a compact subset of Ω . If $(\rho_k)_{k \in \mathbb{N}}$ is a δ -sequence in $C_c^\infty(\mathbb{R}^n)$, then one concludes that $\operatorname{div}(\rho_k * u_\lambda) = 0$ for all $k \in \mathbb{N}$, $\rho_k * u_\lambda \in C_c^\infty(\Omega; \mathbb{K}^n)$ for large k , and $\rho_k * u_\lambda \rightarrow u_\lambda$ in $H_0^1(\Omega; \mathbb{K}^n)$ as $k \rightarrow \infty$. As a consequence, $u_\lambda \in H_{\sigma,0}^1(\Omega; \mathbb{K}^n)$ for all $\lambda \in (0, 1)$. Taking $\lambda \rightarrow 1$ one obtains $u_\lambda \rightarrow u$; hence $u \in H_{\sigma,0}^1(\Omega; \mathbb{K}^n)$.

So we have shown that $H_{\sigma,0}^1(\Omega; \mathbb{K}^n) = H_{0,\sigma}^1(\Omega; \mathbb{K}^n)$. Theorem 13.11 shows that (H) is satisfied; hence (H') is satisfied.

(ii) Without loss of generality we assume that Ω is connected. From the proof of Theorem 13.9 we recall that there exists an open covering $(\Omega_j)_{j=1,\dots,m}$ of Ω , where each Ω_j is star-shaped with respect to the points in a ball $B(x^j, r_j) \subseteq \Omega_j$.

Let $\eta \in H_{\sigma,0}^1(\Omega; \mathbb{K}^n)^\circ$. Then clearly $\eta^j := \eta|_{H_{\sigma,0}^1(\Omega_j; \mathbb{K}^n)} \in H_{\sigma,0}^1(\Omega_j; \mathbb{K}^n)^\circ$, and from (i) we conclude that there exists $f_j \in L_2(\Omega_j)$ such that $\nabla f_j = \eta^j$, for all $j = 1, \dots, m$.

Note that each function f_j is only determined up to a constant (see Remark 8.3), and we have to “glue together” suitable versions of these functions.

(iii) We show that for all $1 \leq j, k \leq m$, $j \neq k$, with the property that $\Omega_j \cap \Omega_k \neq \emptyset$ there exists $c_{kj} \in \mathbb{K}$ such that $f_k = f_j + c_{kj}$ on $\Omega_j \cap \Omega_k$. (Note that $\nabla(f_k - f_j) = 0$ on $\Omega_j \cap \Omega_k$ alone does not imply that $f_k - f_j$ is constant since $\Omega_j \cap \Omega_k$ need not be connected.)

Let $y \in \Omega_j \cap \Omega_k$, $r_y > 0$ be such that $B(y, r_y) \subseteq \Omega_j \cap \Omega_k$. Then $\nabla(f_k - f_j) = 0$ on $B(y, r_y)$ implies that there exists $c_{kj} \in \mathbb{K}$ such that $f_k - f_j = c_{kj}$ on $B(y, r_y)$. We are going to show that each $y' \in \Omega_j \cap \Omega_k$ has a neighbourhood where $f_k - f_j = c_{kj}$; then it follows that $f_k - f_j = c_{kj}$ on $\Omega_j \cap \Omega_k$.

Let $y' \in \Omega_j \cap \Omega_k$, and let $0 < r \leq \min\{r_j, r_k, r_y\}$ be such that $B(y', r) \subseteq \Omega_j \cap \Omega_k$. Then there exists $c'_{kj} \in \mathbb{K}$ such that $f_k - f_j = c'_{kj}$ on $B(y', r)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)_+$, $\operatorname{spt} \varphi \subseteq B(0, r)$, $\int \varphi dx = 1$. Then $\varphi(\cdot - y) - \varphi(\cdot - x^j)$ is the divergence of a function $\Phi_{yx^j} \in C_c^\infty(\Omega_j)^n$. (See Exercise 13.3; in fact $\operatorname{spt} \Phi_{yx^j}$ is a subset of the convex hull $\operatorname{co}(B(x^j, r) \cup B(y, r)) \subseteq \Omega_j$ of $B(x^j, r) \cup B(y, r)$.) Similarly one obtains $\Phi_{yx^k} \in C_c^\infty(\Omega_k)^n$ such that $\operatorname{div} \Phi_{yx^k} = \varphi(\cdot - y) - \varphi(\cdot - x^k)$. Note that $\nabla f_j = \eta^j$ implies

$$\langle \eta, \Phi_{yx^j} \rangle = \langle \eta^j, \Phi_{yx^j} \rangle = - (f_j | \operatorname{div} \Phi_{yx^j}) = (f_j | \varphi(\cdot - x^j) - \varphi(\cdot - y)),$$

and similarly for k . This implies

$$\begin{aligned} \langle \eta, \Phi_{yx^j} - \Phi_{yx^k} \rangle &= (f_j | \varphi(\cdot - x^j) - \varphi(\cdot - y)) - (f_k | \varphi(\cdot - x^k) - \varphi(\cdot - y)) \\ &= (f_j | \varphi(\cdot - x^j)) - (f_k | \varphi(\cdot - x^k)) + c_{kj}. \end{aligned} \quad (13.7)$$

Similarly, “connecting” x^j with x^k via y' and using the corresponding functions, one obtains

$$\langle \eta, \Phi_{y'x^j} - \Phi_{y'x^k} \rangle = (f_j | \varphi(\cdot - x^j)) - (f_k | \varphi(\cdot - x^k)) + c'_{kj}.$$

The above definitions imply that $\operatorname{div}((\Phi_{yx^j} - \Phi_{yx^k}) - (\Phi_{y'x^j} - \Phi_{y'x^k})) = 0$, and therefore

$$0 = \langle \eta, \Phi_{yx^j} - \Phi_{yx^k} \rangle - \langle \eta, \Phi_{y'x^j} - \Phi_{y'x^k} \rangle = c_{kj} - c'_{kj}.$$

So, we obtain $f_k - f_j = c_{kj}$ on $B(y', r)$.

(iv) From step (iii) we conclude that there exists $0 < r \leq \min\{r_1, \dots, r_m\}$ such that, given a function $\varphi \in C_c^\infty(\mathbb{R}^n)_+$ with $\text{spt } \varphi \subseteq B(0, r)$ and $\int \varphi = 1$, the following property holds: for all $1 \leq j, k \leq m$ with $j \neq k$, $\Omega_j \cap \Omega_k \neq \emptyset$ there exists a function $\Phi_{kj} \in C_c^\infty(\Omega_j \cup \Omega_k)^n$ with $\text{div } \Phi_{kj} = \varphi(\cdot - x^k) - \varphi(\cdot - x^j)$, and for this function one has

$$\langle \eta, \Phi_{kj} \rangle = \int f_j \varphi(\cdot - x^j) - \int f_k \varphi(\cdot - x^k) + c_{kj}.$$

(Indeed, use $\Phi_{kj} := \Phi_{yx^j} - \Phi_{yx^k}$ from (iii) and recall (13.7).)

Now let $(j_0, j_1, \dots, j_\ell = j_0)$ be a ‘closed chain’ in $\{1, \dots, m\}$ of ‘neighbouring indices’ (i.e., $j_{k-1} \neq j_k$ and $\Omega_{j_{k-1}} \cap \Omega_{j_k} \neq \emptyset$ for all $1 \leq k \leq \ell$). Then $\text{div}(\sum_{k=1}^\ell \Phi_{j_k j_{k-1}}) = 0$, hence

$$0 = \left\langle \eta, \sum_{k=1}^\ell \Phi_{j_k j_{k-1}} \right\rangle = \sum_{k=1}^\ell c_{j_k j_{k-1}}.$$

Thus we obtain a well-defined function $f \in L_2(\Omega)$ if we define $f := f_1$ on Ω_1 , and for $j \in \{2, \dots, m\}$ we choose a ‘chain’ $(j_0 = 1, j_1, \dots, j_{\ell-1}, j_\ell = j)$ of neighbouring indices (which can be found because Ω is connected) and define $f := f_j - \sum_{k=1}^\ell c_{j_k j_{k-1}}$ on Ω_j . It is clear that then $\eta = \nabla f$ locally on Ω , and this implies $\eta = \nabla f$.

(v) From what is shown above we know that Ω satisfies condition (H'), i.e., $\text{ran}(\nabla) = H_{\sigma,0}^1(\Omega; \mathbb{K}^n)^\circ$. Together with the property $\text{ran}(\nabla) \subseteq H_{0,\sigma}^1(\Omega; \mathbb{K}^n)^\circ \subseteq H_{\sigma,0}^1(\Omega; \mathbb{K}^n)^\circ$ we obtain equality in the last inclusions. Therefore the reflexivity of $H_0^1(\Omega; \mathbb{K}^n)$ implies $H_{0,\sigma}^1(\Omega; \mathbb{K}^n) = H_{\sigma,0}^1(\Omega; \mathbb{K}^n)$. \square

Note that Theorem 13.13 strengthens Theorem 13.11. In step (i) of the proof we used Theorem 13.11, but only for the case of ‘strongly star-shaped’ Ω .

13.14 Remarks. In these remarks we sketch how Theorems 13.11 and 13.13 are obtained in [Tem77; I, § 1].

(a) The basis is Nečas’ inequality, stating that there exists $c > 0$ such that

$$\|f\|_{L_2} \leq c \left(\sum_{j=1}^n \|\partial_j f\|_{H^{-1}} + \|f\|_{H^{-1}} \right) \quad (f \in L_2(\Omega)),$$

if Ω has Lipschitz boundary, asserted in [Neč12; Lemma 7.1, p. 186]. Using the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ one can show that this inequality implies

$$\|f\|_2 \leq c \sum_{j=1}^n \|\partial_j f\|_{H^{-1}} \quad \left(f \in L_2(\Omega) \text{ with } \int_\Omega f \, dx = 0 \right) \quad (13.8)$$

if Ω is bounded with Lipschitz boundary. From (13.8) one obtains that $\text{ran}(\nabla)$ is closed (where $\nabla: L_2(\Omega) \rightarrow H^{-1}(\Omega)^n$); see Exercise 13.4. As also $\text{ran}(\nabla)$ is dense in $H_{0,\sigma}^1(\Omega; \mathbb{K}^n)^\circ$, Theorem 13.11 is proved.

(b) Theorem 13.13 is also derived in [Tem77; I, Remark 1.4]. The argument there is based on a stronger version of (13.8), where f is not a priori in $L_2(\Omega)$. The source [Neč66] for this version, quoted in [Tem77], was not available to us.

Notes

Our introduction and presentation of the Stokes operator follows [Mon06] and [AE12]. Working out details we realised that we needed properties of Sobolev spaces of divergence free vector fields. These spaces are also of importance for the treatment of the Navier-Stokes equation.

The treatment given for the space $H_{0,\sigma}^1(\Omega; \mathbb{K}^n)$ in Section 13.3 can be found to a large part in [Gal11; Section III.3]. The *derivation* of the Bogovskiĭ formula presented at the beginning of Section 13.3 was found by the authors, and the same holds for the proof of Theorem 13.13. We mention that Bogovskiĭ's operator is also treated in the L_p -context in [Gal11] and in Sobolev spaces of negative order in [GHH06]. The treatment in Section 13.4 uses ideas contained in [Tem77; Ch. I, § 1].

Exercises

13.1 Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Show that the mapping $I - \Delta: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is an isometric isomorphism.

(b) Assume additionally that Ω is bounded, and provide $H_0^1(\Omega)$ with the scalar product $(u, v) \mapsto \int \nabla u \cdot \overline{\nabla v}$. Show that $-\Delta: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is an isometric isomorphism.

13.2 Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded.

(a) Show that $L_2^0(\Omega)$ is a closed subspace of $L_2(\Omega)$ and that $L_{2,c}^0(\Omega) := \{f \in L_2^0(\Omega); \text{spt } f \text{ compact}\}$ is dense in $L_2^0(\Omega)$.

(b) Assume additionally that Ω is connected. Let $(\Omega_j)_{j=1,\dots,m}$ be a finite open covering of Ω . Let $f \in L_2^0(\Omega)$.

Show that there exist functions $f_1, \dots, f_m \in L_2^0(\Omega)$, $[f_j \neq 0] \subseteq \Omega_j$ for all $1 \leq j \leq m$ such that $f = \sum_{j=1}^m f_j$.

13.3 Let $\varphi \in C_c^\infty(\mathbb{R}^n)$, $y, z \in \mathbb{R}^n$. Define $\Phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\Phi(x) := \int_{-\infty}^0 (\varphi(\cdot - z) - \varphi(\cdot - y))(x + t(z - y)) dt (z - y).$$

Show that $\Phi \in C_c^\infty(\mathbb{R}^n)$, $\text{div } \Phi = \varphi(\cdot - z) - \varphi(\cdot - y)$ and that $\text{spt } \Phi$ is contained in the convex hull of $(\text{spt } \varphi + y) \cup (\text{spt } \varphi + z)$.

13.4 Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and with Lipschitz boundary. Show that the inequality (13.8) implies that $\text{ran}(\nabla)$ is closed, for the mapping $\nabla: L_2(\Omega) \rightarrow H^{-1}(\Omega)^n$.

13.5 Let Ω and the Bogovskiĭ operator B be as in Theorem 13.8. Show that there exists $c > 0$ such that $\|Bf\|_2 \leq c\|f\|_2$ for all $f \in C_c^\infty \cap L_2^0(\Omega)$.

Hint: Recall the dominating function indicated in the proof of Theorem 13.6. Then either use Proposition 4.3(b), or else show an L_1 - L_1 bound and an L_∞ - L_∞ bound for B , and then use Riesz-Thorin.

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Lecture 14

Non-autonomous equations

So far we have considered forms that do not depend on time. The aim of this lecture is to prove an elegant theorem of Lions that establishes well-posedness of the non-autonomous equation

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0$$

where each operator $\mathcal{A}(t)$ is associated with a form $a(t, \cdot, \cdot)$. In contrast to previous results, the existence of solutions will be investigated in a larger space V^* . In fact, the situation is slightly more delicate and interesting: the solution will take its values in V , but the equation will be solved only in V^* .

We start by introducing the setup for V^* in the first section of the lecture. The next sections are devoted to the Bochner integral for Hilbert space valued functions and to Hilbert space valued Sobolev spaces. Of special interest are the maximal regularity spaces – as we call them –, certain mixed Sobolev spaces which have interesting properties. Lions' representation theorem extends the Lax-Milgram lemma and yields an elegant proof of the final result.

14.1 Gelfand triples

Let V, H be Hilbert spaces over \mathbb{K} such that $V \xhookrightarrow{d} H$. We have frequently encountered this situation in previous lectures; typical examples are $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ or $H^1(\Omega) \hookrightarrow L_2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is open. In this section we want to add the antidual V^* to our considerations. For $\eta \in V^*, u \in V$ we let

$$\langle \eta, u \rangle := \eta(u).$$

Note that V^* is a Hilbert space (being isomorphic to V). Given $f \in H$ we define $\eta_f \in V^*$ by

$$\langle \eta_f, u \rangle := (f | u)_H \quad (u \in V).$$

It is not difficult to see that the mapping $H \ni f \mapsto \eta_f \in V^*$ is linear, injective, continuous and has dense range. We will identify η_f and f and thus consider H as a subspace of V^* . This yields the following consistency identity:

$$\langle f, u \rangle = (f | u)_H \quad (f \in H, u \in V).$$

Thus we now have a triple of injected spaces

$$V \xhookrightarrow{d} H \xhookrightarrow{d} V^*,$$

called a **Gelfand triple**. We have encountered this situation already in Lecture 13 – see Remark 13.1 – and illustrate it with another example.

14.1 Example. Let (Ω, μ) be a measure space and let $m: \Omega \rightarrow [\delta, \infty)$ be measurable, where $\delta > 0$. Let

$$H := L_2(\Omega, \mu), \quad V := L_2(\Omega, m\mu) = \{u: \Omega \rightarrow \mathbb{K} \text{ measurable; } \int |u|^2 m \, d\mu < \infty\}.$$

Then $V \xrightarrow{d} H$. We can identify V^* with $L_2(\Omega, \frac{1}{m}\mu)$, where the duality is given by

$$\langle f, u \rangle = \int_{\Omega} f(x) \overline{u(x)} \, d\mu(x) \quad (f \in L_2(\Omega, \frac{1}{m}\mu), u \in V).$$

One can easily see this by using that $\Phi f := mf$ defines an isometric isomorphism $\Phi: L_2(\Omega, m\mu) \rightarrow L_2(\Omega, \frac{1}{m}\mu)$.

14.2 Remark. Example 14.1 is generic. Whenever $V \xrightarrow{d} H$, there exist a measure space (Ω, μ) , a unitary operator $U: H \rightarrow L_2(\Omega, \mu)$ and a measurable function $m: \Omega \rightarrow [\delta, \infty)$ (with $\delta > 0$) such that $U|_V$ is a unitary mapping from V onto $L_2(\Omega, m\mu)$. This can be proved with the help of the spectral theorem, applied to the accretive self-adjoint operator A in H associated with the form $a: V \times V \rightarrow \mathbb{K}$, $a(u, v) := (u | v)_V$; see also Exercise 14.1.

Now let $a: V \times V \rightarrow \mathbb{K}$ be a continuous form. Denote by $\mathcal{A} \in \mathcal{L}(V, V^*)$ the operator given by $\langle \mathcal{A}u, v \rangle = a(u, v)$. Let A be the operator in H associated with a . It is easy to see that A is the part of \mathcal{A} in H , i.e.,

$$A = \mathcal{A} \cap (V \times H),$$

where we identify $V \hookrightarrow H \hookrightarrow V^*$ as before. We will illustrate this situation below in Example 14.3. If a is H -elliptic, then we know that $-A$ generates a holomorphic C_0 -semigroup on H . One can show that $-\mathcal{A}$ generates a holomorphic C_0 -semigroup on V^* ; we will not pursue this here. However, our final aim will be to study the inhomogeneous Cauchy problem

$$\begin{aligned} u' + \mathcal{A}u &= f, \\ u(0) &= u_0 \end{aligned}$$

when \mathcal{A} depends on time; see Section 14.5.

14.3 Example. Consider the situation described in Example 14.1. Let $a: V \times V \rightarrow \mathbb{K}$ be given by

$$a(u, v) = \int_{\Omega} mu\bar{v} \, d\mu.$$

Then a is continuous and coercive. The operator \mathcal{A} in V^* is given by

$$\mathcal{A}u = mu \text{ with } \text{dom}(\mathcal{A}) = V,$$

and the operator A in $L_2(\Omega, \mu)$ is given by

$$\begin{aligned} \text{dom}(A) &= \{u \in L_2(\Omega, \mu); mu \in L_2(\Omega, \mu)\}, \\ Au &= mu \quad (u \in \text{dom}(A)). \end{aligned}$$

14.2 Interlude: The Bochner integral for Hilbert space valued functions

There is a general theory extending the Lebesgue integral to Banach space valued functions (the Bochner integral, see [ABHN11; Section 1.1]). On separable Hilbert spaces one may use a more elementary approach, which we will present here (cf. [AU10; Section 8.5]).

Let H be a separable Hilbert space, and let $-\infty \leq a < b \leq \infty$. A function $f: (a, b) \rightarrow H$ is called **measurable** if $(f(\cdot) | v)$ is measurable for all $v \in H$. If (v_n) is a dense sequence in $B_H(0, 1)$, then $\|f(t)\| = \sup_{n \in \mathbb{N}} |(f(t) | v_n)|$ for all $t \in (a, b)$, and this implies that $\|f(\cdot)\|: (a, b) \rightarrow \mathbb{R}$ is measurable. We let

$$L_1(a, b; H) := \left\{ f: (a, b) \rightarrow H \text{ measurable; } \int_a^b \|f(t)\| dt < \infty \right\},$$

where the elements of $L_1(a, b; H)$ are to be understood as equivalence classes of a.e. equal functions.

14.4 Lemma. *Let $f \in L_1(a, b; H)$. Then there exists a unique $w \in H$ such that*

$$\int_a^b (f(t) | v)_H dt = (w | v)_H \quad (v \in H),$$

and we define $\int_a^b f(t) dt := w$. The mapping $L_1(a, b; H) \ni f \mapsto \int_a^b f(t) dt \in H$ is linear and continuous.

Lemma 14.4 is an easy consequence of the theorem of Riesz-Fréchet; see Exercise 14.2. We also introduce the space

$$L_2(a, b; H) := \left\{ f: (a, b) \rightarrow H \text{ measurable; } \int_a^b \|f(t)\|^2 dt < \infty \right\},$$

again identifying a.e. equal functions. Note that $L_2(a, b; H) \subseteq L_1(a, b; H)$ if (a, b) is a bounded interval.

14.5 Proposition. *The space $L_2(a, b; H)$ is a Hilbert space for the scalar product*

$$(f | g)_{L_2(a, b; H)} := \int_a^b (f(t) | g(t))_H dt.$$

In order to see that the function $t \mapsto (f(t) | g(t))$ in Proposition 14.5 is measurable we note the following denseness property.

14.6 Lemma. *For each $f \in L_2(a, b; H)$ there exists a sequence (f_n) in*

$$L_2(a, b) \otimes H := \text{lin}\{\varphi(\cdot)v; \varphi \in L_2(a, b), v \in H\}$$

such that $\|f_n(\cdot)\| \leq \|f(\cdot)\|$ for all $n \in \mathbb{N}$ and $f_n \rightarrow f$ a.e.

In particular, $L_2(a, b) \otimes H$ is dense in $L_2(a, b; H)$.

Proof. This is obvious if $\dim H < \infty$; assume that H is infinite-dimensional. Let $f \in L_2(a, b; H)$. Let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis of H . Then, with $f_n := \sum_{k=1}^n (f(\cdot) | e_k) e_k$, the sequence (f_n) has the required properties. The dominated convergence theorem implies that $f_n \rightarrow f$ in $L_2(a, b; H)$. \square

Proof of Proposition 14.5. We prove that $t \mapsto (f(t) | g(t))_H$ is measurable.

If $f \in L_2(a, b) \otimes H$, $f = \sum_{j=1}^n \varphi_j(\cdot) v_j$, then $(f(\cdot) | g(\cdot)) = \sum_{j=1}^n \varphi_j(\cdot) (v_j | g(\cdot))$ is measurable. Every $f \in L_2(a, b; H)$ can be approximated pointwise a.e. by a sequence in $L_2(a, b) \otimes H$, by Lemma 14.6, and this implies the measurability of $(f(\cdot) | g(\cdot))$.

The completeness is proved in the same way as the completeness of the scalar valued $L_2(a, b)$. (Show that every absolutely convergent series is convergent.) \square

14.3 Vector valued Sobolev spaces

We now define Hilbert space valued Sobolev spaces. As before, let H be a separable Hilbert space over \mathbb{K} , and let $-\infty \leq a < b \leq \infty$.

Given $u \in L_2(a, b; H)$, a function $u' \in L_2(a, b; H)$ is called **weak derivative** of u if

$$-\int_a^b u(s) \varphi'(s) ds = \int_a^b u'(s) \varphi(s) ds \quad (\varphi \in C_c^\infty(a, b)).$$

Such a weak derivative is unique whenever it exists; see Exercise 14.3(a). We let

$$H^1(a, b; H) := \{u \in L_2(a, b; H); u \text{ has a weak derivative } u' \text{ in } L_2(a, b; H)\}.$$

It is easy to see that $H^1(a, b; H)$ is a Hilbert space for the scalar product

$$(u | v)_{H^1} = \int_a^b ((u(t) | v(t))_H + (u'(t) | v'(t))_H) dt$$

(cf. Theorem 4.8 and Proposition 14.5).

The next result can be proved essentially as Proposition 4.6.

14.7 Proposition. *Let $-\infty < a < b < \infty$.*

(a) *Let $v \in L_2(a, b; H)$, $u_0 \in H$, $u(t) := u_0 + \int_a^t v(s) ds$ ($t \in (a, b)$). Then $u \in H^1(a, b; H)$ and $u' = v$.*

(b) *Conversely, let $u \in H^1(a, b; H)$. Then there exists $u_0 \in H$ such that*

$$u(t) = u_0 + \int_a^t u'(s) ds \quad (\text{a.e. } t \in (a, b)).$$

In particular, each function $u \in H^1(a, b; H)$ has a representative in $C([a, b]; H)$, and with this representative one has $\int_a^b u'(s) ds = u(b) - u(a)$.

We will always use the continuous representative for functions $u \in H^1(a, b)$.

14.8 Remark. We note the following product rule for differentiation: if $u \in H^1(a, b; H)$ and $\varphi \in C^\infty[a, b]$, then $\varphi u \in H^1(a, b; H)$, and $(\varphi u)' = \varphi' u + \varphi u'$.

Indeed, for $\psi \in C_c^\infty(a, b)$ one has

$$\begin{aligned} \int_a^b (\varphi u) \psi' dt &= \int_a^b (\varphi \psi)' u dt - \int_a^b \varphi' \psi u dt \\ &= - \int_a^b \varphi \psi u' dt - \int_a^b \varphi' u \psi dt = - \int_a^b (\varphi u' + \varphi' u) \psi dt. \end{aligned}$$

Next we suppose that V is a separable Hilbert space such that $V \xrightarrow{d} H$. Then we identify H with a dense subspace of V^* as in Section 14.1. The following mixed Sobolev space plays an important role for evolutionary problems. Let $-\infty < a < b < \infty$ and

$$MR(a, b) := H^1(a, b; V^*) \cap L_2(a, b; V).$$

Here the symbol “ MR ” stands for “maximal regularity”. It is easy to see that $MR(a, b)$ is a Hilbert space for the norm

$$\|u\|_{MR(a,b)} := \left(\|u\|_{L_2(a,b;V)}^2 + \|u'\|_{L_2(a,b;V^*)}^2 \right)^{1/2}.$$

By Proposition 14.7 each $u \in MR(a, b)$ has a representative $u \in C([a, b]; V^*)$. The following result shows that in fact the representative is even continuous with values in the smaller space H .

14.9 Proposition. (a) *One has $MR(a, b) \hookrightarrow C([a, b]; H)$.*

(b) *If $u \in MR(a, b)$, then the function $\|u(\cdot)\|_H^2$ is in $W_1^1(a, b)$ and*

$$\left(\|u(\cdot)\|_H^2 \right)' = 2 \operatorname{Re} \langle u'(\cdot), u(\cdot) \rangle. \quad (14.1)$$

In the following remarks we explain the assertions of Proposition 14.9.

14.10 Remarks. (a) We define the scalar Sobolev space

$$W_1^1(a, b) := \{u \in L_1(a, b); u' = \partial u \in L_1(a, b)\}$$

(cf. Subsection 4.1.2). Using Proposition 4.6 one can see that $W_1^1(a, b) \hookrightarrow C[a, b]$ and that $u(t) = u(a) + \int_a^t u'(s) ds$ for all $u \in W_1^1(a, b)$. In particular, u is decreasing if $u' \leq 0$ a.e.

(b) From Section 14.1 recall the embeddings $V \hookrightarrow H \hookrightarrow V^*$ and the dual pairing $\langle \cdot, \cdot \rangle$.

(c) The measurability of $t \mapsto f(t) := \langle u'(t), u(t) \rangle$ in Proposition 14.9(b) is shown as in the proof of Proposition 14.5. Since $|f(t)| \leq \|u'(t)\|_{V^*} \|u(t)\|_V$ for all $t \in (a, b)$, the Schwarz inequality implies that $f \in L_1(a, b)$.

For the proof of Proposition 14.9 we need the following denseness property.

14.11 Proposition. *The space $C^\infty([a, b]; V)$ is dense in $MR(a, b)$.*

Proof. In several arguments in the following proof it is used without mentioning that $L_2(\mathbb{R}) \otimes V$ is dense in $L_2(\mathbb{R}; V)$, and similarly for V^* .

(i) Let $u \in MR(a, b)$, assume that $u = 0$ near b , and extend u by 0 to \mathbb{R} , keeping the notation u . For $\tau > 0$ we define the left translate $u_\tau := u(\cdot + \tau)$; then $u_\tau \in MR(a - \tau, b)$ and $u_\tau|_{(a, b)} \rightarrow u$ in $MR(a, b)$ as $\tau \rightarrow 0$. (In order to see the latter, note that $u_\tau|_{(a, b)} \rightarrow u$ in $L_2(a, b; V)$ and $(u_\tau)' = (u')_\tau \rightarrow u'$ in $L_2(a, b; V^*)$.)

Now we fix $\tau > 0$ and choose $\chi \in C_c^\infty(\mathbb{R})$ with $\text{spt } \chi \subseteq (a - \tau, \infty)$ and $\chi = 1$ on $[a, b]$; then $\chi u_\tau \in L_2(\mathbb{R}; V)$ with $(\chi u_\tau)' \in L_2(\mathbb{R}; V^*)$, by Remark 14.8. Let (ρ_k) be a δ -sequence in $C_c^\infty(\mathbb{R})$. Then $\rho_k * (\chi u_\tau) \in C_c^\infty(\mathbb{R}; V)$ for $k \in \mathbb{N}$, $\rho_k * (\chi u_\tau) \rightarrow \chi u_\tau$ in $L_2(\mathbb{R}; V)$, $(\rho_k * (\chi u_\tau))' = \rho_k * (\chi u_\tau)' \rightarrow (\chi u_\tau)'$ in $L_2(\mathbb{R}; V^*)$ as $k \rightarrow \infty$, by Proposition 4.3. In consequence, $(\rho_k * (\chi u_\tau))|_{(a, b)} \rightarrow (\chi u_\tau)|_{(a, b)} = u_\tau|_{(a, b)}$ in $MR(a, b)$, as $k \rightarrow \infty$.

This double approximation procedure shows that u can be approximated as asserted.

(ii) The corresponding procedure yields the approximation if $u = 0$ near a . For general $u \in MR(a, b)$ choose a function $\alpha \in C^\infty(\mathbb{R})$ with $\alpha = 1$ in a neighbourhood of $(-\infty, a]$, $\alpha = 0$ in a neighbourhood of $[b, \infty)$ and apply the result shown so far to αu and $(1 - \alpha)u$. (Note that Remark 14.8 implies that $\alpha u, (1 - \alpha)u \in MR(a, b)$.) \square

We mention that the proof given above is analogous to the proof of Theorem 7.7.

Proof of Proposition 14.9. If $u \in C^1([a, b]; V)$, then it is immediate that $\frac{d}{dt} \|u(t)\|_H^2 = (u'(t) | u(t))_H + (u(t) | u'(t))_H = 2 \text{Re} \langle u'(t), u(t) \rangle$.

(a) For $u \in C^1([a, b]; V)$ we deduce (as in the proof of Theorem 4.9) that

$$\begin{aligned} \|u\|_{C^1([a, b]; H)}^2 &\leq \inf_{t \in (a, b)} \|u(t)\|_H^2 + \int_a^b \frac{d}{dt} \|u(t)\|_H^2 dt \\ &\leq \frac{1}{b-a} \int_a^b \|u(t)\|_H^2 dt + 2 \int_a^b \|u'(t)\|_{V^*} \|u(t)\|_V dt \\ &\leq \frac{c}{b-a} \|u\|_{L_2(a, b; V)}^2 + 2 \|u'\|_{L_2(a, b; V^*)} \|u\|_{L_2(a, b; V)}, \end{aligned}$$

with the embedding constant $c \geq 0$ of $V \xrightarrow{d} H$. As $C^1([a, b]; V)$ is dense in $MR(a, b)$, by Proposition 14.11, this inequality shows that $MR(a, b) \hookrightarrow C([a, b]; H)$, and the inequality carries over to all $u \in MR(a, b)$.

(b) Initially we have shown (14.1) for $u \in C^1([a, b]; V)$. Let now $u \in MR(a, b)$. By Proposition 14.11 there exists a sequence (u_n) in $C^1([a, b]; V)$ converging to u in $MR(a, b)$. Then

$$(\|u_n(\cdot)\|_H^2)' = 2 \text{Re} \langle u_n'(\cdot), u_n(\cdot) \rangle \rightarrow 2 \text{Re} \langle u'(\cdot), u(\cdot) \rangle$$

in $L_1(a, b)$. Moreover, $u_n \rightarrow u$ in $C([a, b]; H)$ by part (a), and therefore $\|u_n(\cdot)\|_H^2 \rightarrow \|u(\cdot)\|_H^2$ in $C[a, b]$. This implies that $2 \text{Re} \langle u'(\cdot), u(\cdot) \rangle$ is the distributional derivative of $\|u(\cdot)\|_H^2$. \square

14.4 Lions' representation theorem

The great importance of Hilbert spaces is the representation theorem of Riesz-Fréchet, which so often gives us weak solutions of partial differential equations. We use it now in order to prove a much more general representation theorem.

14.12 Theorem. (*Lions' representation theorem*) Let \mathcal{V} be a Hilbert space, \mathcal{W} a pre-Hilbert space such that $\mathcal{W} \hookrightarrow \mathcal{V}$. Let $E: \mathcal{V} \times \mathcal{W} \rightarrow \mathbb{K}$ be sesquilinear such that

(i) $E(\cdot, w) \in \mathcal{V}'$ for all $w \in \mathcal{W}$,

(ii) $|E(w, w)| \geq \alpha \|w\|_{\mathcal{W}}^2$ for all $w \in \mathcal{W}$, with some $\alpha > 0$.

Let $L \in \mathcal{W}^*$. Then there exists $u \in \mathcal{V}$ and such that $L(w) = E(u, w)$ for all $w \in \mathcal{W}$.

The fact that \mathcal{W} may not be complete makes Lions theorem more widely applicable than the Lax-Milgram lemma. To say that $\mathcal{W} \hookrightarrow \mathcal{V}$ means that

$$\|w\|_{\mathcal{V}} \leq c \|w\|_{\mathcal{W}} \quad (w \in \mathcal{W}) \quad (14.2)$$

for some constant c . The larger the norm on \mathcal{W} the less restrictive is the assumption on L to be continuous in the norm of \mathcal{W} . On the other hand the coercivity hypothesis (ii) becomes more restrictive if we take larger norms. Even if we choose as norm on \mathcal{W} the norm of \mathcal{V} the fact that E need not be defined on all of \mathcal{V} in the second variable is an advantage; note that there is no continuity requirement on E with respect to the second variable.

Proof of Theorem 14.12. By the Riesz-Fréchet representation theorem there exists a linear operator $T: \mathcal{W} \rightarrow \mathcal{V}$ such that $E(v, w) = (v | Tw)_{\mathcal{V}}$ for all $v \in \mathcal{V}$, $w \in \mathcal{W}$.

It follows from property (ii) that

$$\alpha \|w\|_{\mathcal{W}}^2 \leq |E(w, w)| = |(w | Tw)_{\mathcal{V}}| \leq \|w\|_{\mathcal{V}} \|Tw\|_{\mathcal{V}} \leq c \|w\|_{\mathcal{W}} \|Tw\|_{\mathcal{V}},$$

where $c > 0$ is the embedding constant from (14.2). Thus $\|Tw\|_{\mathcal{V}} \geq \frac{\alpha}{c} \|w\|_{\mathcal{W}}$ for all $w \in \mathcal{W}$. This implies that T is injective and that $T^{-1}: T(\mathcal{W}) \rightarrow \mathcal{W}$ is linear and continuous, where $T(\mathcal{W})$ is provided with the norm of \mathcal{V} . Since $L \in \mathcal{W}^*$, the functional $L \circ T^{-1}$ extends to a continuous antilinear functional $\ell \in \mathcal{V}'$, and by the Riesz-Fréchet theorem there exists $u \in \mathcal{V}$ such that $\ell(v) = (u | v)_{\mathcal{V}}$ for all $v \in \mathcal{V}$. In particular, for $w \in \mathcal{W}$ one has

$$L(w) = \ell(Tw) = (u | Tw)_{\mathcal{V}} = E(u, w). \quad \square$$

14.13 Remark. (Uniqueness in Theorem 14.12) The vector $u \in \mathcal{V}$ is unique if and only if

$$v \in \mathcal{V}, E(v, w) = 0 \text{ for all } w \in \mathcal{W} \text{ implies } v = 0.$$

This is the same as saying that $T(\mathcal{W})$ is dense in \mathcal{V} , with the operator T from the proof.

14.5 The non-autonomous equation

Now we come to the main result of this lecture. We study the non-autonomous inhomogeneous evolution equation

$$\begin{aligned} u'(t) + \mathcal{A}(t)u(t) &= f(t), \\ u(0) &= u_0. \end{aligned} \tag{14.3}$$

Our assumptions are as follows. Let V, H be separable Hilbert spaces, $V \xrightarrow{d} H$. With these spaces we form the Gelfand triple $V \xrightarrow{d} H \xrightarrow{d} V^*$ and use the notation introduced in Section 14.1.

Let $\tau > 0$ and let $a: [0, \tau] \times V \times V \rightarrow \mathbb{K}$ be a mapping such that

- (i) $a(t, \cdot, \cdot): V \times V \rightarrow \mathbb{K}$ is sesquilinear for all $t \in [0, \tau]$;
- (ii) $|a(t, u, v)| \leq M \|u\|_V \|v\|_V$ for all $t \in [0, \tau]$, $u, v \in V$, with some $M \geq 0$;
- (iii) $\operatorname{Re} a(t, u, u) \geq \alpha \|u\|_V^2$ for all $t \in [0, \tau]$, $u \in V$, with some $\alpha > 0$;
- (iv) $a(\cdot, u, v)$ is measurable for all $u, v \in V$.

We point out that α and M are independent of t . Below we will repeatedly need that $t \mapsto a(t, u(t), v(t))$ is measurable for all $u, v \in L_2(0, \tau; V)$. This property follows from Lemma 14.6 and the conditions (iv) and (ii).

For each $t \in [0, \tau]$ we denote by $\mathcal{A}(t) \in \mathcal{L}(V, V^*)$ the operator given by

$$\langle \mathcal{A}(t)u, v \rangle = a(t, u, v).$$

First we want to interpret \mathcal{A} as an operator in the following way.

14.14 Proposition. *Defining*

$$(\mathcal{A}u)(t) := \mathcal{A}(t)u(t) \quad (t \in (0, \tau))$$

for $u \in L_2(0, \tau; V)$, one obtains a bounded linear operator $\mathcal{A}: L_2(0, \tau; V) \rightarrow L_2(0, \tau; V^*)$.

Proof. Let $u \in L_2(0, \tau; V)$. Recalling that the function $t \mapsto \langle \mathcal{A}u(t), v \rangle = a(t, u(t), v)$ is measurable for all $v \in V$ one deduces that $\mathcal{A}u$ is measurable. The estimate

$$\|\mathcal{A}(t)u(t)\|_{V^*} \leq M \|u(t)\|_V \quad (t \in [0, \tau]),$$

shows that $\mathcal{A}u = \mathcal{A}(\cdot)u(\cdot) \in L_2(0, \tau; V^*)$.

Obviously the operator \mathcal{A} is linear, and the previous inequality shows that \mathcal{A} is bounded. \square

In order to motivate our definition of E in the proof of Theorem 14.16 below we show the following proposition. We will use the notation $C_c^\infty[0, \tau] := \{\varphi \in C^\infty[0, \tau]; \operatorname{spt} \varphi \text{ compact}\}$.

14.15 Proposition. *Let $u_0 \in H$, $f \in L_2(0, \tau; V^*)$, $u \in L_2(0, \tau; V)$. Then the following statements are equivalent.*

- (i) $u \in MR(0, \tau)$ and $u' + \mathcal{A}u = f$, $u(0) = u_0$.
- (ii) For all $\varphi \in C_c^\infty[0, \tau]$, $v \in V$ one has

$$-\int_0^\tau (u(t) | \varphi'(t)v)_H dt + \int_0^\tau a(t, u(t), \varphi(t)v) dt = \int_0^\tau \langle f(t), \varphi(t)v \rangle dt + (u_0 | \varphi(0)v)_H.$$

Since $u \in MR(0, \tau) \subseteq C([0, \tau]; H)$, the initial condition $u(0) = u_0$ in statement (i) makes sense.

Proof of Proposition 14.15. Note that condition (ii) is equivalent to

$$-\int_0^\tau \varphi'(t)u(t) dt + \int_0^\tau \varphi(t)(\mathcal{A}u)(t) dt = \int_0^\tau \varphi(t)f(t) dt + \varphi(0)u_0 \quad (14.4)$$

for all $\varphi \in C_c^\infty[0, \tau]$. This equality holds in V^* ; recall that $\mathcal{A}u = \mathcal{A}(\cdot)u(\cdot) \in L_2(0, \tau; V^*)$.

(i) \Rightarrow (ii). Let $\varphi \in C_c^\infty[0, \tau]$. Using $\mathcal{A}u = f - u'$ and Remark 14.8 we obtain

$$-\int_0^\tau \varphi'(t)u(t) dt + \int_0^\tau \varphi(t)(\mathcal{A}u)(t) dt = -\int_0^\tau (\varphi u)'(t) dt + \int_0^\tau \varphi(t)f(t) dt.$$

Since $\varphi(\tau) = 0$ and $u(0) = u_0$, it follows from Proposition 14.7(b) that (14.4) holds.

(ii) \Rightarrow (i). Using (14.4) with $\varphi \in C_c^\infty(0, \tau)$ we obtain

$$-\int_0^\tau \varphi'(t)u(t) dt = \int_0^\tau \varphi(t)(f(t) - (\mathcal{A}u)(t)) dt.$$

This implies that $u \in H^1(0, \tau; V^*)$ and $u' = f - \mathcal{A}u$.

In order to show that $u(0) = u_0$ we choose $\varphi \in C_c^\infty[0, \tau]$ with $\varphi(0) = 1$. From (i) \Rightarrow (ii) it follows that (14.4) holds with $u(0)$ in place of u_0 . Therefore $u(0) = u_0$. \square

Now we can formulate and prove the main result of this lecture.

14.16 Theorem. *Let $u_0 \in H$, $f \in L_2(0, \tau; V^*)$. Then there exists a unique $u \in MR(0, \tau)$ such that*

$$\begin{aligned} u' + \mathcal{A}u &= f, \\ u(0) &= u_0. \end{aligned} \quad (14.5)$$

Note that both terms u' , $\mathcal{A}u$ are in the same space $L_2(0, \tau; V^*)$ as f . For this reason we say that the problem has **maximal regularity in V^*** .

Proof of Theorem 14.16. To show existence we apply Lions' representation theorem 14.12 with $\mathcal{V} := L_2(0, \tau; V)$ and the pre-Hilbert space

$$\mathcal{W} := C_c^\infty[0, \tau] \otimes V = \text{lin}\{\varphi v; \varphi \in C_c^\infty[0, \tau], v \in V\}$$

with norm $\|w\|_{\mathcal{W}} := (\|w\|_{\mathcal{V}}^2 + \|w(0)\|_H^2)^{1/2}$. We define $E: \mathcal{V} \times \mathcal{W} \rightarrow \mathbb{K}$ by

$$E(v, w) = -\int_0^\tau (v(t) | w'(t))_H dt + \int_0^\tau a(t, v(t), w(t)) dt.$$

For $w \in \mathcal{W}$, one has

$$|E(v, w)| \leq \|v\|_{\mathcal{V}} \|w'\|_{L_2(0, \tau; V^*)} + M \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}}$$

for all $v \in \mathcal{V}$. Thus condition (i) of Theorem 14.12 is satisfied.

In order to show (ii) let $w \in \mathcal{W}$. By Proposition 14.9(b) we have $\|w(\cdot)\|_H^2 \in W_1^1(0, \tau)$ and $\frac{d}{dt}\|w(t)\|_H^2 = 2 \operatorname{Re}(w'(t) | w(t))_H$. Thus

$$-\int_0^\tau \operatorname{Re}(w'(t) | w(t))_H dt = -\frac{1}{2}\|w(\tau)\|_H^2 + \frac{1}{2}\|w(0)\|_H^2 = \frac{1}{2}\|w(0)\|_H^2.$$

By the coercivity condition (iii) it follows that

$$\operatorname{Re} E(w, w) \geq \frac{1}{2}\|w(0)\|_H^2 + \alpha\|w\|_{\mathcal{V}}^2 \geq \min\{1/2, \alpha\}\|w\|_{\mathcal{W}}^2.$$

Thus (ii) is satisfied.

Define $L \in \mathcal{W}^*$ by

$$Lw = \int_0^\tau \langle f(t), w(t) \rangle dt + (u_0 | w(0))_H.$$

By Lions' representation theorem, there exists $u \in \mathcal{V}$ such that $E(u, w) = Lw$ for all $w \in \mathcal{W}$. This statement is equivalent to the validity of the statement (ii) of Proposition 14.15. Therefore Proposition 14.15 implies that $u \in MR(0, \tau)$ and that u is a solution of (14.5).

In order to show uniqueness, let $u \in MR(0, \tau)$ such that $u' + \mathcal{A}u = 0$ and $u(0) = 0$. Then by Proposition 14.9(b) we obtain

$$(\|u(\cdot)\|_H^2)'(t) = 2 \operatorname{Re} \langle u'(t), u(t) \rangle = -2 \operatorname{Re} \langle (\mathcal{A}u)(t), u(t) \rangle = -2 \operatorname{Re} a(t, u(t), u(t)) \leq 0$$

for a.e. $t \in [0, \tau]$. Remark 14.10(a) implies that $\|u(\cdot)\|_H$ is decreasing, so $u(t) = 0$ for all $t \in [0, \tau]$. \square

14.17 Remark. The notion of solution we use here is rather weak. This is reflected by the fact that our space of test functions \mathcal{W} is small and consists of very regular functions. Of course, if we have a weak notion of solution, existence of solutions might become easy to prove (at the extreme case we might call every function a solution). But uniqueness may become very hard to prove. Here for existence we need the key equality (14.1) only for functions in \mathcal{W} , for which it is in fact trivial. With the help of Lions' representation theorem the proof of existence is indeed rather elegant and easy. In contrast, uniqueness is more technical, since (14.1) is needed for a rather general class of functions.

Notes

The results of Sections 14.4 and 14.5 go back to J. L. Lions and his school in the 1950s and 1960s. Lions' representation theorem, Theorem 14.12, is first proved in [Lio57]. The existence and uniqueness result Theorem 14.16 is contained in Lions' book ([Lio61; Théorème 1.1, p. 46]), with a slightly different formulation. Lions quotes I. M. Vishik [Vis56] and J. L. Lions [Lio59] for the first proof of existence. Uniqueness is due to J. L. Lions [Lio59]; this had been an open problem for a while. It is also possible to prove Theorem 14.16 by the Galerkin method; see [DL92]. This is of great interest for the numerical treatment but less elegant than via the representation theorem.

A revival of the subject is due to the interest in maximal regularity. We refer to the recent thesis [Die14] by D. Dier and to [ADLO14] for further information, in particular for the discussion of maximal regularity in H .

Exercises

14.1 Let V, H be separable Hilbert spaces such that $V \overset{d}{\hookrightarrow} H$, $\dim H = \infty$. Assume that the injection of V into H is compact. Let $a: V \times V \rightarrow \mathbb{K}$ be continuous, coercive and symmetric.

(a) Show that there exist an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of H and an increasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ satisfying $\lim_{n \rightarrow \infty} \lambda_n = \infty$ such that

$$V = \left\{ u \in H; \sum_{n=1}^{\infty} \lambda_n |(u | e_n)_H|^2 < \infty \right\}$$

and $a(u, v) = \sum_{n=1}^{\infty} \lambda_n (u | e_n)_H (e_n | v)_H$ for all $u, v \in V$.

Hint: Use Theorem 6.17 and Corollary 6.18.

(b) Show that V^* can be identified with the space $\{(y_n) \in \mathbb{K}^{\mathbb{N}}; \sum_{n=1}^{\infty} \lambda_n^{-1} |y_n|^2 < \infty\}$, with the canonical mapping $H \hookrightarrow V^*$ given by $u \mapsto ((u | e_n)_H)_{n \in \mathbb{N}}$.

(c) Show that the operator $\mathcal{A} \in \mathcal{L}(V, V^*)$ from Section 14.1, considered as an operator in V^* , is given by

$$\begin{aligned} \text{dom}(\mathcal{A}) &= \{y \in V^*; (\lambda_n y_n)_{n \in \mathbb{N}} \in V^*\}, \\ \mathcal{A}y &= (\lambda_n y_n)_{n \in \mathbb{N}} \quad (y \in \text{dom}(\mathcal{A})). \end{aligned}$$

Show that $-\mathcal{A}$ generates a C_0 -semigroup on V^* , given by

$$e^{-t\mathcal{A}}y = (e^{-t\lambda_n} y_n)_{n \in \mathbb{N}}.$$

14.2 Prove Lemma 14.4.

14.3 Let $(a, b) \subseteq \mathbb{R}$ be an interval.

(a) Let H be a separable Hilbert space, $u \in L_1(a, b; H)$ with the property that

$$\int_a^b \varphi(t)u(t) dt = 0$$

for all $\varphi \in C_c^\infty(a, b)$. Show that $u = 0$ a.e.

Deduce the uniqueness of the weak derivative defined in Section 14.3.

(b) Find a (non-separable) Hilbert space H and a function $u: (a, b) \rightarrow H$ with $\|u(t)\| = 1$ for all $t \in (a, b)$ and $(u(\cdot) | v) = 0$ a.e. for all $v \in H$. (Then $\int_a^b \varphi(t)u(t) dt = 0$ for all $\varphi \in C_c^\infty(a, b)$, if the integral would be defined as in Lemma 14.4.)

14.4 Prove Theorem 14.16 in the case where the form a is not necessarily coercive but H -elliptic, i.e.,

$$\text{Re } a(t, u, u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad (t \in [0, \tau], u \in V)$$

holds for some $\omega \geq 0$, $\alpha > 0$.

Hint: Solve $v' + (\mathcal{A} + \omega)v = e^{-\omega \cdot} f$.

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