Abstracts

Days in Logic 2024

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Focusing Gentzen's Sequent Calculus

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Gerhard Gentzen introduced the sequent calculus in 1935 to provide a proof theory for classical and intuitionistic logics. Since then, many innovations in that calculus have been forced by new application areas (e.g., Curry-Howard correspondence and logic programming) and new insights (e.g., linear logic and polarity). I will describe one of those innovations—focused proof systems—and present the LJF and LKF focused proof systems based on Gentzen's original LJ and LK proof systems. I will also use the completeness of these proof systems to prove several other results, such as Herbrand's Theorem and the completeness of several specialized proof systems.

O-minimality and Combinatorics

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We will explore various model-theoretic tameness conditions, ranging from o-minimality all the way up to NIP (not the independence property), and their connection to graph-combinatorial problems (such as Zarankiewicz's problem on bounding the number of edges in certain classes of graphs). In particular, we will see how the combinatorial bounds can be improved if we assume that the class of graphs we consider is definable in a structure satisfying some tameness condition.

Type Theory

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In this course we will give an introduction to Type Theory as an alternative foundation of Mathematics using cubical Agda as a vehicle. In particular we will highlight the differences to using set theory.

We plan to cover the following topics:

- Functions, Products and Sums
- Dependent Types, Π- and Σ-types
- Equality Types, Univalence
- Propositions as Types
- Inductive and Coinductive Types
- Higher Inductive Types
- Type Theory in Type Theory

Regular typed unification

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This talk focuses on unification in typed first order theories, where types are described by regular grammars [1, 2, 3, 4]. We present a term unification algorithm which returns one of three different results: a *most general unifier*, *failure* or *wrong*. This last value *wrong* is inspired by a similar notion used by Robin Milner to denote run-time type errors in a functional programming language [5] and, in our framework, it corresponds to the unification of terms that can never belong to the same semantic domain. We generalize a typed unification algorithm previously defined by the authors in [4] where function symbols f of arity n, had co-domains which were always sets of terms of the form $f(t_1, \ldots, t_n)$ where t_i belongs to the corresponding domain of f. This induced a partition of the Herbrand domain into sets of trees. Here, we extend this notion, enabling the use of arbitrary semantic domains and co-domains for functors. Consider an algebra of lists where *cons* is the list constructor with type $cons :: \alpha \times list(\alpha) \rightarrow list(\alpha)$ and $list(\alpha) = nil + cons(\alpha, list(\alpha))$ (+ denotes disjoint type union). Suppose we have terms $t_1 = cons(1, X)$ and $t_2 = cons(Y, 1)$. These terms unify using first-order (untyped) unification, but they do not have a correct type, since the second argument of *cons* must be a list. This ill-typing is captured by our algorithm that, in this case, outputs *wrong*. We have proved termination but correctness and completeness of the algorithm is now ongoing work. The most related unification problem is many-sorted unification with a forest-structured sort hierarchy [6], but compared with this strong restricted unification problem, our work gives easier and nicer results, mostly due to the use of an expressive universe partition based on regular types but with no underlying hierarchy on the domains.

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Hypothetical query answering over continuous data streams

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Many modern-day reasoning systems need to react based on information that is received in real-time from different sources. One of the formalisms developed to model this problem from a theoretical perspective uses ideas and techniques from logic programming [4]. In this model, information (facts) arrives at discrete time points via a *datastream*, while the reasoning engine is formalised as a program in Temporal Datalog – a variant of Datalog where all predicates are annotated with a timestamp corresponding to the point of time where they hold. The reasoning tasks are then formulated as queries that need to be answered as time flows.

The formalism of hypothetical answers [1], which we presented at Days in Logic in 2020, extends this framework by defining a hypothetical semantics of answers that are dependent on some information still arriving at the data stream in the future. Our initial work showed that the intuitive approach, using techniques similar to those in abductive logic programming, could be turned into an algorithm that continuously updates a list of hypothetical answers to queries.

Our original work considered only a language without negation, and assumed perfect and instantaneous communication. In this talk, we show how the framework can be extended to deal with both of these aspects, and how their interaction poses challenging theoretical problems [2, 3].

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Towards a specification theory for paraconsistent logic*

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Paraconsistent logics distinguish themselves from classical ones by their ability to navigate through inconsistent information without slipping into absurdity. As interest in these logics grows, various applications and perspectives are emerging as evidenced by works like [1].

This talk explores a paraconsistent modal logic introduced in [3], which generalizes the Belnap-Dunn four-valued logic. Transitions in these paraconsistent models are labeled with pairs of weights—one indicating evidence for the transition's existence, and the other weighting its potential non-existence. Throughout the talk, we'll continually work with these pairs of weights, allowing us to capture different scenarios (consistency, vagueness, and even inconsistency) in a controlled manner.

The logic is later formalized as an institution for reasoning about paraconsistent transition systems and their corresponding processes. This lays the groundwork for developing an associated theory of structured specification, documented in [2], following the stepwise implementation process by Sannella and Tarlecki.

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Nonstandard analysis meets category theory

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Classical set theory is usually built over ZFC. The language of ZFC consists of a single symbol $\in$ indicating set membership. In [1] F. Lawvere proposed a new foundation for mathematics using Category theory, where he replaces $\in$ by the existence of maps from terminal objects. In his work, the axioms of ZFC were replaced by axioms relating to mappings and objects. For example, the existence of a Dedekind-Pierce object is axiomatized, such object plays the role of the axiom of axiom of infinity and can be seen as the natural numbers. As in algebra and topology, this formalization allows us to see the Form of classical set theory without dependence on the symbol $\in$.

In [2], H. Saigo and J. Nohmi also proposed a new axiomatization of E. Nelson Nonstandard Analysis, IST (see [3]) in terms of category theory. Such new axiomatization is built over the axiomatization of F. Lawvere.

In this talk, I will give a brief description of the new axiomatization and talk about ongoing work with B. Dinis, on an axiomatization of Nonstandard Analysis using category theory based on Elementary Nonstandard Analysis (see [4]).

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Vagueness and transitivity

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According to some recent accounts of vagueness [1, 6], transitivity is to be rejected as it is seen as the cause of the so-called Sorites paradox. A typical example of this paradox consists of a finite sequence of people ordered from smallest to tallest, with clear cases of small and tall people among its members, and such that adjacent members in the series differ by a negligible amount, say by not more than 0.5cm in height.

Even more recently, accounts of vagueness relying on nonstandard analysis were introduced [2, 5, 3]. In particular, the so-called *nonstandard primitivist* account [3, 4] embraces transitivity for marginal differences, but not for large differences in a soritical series.

We aim at comparing and assessing the relative merits of nonstandard primitivism with those accounts that reject transitivity altogether.

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Atomization in the polymorphic Lambda-calculus: exploring Russell-Prawitz translation variants

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This presentation embarks on an exploration of variants of the Russell-Prawitz translation of intuitionistic propositional calculus into second-order propositional logic, a system independently introduced by Jean-Yves Girard and John Reynolds, also commonly known as System F, or polymorphic lambda-calculus. Analyzing some properties of system F and of the translation itself we delve into atomic polymorphism and atomization conversions. Our tour encompasses the impact of introducing new (atomization) conversions to system F [2] and the more radical option of replacing system F by an atomic polymorphic target system. Within this exploration, we travel through the original atomic polymorphic variant of the Russell-Prawitz translation based on instantiation overflow [4, 5] and the subsequent efforts to enhance such translation: including a variant based on instantiation overflow with gains in proof and reduction sequences sizes [6]; a variant based on the admissibility of disjunction and absurdity elimination rules, yielding more proof size gains and less use of administrative reductions [1], and a variant obtaining an image of intuitionistic propositional calculus truly free from commuting conversions [3]. A significant part of the journey reports joint work with José Espírito Santo.

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Robust non-computability of the reachability problem and stability of dynamical systems

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The reachability problem consists in deciding whether a certain state of a system can be reached from a given initial state. This problem has applications in verification theory and can be considered both over discrete and continuous state spaces. In this talk we will consider the reachability problem defined by ordinary differential equations and we study the reachability problem of attracting equilibrium points. We will show that the computability of this problem is connected with the global stability of the system by showing that the basin of attraction of an attracting equilibruim point (i) can be robustly non-computable, i.e. the existence of the attracting equilibruim point and the non-computability of its basin of attraction persist under small perturbations of the system and (ii) is computable over a planar compact if the system is structurally stable – globally stable. This talk describes joint work with Ning Zhong [GZ23].

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Deciding intuitionistic propositional logic using a 2-valued non-deterministic logical matrix

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In recent years, Kearn's approach [1] to level valuations as applied to non-deterministic matrices (Nmatrices) has led to interesting and sometimes effective semantics for various logics not directly characterizable via finite Nmatrices, including various modal logics (see [2, 3, 4, 5]).

Abstracting from the particular cases, we present a general approach to this method and sufficient conditions under which it leads to decision procedures for the characterized logics. The underlying idea is to depart from a simpler finite-valued base logic whose closure by some envisaged metaproperties coincides with the target logic. When the relevant metaproperties can be imposed analytically we obtain decision procedures for the target logic by recursively sieving out unsound (partial) valuations. The complexity analysis of these procedures yields space and time complexity upperbounds for deciding the target logic based on the underlying complexity of the base logic.

We illustrate these ideas by obtaining a new PSPACE decision algorithm for intuitionistic propositional logic.

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An ecumenical view of proof-theoretic semantics

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Debates concerning philosophical grounds for the validity of classical and intuitionistic logics often have the very nature of logical proofs as one of the main points of controversy. The intuitionist advocates for a strict notion of constructive proof, while the classical logician advocates for a notion which allows non-construtive proofs through *reductio ad absurdum*. A great deal of controversy still subsists to this day on the matter, as there is no agreement between disputants on the precise standing of non-constructive methods. Two very distinct approaches to logic are currently providing interest- ing contributions to this debate. The first, oftentimes called logical ecumenism [1], aims to provide a unified framework in which two "rival" logics may peacefully coexist, thus providing some sort of neutral ground for the contestants. The second, proof-theoretic semantics [2], aims not only to elucidate the meaning of a logical proof, but also to provide means for its use as a basic concept of semantic analysis. Logical ecumenism thus provides a medium in which meaningful interactions may occur between classical and intuitionistic logic, whilst proof-theoretic semantics provides a way of clarifying what is at stake when one accepts or denies reductio ad absurdum as a meaningful proof method. We show how to coherently combine both approaches by providing not only a medium in which classical and intuitionistic logics may coexist, but also one in which classical and intuitionistic notions of proof may coexist.

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Decidability of modal logic **S4** through a labelled proof system

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Modal logics extends classical propositional logic to include operators expressing modality, representing concepts such as necessity ('$\Box$') and possibility ('$\Diamond$'). Amongst the most well known modal logics there is **S4**, obtained from the basic modal logic **K** by considering additional an additional pair of axioms $\Box p \to p$ and $\Box p \to \Box\Box p$, expressing reflexivity and transitivity. It is well known that **S4** is decidable (cf. [2], [1]), i.e. that there is a recursive procedure which determines whether a given formula is a theorem or not in this system.
In this work, we focus on a proof-theoretical approach to this problem, investigating termination strategies through proof search in a cut-free labelled proof system, as presented in [3], taking **S4** as a first case study. We provide an algorithm such that, given a sequent $\Gamma \Rightarrow \Delta$, either returns a derivation or can be used to extract a finite Kripke-model where it is not satisfiable. As a consequence of proof-search termination we are able to establish that **S4** is decidable and prove the finite model property for this logic, via purely syntactical methods. This work is being developed as part of my Master Thesis, supervised by Dr. Marianna Girlando (University of Amsterdam).

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Approaches and metrics in automated theorem generation/finding in geometry

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The pursue of what are properties that can be identified to permit an automated reasoning program to generate and find new and interesting theorems is an interesting research goal (pun intended). The automatic discovery of new theorems is a goal in itself, and it has been addressed in specific areas, with different methods. The separation of the “weeds”, uninteresting, trivial facts, from the “wheat”, new and interesting facts, is much harder, but is also being addressed by different authors using different approaches. We will focus on geometry. We present and discuss different approaches for the automatic discovery of geometric theorems (and properties), and different metrics to find the interesting theorems among all those that were generated. After this description an undecidability result is presented, proving that having an algorithmic procedure that decides for every possible Turing Machine that produces theorems, whether it is able to produce also interesting theorems, is an undecidable problem. Consequently, we will argue that judging whether a theorem prover is able to produce interesting theorems remains a non deterministic task, at best a task to be addressed by program based in an algorithm guided by heuristics criteria. Therefore, as a human, to satisfy this task two things are necessary: an expert survey that sheds light on what a theorem prover/finder of interesting geometric theorems is, and—to enable this analysis—other surveys that clarify metrics and approaches related to the interestingness of geometric theorems. The structure of two of these surveys are introduced and some future work is discussed.

Event-based time-stamped claim logic

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The *Event-based time-stamped claim logic* that we define in this paper allows one to reason about distributed time-stamped claims that can change through time by the occurrence of events. Such a logic is interesting for theoretical reasons, i.e., as a logic *per se*, but also because it can be applied in a number of different disciplines and application domains (e.g., history, crime forensics or cyber forensics) as it allows one to reason about a huge amount of pieces of evidence collected from different sources over time, where some of the pieces of evidence may be contradictory and some sources considered to be more trustworthy than others. We formalize the language and the semantics of the Event-based time-stamped claim logic, provide a sound and complete Hilbert calculus, and consider some concrete examples. We also show that the validity problem for the logic is decidable by providing a tableau-like decision algorithm.

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Labelled proof systems for existential reasoning

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Usually in logic, proof systems are defined having in mind proving properties like validity and semantic consequence. It seems worthwhile to address the problem of having proof systems where satisfiability is a primitive notion in the sense that a formal derivation means that a finite set of formulas is satisfiable. Moreover, it would be useful to cover within the same framework as many logics as possible. We consider Kripke semantics where the properties of the constructors are provided by valuation constraints as the common ground of those logics. This includes for instance intuitionistic logic, paraconsistent Nelson's logic N4, paraconsistent logic imbC and modal logics among others. After specifying a logic by those valuation constraints we show how to induce automatically and from scratch an existential proof system for that logic. General results of soundness and completeness are proved and then applied to the logics at hand.

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On the use of proof terms in the study of proof search

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We propose a methodology for the study of proof search employing *partial proof terms*, which are proof terms containing *formal sequents*, that is, sequents occurring as first-class terms. We can specify proof search procedures as rewri-ting systems acting on partial proof terms; and we can extend the original logical system to one dealing with partial proof terms, becoming a calculus whose sequents express proof states, comprising a goal sequent, a record of the history of the search, and a list of proof obligations. The methodology is illustrated with two simple examples: the focused sequent calculus LJT [1, 2] and a bidirectional natural deduction system we call NJT, both dealing with implication only. In NJT the proof search procedure follows the ideas of the intercalation calculus [3, 4], and we prove this procedure isomorphic to focusing using the tools we propose. (This is work in progress. Extension of these ideas to classical logic is joint work with Ana Catarina Sousa.)

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General notions of consistency

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We explore *general notions of consistency*. These notions are formulas $\mathcal{C}_\alpha$ (they depend on numerations α of a certain theory) that generalize the usual features of consistency statements. The following forms of consistency fit the definition of general notions of consistency ($\mathtt{Pr}_\alpha$ denotes the provability predicate for the numeration α): $\neg\mathtt{Pr}_\alpha(\ulcorner\bot\urcorner)$, ω-$\mathtt{Con}_\alpha$ (the formalized ω-consistency), $\neg\mathtt{Pr}_\alpha(\ulcorner\mathtt{Pr}_\alpha(\ulcorner\cdots\mathtt{Pr}_\alpha(\ulcorner\bot\urcorner)\cdots\urcorner)\urcorner)$, and n-$\mathtt{Con}_\alpha$ (the formalized n-consistency of Kreisel).

We generalize the former notions of consistency while maintaining two important features, to wit: Gödel's Second Incompleteness Theorem, i.e. $T \nvdash \mathcal{C}_\xi$ (with ξ some standard $\Delta_0(T)$-numeration of the axioms of T), and a result by Feferman [1] that guarantees the existence of a numeration τ such that $T \vdash \mathcal{C}_\tau$.

We encompass slow consistency into our framework. We create a *notion of provability* from a given $\mathcal{C}_\alpha$, we call it $\mathcal{P}_{\mathcal{C}_\alpha}$, and we present sufficient conditions on $\mathcal{C}_\alpha$ for the notion $\mathcal{P}_{\mathcal{C}_\alpha}$ to satisfy the standard derivability conditions. Moreover, we also develop a *notion of interpretability* from a given $\mathcal{C}_\alpha$, we call it $\rhd_{\mathcal{C}_\alpha}$, and we study some of its properties.

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On algebraic closedness of the structure of external numbers

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We generalize the properties of algebraic closeness of the real and complex numbers to the structure $\mathbb{E}$ of external numbers of non-standard analysis. Polynomials will be words with two types of elements, regular elements a (also the variable x will be regular) and non-regular elements A. The latter are called neutrices, contain regular elements, are - like zero - stable under addition and negation, but in general do not have the distributive property with respect to the regular elements. As a consequence polynomials cannot always be *structured*, i.e., reduced to sums of monomials. In this context a polynomial equation becomes in fact an inclusion. It cannot always be reduced to zero, still it can be reduced to a neutrix. A *root* is a maximal convex set satisfying such a polynomial inclusion.

We show that a root of a polynomial inclusion over $\mathbb{E}$ must always be an external number. The root satisfies the polynomial inclusion with equality if all coefficients are real. Moreover, $\mathbb{E}$ can be extended to a complex structure $\mathbb{E}+i\mathbb{E}$, on which a complex structured polynomial of standard degree n has n roots (counting multiplicity). The proofs need full Saturation [1][2].

Joint work with João Horta, Universidade de Cabo Verde.

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