Methods of Nonlinear Dynamical Systems Theory

Luis T. Magalhães and Carlos Rocha

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Departamento de Matemática • Instituto Superior Técnico • Lisboa

Chapter 1

Phase spaces and dynamical systems

1.1 Determinism, phase space, group, vector field

Dynamical systems describe deterministic evolution processes with continuous dependence on time and on the initial condition. The idea of determinism involves the consideration of an evolution in a **phase space** or **state space** in such a way that all past and future of the evolution is determined by the present **state** of the system, and the state reached from a given state y after t units of time is the same, no matter the evolution from y developed between instants 0 and t or between instants s and $s+t$ (see Figure 1.1). Determinism can be described precisely by the concept of dynamical system or flow. Let I be either $\mathbb{R}$ or $\mathbb{Z}$ and let X be a topological space. A **dynamical system** or **flow** in X is a continuous function $\varphi : I \times X \to X$ such that

i) $\varphi(0, x) = x, \qquad x \in X$

ii) $\varphi(t + s, x) = \varphi(t, \varphi(s, x)), \qquad x \in X, \qquad t, s \in I$

The variable t in $\varphi(t, x)$ is known as the time variable and the variable x as the state variable. A dynamical system is said to be **finite dimensional** if its **phase space** X is finite dimensional and **infinite dimensional** if that is not the case, it is said to be **continuous** if $I = \mathbb{R}$ and it is said to be **discrete** if $I = \mathbb{Z}$. For a continuous dynamical system it is also usual to write $\varphi_t x = \varphi(t, x)$ or $T(t)x = \varphi(t, x)$ and refer to the family $\varphi_t, t \in \mathbb{R}$ or $T(t), t \in \mathbb{R}$ as a **one-parameter group of transformations** in X or, if the context is clear, a **group** in X. This comes naturally because the map $t \to \varphi_t$

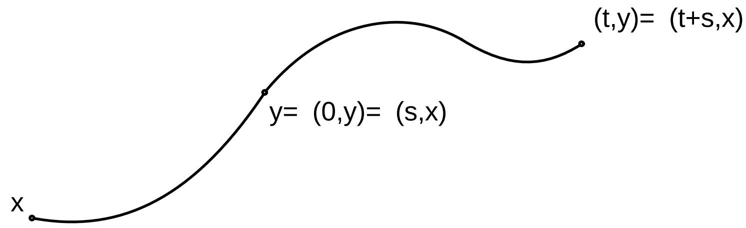

Figure 1.1:

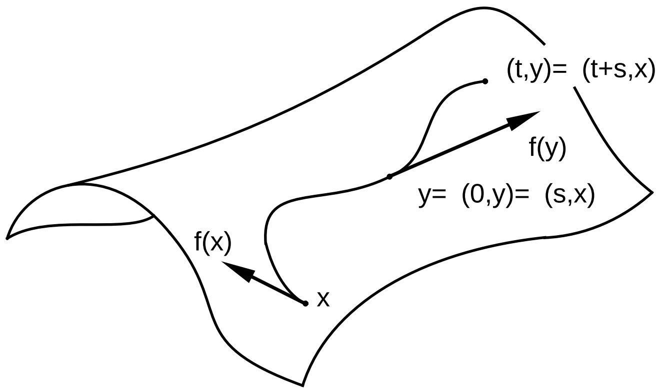

Figure 1.2:

defines a group isomorphism from the real additive group to a subgroup of the group of homeomorphisms (continuous functions with continuous inverse) on X with the composition of functions as operation.

It is clear from the previous discussion that, when modeling an evolution process as a dynamical system, we must consider for the state of the system only information from which all the past and future evolution of the process, for the purposes desired from the model, would be uniquely determined. In any particular instance, there will be infinitely many possible choices for the model phase space and we should choose one that is as simple as possible for the desired purposes; for example, it may be appropriate to take a phase space with minimal dimension.

Sometimes it does not make sense to define the evolution $\varphi(t, x)$ for all $t \in \mathbb{R}$ (e.g, the state may evolve to infinity in finite time), but it is possible for every $x \in X$ to define $\varphi(t, x)$ for t in an interval I_X which is the intersection of I with an open interval containing zero and depending on x; we then say that φ is a **local dynamical system** or a **local flow** in X.

A continuous dynamical system φ is said to be **differentiable** or **smooth** if X is a differential manifold and $\varphi(t, x)$ is differentiable relative to t at $t = 0$; if this is the case and the system is finite dimensional, we call the function $f(x) = \partial\varphi(t,x)/\partial t|_{t=0}$ the **vector field** associated with the dynamical system φ (see Figure 1.2), and the differential equation $\dot{x}(t) = f(x(t))$ or $\dot{x} = f(x)$, where the dot stands for derivative relative to t, is known as the **Ordinary Differential Equation (ODE)** associated with the dynamical system φ.

In certain applications we need to consider processes that are semideterministic, in the sense that the present state determines the future but not the past. This situation may be described by the concept of **semidynamical system** or **semiflow** which amounts to modifying the concept of dynamical system or flow, introduced above, by only requiring $\varphi(t,x)$ to be defined for $t \geq 0$. For a semidynamical continuous system, the family φ_t, $t \geq 0$ or $T(t)$, $t \geq 0$ is called a **one-parameter semigroup of transformations** in X or, if the context is clear, a **semigroup** in X. The other concepts refered above for dynamical systems can be defined for semidynamical systems just by restricting to $t \geq 0$.

Even if a semigroup $T(t)$, $t \geq 0$ in a linear phase space X is not differentiable, it may be useful to consider the function

$$Ax = \lim_{h\to 0^+} \frac{1}{h}[T(h)x - x],$$

defined in the set $D(A)$ where the limit exists. The function A, which takes the role played above by the vector field, is then known as the **infinitesimal generator** of the semigroup $T(t)$, and $D(A)$ is called the **domain of the infinitesimal generator**. This is particularly useful for infinite dimensional systems.

1.2 Systems defined by ODEs in $\mathbb{R}^n$

We saw in the preceding section that a smooth local dynamical system in $\mathbb{R}^n$ defines an ODE. Inversely, under very broad conditions, an ODE $\dot{x} = f(x)$ in $\mathbb{R}^n$ defines a smooth local dynamical system.

An ODE of the type $\dot{x} = f(x)$ is called **autonomous** because the vector field f does not depend explicitly on t. We will also have to study **nonautonomous** ODEs $\dot{x} = f(t,x)$ and their dependence on parameters. Therefore, we consider here some facts from the elementary theory of ODEs of the form $\dot{x} = f(t,x,\lambda)$. It is known that if $f : D \to \mathbb{R}^n$ is continuous in an open set $D \subset \mathbb{R}^{1+n+p}$, with $f(t,x,\lambda)$ locally lipschitzian[1] in x for

[1] i.e., for every compact set $K \subset D$ there exists a constant C_K such that $|f(t,x,\lambda) - f(t,y,\lambda)| \leq C_K|x-y|$, $(t,x,\lambda),(t,y,\lambda) \in K$; note this necessarily holds if the derivative of $f(t,x,\lambda)$ relative to x exists and is continuous in D.

$(t, x, \lambda) \in (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p) \cap D$, then for each $(t_0, x_0, \lambda_0) \in D$ the initial value problem

$$\dot{x} = f(t, x, \lambda_0), \qquad x(t_0) = x_0$$

has a unique solution $x(t) = x(t; t_0, x_0, \lambda_0)$ defined for t in an open interval depending on t_0, x_0, λ_0 and containing t_0; this solution is continuous in t_0, x_0, λ_0 and continuously differentiable in t. Furthermore, if f is C^k then the solution is C^k in t_0, x_0, λ_0 and C^{k+1} in t. Solutions can be extended to maximal intervals of definition $I(t_0, x_0, \lambda_0)$ and, when t approaches one of the endpoints of this interval, either $(t, x(t; t_0, x_0, \lambda_0), \lambda_0)$ approaches the boundary of the domain of definition D or else goes to infinity in norm; in this last case, if the maximal interval of definition is $(-\infty, +\infty)$ we say the solution is **global**, and if one of the endpoints of the interval is finite then necessarily $|x(t; t_0, x_0, \lambda_0)| \to \infty$ as t approaches the endpoint and we say that the solution **blows up** at that point.

Applying the results above to an autonomous ODE $\dot{x} = f(x)$ in $\mathbb{R}^n$, with f locally lipshitzian, it follows that the solutions $x(t; t_0, x_0)$ of initial value problems for this equation define a smooth local dynamical system φ in $\mathbb{R}^n$ by $\varphi(t, y) = x(t; 0, y)$; if, in addition, all the solutions of the equation are global then they define a dynamical system. This shows that, except for the technical assumption of f being lipschitzian, specifying a smooth (local) dynamical system in $\mathbb{R}^n$ amounts to specifying an autonomous ODE in $\mathbb{R}^n$.

1.3 Systems defined by ODEs in differential manifolds

Frequently, in applications, the natural phase space for a system is a differential manifold. This is illustrated here with two examples.

As a first example, let us consider a simple pendulum without friction of length L and mass m, under the action of a constant vertical force (Figure 1.3), modeled with basis on Newton's law of motion by $mL\ddot{x} = F \sin x$. We can take for state variables the angular position x and the angular velocity $y = \dot{x}$ of the pendulum and choose $\mathbb{R}^2$ for phase space, writing the corresponding differential equation as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\tfrac{F}{mL} \sin x. \end{aligned}$$

Clearly, modifying the angle x by a multiple of 2π should not modify the state

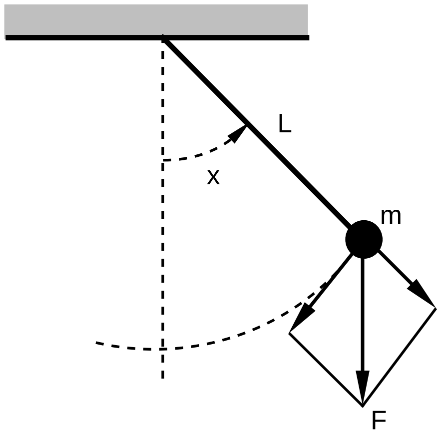

Figure 1.3:

of the pendulum. It is then natural to take for phase space the cylindrical surface $[0, 2\pi) \times \mathbb{R}$, with 0 identified with 2π.

Perhaps a more clear example is that of a Newtonian motion of a rigid body with a point fixed. The position of the body can be specified by the position of an orthonormal frame attached to the body relative to a reference frame. As for the preceding example, we can take as state variables the position of the body and components specifying its velocity. The position can be specified by the orthogonal transformations preserving orientation, i.e., by points of the special orthogonal group $SO(3) = \{A : A \text{ is a } 3 \times 3 \text{ real matrix with } AA^t = I \text{ and } \det A = 1\}$, which is a differential manifold of dimension 3. In this case the natural phase space is $SO(3) \times \mathbb{R}^3$, a differential manifold of dimension 6.

Since differential manifolds are locally euclidean, the concepts and methods of the local theory of dynamical systems in euclidean spaces can be modified to apply to dynamical systems in differential manifolds. We shall keep this in mind, but will restrict our attention to dynamical systems in euclidean phase spaces for simplicity.

1.4 Infinite dimensional systems

Infinite dimensional systems, sometimes called **distributed systems** arise frequently in applications. Particular cases of interest are systems defined by Retarded Functional Differential Equations and by Partial Differential Equations. We shall introduce below these types of systems through examples.

Retarded Functional Differential Equations (FDEs) occur in many situations in applications where one has to consider delays when modeling evolution processes. This happens, in particular, in control theory, transmission systems, viscoelasticity, chemical and nuclear reactions, population dynamics, spread of diseases, biological systems, economics, etc.

Evolution **Partial Differential Equations (PDEs)** provide the fundamental modeling tool for continuum mechanics in areas like fluid dynamics, elasticity, electromagnetism, magnetohydrodynamics, combustion theory, chemical and nuclear reactions, population dynamics, biological systems, etc. They are also important in quantum mechanics, particle physics, economics, and can be used to describe the evolution of certain probabilistic processes. The above areas are central to almost all aspects of Science and Engineering.

Because of its wide area of application, the study of the dynamics of infinite dimensional systems is of great practical importance. It happens at the interface between the theory of dynamical systems which developed from the classical theory of Ordinary Differential Equations and the theory of Partial Differential Equations and Functional Differential Equations on questions of existence and uniqueness of solutions. Due to their own overwhelming technical difficulties, these two fields have mostly grown separately, with the result that the systematic study of the dynamics of infinite dimensional systems began only recently, about the mid 60's, and is currently playing a growing role in mathematics research.

1.4.1 Retarded Functional Differential Equations

Examples of differential equations with delays are

$$\dot{x}(t) = f(x(t), x(t-1), x(t-2))$$
$$\dot{x}(t) = \int_{-1}^{0} f(\theta, x(t+\theta))d\theta.$$

In both cases $\dot{x}(t)$ depends on past values of $x(t)$. In the first equation this dependence is on values at a finite number of past instants of time and in the second equation it is on past values along a whole interval of time; we say that the first equation has **concentrated delays** and the second **distributed delays**. The first type of equations is also known as **differential delay equations** or **differential difference equations** and the second type as **integro-differential equations**.

In order to consider more general dependences on past values, given a function defined on an interval of real numbers $x : I \to \mathbb{R}^n$, we denote its translation of t units of time by x_t , with $x_t(\theta) = x(t+\theta)$; if we want to consider delays in an interval $[-r, 0]$, $x_t(\theta)$ should be taken with $\theta \in [-r, 0]$.

We can then consider an autonomous **Retarded Functional Differential Equation (RFDE)** as an equation of the form

$$\dot{x}(t) = f(x_t).$$

In attempting to choose a phase space for a system defined by solutions of this equation, we note that the values of a solution after the instant of time t depend on the values it assumed in the interval of length r ending at t and, consequently, the phase space should consist of functions defined on $[-r, 0]$ (see Figure 1.4). A simple possibility is to take for phase space the space $X = C([-r, 0], \mathbb{R}^n)$ of the continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ with the uniform norm $||\xi|| = \sup\{|\xi(\theta)| : \theta \in [-r, 0]\}$. We should then consider initial value problems of the form $\dot{x}(t) = f(x_t)$, $x_0 = \xi$. If $f : X \to \mathbb{R}^n$ is continuous and $x(t)$, $t \in [0, b)$, is a solution, it follows from the equation that $\dot{x}(t)$ is continuous on $t \in [0, b)$; consequently, if the initial condition ξ is not continuously differentiable it is impossible to extend x_t for $t < 0$. More dramatically, solutions with different initial conditions may come together and coincide after some time, as it happens for the delay differential equation $\dot{x}(t) = -x(t-1)[1 + x(t)]$ since all solutions with initial condition ξ satisfying $\xi(0) = -1$ are identically -1 for $t > 0$. The two preceding observations indicate that we can only expect that solutions of initial value problems for RFDEs will define a semidynamical system in X. This is indeed the case if $f : X \to \mathbb{R}^n$ is locally lipschitzian. Similarly to what happens for ODEs, it can then be shown that solutions to initial value problems for the equation considered exist for $t > 0$, are unique, depend continuously on $t > 0$ and $\xi \in X$, are continuously differentiable on $t > 0$, and can be extended to maximal intervals of existence; if, in addition, f maps bounded sets into bounded sets and the maximal interval of existence $[0, b)$ is bounded then the solution blows up at $t = b$, i.e., $x(t) \to \infty$ as $t \to b^-$. This implies that the solutions $x(t; \xi)$ of initial value problems at $t_0 = 0$ for this equation define a smooth local semidynamical system φ in X by $\varphi(t, \xi) = x_t(\xi)$, $t \geq 0$. The infinitesimal generator of the associated semigroup $T(t)\xi = x_t(\xi)$, $t \geq 0$, is

$$(A\xi)(\theta) = \lim_{h \to 0^+} \frac{1}{h} \left[T(h)\xi - \xi\right](\theta) = \begin{cases} f(\xi) & ,\theta = 0 \\ \xi'(\theta) & ,\theta < 0 \end{cases}$$

for $\xi \in D(A) = \{\xi \in C^1([-r, 0], \mathbb{R}^n) : \xi'(0) = f(\xi)\}$, which is a dense set in X with empty interior.

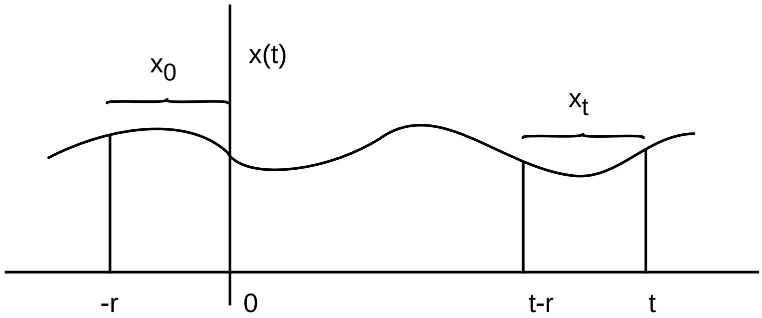

Figure 1.4:

1.4.2 Partial Differential Equations

As the general theory of PDEs is quite complicated we give two simple examples.

Let us first consider a scalar equation obtained by subtracting to the wave equation a nonlinear term corresponding to a reactive process:

$$u_{tt} = u_{xx} - f(u), \qquad 0 < x < 1,$$

with $u(t,x)$ satisfying Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$ and $f : \mathbb{R} \to \mathbb{R}$ in C^2 satisfying $f(u)/u > 0$ for $u \neq 0$. Similarly to what was done above for the pendulum, we can write this equation as a system

$$u_t = v$$

$$v_t = u_{xx} - f(u).$$

It is clear that points in the phase space must be pairs of functions $\xi = (u,v)$ on the interval $[0,1]$ such that $u(t,0) = u(t,1) = 0$. One possibility, would be to try to use pairs of C^2 functions on $[0,1]$ with the first element vanishing at 0 and 1, a space we denote by $C_0^2 \times C^2$, taken with the uniform C^2 norm, $||\xi|| = \sup\{|\xi(x)|, |\xi'(x)|, |\xi''(x)| : x \in [0,1]\}$. We should then consider initial value problems for the equation with initial condition $(u(0,x), v(0,x)) = \xi(x)$, with $\xi \in C_0^2 \times C^2$. Although the solutions to initial value problems for this equation can be shown to define a local dynamical system in $C_0^2 \times C^2$, this will not happen for x in higher dimensional spaces. A natural approach is to use a phase space based on the energy associated with the equation, given by

$$E(u,v) = \int_0^1 \left[\frac{1}{2}v^2 + \frac{1}{2}(u_x)^2 + F(u)\right] dx,$$

where $F' = f$. It is easy to verify that this quantity is conserved along solutions, by taking its derivative relative to t and using the equation and an integration by parts. This suggests taking as points in the phase space pairs (u, v) such that v is a square integrable function in $I = [0, 1]$ and u is a function with square integrable derivative in the same interval and vanishing at 0 and 1; these spaces are usually denoted by L^2 and H_0^1 when one uses Lebesgue integrals and the norms $||v||_{L^2} = [\int_I v^2]^{\frac{1}{2}}$ and $||u||_{H_0^1} = \max\ \{||u||_{L^2}, ||u_x||_{L^2}\}$, and they are, respectively, the closure in the corresponding norm of the set of continuous functions and the set of continuously differentiable functions vanishing at 0 and 1. We denote this phase space by $X = H_0^1 \times L^2$. It can be shown that solutions to initial value problems for the equation considered do, indeed, define a local dynamical system in X. The infinitesimal generator of the associated group $T(t)$ is

$$A \begin{pmatrix} u \\ v \end{pmatrix} = \lim_{h \to 0^+} \frac{1}{h} \left[T(h) \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} \right] = \begin{pmatrix} u_{xx} \\ v - f(u) \end{pmatrix}.$$

for $(u, v) \in D(A) = \{(u, v) \in X : u_{xx} \in L^2, v \in H_0^1\}$, which is a dense set in X with empty interior. It is sometimes useful to consider the effect of a linear damping term added to the previous equation modifying it to

$$u_{tt} + \alpha u_t = u_{xx} - f(u), \qquad 0 < x < 1,$$

where $\alpha > 0$. In this case, computing the time derivative of the energy along solutions we obtain

$$\frac{d}{dt} E(u, v) = \frac{d}{dt} \int_0^1 \left[\frac{1}{2} v^2 + \frac{1}{2} (u_x)^2 + F(u) \right] dx = -\alpha \int_0^1 v^2 dx.$$

The time derivative of the energy is negative along solutions and, therefore, energy is dissipated. As in the preceding example, we can take for phase space $X = H_0^1 \times L^2$ and it can be shown that solutions to initial value problems for the equation considered also define a local dynamical system in X.

We now consider a scalar reaction-diffusion equation, obtained by adding to the linear heat equation, which by itself describes a diffusion process, a nonlinear term corresponding to a reactive process:

$$u_t = u_{xx} - f(u), \quad 0 < x < 1,$$

with $u(t, x)$ satisfying Dirichlet boundary conditions $u(t, 0) = u(t, 1) = 0$ and $f : \mathbb{R} \to \mathbb{R}$ being C^2 and satisfying $f(u)/u > 0$ for $|u|$ large. Reaction-diffusion equations are used as models for processes occurring in chemical and nuclear reactions, combustion, population dynamics, genetics, etc. Similarly to the energy function considered in the preceding example, we define the function

$$E(u) = \int_0^1 \left[\frac{1}{2}(u_x)^2 + F(u)\right] dx,$$

where $F' = f$. Its time derivative along solutions satisfies

$$\frac{d}{dt}E(u) = \frac{d}{dt}\int_0^1 \left[\frac{1}{2}(u_x)^2 + F(u)\right] dx = -\int_0^1 (u_t)^2 dx.$$

This derivative is negative for all time and, therefore, the function $E(u)$ decreases. A natural phase space in this case is $X = H_0^1$. Similarly to what happens for RFDEs, we should not expect to determine past evolution from a given initial state; as a matter of fact, like for the heat equation $u_t = u_{xx}$, initial conditions are smoothed instantaneously due to the diffusion effect, in the sense that they become C^∞ for $t > 0$ even if they are not smooth at $t = 0$. We can only hope for solutions to initial value problems in X to define a semidynamical system. It can be shown that this is, indeed, the case, and we obtain a semigroup $T(t)$ defined for all $t \geq 0$. Its infinitesimal generator is given by

$$Au = \lim_{h \to 0^+} \frac{1}{h}[T(h)u - u] = u_{xx} - f(u),$$

for $u \in D(A) = \{u \in X : u_{xx} \in L^2\}$, which is a dense set in X with empty interior.

Dissipative infinite dimensional systems like those associated with RFDEs with f bounded, and with the above reaction-diffusion equations, have in common the property of defining a semigroup $T(t)$, $t \geq 0$, that compactifies, in the sense that it maps closed bounded sets of the phase space into compact sets (for all $t > 0$ in the case of reaction-diffusion equations, and for all $t \geq r$ in the case of RFDEs). The damped nonlinear wave equation considered above also compactifies, but only in the limit $t \to +\infty$. The asymptotic behavior of such dissipative systems, as $t \to +\infty$, and therefore their stationary dynamics, can be studied by attempting to understand the evolution they define in a suitable finite dimensional set. Therefore, much of the methods used for the study of dynamical systems defined in finite dimensional spaces can be used in studying a large class of infinite dimensional dissipative systems. We will restrict our attention to finite dimensional systems, but should keep in mind that their study is useful for understanding the dynamics in the infinite dimensional case.

1.5 Discrete systems

Frequently in the study of continuous systems it is useful to consider associated discrete systems. This happens, for instance, when a continuous system is discretized in order to compute numerically its solution by a procedure that can be programmed in a digital computer. For example, if we want to solve numerically an initial value problem for an ODE in $\mathbb{R}^n$, $\dot{x} = f(x)$, $x(t_0) = x_0$ with the Euler method, using interval steps of length Δ, we obtain the recurrence relation

$$x_{i+1} = x_i + \Delta f(x_i)$$

which is to be solved successively for x_1, x_2, x_3,... beginning from x_0, and we expect that this sequence of values is a good approximation for the values of the solution $x(1\Delta)$, $x(2\Delta)$, $x(3\Delta)$,..., provided Δ is sufficiently small. Clearly, the recurrence relation above defines a local semidynamical discrete system in $\mathbb{R}^n$. As another example, when studying a nonautonomous ODE in $\mathbb{R}^n$ $\dot{x} = f(t,x)$ with $f(t,x)$ periodic and C^1 in t, say $f(t+T,x) = f(t,x)$ with $T > 0$, it is frequently useful to consider a local discrete dynamical system φ in $\mathbb{R}^n$ taking only the evolution of the values of solutions of the ODE sampled after time intervals of length T, as $x_i = \varphi(i,x_0) = x(iT;0,x_0)$, where $x(t;t_0,x_0)$ denotes de value at t of the solution of the ODE which satisfies the initial condition $x(t_0) = x_0$; T-periodic solutions of the ODE are just fixed points of the discrete system and the identification of the solutions of the ODE approaching, as $t \to +\infty$, a certain T-periodic solution amounts to the identification of initial conditions x_0 for the associated discrete system such that $\varphi(i,x_0)$ converges, as $t \to +\infty$,to the fixed point of the discrete system corresponding to the periodic solution of the ODE considered.

Of course, discrete systems are also of interest by themselves and not only as discretizations of continuous systems. In fact, it is sometimes appropriate to model particular situations of various areas of application directly in terms of discrete systems (a simple example that matters: interest rates in bank deposits are computed just once daily and not continuously in time). Furthermore, as mathematical models for many situations are to be simulated in digital computers, it has been advocated that those situations should be directly modeled by discrete systems, whenever that is reasonable.

As a specific illustration, we take the application of the Euler method to the scalar ODE $\dot{x} = x - x^2$. It is clear that this equation has two constant solutions, $x = 0$ and $x = 1$, and that all other solutions are monotonous and converge, as $t \to +\infty$, to $-\infty$ if the initial condition is negative and to 1 if the initial condition is positive. The recurrence relation corresponding to the application of the Euler method is $x_{i+1} = x_i + \Delta f(x_i) = (1+\Delta)x_i - \Delta x_i^2$.

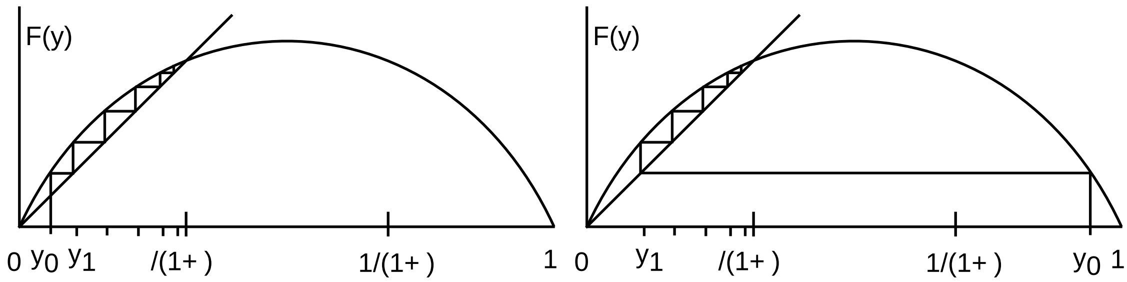

Figure 1.5:

There are two points that are mapped to the origin, namely $x = 0$ and $x = (1+\Delta)/\Delta$. It is convenient to change variables by setting $x = y(1+\Delta)/\Delta$, so that in the new variable these points do not depend on the time step Δ. That leads to the recurrence relation $y_{i+1} = F(y_i)$, with $F(y) = (1+\Delta)(y - y^2)$. Now the points mapped to the origin are $y = 0$ and $y = 1$. The evolution from any initial condition y_0 is given by successive compositions of the function F with itself $y_i = F^i(y_0)$ and can be very easily described geometrically, as shown in Figure 1.5.

If we take a small fixed time step $\Delta > 0$ and observe the evolution of the discrete system beginning at different initial conditions, we discover that there are two fixed points, $y = 0$ and $y = \Delta/(1+\Delta)$ (in the original variable: $x = 0$ and $x = 1$), and that the evolutions beginning at initial conditions $y_0 \in (0,1)$ all approach this last fixed point, monotonically if $y_0 \leq 1/(1+\Delta)$ and nonmonotonically if $y_0 > 1/(1+\Delta)$, while the evolutions beginning at initial conditions outside the interval $[0,1]$ all go to $-\infty$ (see Figure 1.6). This contrasts with the observations that all solutions of the ODE considered are monotonous and that all solutions with initial condition larger than 1 converge to 1 , as $t \to +\infty$.

It is also interesting to study what happens as the time step increases. Now, we will be interested only in initial conditions $y \in [0,1]$ and, since the maximum value of F is $(1+\Delta)/4$, it follows that the successive values assumed throughout the evolution remain in the interval $[0,1]$, provided $\Delta \leq 3$. For these values of Δ, $y_i = F_i(y_0)$ defines a semidynamical system; in the interval $X = [0,1]$. It is not difficult to see that the evolution described above for the discrete system is similar for all $\Delta \in (0, \Delta_1)$, with $\Delta_1 = 2$, in particular evolutions with initial conditions different from the two fixed points, $y = 0$ and $y = \Delta/(1+\Delta)$ approach the latter. However, for $\Delta > \Delta_1$ that is no longer true: none of the evolutions beginning at points different from the

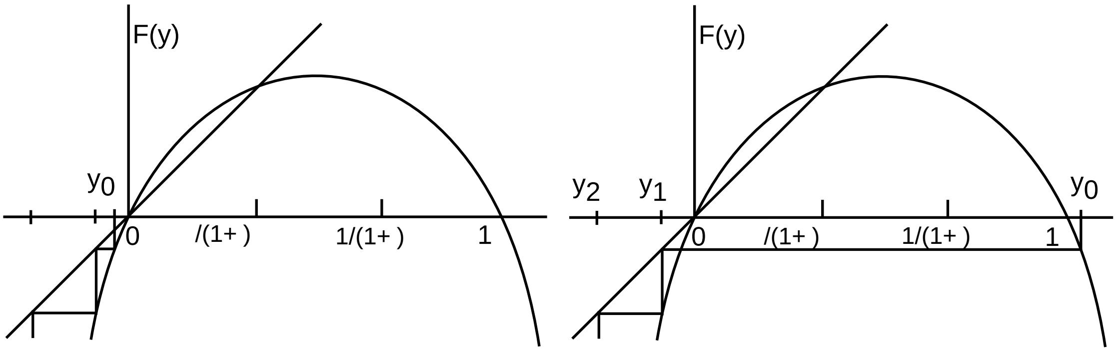

Figure 1.6:

fixed points approach one of them. For $\Delta \in (\Delta_1, \Delta_2)$, with $\Delta_2 \approx 2.45$, such evolutions approach, instead, an evolution that alternates between two different points which, for that reason, is said to be periodic of period 2 (see Figure 1.7); notice that evolutions of period 2 alternate between fixed points of F^2 and, similarly, evolutions of period k alternate between fixed points of F^k.

It is possible to continue this analysis and identify an increasing sequence of numbers Δ_k such that for $\Delta \in (\Delta_k, \Delta_{k+1})$ all evolutions beginning at points in $(0, 1)$ outside some finite set approach a periodic evolution of period 2^k (see Figure 1.8 for $k = 2$); the periodic evolutions of periods 2^i $(i = 1, 2, ..., k-1)$ that appeared for smaller values of Δ are also present as well as the two fixed points, but instead of attracting points on the nearby they repell them. Furthermore, the sequence $\{\Delta_k\}$ is bounded and approaches $\Delta_\infty \approx 2.57$. One can find values of Δ larger and arbitrarily close to Δ_∞ such that the infinite number of periodic evolutions with arbitrarily long periods which appeared for lower values of Δ and became repelling influence the evolutions beginning at almost any point in the interval to be highly irregular and strongly sensitive to changes in initial conditions, rendering unpractical to predict the future evolution from an initial point that is known only approximately.

The preceding illustration shows that a discretization of a given continuous dynamical system may define a discrete dynamical system whose dynamics can be much different from that of the given system and, therefore, must be studied on its own in order to understand its behavior and, eventually, to decide what it means regarding the original system. In particular, the study of the dynamics of numerical schemes is important.

If you consider the above example too artificial because, after all, the

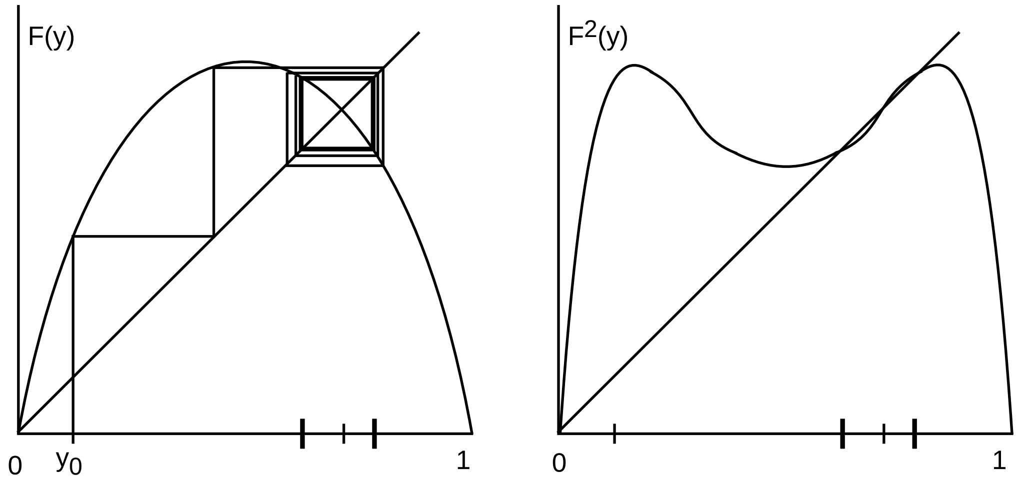

Figure 1.7:

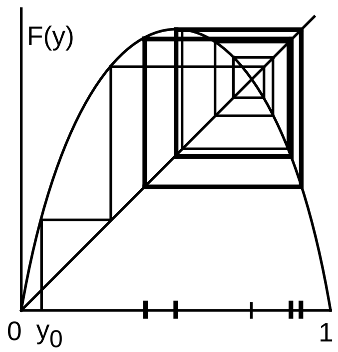

Figure 1.8:

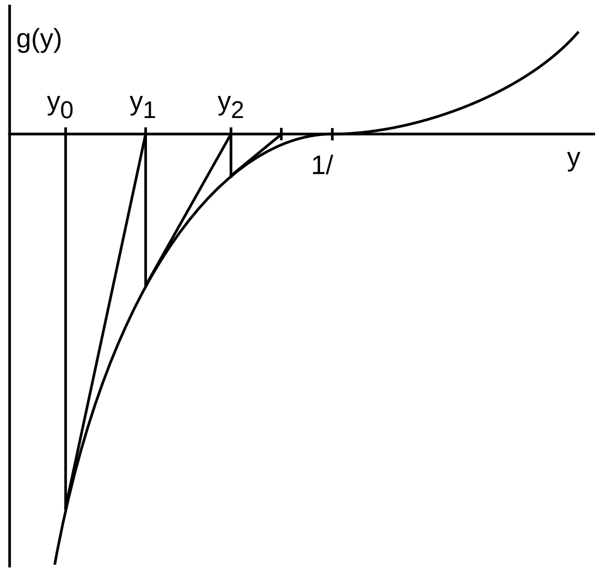

Figure 1.9:

Euler method should be applied with Δ sufficiently small, you may prefer to consider the application of the method of Newton (see Figure 1.9) for computing numerically the zeros of the function $g(y) = [1-1/(\lambda y)]^{\lambda-1}$. That amounts to considering the recurrence relation $y_{i+1} = y_i - f(y_i)/f'(y_i) = \lambda/(\lambda-1)(y_i - y_i^2)$ which is the same as that considered above with $\Delta = 1/(\lambda-1)$.

In the illustration with the Euler method presented above, for some values of Δ larger but arbitrarily close to Δ_∞ the evolution is highly irregular and up to a certain extent undistinguishable from random fluctuations. For this reason, it has been called chaotic. The occurrence of chaos in deterministic systems has received a lot of attention, specially in the past 15 years. It is a topic attracting the interest of many researchers in different fields of science, in part because of its implications for modeling (e.g., probabilistic v.s. deterministic models). It is remarkable that such a complicated behavior can occur for such simple dynamical systems as those defined in a real interval by the iteration of a continuous map. Not surprisingly, part of the research on discrete dynamical systems has been addressed to systems defined by **interval maps**. Dynamical systems defined by maps in the real plane or by analytic maps in the complex plane are also being extensively studied; the study of the latter is a fascinating topic known as **analytical dynamics**.

In the following chapters we concentrate on continuous systems and use discrete systems only occasionally. This is not an excessive particularization, since most of the methods and results presented can be readily adapted to apply to discrete systems.

1.6 Systems defined by nonautonomous equations

We consider here only nonautonomous ODEs in $\mathbb{R}^n$ since the situation is similar for equations defined in other spaces. The solutions to initial value problems for a nonautonomous ODE in $\mathbb{R}^n$

$$\dot{x} = f(t, x)$$

with $f(t, x)$ bounded and C^1 in $\mathbb{R} \times \mathbb{R}^n$, do not define dynamical systems in $\mathbb{R}^n$ because a solution is determined not only by its initial value in $\mathbb{R}^n$ but also by the initial instant of time. There are, however, several ways of associating a dynamical system to a nonautonomous equation.

We have already encountered one of them in the previous section when it was shown how we can associate a discrete dynamical system to a periodic ODE, say $f(t+T, x) = f(t, x)$ with $T > 0$. In fact, taking only the evolution of the values of solutions of the ODE sampled after time intervals of length T, we obtain a local discrete dynamical system φ in $\mathbb{R}^n$ in the form $\varphi(i, y) = x(iT; 0, y, f)$, where $x(iT; s, y, f)$ denotes the value at t of the solution of the ODE which satisfies the initial condition $x(s) = y$.

Another simple possibility, which holds even if the ODE is not periodic, is to add time to the state variables, defining a local dynamical system φ with phase space $\mathbb{R}^n \times \mathbb{R}$ by $\varphi(t, (y, s)) = (x(t; s, y, f), t)$. This describes in the phase space the simultaneous evolution of points in $\mathbb{R}^n$ and of time. It amounts to consider the local dynamical system associated with the autonomous ODE in $\mathbb{R}^n \times \mathbb{R}$

$$\begin{aligned} \dot{x} &= f(t, x) \\ \dot{t} &= 1. \end{aligned}$$

The last possiblity has a serious disadvantage for the study of certain asymptotic properties of the evolution, as $t \to \pm\infty$. In fact, even if a solution in $\mathbb{R}^n$ of the original nonautonomous ODE is bounded, the corresponding solution in $\mathbb{R}^n \times \mathbb{R}$ of the associated autonomous system is unbounded. Accordingly, the closure of the set of values assumed by solutions of the nonautonomous equation may be a compact set, while at the same time the closure of the set of values of the corresponding solution of the associated autonomous system cannot be compact, preventing the use of methods based on compactness for the analysis of asymptotic behavior. In order to avoid this drawback, we observe that, for the determination of the future and past evolution of solutions, it is not important to know the specific instants of time throughout the evolution but only the values of the function f. We could, therefore, think of considering a dynamical system which accounts for the simultaneous evolution through time of points in $\mathbb{R}^n$ and of the function f. The evolution of f is just a translation in time that can be easily expressed in terms of the notation $f_t(s,x) = f(s+t,x)$. We are led to consider a local dynamical system φ such that $\varphi(t,(y,g)) = (x(t;0,y,g), g_t)$. Of course, before we can refer to φ as a dynamical system we have to specify the phase space and its topology. One possibility would be to consider the set of bounded C^1 functions in $\mathbb{R}^n$ with the topology corresponding to uniform convergence (known as the compact open topology), define the set $H(f)$ as the closure in this topology of the set $\{f_t : t \in \mathbb{R}\}$ of the translates of f, and take for phase space $\mathbb{R}^n \times H(f)$. It can be easily shown that $\varphi(t,g) = g_t$ defines a dynamical system in $H(f)$ and that the function φ described above defines a local dynamical system in $\mathbb{R}^n \times H(f)$. Furthermore, since $H(f)$ is a compact set in the topology considered, if a solution of the given nonautonomous equation is bounded not only the corresponding evolution in $\mathbb{R}^n \times H(f)$ is bounded but its closure is compact, bypassing the difficulty previously encountered. A flow defined in a product space $X = A \times B$, as φ above, in the form $\varphi = (\sigma, \psi)$, where ψ is a flow in B is called a **skew-product flow**.

1.7 Control systems

Control systems are important in many areas of technology and science, such as machine-tool control, automated assembly lines, robotics, manipulators control, quality control of production, transportation distribution and power systems, aeroplanes and space vehicles, chemical and combustion processes, biology, economics, etc. They may be continuous or discrete, finite or infinite dimensional, defined in a linear space or in a manifold, etc. For simplicity, we only introduce here continuous systems defined by ODEs in $\mathbb{R}^n$.

The distinctive feature of control systems is that their dynamics can be changed in the course of time through a control function called the **input** of the system. When they are defined by an ODE in $\mathbb{R}^n$ and the input has values in $\mathbb{R}^m$ control systems have the form

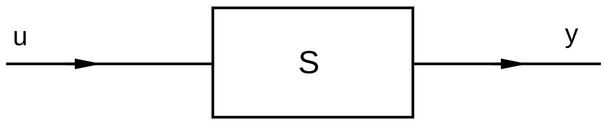

Figure 1.10:

$$\dot{x}(t) = f(t, x(t), u(t)),$$

where u is a function of real variable with values in $\mathbb{R}^m$. In control system theory, it is usual to refer to the value $x(t)$ as the **state** of the system at time t, and it is also usual to consider an **output** of the system which may depend on the state and the input in the form

$$y(t) = h(t, x(t), u(t)).$$

A system S of this type is frequently represented in diagrams as in Figure 1.10. The system is said to be **time-invariant** if $f(t, x, u)$ and $h(t, x, u)$ are independent of t.

In the most simple situation one wants to build from scratch a control system to achieve a certain input-output behavior, but frequently control systems are designed in order to change the input-output behavior of a given system S which cannot be altered internally. In this case, one wants to specify a system C, called the controller, which uses an external command signal v and the output of S to generate an adequate input u for S (see Figure 1.11). Often, the inputs u and v are restricted to lie in certain prescribed sets. As simple examples of control design objectives we mention guaranteeing that: bounded inputs v will result in bounded outputs y (stabilization), the output y will follow the input command v in the sense of minimizing an appropriate distance between the two (command following), the effect of external disturbances on the input-output behavior of the system will be minimized (disturbance rejection), the output will be driven to a target value in minimal time (time-optimal steering), a performance functional like energy consumption associated with the execution of a certain task will be minimized (optimal control). One should notice that in a practical situation, the system S should be viewed as a model for a physical process whose parameters have to be estimated; therefore, control analysis and design must be preceded by modeling and parameter identification.

In order to illustrate how a dynamical system can be associated to a control system we consider, for simplicity, time invariant systems of the form

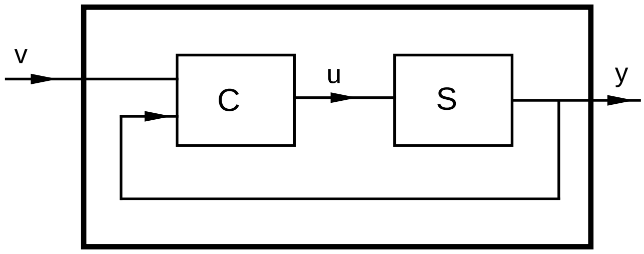

Figure 1.11:

$$\dot{x} = f(x, u),$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1 and $u : \mathbb{R} \to \mathbb{R}^n$ is bounded and C^1. The solutions $x(t; t_0, x_0, u)$ of initial value problems $x(t_0) = x_0$ for this equation, with u fixed, do not define a dynamical system in $\mathbb{R}^n$ because a solution is determined not only by its initial value in $\mathbb{R}^n$ but also by the values taken by the input u along time. However, one can associate dynamical systems to such a control system in similar ways as done above for nonautonomous ODEs in $\mathbb{R}^n$. In particular, we could add time to the state variables, considering a local dynamical system φ in a phase space $\mathbb{R}^n \times \mathbb{R}$ defined by $\varphi(t, (y, s)) = (x(t; s, y, u), t)$. Alternatively, we could use a skew-product flow accounting for the simultaneous evolution through time of points in $\mathbb{R}^n$ and of the input function u as $\varphi(t, (y, u)) = (x(t; 0, y, u), u_t)$ which is a local dynamical system in the phase space $\mathbb{R}^n \times H(u)$. Here $H(u)$ is taken with the compact open topology and is defined as the closure in this topology of the set $\{u_t : t \in \mathbb{R}\}$ of the translates of u. Similarly to what happens for nonautomomous ODEs in $\mathbb{R}^n$, the second approach enables the use of methods based on compactness for the analysis of the asymptotic behavior of solutions of the equation.

Chapter 2

Geometry of the phase space

2.1 Orbits, phase portrait

In the following let φ denote a local dynamical system defined on a topological space X. For each $x \in X$ the set $\gamma_x = \cup_{t\in I_x}\{\varphi(t,x)\}$ is called the **orbit** of φ passing through x. It is a point or a curve in the phase space X, the first case occurring if and only if $x = \varphi(t,x)$. We also define the **trajectory** corresponding to this orbit as the set $\cup_{t\in I_x}\{(t,\varphi(t,x))\}$ which is a subset of $I\times X$. The set of orbits of φ, oriented in the direction corresponding to increasing time, is called the **phase portrait**.

To exemplify these concepts we consider a mechanical system described by the following second order differential equation

$$\ddot{x} - x(x-a)(x-1) = 0, \quad 0 < a < 1/2 \ .$$

This equation describes the one dimensional motion of a body with unit mass under the action of the nonlinear force $g(x) = x(x-a)(x-1)$ without friction. As before, we take for state variables the position x and the velocity $y = \dot{x}$ and choose $\mathbb{R}^2$ for phase space. The mechanical energy for this system is given by

$$E(x,y) = \frac{1}{2}y^2 + G(x),$$

where the potential $G(x) = -\int_0^x g(s)ds$ is given by $G(x) = -x^2[3x^2 - 4(1+a)x + 6a]/12$. It is easy to check that $\frac{d}{dt}E(x(t),\dot{x}(t)) = 0$ and, therefore, each orbit of the system is contained in one level curve of E. Conversely, each level curve of E is entirely composed of orbits of the system, thus by tracing these level curves one obtains the phase portait of the system (see Figure 2.1). When, as in this case, there exists a C^1 function defined on the phase

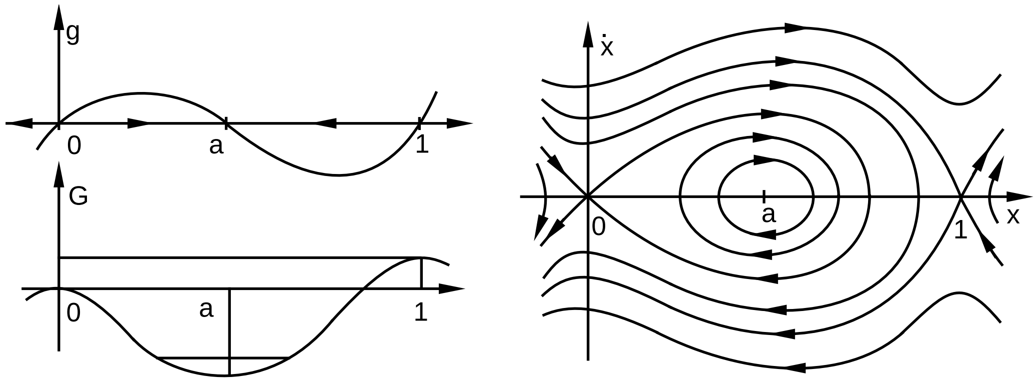

Figure 2.1:

space, not constant on any open set but constant on orbits, the system is said to be **conservative**. In the case of a differential equation any such function is called an **integral** of the equation.

The phase portrait of the preceding example has three different kinds of orbits: (a) orbits consisting of only one point (those corresponding to the constant solutions with values 0, a or 1); (b) orbits homeomorphic to a circle (all those in the interior of the teardrop shaped region excluding the point $(a, 0)$); and (c) orbits homeomorphic to $\mathbb{R}$ (all other orbits). In fact, these are the only possibilities as asserted by the following proposition.

(2.1) **Proposition:** *For $x \in X$ let I_x denote the interval of definition of the map $\Phi_x(t) = \varphi(t, x)$. Then, one of the following alternatives holds:*

(i) $I_x = I$ and Φ_x is constant;

(ii) $I_x = I$ and Φ_x is periodic and not constant;

(iii) Φ_x is one-to-one on I_x.

Proof. To prove this proposition we start by noticing that the evolution Φ_x is either one-to-one on I_x or else there exist r, s in I_x such that $r \neq s$ and $\Phi_x(r) = \Phi_x(s)$. In the second case we immediately conclude that $I_x = I$, and taking $c = r - s \neq 0$ we have $\Phi_x(t + c) = \Phi_x(t)$ for all $t \in I$. If the relation holds for a sequence $\{c_k\}$ and $c_k \to c$ then

$$\Phi_x(t + c) = \Phi_x(t + \lim_{k\to\infty} c_k) = \lim_{k\to\infty} \Phi_x(t + c_k) = \Phi_x(t),$$

proving that the set of constants $c \in I$ for which the preceding relation holds is a closed subset of I. Let $T = \inf |c|$ for $c \neq 0$ in this set. If $I = \mathbb{R}$ then either $T > 0$ and Φ_x is periodic of minimum period T, or else $T = 0$ and Φ_x is constant. An analogous conclusion is reached if $I = \mathbb{Z}$ in which case $T \geq 1$.

QED

Points in the phase space for which the first alternative in the proposition holds (the corresponding orbit is the point itself) are called **critical point** or **equilibrium points** of the dynamical system and noncritical points are called **regular points**. In the case of orbits homeomorphic to a circle, these are said to be **closed** or **periodic orbits**.

In the above example we also notice the existence of an orbit which together with the critical point $(0,0)$ forms the boundary of the teardrop shaped region separating the periodic orbits from all the other nonperiodic nonconstant orbits. Orbits like this one, connecting a critical point to itself, are called **homoclinic orbits**, and orbits connecting different critical points are called **heteroclinic orbits**.

An orbit γ_x is said to be **global** if $I_x = I$. Clearly, equilibrium points and periodic orbits are always global orbits but there may be others. For instance, in the example just considered the global orbits are those in the closure of the teardrop and the orbit consisting of the point $(1,0)$. In fact, it is very simple to prove that all other orbits exhibit blow up in finite time. To see this, let $x = x(t)$ denote a solution of the equation satisfying $x(t_0) = x_0$ and $x(t_1) = x_1$ where $t_0 < t_1$. Then, considering again the integral E, we can compute the time $t_1 - t_0$ in terms of x_0 and x_1 by performing a quadrature. To exemplify, we assume that the solution satisfies $x(t) > 0$ for $t_0 < t < t_1$ and let $E_0 \equiv E(x(t_0), \dot{x}(t_0))$. As $E(x(t), \dot{x}(t)) = E_0$ we have $(\dot{x}(t))^2 = 2[E_0) - G(x(t))]$ and, consequently,

$$t_1 - t_0 = \int_{x_0}^{x_1} \left(\frac{dx}{dt}\right)^{-1} dx = \int_{x_0}^{x_1} \frac{dx}{\sqrt{2[E_0 - G(x)]}}.$$

If the solution $x = x(t)$ is unbounded, then we can take the limit $x_1 \to \infty$ and since G is a fourth order polynomial in x we conclude that the time ellapsed for the orbit through x_0 to reach infinity is

$$\tau \equiv \int_{x_0}^{\infty} \frac{dx}{\sqrt{2[E_0 - G(x)]}} < \infty$$

therefore the solution blows up at time $t_1 = t_0 + \tau$. This is a simple application of **Wintner's method** for studying the blow up of solutions.

Notice that for a dynamical system φ all orbits are global since φ is defined on $I \times X$.

For a particular differential equation it may be of great interest to determine which solutions are global and which blow up. However, if we are interested in studying a local dynamical system defined by a differential equation only in a compact set of the phase space, we can always find a differential equation with all solutions global and defining a dynamical system whose orbits restricted to the compact set considered coincide, in this set, with the orbits of the original system.

(2.2) **Proposition:** *If $K \subset \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}^n$ is continuous, then there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that all solutions of $\dot{x} = g(x)$ are global and their restrictions to intervals of time for which they lie entirely in K agree with solutions of $\dot{x} = f(x)$.*

Proof. Let $(f_1, ..., f_n)$ denote an arbitrary continuous extension of the given f to $\mathbb{R}^n$ and $M = \sup\{|f_i(x)| : x \in K, i = 1, \ldots, n\}$. Define $g = (g_1, \ldots g_n)$ in $\mathbb{R}^n$ by

$$g_i(x) = \begin{cases} f_i(x) & , \quad \text{if } |f_i(x)| \leq M \\ M & , \quad \text{if } f_i(x) > M \\ -M & , \quad \text{if } f_i(x) < -M \end{cases}$$

Since $|g_i| \leq M$, the components of any solution of $\dot{x} = g(x)$ satisfy $|x_i(t) - x_i(t_0)| \leq M\,|t - t_0|$ for t, t_0 in its interval of definition; consequently solutions cannot blow up and all solutions are global. As $g = f$ on K the relationship between the solutions of the two equations follows.

QED

It is convenient to consider here more examples of phase portraits of local dynamical systems. Clearly, in the study of qualitative properties it is not interesting to distinguish situations that differ only by smooth deformations of the phase space that can be seen as possibly nonlinear changes of coordinates. For example, it is appropriate to consider equivalent the dynamical systems defined by the differential equations

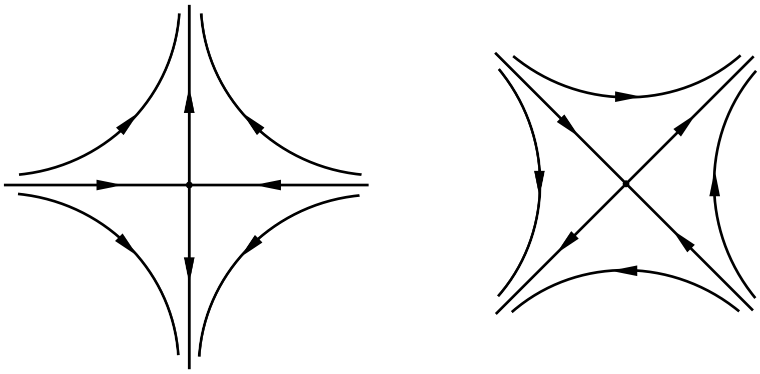

Figure 2.2:

$$\begin{cases} \dot{x} &= -2x \\ \dot{y} &= +3y \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{x} &= y \\ \dot{y} &= x \end{cases}$$

whose phase portraits are sketched in Figure 2.2. More generally, we say that two dynamical systems or two vector fields are **equivalent** in subsets S_1 and S_2 of their phase spaces if there exists a homeomorphism between these sets which maps orbits of one of the systems onto orbits of the other and preserves the direction of time. In the following examples, we consider families of differential equations depending on parameters and identify the phase portrait of each one of their equivalence classes. In particular, it is observed that for some equations arbitrary small changes in the parameters give systems in the same equivalence class while for others arbitrarily small changes in the parameters can give systems in different equivalence classes. The identification of these two situations plays an important role in the qualitative theory of differential equations; in the first case, we say the system is **structurally stable** and in the second we say it is a **bifurcation**.

(2.3) **Examples:**

1. We consider the scalar differential equation $\dot{x} = \lambda x$ for $\lambda \in \mathbb{R}$. The zero function is always a solution and, consequently, $x = 0$ is always an equilibrium point. When $\lambda \neq 0$ the other solutions of this equation are exponentials which decrease for $\lambda < 0$ and increase for $\lambda > 0$; all the systems corresponding to $\lambda < 0$ are equivalent, as well as all those corresponding

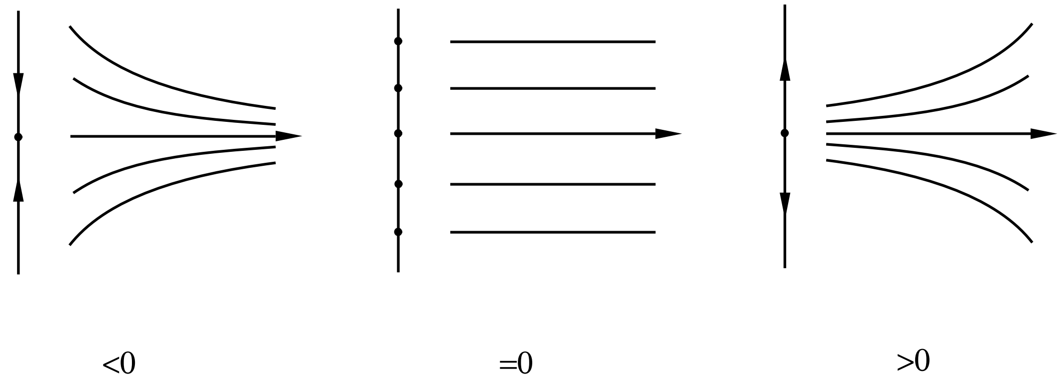

Figure 2.3:

to $\lambda > 0$, and we have two different equivalence classes. When $\lambda = 0$ all solutions are constant and, therefore, all points in the phase space are equilibria. The trajectories and the phase portraits are sketched in Figure 2.3. The system corresponding to $\lambda = 0$ is a bifurcation point. As λ is decreased through zero the orbits approaching the equilibrium, become stationary and then pull apart from the equilibrium.

2. We consider the scalar differential equation $\dot{x} = -(x^2 - \lambda)$ for $\lambda \in \mathbb{R}$. For $\lambda < 0$ there are no equilibria, for $\lambda = 0$ the only equilibrium point is $x = 0$ and for $\lambda > 0$ there are two equilibria $x = \pm\sqrt{\lambda}$. Analysing the sign of the right hand side of the equation, we conclude that for $\lambda < 0$ all solutions decrease with time, for $\lambda = 0$ all solutions except the zero solution decrease with time and for $\lambda > 0$ the solutions with values below $-\sqrt{\lambda}$ or above $+\sqrt{\lambda}$ decrease with time while those with values between $-\sqrt{\lambda}$ and $+\sqrt{\lambda}$ increase. Again, there are three equivalence classes of the given systems: for $\lambda < 0$, for $\lambda = 0$, and for $\lambda > 0$. The trajectories and the phase portraits are sketched in Figure 2.4. The system corresponding to $\lambda = 0$ is a bifurcation point. As λ is decreased through zero the two equilibria collide at $\lambda = 0$ an then disapear for negative λ.

3. We consider the scalar differential equation $\dot{x} = -x(x^2 - \lambda)$ for $\lambda \in \mathbb{R}$. For all $\lambda \in \mathbb{R}$ the point $x = 0$ is an equilibrium, for $\lambda \leq 0$ there are no other equilibria and for $\lambda > 0$ there are two other equilibria $x = \pm\sqrt{\lambda}$. Analysing the sign of the righthand side of the equation, we conclude that for $\lambda \leq 0$ all solutions different from zero approach zero as time increases and for $\lambda > 0$ the solutions with values smaller than zero approach $-\sqrt{\lambda}$ while those with values larger than zero approach $+\sqrt{\lambda}$ as time increases. Once again, there

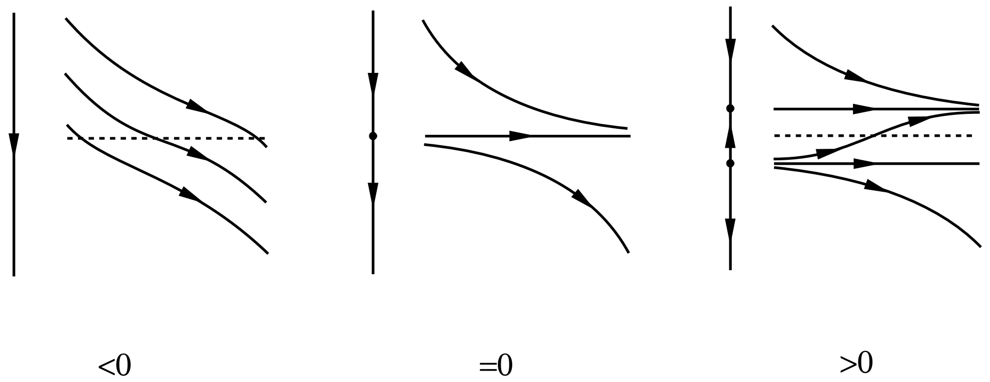

Figure 2.4:

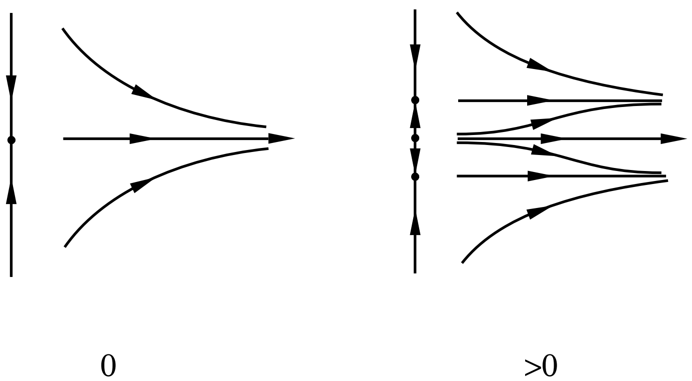

Figure 2.5:

are three equivalence classes of the given systems: for $\lambda < 0$, for $\lambda = 0$, and for $\lambda > 0$. The trajectories and the phase portraits are sketched in Figure 2.5. The system corresponding to $\lambda = 0$ is a bifurcation point. As λ is decreased through zero the three equilibria collide at $\lambda = 0$, a pair of them is annihilated and only one remains for negative λ.

4. We consider the differential equation in $\mathbb{R}^2$

$$\begin{aligned} \dot{x} &= y - x(x^2 + y^2 - \lambda) \\ \dot{y} &= -x - y(x^2 + y^2 - \lambda) \end{aligned}$$

for $\lambda \in \mathbb{R}$. For all $\lambda \in \mathbb{R}$ the origin $(x, y) = (0, 0)$ is the only equilibrium point. However, the orbits of points in a neighbourhood of the origin approach the origin as time increases for $\lambda \leq 0$, and draw apart from it for $\lambda > 0$. In fact, the square of the distance to the origin $V(x, y) = x^2 + y^2$ satisfies along solutions $dV/dt = -2V(V - \lambda)$ which is an equation similar to that considered in the preceding example. Therefore, for $\lambda \leq 0$, $V(x, y) = x^2 + y^2$

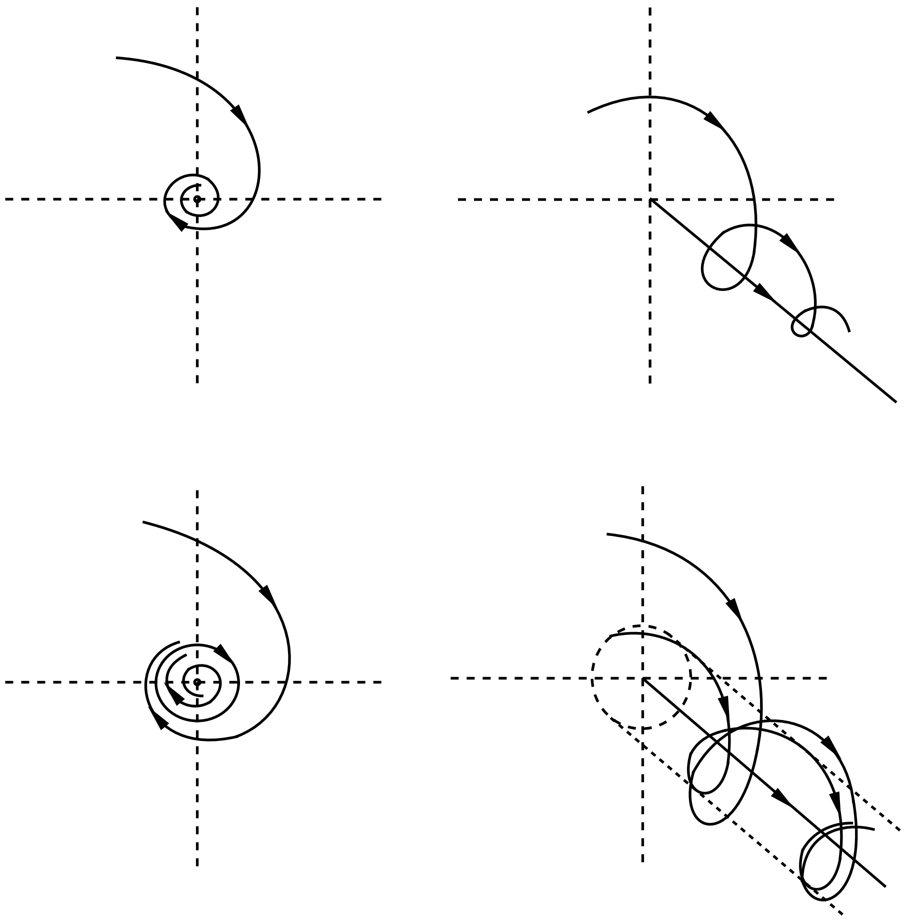

Figure 2.6:

decreases along solutions at points $(x, y) \neq (0, 0)$, and it increases for $\lambda > 0$ and $x^2 + y^2$ small. From this observation we also conclude that for $\lambda > 0$ there exists a periodic orbit which is a circle of radius $\sqrt{\lambda}$ with all other orbits except the equilibrium point at the origin approaching the periodic orbit as time increases, while for $\lambda \leq 0$ there are no periodic orbits and all the orbits approach the equilibrium as time increases. Once more, there are three equivalence classes of the given systems: for $\lambda < 0$, for $\lambda = 0$, and for $\lambda > 0$.

The trajectories and the phase portraits are sketched in Figure 2.6. As λ is decreased through zero the periodic orbit collides with the equilibrium at $\lambda = 0$ and disapears remaining only the equilibrium for negative λ .

5. We consider the differential equation in $\mathbb{R}^2$

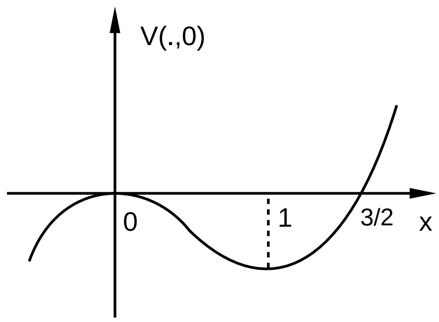

Figure 2.7:

$$\dot{x} = y$$
$$\dot{y} = x - x^2 + \lambda y$$

for $\lambda \in \mathbb{R}$. For all $\lambda \in \mathbb{R}$ the equilibria are $(0,0)$ and $(1,0)$. The function $V(x,y) = y^2 - x^2 + 2x^3/3$ satisfies along solutions $dV(x(t), y(t))/dt = 2\lambda y^2(t)$. Consequently, for $\lambda = 0$ the orbits lie in level curves of V, and for $\lambda \neq 0$ the orbits cross the level curves of V as time increases, in the direction of decreasing values of V for $\lambda < 0$, or in the direction of increasing values of V for $\lambda > 0$. Each one of the level curves $V(x,y) = C$ is the union of the graphs of the functions $y(x) = \pm\sqrt{C - V(x,0)}$. As the graph of the function $x \to V(x,0)$ is as sketched in Figure 2.7, we conclude that the phase portraits for $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ are as drawn in Figure 2.8.

For $\lambda = 0$ there is a homoclinic orbit connecting connecting the origin to itself and all the orbits in the region bounded by the union of this homoclinic with the origin are periodic and encircle the equilibrium $(1,0)$, except for this equilibrium itself. For $\lambda < 0$ all the orbits passing through points in this region approach the equilibrium $(1,0)$ and for $\lambda > 0$ they draw appart from this equilibrium; in either case there exists no homoclinic orbit and instead there exists a heteroclinic orbit connecting the two equilibria from $(0,0)$ to $(1,0)$ for $\lambda < 0$ and the reverse for $\lambda > 0$. As before, we have three equivalence classes and the system corresponding to $\lambda = 0$ is a bifurcation point. As λ is decreased through zero, the heteroclinic connecting $(1,0)$ to $(0,0)$ collides with an orbit approaching the origin as time goes to $-\infty$ to form an homoclinic to the origin at $\lambda = 0$ which disapears for negative λ breaking up in a heteroclinic connecting $(0,0)$ to $(1,0)$ and an orbit approaching the origin as time goes to $+\infty$.

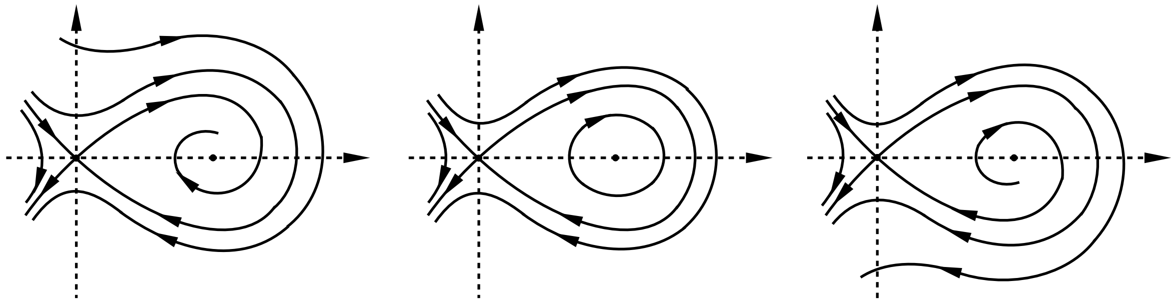

Figure 2.8:

6. To each point $(x,y,z) \in \mathbb{R}^3$ we assign coordinates $(r,\theta,\varphi) \in [0,+\infty[\times [0,2\pi[\times[0,2\pi[$ according to Figure 2.9, where $R > 2$. This assignment defines a one-to-one correspondence except for points in the z-axis and in the circle of radius R and centered at the origin in the plane xy. Each one of the points in the z-axis has unique coordinates $r = -R\cos\theta$ and $\theta \in]\pi/2, 3\pi/2[$ but φ is arbitrary, on the other hand each one of the points in the mentioned circle has unique coordinates $r = 0$ and $\varphi \in [0,2\pi[$ but θ is arbitrary. Let $g(r,\theta) = r + R/\cos\theta$ if $\theta \in]\pi/2, 3\pi/2[$ and $g(r,\theta) = 0$ otherwise. Let $B : \mathbb{R} \to \mathbb{R}$ be a C^∞ bump function monotonically decreasing on $[0,+\infty[$ and satisfying $B(s) = B(-s)$ for all $s \in \mathbb{R}$, $B(0) = 1$, $B(s) = 0$ for $|s| \geq 1$, and consider the differential equation

$$\begin{aligned}\dot{r} &= [1 - B(g(r,\theta))]r(1-r) - B(g(r,\theta))R\frac{\sin\theta}{\cos^2\theta}\\ \dot{\theta} &= 1\\ \dot{\varphi} &= \lambda\end{aligned}$$

where $R = 4$ and $\lambda \in \mathbb{R}$.

We begin by analysing the two dimensional system defined by the first two equations. On the z-axis $r = -R/\cos\theta$ and $r = -R[\sin\theta/\cos^2\theta]$; therefore this axis is itself an orbit. The function g has level curves which approach the z-axis at infinity bounding regions along this axis as shown in Figure 2.10. Along solutions we have

$$\frac{d}{dt}[g(r,\theta)] = \dot{r} + R\frac{\sin\theta}{cos^2\theta} = [1 - B(g(r,\theta))]\Big(r(1-r) + R\frac{\sin\theta}{\cos^2\theta}\Big).$$

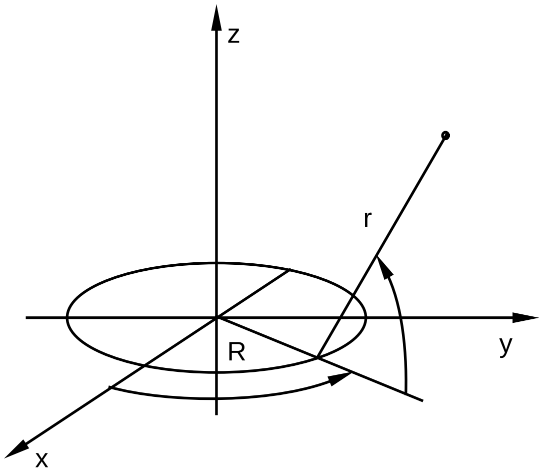

Figure 2.9:

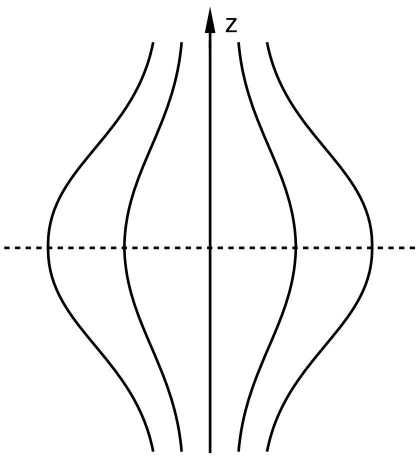

Figure 2.10:

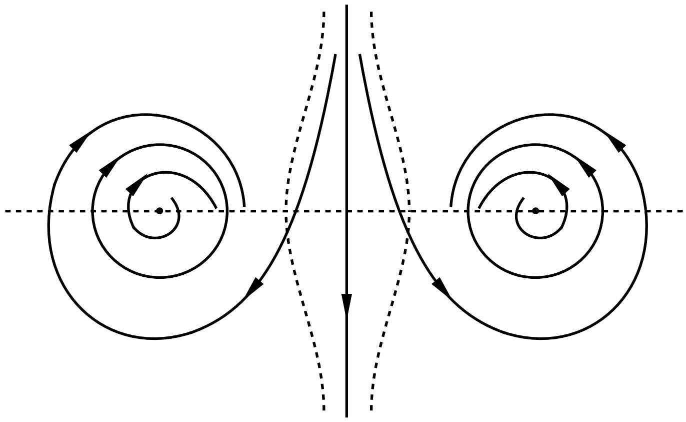

Figure 2.11:

For points in the region along the z-axis bounded by the curve $g(r,\theta) = -1$ we have $\theta \in (\pi/2, 3\pi/2)$. Along a solution passing through any point of that curve we have $\dot{g} = -[1+R/\cos\theta][2+R/\cos\theta]+R[\sin\theta/\cos^2\theta]$. One can check that the function on the right-hand side is negative for $\theta \in (\pi/2, 3\pi/2)$ and, consequently, orbits starting at points outside the region considered cannot enter this region for positive time. On the other hand, as θ increases proportionally to time, an orbit starting at a point in the part of the region with $\theta \in (\pi/2, 5\pi/4)$ will eventually either leave the region through the bounding curve or else enter at some point the part of the region where $\theta \in (5\pi/4, 3\pi/2)$. We denote by g_0 the value of g at that point. Then along the solution we have $\dot{g} < 0$ and, consequently, $\dot{g} < -[1 - B(g_0)]g$. It follows that g decreases at least exponentially in that part of the region bounded by the curve $g(r,\theta) = -1$ and must cross it after some time. Outside the region bounded by the curves $g(r,\theta) = -1$ we have $B(r + R/\cos\theta) = 0$ and, consequently, $\dot{r} = r(1-r)$, implying that the points with $r = 0$ are equilibria, the solutions passing through points outside the circles $r = 1$ approach them as $t \to +\infty$ and the solutions passing through points inside those circles approach them as $t \to +\infty$ and approach the equilibrium at the center of the circle as $t \to -\infty$. The corresponding phase portrait is sketched in Figure 2.11.

The phase portrait for the three dimensional system is now easy to visualize. The z-axis is one of the orbits, there is one circular periodic orbit in the xy-plane centered at the origin and with radius 4, other orbits outside the surface of the torus defined by the equation $r = 1$ spiral towards this

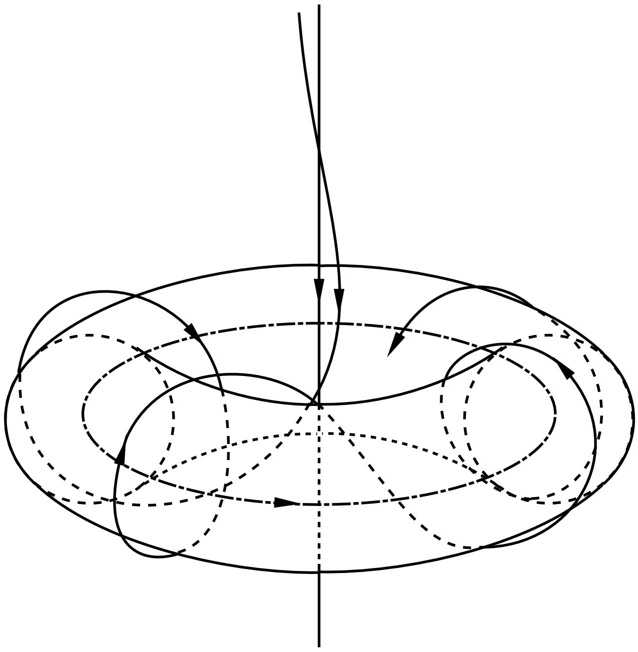

Figure 2.12:

surface as $t \to +\infty$, and the orbits inside the torus surface spiral as $t \to -\infty$ to the periodic orbit in the xy-plane (see Figure 2.12). What happens on the surface of the torus depends on the value of the parameter λ. If the parameter is a rational number, $\lambda = m/n$ in lowest terms, after each time interval of length $2\pi n$ each solution passing through a point in the torus surface returns to the same point after m complete turns in the angle φ around the torus; therefore, for λ rational all orbits passing through a point in the torus surface remain for all time in this surface and are periodic. On the other hand, if λ is an irrational number, orbits passing through a point in the torus surface also remain in this surface but they never close, being, however, dense in the torus surface. These situations, respectively known as the **rational flow** on the torus and the **irrational flow** on the torus, are illustrated in Figure 2.13. In this example there are no isolated bifurcation points: the system is a bifurcation point for all $\lambda \in \mathbb{R}$.

2.2 Invariant sets and limit sets

A set $\Gamma \subset X$ is said to be an **invariant set** of a local dynamical system φ if for every $x \in \Gamma$ the orbit through x is global and it is contained in Γ.

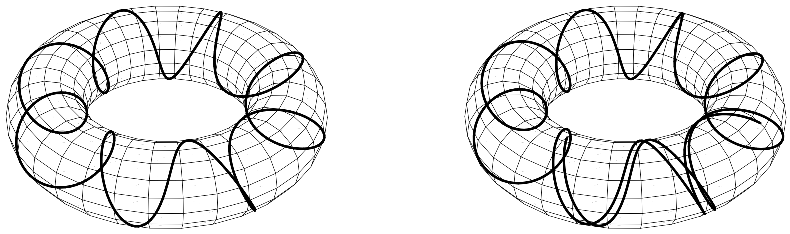

Figure 2.13:

One immediately verifies that every single global orbit in the phase portrait of a local dynamical system is an invariant set. On the other hand, the importance of the notion of invariance stems from the fact that the restriction of any local dynamical system to an invariant set with the induced topology is a dynamical system. This is, for instance, what happens for the first example in the preceding section when the local dynamical system defined by the given ODE is restricted to the teardrop. In Example (2.3)-4 for $\lambda > 0$ one of the invariant sets is a disc, in Example(2.3)-5 for $\lambda = 0$ one of the invariant sets is the teardrop region bounded by the homoclinic orbit, in Example(2.3)-6 the surface $r = 1$ as well as the solid torus it bounds are invariant sets.

It is easy to verify that $\Gamma \subset X$ is an invariant set for a dynamical system if and only if $\varphi_t(\Gamma) = \Gamma$ for all $t \in I$. In fact, $\varphi_{-t}(\Gamma) \subset \Gamma$ and $\varphi_t(\Gamma) \subset \Gamma$ and, consequently, $\Gamma = \varphi_t(\varphi_{-t}(\Gamma)) \subset \varphi_t(\Gamma) \subset \Gamma$, implying that $\varphi_t(\Gamma) = \Gamma$ for all t.

For each $x \in X$ we define the **positive semiorbit** through x as the set $\gamma_x^+ = \{\varphi(t,x) : t \in I_x, t \geq 0\}$ and, similarly, the **negative semiorbit** through x is $\gamma_x^- = \{\varphi(t,x) : t \in I_x, t \leq 0\}$. Naturally, $\gamma_x = \gamma_x^+ \cup \gamma_x^-$. This notion of semiorbit is useful in the study of the limiting behaviour of the evolution as $t \to \pm\infty$. With this objective we define the **α-limit** and **ω-limit** sets of a global orbit γ_x as the sets

$$\alpha(\gamma_x) = \bigcap_{y \in \gamma_x} \text{cl } \gamma_y^- \quad \text{and} \quad \omega(\gamma_x) = \bigcap_{y \in \gamma_x} \text{cl } \gamma_y^+$$

where cl denotes the closure in X. Equivalently this can be written as

$$\alpha(\gamma_x) = \bigcap_{s \in I} \text{cl} \bigcup_{t \leq s} \varphi(t,x) \quad \text{and} \quad \omega(\gamma_x) = \bigcap_{s \in I} \text{cl} \bigcup_{t \geq s} \varphi(t,x),$$

and it is a simple exercise to verify that a point $y \in X$ is an element of $\omega(\gamma_x)$ if and only if there exists a sequence $\{t_k\}$ in I such that $t_k \to +\infty$ and $\varphi(t_k, x) \to y$ as $k \to \infty$ and similarly for $\alpha(\gamma_x)$ with $t_k \to -\infty$.

In order to illustrate these concepts we refer again to the first example in the previous section. One immediately verifies that the α and ω-limit sets of the critical points are the critical points themselves. Similarly, the α and ω-limit sets of a periodic orbit are the orbit itself. For the homoclinic orbit one verifies that its α and ω-limit sets coincide with the critical point $(0, 0)$. It is instructive at this point to return to Example (2.3) and identify the limit sets for each one of the systems.

In the following, to avoid unnecessary complications we consider only dynamical systems and establish some of the most important properties of the limit sets.

(2.4) **Theorem:** *α and ω-limit sets of any orbit γ_x of a dynamical system φ are closed and invariant sets.*

Proof. Closure follows from the definition. Furthermore, taking a point y in $\omega(\gamma_x)$ there exists a sequence $\{t_k\}$ such that $t_k \to +\infty$, $\varphi(t_k, x) \to y$ as $k \to +\infty$ and from continuity we have that $\varphi(t + t_k, x) = \varphi(t, \varphi(t_k, x)) \to \varphi(t, y)$ concluding that $\gamma_y \subset \omega(\gamma_x)$ as desired.

QED

As pointed out earlier, for the sake of simplicity we will restrict our attention to finite dimensional systems although the methods and results, when carefully reformulated, do not essentially differ in infinite dimensions. From here on we let φ denote a dynamical system in a closed set $X \subset \mathbb{R}^n$, and let $d(p, S)$ denote the usual distance between a point p and a compact set S in $\mathbb{R}^n$.

(2.5) **Theorem:** *If the positive semiorbit γ_x^+ is bounded then the ω-limit set $\omega(\gamma_x)$ is nonempty, compact and $d(\varphi(t, x), \omega(\gamma_x)) \to 0$ as $t \to +\infty$. Similarly, if γ_x^- is bounded then $\alpha(\gamma_x)$ is also nonempty, compact and $d(\varphi(t, x), \alpha(\gamma_x)) \to 0$ as $t \to -\infty$.*

Moreover, for continuous dynamical systems these limit sets are connected.

Proof. We prove these results for $\omega(\gamma_x)$ only. Since γ_x^+ is bounded the set $K = \text{cl } {\gamma_x}^+ \subset X$ is compact. The sequence $\{\varphi(k, x)\}$ is contained in K

and taking a converging subsequence to some point $y \in K$ we conclude that $y \in \omega(\gamma_x)$, hence $\omega(\gamma_x)$ is nonempty. Since this set is also contained in K, it is bounded and from the previous theorem we conclude that it is compact.

It is also clear that $d(\varphi(t,x), \omega(\gamma_x)) \to 0$ as $t \to +\infty$ since otherwise for some $\epsilon > 0$ we could find a sequence t_k such that $t_k \to +\infty$, $d(\varphi(t_k,x), \omega(\gamma_x)) > \epsilon$ and taking a converging subsequence of $\varphi(t_k,x)$ we could obtain a point $y \in \omega(\gamma_x)$ satisfying $d(y, \omega(\gamma_x)) \geq \epsilon$.

Finally, to show that $\omega(\gamma_x)$ must be connected for a continuous dynamical system, let us assume that there exist disjoint, closed, nonempty sets A and B such that $\omega(\gamma_x) = A \cup B$ and define $\delta = \min\{d(y,A) : y \in B\}$. Then $\delta > 0$, and since both sets A and B are contained in $\omega(\gamma_x)$, we can construct an increasing sequence $\{t_k\}$ such that $t_k \to +\infty$, $d(\varphi(t_k,x),A) < \delta/2$ for k even and $d(\varphi(t_k,x),A) > \delta/2$ for k odd. From the continuity there is a sequence $\{t_k^{'}\}$ for which $t_k^{'} \to +\infty$ and $d(\varphi(t_k^{'},x),A) = \delta/2$. But as the set $\{x \in X : d(x,A) = \delta/2\}$ is compact, taking a converging subsequence of $\varphi(t_k^{'},x)$ we obtain a point $z \in \omega(\gamma_x)$ such that $z \notin A \cup B$, which is a contradiction.

QED

The notion of limit set was introduced for the study of the limiting behaviour of the evolution $\varphi(t,x)$ as $t \to \pm\infty$. This notion was defined for orbits of the phase portrait, but it is sometimes useful to extend it to general subsets of the phase space. If S denotes a subset of the phase space X, it is natural to define its α and ω -limit sets as

$$\alpha(S) = \bigcap_{s \in I} \mathrm{cl} \bigcup_{t \leq s} \varphi(t,S); \qquad \omega(S) = \bigcap_{s \in I} \mathrm{cl} \bigcup_{t \geq s} \varphi(T,S)$$

where $\varphi(t,S) = \bigcup_{y \in S} \varphi(t,y)$. Again it is a simple exercise to verify that a point $y \in X$ is an element of $\omega(S)$ if and only if there exist sequences $\{t_k\}$ in I and $\{y_k\}$ in S such that $t_k \to +\infty$ and $\varphi(t_k,y_k) \to y$ as $k \to +\infty$ and similarly for $\alpha(S)$ with $t_k \to -\infty$. As before we have the following result.

(2.6) **Theorem:** *If S is a subset of X such that the set of positive semiorbits passing through points of S is bounded then the limit set $\omega(S)$ is nonempty, invariant, compact and for every $y \in S$ we have that $d(\varphi(t,y), \omega(S)) \to 0$ as $t \to +\infty$. Similarly, if the set of negative semiorbits passing through points of S is bounded then the limit set $\alpha(S)$ is nonempty, invariant, compact and for every $y \in S$ we have that $d(\varphi(t,y), \alpha(S)) \to 0$ as $t \to -\infty$.*

Moreover, for continuous dynamical systems, if S *is connected these limit sets are also connected.*

Proof. From the previous theorems it follows that $\alpha(S)$ and $\omega(S)$ are nonempty and invariant. Again compactness follows from the definition. Finally, the limiting behaviour and the connectedness follow from a reasoning similar to that used in the proof of the preceding theorem.

QED

In order to illustrate these notions we consider the dynamical system defined in $\mathbb{R}^2$ by the following ordinary differential system of equations

$$\dot{x} = \sin x, \qquad\qquad \dot{y} = -y.$$

The solutions of this system have the form $x(t) = 2\arctan(k_1 e^t), y(t) = k_2 e^{-t}$ and the corresponding phase portrait is shown in Figure 2.14. The equilibrium points are the points of the form $(k\pi, 0)$ with integer k and every orbit has one equilibrium point for ω-limit set. On the other hand only the orbits contained in the x-axis have a nonempty α-limit set which is also an equilibrium point. To distinguish the sets of orbits with the same ω-limit set we introduce the notion of **basin of attraction** of an equilibrium point e as the set of points z in the phase space for which $\omega(\gamma_z) = e$. For instance, in the above example the basin of attraction of the equilibrium $(\pi, 0)$ is the open strip $\{(x, y) : 0 < x < 2\pi, y \in \mathbb{R}\}$ and the basin of attraction of the origin is the y-axis. Naturally, from the point of view of specific applications it is important to consider the relative size of the different basins of attraction; for instance, in the example the basin of attraction of $(\pi, 0)$ is a surface and the basin of attraction of $(0, 0)$ is a line. For dynamical systems in the plane it is also useful to take into consideration the curves, consisting of orbits, which separate the plane into two parts and such that not all orbits of points on each neighbourhood of this curve exhibit the same limiting behaviour. Such a curve is called a **separatrix** of the phase plane. In the above example the x-axis and the vertical lines with abscisae $2k\pi$ with integer k are separatrices.

To exemplify the notion of limiting set of a set $S \subset X$ we present in Figure 2.14 the disk $S = \{(x, y) : x^2 + (y-2)^2 \leq 1\}$ and its ω-limit set which is the line segment $\omega(S) = \{(x, 0) : -\pi \leq x \leq \pi\}$. This example shows that in general the set $\omega(S)$ is larger than the union of the ω-limit sets of the points of S. In fact, if $z \in S$ then $\omega(\gamma_z)$ is the equilibrium $(\pi, 0)$ if z is in the right half plane, is $(-\pi, 0)$ if z is in the left half plane and is $(0, 0)$ if z

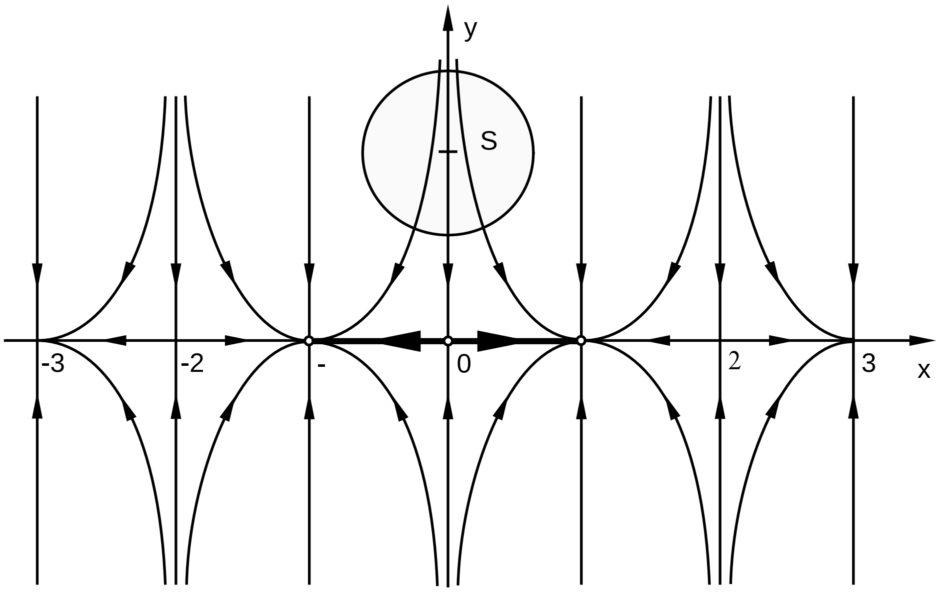

Figure 2.14:

is in the y-axis. Notice that the union of these three equilibria fails to be a connected set, even though S is connected.

We have already pointed out that single orbits of a dynamical system are invariant sets. Since the union of invariant sets is also invariant it is important to consider the possibility of decomposing the invariant sets into minimal invariant sets. A subset $S \subset X$ is said to be a **minimal invariant set** of the dynamical system φ if it is a nonempty compact invariant set of φ and contains no proper subset with these properties. Then, any nonempty compact invariant set contains at least one minimal invariant set. This is sometimes useful for obtaining the desired decomposition, yielding information about the limiting behaviour of a dynamical system. On the other hand, to take into account all the possible limiting behaviour it is important to search also for maximal invariant sets, and in order to avoid the uninteresting invariant set consisting of all the phase space X this search is restricted to compact sets. Then, a subset $A \subset X$ is said to be a **maximal compact invariant set** of the dynamical system φ if it is compact, invariant and contains every compact invariant set of φ. Example (2.3) shows there are systems with minimal invariant sets equal to a point, a periodic orbit or the surface of a torus, and there are systems with maximal compact invariant sets equal to a point, a disc, a teardrop or a solid torus.

It is clear that the existence of a maximal compact invariant set is not ensured for every dynamical system (consider, for instance, the system defined

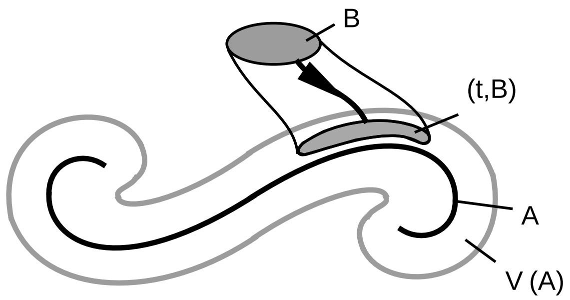

Figure 2.15:

by the linear differential equation $\ddot{x} + x = 0$). In the following section we introduce a special class of dynamical systems, called dissipative, for which the existence of this set is always ensured.

2.3 Dissipative systems and global attractors

Consider again a dynamical system φ in a closed set $X \subset \mathbb{R}^n$. For a subset $A \subset X$ let $V_\epsilon(A)$ denote the ϵ-neighbourhood $V_\epsilon(A) = \{y \in X : |x - y| < \epsilon, x \in A\}$. A set A is said to attract a set B under the flow φ if for any $\epsilon > 0$ there exists $t_0 > 0$ such that $\varphi(t, B) \subset V_\epsilon(A)$ for every $t \geq t_0$ (see Figure 2.15). The dynamical system φ is said to be **dissipative** if there is a bounded set A attracting each point of X under the flow.

From the point of view of the applications, adequate models are either conservative, like the Hamiltonian systems, or else they involve some form of dissipation. Furthermore, good models are sometimes obtained by considering a Hamiltonian system and adding some dissipative terms. It turns out that in this way one usually obtains a dissipative system. Hence, the applications provide the major motivation for the study of dissipative systems. For these systems we can determine a maximal compact invariant set and show that it characterizes every limiting behaviour as $t \to +\infty$.

(2.7) **Theorem:** *If φ is a dissipative system then there exists a compact set K that attracts every compact set under the flow φ. The set $J = \bigcap_{i \geq 0} \varphi(i, K)$ is the ω-limit set $\omega(K)$ and is nonempty, compact and invariant. Furthermore, J does not depend on the set K attracting all the compact sets and it is the maximal compact invariant set of φ.*

Moreover, if φ is a continuous dynamical system then J is connected.

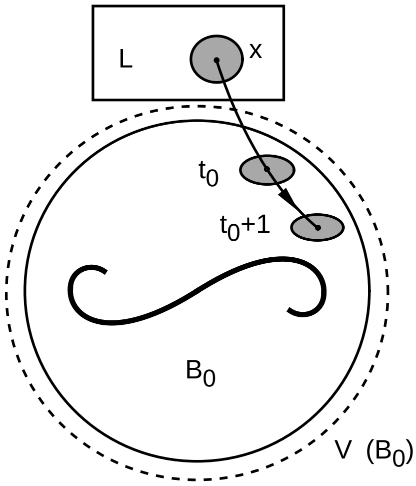

Figure 2.16:

Proof. To prove this theorem let B_0 denote a bounded set attracting each point of X under the flow φ. Then, given $\epsilon > 0$ and for each point $x \in X$ there is a $t_0 = t_0(x)$ such that $\varphi(t, x) \subset V_\epsilon(B_0)$ for $t \geq t_0$. By continuity there is an open neighbourhood $V(x)$ of x such that $\varphi(t, V(x)) \subset V_\epsilon(B_0)$ for $t_0(x) \leq t \leq t_0(x) + 1$ (see Figure 2.16). Then, the set $B = \varphi(1, V_\epsilon(B_0))$ is bounded and contains $\varphi(t + 1, V(x))$ for every $t_0(x) \leq t \leq t_0(x) + 1$. Since the neighbourhoods $V(x)$ cover the entire space X, if L denotes any arbitrary compact subset of X there is a finite subcovering of L by open neighbourhoods $V(x_i)$ with $x_i \in L$ and we let $N(L)$ denote the smallest integer greater than or equal to max $\{1 + t_0(x_i)\}$. For $m = N(\text{cl } B)$ we define the set $K = \bigcup_{i=0}^{m} \varphi(i, \text{cl } B)$ (see Figure 2.17). Then, K is compact, $\varphi(j, \text{cl}B) \subset K$ for $j \geq m$ and $\varphi(t, L) \subset \varphi(t, \bigcup_i V(x_i)) \subset K$ for $t \geq N(L)$. We conclude that K attracts every compact set L under the flow φ.

The compact set $J = \bigcap_{i \geq 0} \varphi(i, K)$ is obviously contained in the limit set $\omega(K)$. On the other hand, taking a point x in $\omega(K)$ there exist sequences $\{t_j\}$ in $\mathbb{R}$ and $\{x_j\}$ in K such that $t_j \to \infty$ and $\varphi(t_j, x_j) \to x$. If for each $i \geq 0$ we consider the sequence $\{\varphi(t_j - i, x_j)\}$, we have that $\varphi(t_j - i, x_j) \in K$ for $t_j \geq i$. Then, there exists a converging subsequence to some point $y_i \in \omega(K) \subset K$. Moreover, $\varphi(i, y_i) = x$ for every $i \geq 0$ implying that x is also in J, and $J = \omega(K)$. The previous theorem then implies that J is nonempty compact and invariant.

Let H be an arbitrary compact invariant set. Since K attracts H, for every $\epsilon > 0$ there exists $t_0 > 0$ such that $\varphi(t, H) \subset V_\epsilon(K)$ for $t \geq t_0$. As H

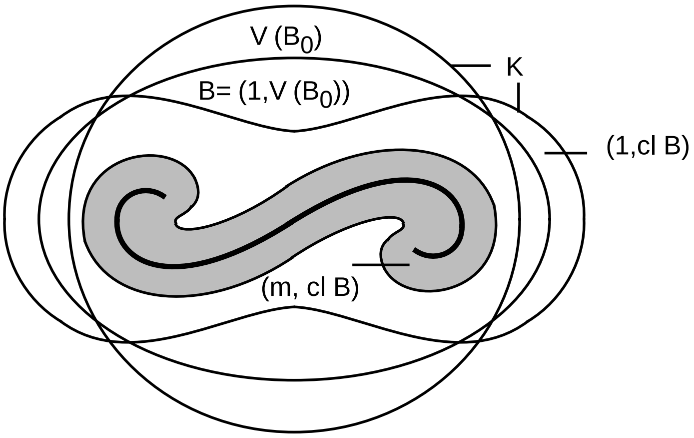

Figure 2.17:

is invariant, we have $H = \varphi(t, H) \subset V_\epsilon(K)$. Since this holds for every $\epsilon > 0$, $H \subset K$, and the invariance of H then implies $H \subset \varphi(i, K)$ for every $i \geq 0$. Hence, $H \subset J$ and J is the maximal compact invariant set.

To show that J does not depend on the set K, we consider another compact set K' also attracting every compact subset of X. We can replace K by K' in all the preceding arguments and conclude that both $J(K)$ and $J(K')$ are equal to the maximal compact invariant set.

Finally, taking a compact connected set L containing K we have that $\varphi(t, L) \subset K$ for $t \geq N(L)$ and we obtain $\omega(L) \subset \omega(K)$. Since $K \subset L$ we obtain $\omega(K) \subset \omega(L)$ and, therefore, $J = \omega(L)$. If φ is a continuous dynamical system, we conclude from the preceding theorem that $J = \omega(L)$ is connected.

QED

The next result identifies the maximal compact invariant set of a dissipative system φ with the set of all bounded orbits of φ and establishes a very strong result on the limiting behaviour of the orbits of the phase portrait.

(2.8) **Theorem:** *The set $A(\varphi)$ of all bounded orbits of a dissipative system φ is the maximal compact invariant set of φ and attracts all bounded sets of X.*

Proof. The previous theorem already asserts the existence of the maximal compact invariant set J of the dissipative system φ. Since any orbit passing through a point of a compact invariant set is bounded we conclude that $J \subset A(\varphi)$. On the other hand, the set $A_\beta(\varphi)$ of orbits bounded in X by a constant $\beta > 0$, $A_\beta(\varphi) = \{x \in X : |\varphi(t,x)| \leq \beta, t \in I\}$, is compact and invariant, hence $A_\beta(\varphi) \subset J$ for every β. But since $A(\varphi) = \bigcup_\beta A_\beta(\varphi)$ we also conclude that $A(\varphi) \subset J$ which implies that $A(\varphi) = J$.

Now, to prove that $A(\varphi)$ attracts all bounded sets it is sufficient to show that it attracts every compact set of X. This proof will be presented in two steps. In the first step we show that, given any $x \in X$ and any constant $\delta > 0$, there is a constant $M(x) > 0$ and a sufficiently small neighbourhood $V(x)$ such that $\varphi(M(x), V(x)) \subset V_\delta(A(\varphi))$. In the second step we show that, given any neighbourhood $V_\epsilon(A(\varphi))$ with $\epsilon > 0$, there is a neighbourhood $V_\delta(A(\varphi))$ with $0 < \delta \leq \epsilon$ such that $\varphi(t, V_\delta(A(\varphi))) \subset V_\epsilon(A(\varphi))$ for every $t \geq 0$. Since, given any compact set L of X we can choose a finite subcovering of L by open sets $V(x_i)$, from the above results we conclude that $\varphi(t, V(x_i)) \subset V_\epsilon(A(\varphi)$ for every $t \geq M(x_i)$ and hence $\varphi(t, L) \subset V_\epsilon(A(\varphi))$ for every $t \geq \max M(x_i)$ proving that $A(\varphi)$ attracts L.

To prove the first step, we consider any given $x \epsilon X$ and let $\delta > 0$. Since $\omega(\{x\}) \subset A(\varphi)$, there is a constant $M(x) > 0$ such that $\varphi(t,x) \in V_{\delta/2} A(\varphi)$ for $t \geq M(x)$. Then, by continuity there is a sufficiently small neighbourhood $V(x)$ such that $\varphi(M(x), V(x)) \subset V_\delta(A(\varphi))$.

We prove the second step by contradiction. Supose then, that there exist $\epsilon > 0$ and sequences $\{t_j\}$, $\{d_j\}$ in $\mathbb{R}$ and $\{y_j\}$ in X, such that $t_j \to +\infty$, $d_j > 0$, $d(y_j, A(\varphi)) \to 0$, $\varphi(t, y_j) \in V_\epsilon(A(\varphi))$ for $0 \leq t < t_j$ and $\varphi(t, y_j) \notin V_\epsilon(A(\varphi))$ for $t_j < t < t_j + d_j$. Since $A(\varphi)$ is compact we can assume that $y_j \to y \in A(\varphi)$. Then, the set $H = \{y, y_j : j \geq 1\}$ is also compact and if K denotes the compact set attracting every compact set under the flow we have that $\bigcup_{t \geq s} \varphi(t, H) \subset V_\epsilon(K)$ for some $s \geq 0$. This means that the positive semiorbits of points in H are bounded and, consequently, their ω -limit sets are nonempty, compact and invariant. Since $A(\varphi)$ is the maximal compact invariant set we conclude that $\omega(H) \subset A(\varphi)$. But since cl $\bigcup_{t \geq s} \varphi(t, H)$ is compact we also conclude that the sequence $\varphi(t_j + d_j/2, y_j)$ has a converging subsequence to some point z satisfying $z \notin V_\epsilon(A(\varphi))$ and $z \in \omega(H) \subset A(\varphi)$ which is a contradiction.

QED

Due to its attracting properties it is natural to call the set $A(\varphi)$ the

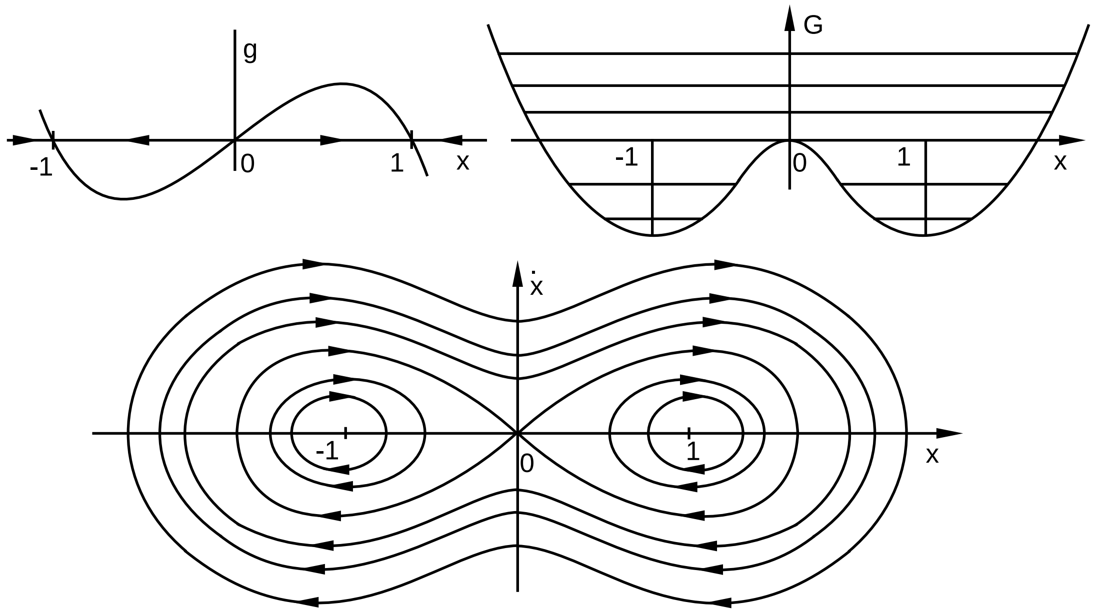

Figure 2.18:

global attractor of the dynamical system and it is certainly the most important object to be searched for in the study of a flow. The objectives of the geometrical theory in what concerns the study of the global attractor are twofold: it is important to study the geometrical and topological properties of all the orbits contained in the attractor in order to understand all the possible limiting behaviour of the system, and it is of interest to study the dependence of $A(\varphi)$ on φ in order to understand the behaviour of the global attractor under perturbations of the dynamical system.

To illustrate here the notions of dissipative system and global attractor we consider again a mechanical system described by the following second order differential equation

$$\ddot{x} + \epsilon\dot{x} - x + x^3 = 0, \qquad \epsilon > 0.$$

As in the previous example, this equation also describes the one dimensional motion of a body with unit mass under the action of a nonlinear force $g(x) = x - x^3$, which is restoring if $|x| > 1$, and with damping proportional to the velocity. As before, we take for state variables the position x and the velocity $y = \dot{x}$ and choose $\mathbb{R}^2$ for phase space. Again the mechanical energy is given by

$$E(x, y) = \frac{1}{2}y^2 + G(x),$$

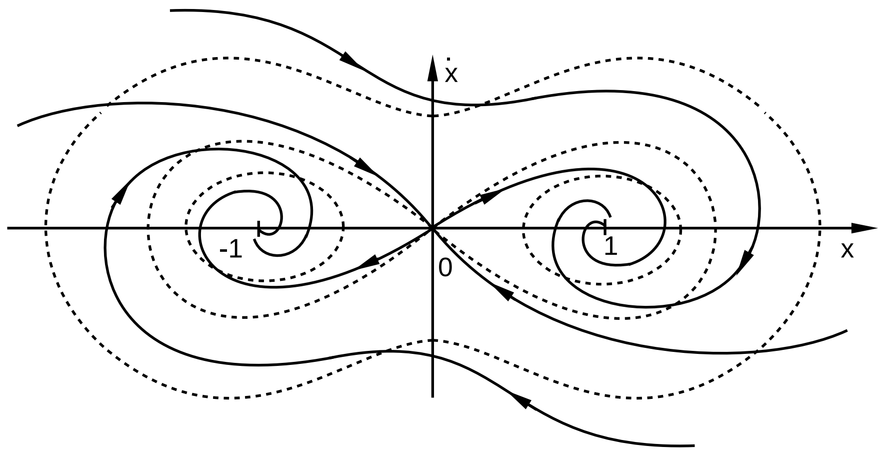

Figure 2.19:

with potential $G(x) = x^4/4 - x^2/2$. Then, we have $\frac{d}{dt}E(x(t), \dot{x}(t)) = -\epsilon(\dot{x}(t))^2$. For $\epsilon = 0$ the system is conservative and the corresponding phase portrait can be sketched as in Figure 2.18. For $\epsilon > 0$ the energy decays with time and all the orbits of the system must cross the level curves of the energy in the direction of decreasing values of energy, leading for $\epsilon > 0$ small to a phase portrait as sketched in Figure 2.19. We consider $\epsilon > 0$ and let c be a positive constant. Then, the level curve corresponding to $E(x, y) = c$ is a simple closed curve that bounds an open connected set B_c (see the largest dotted line in the phase portrait). Any solution starting in B_c cannot leave this set. Furthermore, the energy is bounded below and nonincreasing along the solutions, therefore, it must approach a constant as $t \to +\infty$. Since the ω-limit set of each orbit is invariant and E is continuous, the ω-limit set must lie on a level curve of E, that is, each orbit $\gamma(t) = (x(t), \dot{x}(t))$ in the ω-limit set must have $0 \equiv \frac{d}{dt}E(x(t), \dot{x}(t)) = -\epsilon(\dot{x}(t))^2$, implying that $\dot{x}(t) = 0$. From the invariance of the ω-limit sets it follows that they can be either the origin or one of the equilibria $(\pm 1, 0)$. This shows that the above dynamical system is dissipative, since any bounded set containing the equilibria attracts each point of $\mathbb{R}^2$ under the flow. The corresponding global attractor $A(\varphi)$ is shown in Figure 2.20 and is composed of the three equilibria and the two heteroclinic orbits connecting them. These are the only bounded orbits of the dynamical system. The preceding reasoning involving the simultaneous consideration of level sets of an energy function and the invariance of limit sets is a simple application of the well known **invariance principle of LaSalle**.

The existence of a compact invariant set implies the existence of minimal sets, but it does not directly imply that there exist minimal sets of the

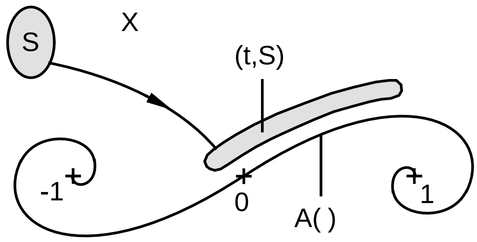

Figure 2.20:

smallest possible type, namely consisting of only one point. However, it turns out that for dissipative systems such minimal sets do indeed exist as a consequence of an interplay between invariance and general topological results on the existence of fixed points of continuous maps.

2.4 Existence of equilibria

It is important to know for a particular dynamical system whether an equilibrium point exists or not. The rational and the irrational flows on the torus of Example (2.3)-6 or the flow defined by the equation $\dot{x} = 1$ show there exist dynamical systems without equilibria. In fact, even if a compact invariant set exists, and therefore there exists a minimal invariant set, it is important to know if any of the minimal sets is just one point. The existence of an equilibrium pointfor a dynamical system is a global question that has been handled by a number of different methods. Here we consider only two situations related to the concepts of invariance and dissipativeness.

The dissipativeness considered in the previous section which implied the existence of a maximal compact invariant set — *a global attractor* — also implies the existence of at least an equilibrium point; and, therefore, of a minimal set which is as small as possible. Before establishing this, we consider the easier question of existence of equilibria in invariant sets homeomorphic to the closed unit ball in $\mathbb{R}^n$. For this it is convenient to use a general topological result establishing the existence of fixed points for continuous functions. We cite the result without proof [1].

[1]A simple proof is given in *John Milnor, The hairy ball theorem, Amer. Math. Month.* 85 (1978), 521-524

(2.9) **Theorem** (Brouwer's Fixed Point Theorem) : *A continuous mapping from the unit closed ball of* $\mathbb{R}^n$ *into itself has at least one fixed point.*

We can now turn to the existence of equilibrium points for dynamical systems.

(2.10) **Theorem**: *If, for a dynamical system* φ *in a topological space* X, A *is a positively invariant set (or negatively invariant set) homeomorphic to the unit closed ball of* $\mathbb{R}^n$*, then there exists at least one equilibrium point of* φ.

Proof. We consider a dynamical system $\varphi : I \times X \to X$ and suppose A is positively invariant. For each $T \in I$ with $T > 0$, the function $x \to \varphi(T, x)$ is a continuous mapping from A into itself. Brouwer's Fixed Point Theorem implies the existence of a fixed point for this mapping $y = \varphi(T, y)$.

In case φ is a discrete system, taking $T = 1$ we obtain an equilibrium of φ. If φ is a continuous system, we choose a sequence T_j with $T_j > 0$ and $T_j \to 0$ and points $x_j = \varphi(T_j, x_j)$. As A is compact, we can suppose, without loss of generality, that x_j converges to a point $x \in A$.

For each $t > 0$ we have

$$|\varphi(t, x) - x| \leq |\varphi(t, x) - \varphi(t, x_j)| + |\varphi(t, x_j) - x_j| + |x_j - x|.$$

The first and the last terms in the right hand side of this inequality approach zero as $j \to +\infty$. For the middle term, as $\varphi(iT_j, x_j) = x_j$ for each positive integer i, we have $|\varphi(t, x_j) - x_j| \leq |\varphi(t, x_j) - \varphi(iT_j, x_j)|$. Taking for each j the integer i_j such that $i_jT_j \leq t \leq (i_j + 1)T_j$, we obtain $i_jT_j \to t$ and, consequently, also this term converges to zero as $j \to +\infty$. We conclude that $\varphi(t, x) = x$ for all $t \geq 0$, proving that x is a equilibrium point.

QED

The existence of equilibrium points for dissipative dynamical systems also stands on a general result on the existence of fixed points which generalizes Brouwer's Fixed Point Theorem by relaxing the hypothesis on the mapping considered through a condition on one of its iterates. The result, which belongs to the class of the so called asymptotic fixed point theorems, is also cited here without proof[2].

(2.11) **Theorem:** *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of $\mathbb{R}^n$ with S_0, S_2 compact and S_1 open relative to S_2. If $f : S^2 \to \mathbb{R}^n$ is a continuous mapping such that for some integer $m > 0$ we have $f^j(S^1) \subset S^2$, for $1 \leq j \leq m-1$, and $f^j(S_1) \subset S_0$, for $m \leq j \leq 2m-1$, then f has a fixed point in S_0.*

The existence of equilibrium points for dissipative systems can be established using this theorem.

(2.12) **Theorem:** *A dissipative dynamical system φ in a set $X \subset \mathbb{R}^n$ has at least one equilibrium point.*

Proof. Let $A(\varphi)$ be the maximal compact invariant set whose existence was established in Theorem (2.7). We denote by S_0 the closure of an open ball containing $A(\varphi)$ and by S_1 an open ball containing S_0. It was established in Theorem (2.8) that $A(\varphi)$ attracts all bounded sets, in particular it attracts S_1 and, consequently, there exists an integer $m > 0$ such that the function $f(x) = \varphi(T, x)$ satisfies $f^j(S_1) \subset S_0$ for $j \geq m$ and $T > 0$. It follows that the union of the sets $f^j(S_1)$, for $j \geq 0$, is bounded and, therefore, it is contained in a closed ball S_2. The function f and the sets S_0, S_1, S_2 satisfy the hypothesis of the preceding theorem and, consequently, we conclude that $f(x) = \varphi(T, x)$ has a fixed point x_T in S_0. As $A(\varphi)$ is the maximal compact invariant set we have $x_1 \in A(\varphi)$.

The existence of an equilibrium point in $A(\varphi)$ can now be established as in the proof of the Theorem (2.10).

QED

[2] A proof can be found in the papers *Felix E. Browder, On a Generalization of the Schauder fixed point theorem, Duke Math J.* **26** *(1959), 291-303 and W.A. Horn, Some fixed point theorems for compact maps and flows in Banach spaces, Trans.Amer.Math.Soc.* **149** *(1970), 391-404*

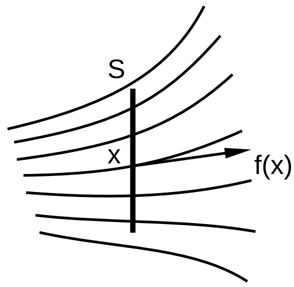

Figure 2.21:

2.5 Flow Box Theorem

To consider more detailed properties of dynamical systems, it is convenient to assume some degree of smoothness. With this purpose in mind, we take $I = \mathbb{R}$ and let φ denote a smooth local dynamical system in an open set $X \subset \mathbb{R}^n$ with an associated vector field of class C^1. Hence, we assume that φ is associated to an ordinary differential equation, $\dot{x} = f(x)$, where $f(x) = \partial\varphi(0, x)/\partial t$ defines a continuously differentiable vector field in X. From the definition of this vector field and the ordinary differential equation $\dot{x} = f(x)$, we conclude that the equilibria of φ correspond to the zeros of the vector field, that is, $\bar{x} \in X$ is an equilibrium point of φ if and only if $f(\bar{x}) = 0$. In this section we show that near regular points the flow of such a dynamical system is completely trivial in the sense that it is homeomorphic to a parallel flow, that is, a flow whose orbits are parallel straight lines.

We say that a subset S of a hyperplane[3] in $\mathbb{R}^n$ which is homeomorphic to a open ball of $\mathbb{R}^{n-1}$ is a **section transverse to the orbits of** φ if for every $x \in S$ the set of vectors $y - x$ for $y \in S$ together with the vector $f(x)$ span the whole space $\mathbb{R}^n$ (see Figure 2.21). For simplicity, we also say that S is a **transversal** to φ.

(2.13) **Proposition:** *If $a \in X$ is a regular point of φ, then there is a tranversal to φ containing a. Moreover, this transversal can be taken to be diffeomorphic[4] to an open ball of* $\mathbb{R}^{n-1}$.

[3] A hyperplane in $\mathbb{R}^n$ is a $(n-1)$-dimensional plane in $\mathbb{R}^n$, that is a translation of a $(n-1)$-dimensional subspace of $\mathbb{R}^n$

[4] A diffeomorphism is a continuously differentiable function with continuosly differentiable inverse. Two sets are said to be diffeomorphic if one is the image of the other under a diffeomorphism.

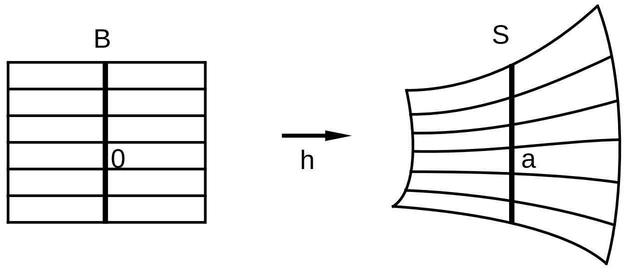

Figure 2.22:

Proof: Let $a \in X$ denote a regular point of φ. Then, the corresponding vector field satisfies $f(a) \neq 0$ and, consequently, there are $n-1$ linearly independent vectors $v_i \in \mathbb{R}^n, i = 1, ..., n-1$, such that the set of vectors $\{f(a), v_1, ..., v_{n-1}\}$ forms a base of $\mathbb{R}^n$. Let $B_\delta = \{y \in \mathbb{R}^{n-1} : |y| < \delta\}$ denote the open ball in $\mathbb{R}^{n-1}$ of radius δ and centered at the origin and define the function $g : B_\delta \to \mathbb{R}^n$ by $g(y_1, ..., y_{n-1}) = a + \sum_i y_i v_i$. Then, g is a diffeomorphism of the open ball $B_\delta \subset \mathbb{R}^{n-1}$ to a hyperplane in $\mathbb{R}^n$. Since $D(x) = [f(x), v_1, \ldots, v_{n-1}]$ is nonsingular for $x = a$, that is, $\det D(a) \neq 0$, and since f is continuous it follows that $\det D(x) \neq 0$ for $x \in S_\delta = g(B_\delta)$, provided δ is taken sufficiently small. For such a δ, the linear span of the vectors $\{f(x), v_1, \ldots, v_{n-1}\}$ is $\mathbb{R}^n$ and, therefore S_δ is a transversal to φ.

QED

The following theorem establishes that the set of orbits in a neighbourhood of a regular point of φ is, apart from a diffeomorphism, a set of parallel line segments filling up an open set in $\mathbb{R}^n$, as depicted in Figure 2.22.

(2.14) **Theorem:** (Flow Box Theorem) *If $a \in X$ is a regular point of φ, then there exists a neighbourhood of a diffeomorphic to the cartesian product of an open interval of $\mathbb{R}$ with an open ball of $\mathbb{R}^{n-1}$ such that the corresponding diffeomorphism maps the orbits of φ to parallel line segments.*

Proof: Let S_δ denote a transversal to φ containing a and diffeomorphic to the open ball $B_\delta \subset \mathbb{R}^{n-1}$. Consider the function h which is the restriction of φ to $(-\delta, \delta) \times S_\delta$. If $\{v_1, \ldots, v_{n-1}\}$ is taken as a basis of the subspace spanned by the vectors $y - a$ for $y \in S_\delta$, then for each $x \in S_\delta$ we have

$$x = a + \sum_{i=1}^{n-1} x_i v_i \qquad \text{and} \qquad Dh(t, x) = \left[\frac{\partial \varphi(t, x)}{\partial t}, \frac{\partial \varphi(t, x)}{\partial x} \frac{\partial x}{\partial x_1}, \ldots, \frac{\partial \varphi(t, x)}{\partial x} \frac{\partial x}{\partial x_{n-1}}\right]$$

Therefore, $h(0, a) = a$ and $Dh(0, a) = [f(a), v_1, ..., v_{n-1}]$ which is nonsingular because S_δ is a section transverse to the orbits of φ. From the inverse function theorem we conclude that the inverse mapping of h exists and is continuously differentiable in a neighbourhood of a. Then, if δ is sufficiently small, h is a diffeomorphism. Clearly, for every $x \in S_\delta$ the image of $(-\delta, \delta) \times \{x\}$ by h is the arc of orbit $\{\varphi(t, x) : -\delta < t < \delta\}$ and the range of h is a neighbourhood U of a. It follows that the inverse diffeomorphism h^{-1}, for each $x \in S_\delta$, maps $\gamma_x \cap U$ to $(-\delta, \delta) \times \{x\}$, and that these sets are parallel line segments in $(-\delta, \delta) \times S_\delta$.

The diffeomorphism from U to $(-\delta, \delta) \times B_\delta$ follows immediately by noticing that S_δ is diffeomorphic to B_δ.

QED

From this theorem it follows that the flow of the dynamical system φ near a regular point is, in fact, diffeomorphic to a parallel flow, hence, from the point of view of a local theory, only the characterization of flows near equilibrium points is missing.

It is a trivial observation that a smooth parallel flow is associated with a vector field of constant direction. It is then easy to conclude that a smooth flow, near a regular point, is diffeomorphic to the flow generated by a constant vector field.

2.6 Poincaré-Bendixson theory

Some of the questions arising in the global characterization of phase portraits involve the important and very difficult problem of finding periodic orbits. This characterization also involves the consideration of geometrical and topological tools which in character are very different from the ones used in the local analysis. This was seen when the existence of equilibria in invariant sets was considered in section 2.4, and, more generally, this is the case when considering the characterization of compact invariant sets. Although adequate tools for this characterization seem to be lacking in general, a complete characterization is available in the case of two dimensional systems, due to the celebrated Jordan curve theorem. For a smooth dynamical system defined on the plane we obtain two important results: the first states that the only minimal invariant sets are equilibrium points and periodic orbits; and the second is the Poincaré-Bendixson theorem, characterizing the limit sets of bounded orbits.

To establish these results let φ denote a smooth dynamical system in $\mathbb{R}^2$ with an associated vector field f of class C^1. Again we assume that φ is associated to the ordinary differential equation in the plane $\dot{x} = f(x)$. Let $p \in \mathbb{R}^2$ denote a regular point of φ and L denote a transversal to φ containing p. In this case L is an open line segment, and to avoid problems in its end points we assume that L is such that cl L is contained in another transversal to φ. To identify the periodic orbits of φ we start by studying the orbits of φ that repeatedly intersect L.

(2.15) **Lemma:**

(i) All the orbits intersecting the transversal L cross it in the same direction.

(ii) For every point $p \in L$ there exists an $\epsilon > 0$ and a neighbourhood U containing p such that for every $q \in U$ the flow $\varphi(t, q)$ intersects L exactly once for $|t| < \epsilon$.

(iii) If the flow $\varphi(t, p)$ intersects L at the successive time instants $0 < t_1 < t_2 < \ldots$, then the intersection points $p_i = \varphi(t_i, p)$ along L occur in the same order as along the orbit γ_p.

Proof:

(i) If there were orbits crossing L in oposite directions, the mean value theorem for continuous functions would imply the existence of a point of tangency of L with an orbit, contradicting the fact of L being a transversal.

(ii) If $p \in L$ then p is a regular point and, since φ is smooth and $f(q) = \partial\varphi(0, q)/\partial t$, for every $\delta > 0$ there is a neighbourhood U of p and an $\epsilon > 0$ such that, for every $q \in U$ and $|t| \leq \epsilon$, we have $|\varphi(t, q) - q - tf(q)| < \delta|t|$. Accordingly, taking δ sufficiently small the corresponding U and ϵ satisfy the desired properties.

(iii) It is sufficient to consider the sequence of points p, p_1, p_2. Consider the Jordan curve formed by the arc of orbit $\varphi(t, p)$ with $0 \leq t \leq t_1$, and the segment of L between p and p_1 (see Figure 2.23). At time $t = t_1$ the flow

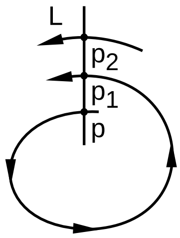

Figure 2.23:

$\varphi(t,p)$ enters one of the regions of the plane defined by this Jordan curve. Since by (i) all the orbits intersecting L must cross in the same direction, we conclude that $\varphi(t,p)$ for $t > t_1$ cannot leave this region of the plane. Therefore, along L the point p_2 must follow the points p and p_1 in the same order as they occur along γ_p.

QED

Due to the last property we say that $\{p_i\}$ is a **monotone sequence along** L.

The following result is useful in the characterization of the limit sets.

(2.16) **Proposition:** *The ω-limit set $\omega(\gamma_p)$ of the orbit γ_p can intersect the transversal L in only one point. If $\omega(\gamma_p) \cap L = \{q\}$, then either:*

(i) $\omega(\gamma_p) = \gamma_p$ and γ_p is a closed orbit; or

(ii) there is a sequence $\{t_k\}$ such that $t_k \to +\infty$ and $\varphi(t_k, p) \neq q$ is a monotone sequence along L.

The same result holds for the α-limit set of γ_p.

Proof: Assume that $q \in \omega(\gamma_p) \cap L$. Then, there is a sequence $\{t_k\}$, such that $t_k \to +\infty$ and $\varphi(t_k, p) \to q$ (see Figure 2.24). From (ii) in the previous lemma, for k sufficiently large, there are points $p_k = \varphi(t'_k, p)$ such that $\{p_k\}$ is in L and $p_k \to q$. From (iii) of the same lemma we conclude that $\{p_k\}$ is monotone along L, therefore, $\omega(\gamma_p) \cap L$ can have at most one point. The alternative follows immediately.

QED

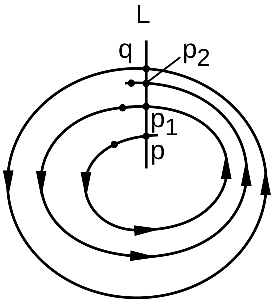

Figure 2.24:

From the previous lemma we also derive the following complete characterization of minimal invariant sets for systems in the plane.

(2.17) **Theorem:** *The minimal invariant sets are either equilibrium points or periodic orbits.*

Proof: Let γ_p denote an orbit contained in the minimal invariant set M. Then, $\alpha(\gamma_p)$ and $\omega(\gamma_p)$ are nonempty and contained also in M. Since the limit sets of bounded orbits are compact and invariant and M is minimal with respect to these properties, it follows that $\alpha(\gamma_p) = \omega(\gamma_p) = M$. If M contains an equilibrium point p, then $\omega(\gamma_p) = p$ and, hence, $M = p$. If $M = \omega(\gamma_p)$ does not contain an equilibrium point, since $\gamma_p \subset \omega(\gamma_p)$ it follows that γ_p and $\omega(\gamma_p)$ have a regular point q in common. Let L denote a transversal containing q. Then, there is a sequence $\{t_k\}$, such that $t_k \to +\infty$, $\varphi(t_k, p) \to q$, and, as in the previous proposition, we can take the sequence $\varphi(t'_k, p)$ of points contained in L, $\varphi(t'_k, p) \to q$. From (ii) of the lemma, this sequence is monotone along L, and since γ_p and $\omega(\gamma_p)$ have a point in common, it cannot be strictly monotone. It follows that all the points $\varphi(t'_k, p)$ are coincident, $\varphi(t'_k, p) = q$, and $\gamma_p = \omega(\gamma_p)$ is a periodic orbit.

QED

(2.18) **Lemma:** *If the ω-limit set of a positive semiorbit γ_p^+ contains a periodic orbit γ then $\omega(\gamma_p^+) = \gamma$.*

Proof: Let q denote a point in γ and L be a transversal to φ containing q.

From the previous proposition we have that $\omega(\gamma_p^+) \cap L = \{q\}$. By (ii) of the previous lemma there is a neighbourhood U of q such that every orbit passing through points of U intersect L. Since $\omega(\gamma_p^+)$ is invariant and $\omega(\gamma_p^+) \cap L = \{q\}$, it follows that every orbit passing through points of $U \cap \omega(\gamma_p^+)$ contains q. Hence, $U \cap \omega(\gamma_p^+) = U \cap \gamma$, and since this holds for every point $q \in \gamma$ we have $\omega(\gamma_p^+) = \gamma$.

QED

We can now establish the famous Poincaré-Bendixson theorem.

(2.19) **Theorem** (Poincaré-Bendixson): *If γ_p^+ is a bounded positive semiorbit and $\omega(\gamma_p^+)$ does not contain equilibrium points, then $\omega(\gamma_p^+)$ is a periodic orbit and, either $\omega(\gamma_p^+) = \gamma_p^+$, or $\omega(\gamma_p^+) = cl\ \gamma_p^+ \backslash \gamma_p^+$.*

An analogous result holds for negative semiorbits and the corresponding α-limit sets.

Proof: Since γ_p^+ is bounded, $\omega(\gamma_p^+)$ is nonempty, compact connected and invariant. Hence, there is a minimal invariant set $M \subset \omega(\gamma_p^+)$. Since $\omega(\gamma_p^+)$ does not contain equilibrium points, the previous theorem implies that M is a periodic orbit γ and the previous lemma implies that $\omega(\gamma_p^+) = \gamma$. Then, if $\gamma_p^+ = \gamma$ we have $\omega(\gamma_p^+) = \gamma_p^+$, otherwise $\omega(\gamma_p^+) = \text{cl}\ \gamma_p^+ \backslash \gamma_p^+$ as stated.

QED

To complete the characterization of the limit sets of bounded orbits we present the following result.

(2.20) **Theorem:** *Let γ_p^+ be a positive semiorbit in a compact subset of the plane containing only a finite number of equilibrium points. Then, one of the following is satisfied (see Figure 2.25):*

(i) $\omega(\gamma_p^+)$ *is an equilibrium point;*

(ii) $\omega(\gamma_p^+)$ *is a periodic orbit;*

(iii) $\omega(\gamma_p^+)$ *contains a finite number of equilibrium points and a set of homoclinic and heteroclinic orbits connecting them.*

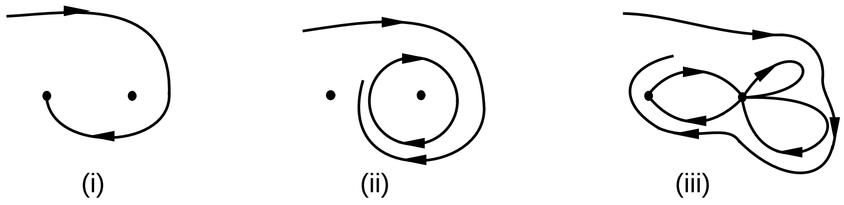

Figure 2.25:

An analogous result holds for negative semiorbits and the corresponding α-limit sets.

Proof: The set $\omega(\gamma_p^+)$ contains at most a finite number of equilibrium points. If $\omega(\gamma_p^+)$ does not contain regular points, then it is a single equilibrium point since $\omega(\gamma_p^+)$ must be connected.

If $\omega(\gamma_p^+)$ contains a closed orbit γ, then the previous lemma implies that $\omega(\gamma_p^+) = \gamma$.

Finally, we assume that $\omega(\gamma_p^+)$ does not contain a periodic orbit but contains regular points. Let γ denote an orbit in $\omega(\gamma_p^+)$. As $\omega(\gamma_p^+)$ is closed, we have $\omega(\gamma) \subset \omega(\gamma_p^+)$. If $q_0 \in \omega(\gamma)$ were a regular point, then considering a transversal L to φ containing q_0, the Proposition 2.16 would imply that $\omega(\gamma_p^+) \cap L = \omega(\gamma) \cap L = \{q_0\}$ and that γ intersect L at the point $\{q_0\}$. For $q \in \gamma$, the same proposition would imply the existence of a sequence $\{t_k\}$ such that $t_k \to +\infty$ and $\varphi(t_k, q)$ is a monotone sequence along L converging to q_0. Since $\gamma \subset \omega(\gamma_p^+)$ it would follow that $\varphi(t_k, q) = q_0$ for every k implying that γ would be a periodic orbit. Consequently, if $\omega(\gamma_p^+)$ does not contain periodic orbits and $\gamma \subset \omega(\gamma_p^+)$, then $\omega(\gamma)$ does not contain regular points. As $\omega(\gamma)$ is connected, it follows that it is an equilibrium point. The same argument applies to α-limit sets. Hence, $\omega(\gamma_p^+)$ is composed of orbits whose α and ω-limit sets are equilibrium points.

QED

As an application of these results we consider the local dynamical system defined by the second order differential equation

$$\ddot{x} + g(x)\dot{x} + f(x) = 0$$

where f and g are C^1 functions satisfying:

(i) f is odd and $xf(x) > 0$ for $x \neq 0$;

(ii) g is even and $g(0) < 0$;

(iii) G defined as $G(x) = \int_0^x g(s)ds$ has a positive zero $x = a$ and is monotone increasing for $x \geq a$;

(iv) $G(x) \to +\infty$ as $x \to +\infty$.

Under these conditions, the equation considered is called a **Liénard equation**. It describes the one dimensional movement of a unit mass subjected, at each point x, to a restoring force $-f(x)$ and to a damping proportional to the velocity with coefficient $g(x)$. Introducing the new variable $z = \dot{x} + G(x)$ we write this differential equation as a system of first order differential equations:

$$\begin{aligned} \dot{x} &= z - G(x) \\ \dot{z} &= -f(x). \end{aligned}$$

From (i) it is easy to conclude that the phase portrait of this system contains only one equilibrium point corresponding to the origin. Moreover, defining the function $V(x, z) = \frac{1}{2}z^2 + \int_0^x f(s)ds$ we have along solutions

$$\frac{d}{dt}V(x, z) = z\dot{z} + f(x)\dot{x} = -f(x)G(x).$$

By (ii) this relation implies that, close to the origin, V is nondecreasing along solutions. Since the level curves of V are closed curves encircling $(0, 0)$ it follows that any solution starting away from this equilibrium point cannot approach the origin in positive time.

Next, we will establish that the origin is contained in a positively invariant region, that is, a region which is invariant for $t \geq 0$. Then, since the ω-limit set of any point different from the origin cannot contain the only equilibrium point $(0, 0)$, the Poincaré-Bendixson theorem implies the existence of a periodic orbit inside this region.

We start by observing that the flow on the phase space (x, z) is to the right if $z > G(x)$ and to the left if $z < G(x)$. Moreover, it is horizontal on the z-axis, and it is decreasing on the right-half plane. By (i) and (ii), if $(x(t), z(t))$ represents a solution then $(-x(t), -z(t))$ is also a solution. From (iv) we conclude that a solution starting at $\alpha = (0, z_0)$ is shaped as sketched in Figure 2.26, intersecting the graph of G exactly at one point and reaching again the z-axis at a point $\beta = (0, -z_1)$. We will show that $z_0 > z_1$ if z_0 is taken sufficiently large. Then, appropriate arcs of orbits of this solution and its symmetric, and line segments in the z-axis define a Jordan curve whose

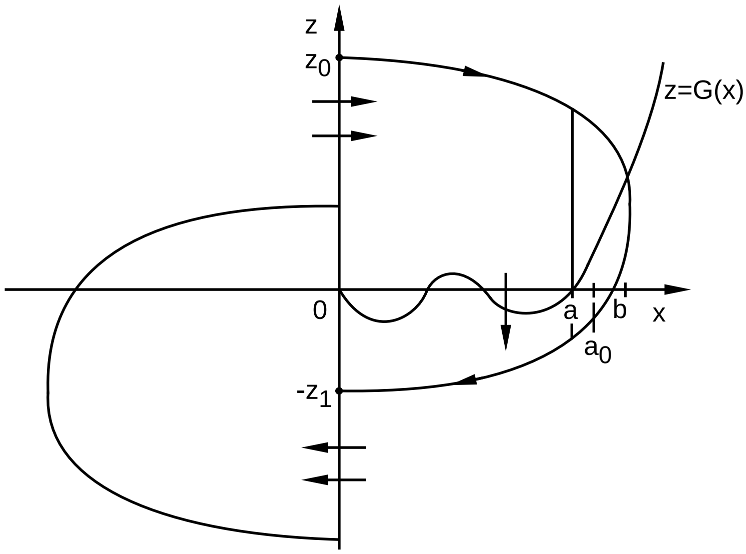

Figure 2.26:

interior is the desired positively invariant region (see Figure 2.26). In fact, a simple computation leads to the relations

$$\frac{dx}{dz} = -\frac{z - G(x)}{f(x)}, \qquad \frac{dV}{dx} = \frac{-f(x)G(x)}{z - G(x)}, \qquad \frac{dV}{dz} = G(x).$$

We denote by $x = \xi(z)$ the function corresponding to the solution for $-z_1 < z < z_0$. Also, for $dx/dz \neq 0$ we can define this solution as a function $z = z(x)$. Let $x = b$ correspond to the intersection point of the solution with the graph of G, that is, $dx(b)/dz = 0$. We define the functions ζ_0 , ζ_1 in $[0, b]$ such that $z = \zeta_0(x)$ and $z = \zeta_1(x)$ correspond to the above solution with $\zeta_0(0) = z_0$ and $\zeta_0(0) = -z_1$ respectively. Then, we have

$$\frac{1}{2}({z_1}^2 - {z_0}^2) = V(0, -z_1) - V(0, z_0) = \int_\alpha^\beta dV$$
$$= \quad -\int_0^a \frac{f(x)G(x)}{\zeta_0(x) - G(x)}\, dx + \int_{\zeta_0(a)}^{\zeta_1(a)} G(\xi(z))\, dz + \int_0^a \frac{f(x)G(x)}{\zeta_1(x) - G(x)}\, dx$$

Since for each x fixed, $\zeta_0(x), \zeta_1(x) \to +\infty$ as $z_0 \to +\infty$, the first and third integrals clearly approach zero as $z_0 \to +\infty$. Moreover, if z_0 is sufficiently large we have that $b > a$ and we can chose a_0 fixed such that $a < a_0 < b$. Then, we obtain

$$
\begin{aligned}
\int_{\zeta_0(a)}^{-\zeta_1(a)} G(\xi(z))\,dz &= -\int_{\zeta_1(a)}^{\zeta_0(a)} G(\zeta(z))\,dz \\
&< -\int_{\zeta_1(a_0)}^{\zeta_0(a_0)} G(\xi(z))\,dz < -G(a_0)[\xi_0(a_0) - \xi_1(a_0)],
\end{aligned}
$$

concluding that the second integral tends to $-\infty$ as $z_0 \to +\infty$. Therefore, if z_0 is taken sufficiently large we will have $z_0 > z_1$ completing the proof of the existence of a periodic orbit for this system.

The equation

$$
\ddot{u} - \mu(1-u^2)\dot{u} + u = 0, \qquad \mu > 0
$$

known as **van der Pol equation** is a famous example which belongs to the above class of equations. It played an important role in the early development of the study of nonlinear systems, providing an example for which traditional methods based on linearizations could not be used to prove the existence of a periodic orbit. It appeared as a model for the study of oscillating solutions of an electronic circuit.

Chapter 3

Linearization: behaviour near equilibria and periodic orbits

3.1 Linear systems and linear variational equations

There exists a definitive and simple theory for linear systems, therefore, in the study of nonlinear smooth systems it is useful to take the most possible advantage from approximating the nonlinear system by a linear one. Of course, we can only expect to obtain in this way information on the local behaviour of a nonlinear system around the points, orbits or sets of orbits where the linearization is considered. Although this technique is quite general, it will be illustrated here mostly for the study of the behaviour of a dynamical system in neighborhoods of equilibria or periodic orbits. One of its consequences is to complete the local analysis of a smooth dynamical system around a point p in the phase space. In fact, as shown in the preceding chapter, if p is a regular point the flow in a neighborhood of it is parallel and, therefore, only the case where p is an equilibrium requires further analysis.

A dynamical system φ in X is said to be a **linear system** if the map $x \to \varphi(t, x)$ is linear. Clearly, if φ is a smooth linear system the vector field associated to φ is linear and the associated ordinary differential equation is $\dot{x} = Ax$ where A represents the linear map $x \to \partial\varphi(0, x)/\partial t$. Thus, smooth linear systems are well understood since linear differential equations form a class for which there exists a definitive theory.

If φ is a smooth nonlinear system and γ_x denotes a particular orbit, one might expect to understand the behaviour of φ in a neighborhood of γ_x by considering its best linear approximation obtained from the first term of the Taylor series expansion of φ around γ_x . This process is called a

linearization and its justification is the subject of this chapter.

For simplicity, we consider $X = \mathbb{R}^n$. The linearization of the flow φ around a point $p \in \gamma_x$ is given by the linear mapping $z \to D_x\varphi(t,p)z$ in $\mathbb{R}^n$. As time evolves, the point where the linearization should be taken changes along the orbit and, therefore, to describe a deterministic evolution process we need to consider pairs $(z,x) \in \mathbb{R}^n \times \mathbb{R}^n$ containing information about the direction of linearization and the point at which the flow is linearized. Accordingly, the linearization of φ around the orbit γ_x can be seen as defining a flow in $\mathbb{R}^n \times \mathbb{R}^n$ by $\pi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ with $\pi(t,z,x) = (\Psi(t,z,x), \varphi(t,x))$, where $\Psi(t,z,x) = D_x\varphi(t,x)z$. Immediately one verifies that π satisfies the definition of a skew product flow. Since $z \to \Psi(t,z,x)$ is linear, π is called a **linear skew product flow** this particular skew product flow is known as the **linearized flow around** γ_x. It is the dynamical system π that is considered in the study of the linearization of φ around orbits.

Since the dynamical system φ is smooth, let f denote its associated vector field, $f(x) = \partial\varphi(0,x)/\partial t$, and let $\dot{x} = f(x)$ be the corresponding ordinary differential equation. In the following we always assume that the vector field f is of class C^1. We notice that, for fixed $(z,x) \in \mathbb{R}^n \times \mathbb{R}^n, \Psi$ satisfies a linear differential equation of the form $\dot{y} = A(t)y$. In fact, computing the derivative of ψ relative to the time variable, and using the fact that $\partial\varphi(t,x)/\partial t = f(\varphi(t,x))$, we obtain

$$
\begin{aligned}
& \frac{\partial}{\partial t}\Psi(t,z,x) = D_{tx}\varphi(t,x)z = D_x f(\varphi(t,x))z \\
= \quad & Df(\varphi(t,x))D_x\varphi(t,x)z = Df(\varphi(t,x))\Psi(t,z,x),
\end{aligned}
$$

and the result follows by taking $A(t) = Df(\varphi(t,x))$. This equation determines the evolution of the linearized flow around γ_x in the direction of linearization which at the point x is given by z. By this reason the above equation is known as the **linear variational equation** of φ around the orbit γ_x. In general this equation is nonautonomous, that is, the corresponding vector field depends explicitly on t. When the orbit λ_x is an equilibrium point, then $\varphi(t,x) = x$ for every t and the linear variational equation becomes $\dot{y} = Ay$ with $A = Df(x)$. Since this equation is autonomous we can define the dynamical system θ on $\mathbb{R}^n$ by taking $\theta(t,z) = \Psi(t,z,x)$ and substitute π by θ in the linearization, calling θ the **linearized flow around the equilibrium** x.

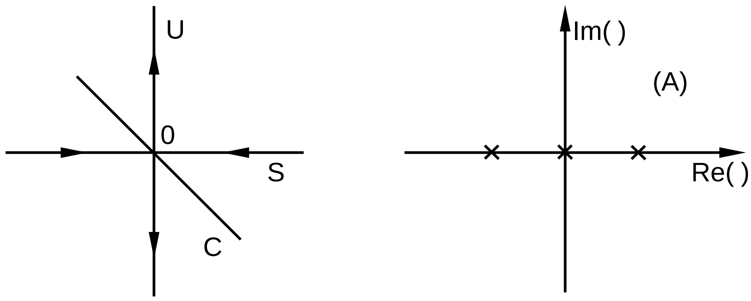

Figure 3.1:

3.2 Stable, unstable and center manifolds for an equilibrium

In order to be specific in a situation as simple as possible, we consider in this section the particular case of linearization around an equilibrium point.

Let x_0 be an equilibrium point for the dynamical system φ and let $\dot{y} = Ay$ denote the linear variational equation corresponding to the linearized flow θ around the equilibrium point x_0. Denote by $\sigma(A) = \{\lambda_1, \ldots, \lambda_m\}$ the set of eigenvalues of A, called the spectrum of A. If we consider the generalized eigenspaces corresponding to the eigenvalues of A, we clearly obtain that the behaviour of $\theta(t, z)$ for z on each of these invariant subspaces is quite different. It is a consequence of the known theory of linear ordinary differential equations that, if z belongs to a generalized eigenspace of A corresponding to $\lambda \in \sigma(A)$ then $\theta(t, z)$ exhibits an exponential rate of approach to zero as $t \to +\infty$ if λ has negative real part, $Re(\lambda) < 0$; $\theta(t, z)$ exhibits an exponential rate of approach to zero as $t \to -\infty$ if $Re(\lambda) > 0$; and $\theta(t, z)$ either does not approach zero or goes to zero with a rate lower than exponential as $t \to +\infty$ or $t \to -\infty$ if $Re(\lambda) = 0$ (in fact, if $\theta(t, z) \to 0$ as $t \to +\infty$ or $t \to -\infty$ it does so with a polynomial rate of approach). The span of the invariant subspaces corresponding to the eigenvalues with negative, positive and zero real parts are linear manifolds which, due to the behaviour of $\theta(t, z)$, are called respectively stable, unstable and center manifolds of the origin for the linear system θ and denoted by S, U and C. Accordingly, the phase space of the linearized flow θ can be decomposed as $\mathbb{R}^n = S \oplus C \oplus U$ (see Figure 3.1).

The exponential behaviour of θ on each of these subspaces motivates the following characterization of the manifolds S, U and C:

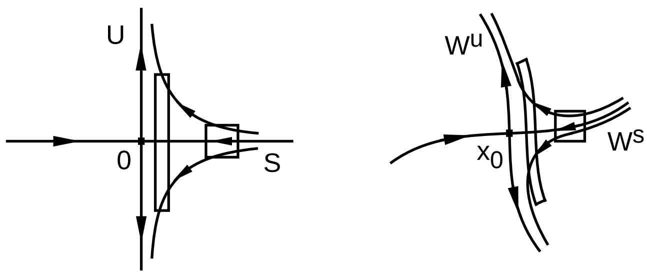

Figure 3.2:

$$
\begin{aligned}
S &= \{z \in \mathbb{R}^n : \text{ there exists } \mu > 0 \text{ such that } e^{\mu t}\theta(t,z) \to 0 \text{ as } t \to +\infty\} \\
U &= \{z \in \mathbb{R}^n : \text{ there exists } \mu > 0 \text{ such that } e^{-\mu t}\theta(t,z) \to 0 \text{ as } t \to -\infty\} \\
C &= \{z \in \mathbb{R}^n \text{ for every } \mu > 0, e^{-\mu|t|}\theta(t,z) \to 0 \text{ as } t \to \pm\infty\}
\end{aligned}
$$

When $\sigma(A)$ does not contain eigenvalues with zero real part the decomposition of the phase space of θ becomes simply $\mathbb{R}^n = S \oplus U$ and the behaviour of θ on each invariant subspace is characterized by an exponential rate of approach to zero either forwards or backwards in time. In this case such a decompositon of the phase space is called an **exponential dichotomy**. Moreover, all the orbits that are neither in S nor in U will leave a neighborhood of the origin both forwards and backwards in time, the phase portrait of θ is said to have a **saddle-point** at the origin and the equilibrium is said to be **hyperbolic**. In this case, we expect the exponential behaviour to persist when the system is subjected to a small perturbation in a neighborhood of the origin. Since the nonlinear flow φ near the equilibrium point x_0 can be considered a small perturbation of the linearized flow θ, we expect φ to exhibit the described exponential behaviour in relation to manifolds of initial conditions corresponding to the perturbation of the linear subspaces S and U. In fact this is the case. We will show in the following sections that under the above conditions the phase portrait of φ at the point x_0 has the same saddle-point property than its linearization θ (see Figure 3.2).

The clear separation between the exponential decay on S and the exponential growth on U is important in the above reasoning. To obtain similar results in more general situations all that is needed is a clear separation between different exponential rates. In fact, performing the change of variables $y \to e^{-\mu t}y$ in the linear variational equation $\dot{y} = Ay$ we obtain $\dot{y} = A_\mu y$

with $A_\mu = A - \mu I$ and the spectrum $\sigma(A_\mu)$ is just a translation by μ of $\sigma(A)$. Hence we can always find $\mu \in \mathbb{R}$ such that $\sigma(A_\mu)$ does not contain eigenvalues with zero real part and obtain for the phase space of θ the decomposition $\mathbb{R}^n = S_\mu \oplus U_\mu$ where S_μ and U_μ are the invariant subspaces corresponding to the eigenvalues in $\sigma(A_\mu)$ with negative and positive real parts respectively. When $\sigma(A)$ contains eigenvalues with zero real part the general decomposition $\mathbb{R}^n = S \oplus C \oplus U$ can be obtained from the above decompositions by taking $\mathbb{R}^n = S_{-\mu} \oplus U_{-\mu} = S_\mu \oplus U_\mu$ for $\mu > 0$ sufficiently small, so that $S = S_{-\mu}$, $U = U_\mu$ and $C = S_\mu \cap U_{-\mu}$. We will show that this type of decomposition also persists under small nonlinear perturbations. With this objective in mind, we define for the nonlinear flow φ the local stable and unstable sets of the equilibrium x_0 that we denote respectively by W^u_{loc} and W^u_{loc} in the following way. If V is neighborhood of x_0 then

$$
\begin{aligned}
W^s_{loc} &= \{x \in V : \varphi(t,x) \in V \text{ for } t \geq 0 \text{ and there exists } \mu > 0 \\
&\quad \text{such that } e^{\mu t}(\varphi(t,x) - x_0) \to 0 \text{ as } t \to +\infty\},
\end{aligned}
$$

$$
\begin{aligned}
W^u_{loc} &= \{x \in V : \varphi(t,x) \in V \text{ for } t \leq 0 \text{ and there exists } \mu > 0 \\
&\quad \text{such that } e^{-\mu t}(\varphi(t,x) - x_0) \to 0 \text{ as } t \to -\infty\}.
\end{aligned}
$$

These sets will be shown to be manifolds diffeomorphic to open subsets of S and U, and tangent at x_0 to the linear manifolds $S + \{x_0\}$ and $U + \{x_0\}$, respectively; they are called, respectively, the **local stable manifold** and the **local unstable manifold** of the equilibrium x_0. Furthermore, we will also prove the existence of a **local center manifold** denoted by W^c_{loc} contained in the set $\{x_0\} \cup \{x \in V : \varphi(t,x) \in V$ for $t \geq 0$ or $t \leq 0$ and for every $\mu > 0$, $e^{\mu|t|}(\varphi(t,x) - x_0) + x_0$ leaves V both for $t \geq 0$ and $t \leq 0\}$ diffeomorphic to C and tangent at x_0 to the linear manifold $C + \{x_0\}$.

The manifold structure mentioned here is important because it provides a local set of coordinates for a neighborhood of x_0 and the qualitative features of the flow near x_0, up to homeomorphisms preserving orbits and the direction in time, are completely determined by the flow restricted to W^c_{loc} and the dimensions of S and U, since the solutions passing through points on W^s_{loc} or W^u_{loc} are known to approach x_0 with exponential rates as $t \to +\infty$ or $t \to -\infty$, respectively, and to leave a sufficiently small neighborhood of x_0 in the opposite time directions. Therefore, for the characterization of the phase portrait locally around an equilibrium in a specific system, besides computing the dimensions of the stable and unstable manifolds, S and U, for the system obtained by linearization around the equilibrium, it is sufficient to characterize the flow restricted to W^c_{loc}. The flow in W^c_{loc} depends strongly on the specific nonlinearities in the system; in general, it may assume any of all the possibilities of the local flow around an equilibrium for an arbitrary system in a phase space with the same dimension as W^c_{loc}. As an example in $\mathbb{R}^3$, we consider a system that locally around the origin is defined by the solutions of the ODE

$$\begin{aligned}\dot{x} &= y - (x^2+y^2)x \\ \dot{y} &= -x - (x^2+y^2)y \\ \dot{z} &= -z - (x^2+y^2)(2x^2+2y^2-1)\end{aligned}$$

Clearly, the origin is an equilibrium and $S = \{(0,0,z) : z \in \mathbb{R}\}$, $U = \{(0,0,0)\}$, $C = \{(x,y,0) : x,y \in \mathbb{R}\}$. It is easy to verify that, in a neighborhood V of the origin, $W^s_{loc} = \{(0,0,z) \in V : z \in \mathbb{R}\}$ and ${W_{loc}}^u = \{(0,0,0)\}$. The function $u(t) = x^2(t) + y^2(t)$, satisfies

$$\dot{u} = 2x\dot{x} + 2y\dot{y} = 2xy - 2(x^2+y^2)x^2 - 2xy - 2(x^2+y^2)y^2 = -2u^2 \quad ,$$

and, therefore, $z(t) = u(t)$ satisfies the last equation in the system. We conclude that if $(x(t), y(t))$ is a solution of the first two equations, then $(x(t), y(t), x^2(t)+y^2(t))$ is a solution of the overall system and the square of its norm is $u^2 + u^4$. Solving $\dot{u} = -2u^2$, we compute $u(t) = 1/(2t+c)$ for an appropriate constant $c \in \mathbb{R}$, and consequently, for every $\mu > 0$, $e^{\mu|t|}u(t)$ is unbounded for $t \to +\infty$ and $t \to -\infty$. Therefore, one possibility for the local center manifold in this case is $W^c_{loc} = \{(x,y,x^2+y^2) \in V : x,y \in \mathbb{R}\}$ and the flow restricted to W^c_{loc} is completely determined by the equation $\dot{u} = -2u^2$ (see Figure 3.3).

The sets W^s_{loc} and W^u_{loc} consist of semiorbits of points in V which are bounded for, respectively, positive and negative time, and similarly, W^c_{loc} is a subset of the points x in V such that $\varphi(t,x)$ is bounded either forwards or backwards in time. Therefore, in order to study $W^s_{loc}, W^u_{loc}, W^c_{loc}$ it is desirable to characterize analytically the solutions of the differential equations corresponding to the flows which are bounded either for positive or for negative time.

3.3 Bounded solutions for forced linear equations with exponential dichotomy

In this section we consider the characterization of solutions which are bounded for positive or negative time for perturbations of the general linear

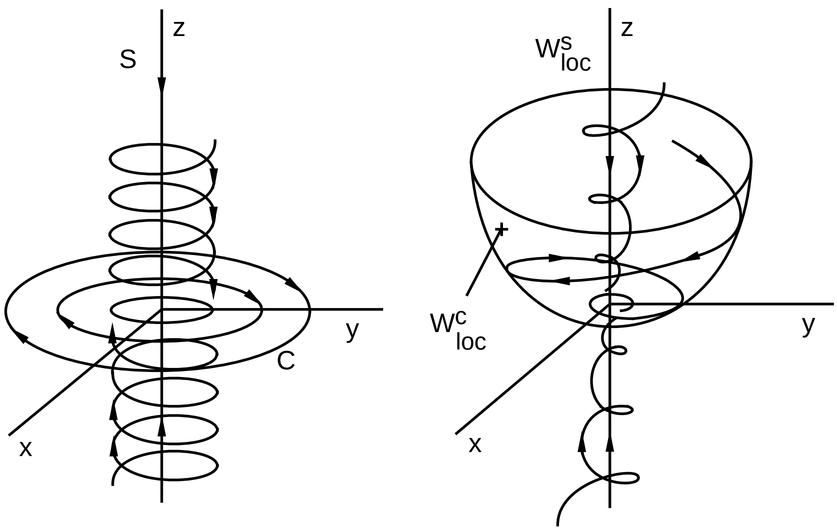

Figure 3.3:

variational equation of the form $\dot{z} = A(t)z$. We begin by considering perturbations of the form $\dot{x} = A(t)x + h(t)$ with h being a bounded continuous function, and afterwards we consider perturbations of the form $\dot{x} = A(t)x + g(t,x)$ with $g(t,0) = 0$ for all $t \in \mathbb{R}$ and $g(t,x)$ globally lipschitzian in x with a sufficiently small lipschitz constant. In order to do the analysis, we first extend to nonautonomous linear variational equations the notion of exponential dichotomy introduced in the previous section to separate the different exponential behaviour in a neighborhood of the origin.

Let $z = z(t,\xi)$ denote the solution of the linear equation $\dot{z} = A(t)z$ satisfying $z(0,\xi) = \xi$ and let $\Psi = \Psi(t)$ denote the linear operator $\Psi(t) : \xi \to z(t,\xi)$. This operator is called the **principal matrix solution** of the linear equation. It clearly satisfies $\Psi(0) = I$, the identity operator, and from the theory of linear differential equations it follows that $\Psi(t)$ is nonsingular for every $t \in \mathbb{R}$.

To define an exponential dichotomy for the nonautonomous linear variational equation it is natural to look for spaces of initial conditions $\xi \in \mathbb{R}^n$ such that the solutions $z(t,\xi) = \Psi(t)\xi$ of this equation exhibit the adequate exponential behaviour. We say that the linear variational equation $\dot{z} = A(t)z$ has an **exponential dichotomy** if there exist a linear projection $P : \mathbb{R}^n \to \mathbb{R}^n$, that is a linear operator P such that $P^2 = P$, and positive constants K and α such that for every $\xi \in \mathbb{R}^n$

$$|\Psi(t)P\xi| \leq Ke^{-\alpha(t-s)}|\Psi(s)\xi| \text{ for every } t \geq s,$$

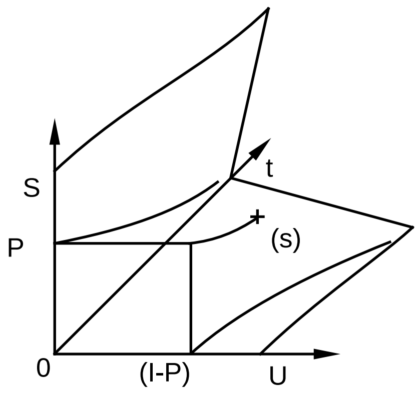

Figure 3.4:

$$|\Psi(t)(I-P)\xi| \leq Ke^{+\alpha(t-s)}|\Psi(s)\xi| \text{ for every } t \leq s.$$

If there is an exponential dichotomy we can define the linear stable and unstable subspaces S and U corresponding to the initial conditions for which the solutions approach zero forwards and backwards in time respectively

$$S = \{\xi \in \mathbb{R}^n : z(t,\xi) \to 0 \quad \text{as} \quad t \to +\infty\},$$
$$U = \{\xi \in \mathbb{R}^n : z(t,\xi) \to 0 \quad \text{as} \quad t \to -\infty\},$$

and again we obtain the decomposition $\mathbb{R}^n = S \oplus U$. Moreover, P is just the projection of $\mathbb{R}^n$ onto S along U and, similarly, $I-P$ is the projection of $\mathbb{R}^n$ onto U along S (see Figure 3.4).

As an example we consider the linear equation $\dot{z} = A(t)z$ defined in $\mathbb{R}^2$ with

$$A(t) = \begin{bmatrix} -1 & a(t) \\ 0 & 1 \end{bmatrix}, \qquad a(t) = \frac{4e^t + 6e^{2t}}{(1+e^t)^2}.$$

Denoting the solution with initial condition $\xi = (\xi_1, \xi_2)$ by $z(t,\xi) = (z_1(t,\xi), z_2(t,\xi))$, we have

$$z_1(t,\xi) = e^{-t}(\xi_1 - \xi_2) + \frac{2e^{2t}}{1+e^t}\xi_2,$$
$$z_2(t,\xi) = e^t \xi_2,$$

from which we can conclude that this equation has an exponential dichotomy with projection P given by $P(\xi_1, \xi_2) = (\xi_1 - \xi_2, 0)$ and constants $K = \alpha = 1$.

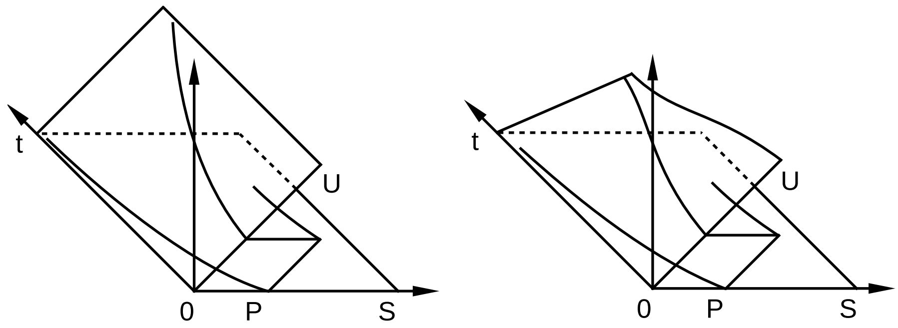

Figure 3.5:

Furthermore, we have $S = \{(\zeta, 0) : \zeta \in \mathbb{R}\}$, $U = \{(\zeta, \zeta) : \zeta \in \mathbb{R}\}$. If for each fixed $t \in \mathbb{R}$ we consider the subspaces $\Psi(t)S$ and $\Psi(t)U$ in $\mathbb{R}^2$ we have that $\Psi(t)S = S$ for every t and $\Psi(t)U$ approaches $\{(2\zeta, \zeta) : \zeta \in \mathbb{R}\}$ as $t \to +\infty$ and $\{(0, \zeta) : \zeta \in \mathbb{R}\}$ as $t \to -\infty$. We remark that, in general, the spaces S and U are different from the stable and unstable manifolds of the linear autonomous equation $\dot{z} = Bz$ obtained by freezing the time dependence on A, that is, by taking $B = A(t_0)$ for some fixed t_0. In fact, taking the above example one easily verifies that the unstable manifold for the equation $\dot{z} = Bz$ with $B = A(0)$ is given by $\{(5\zeta, 4\zeta) : \zeta \in \mathbb{R}\}$. For this example we sketch in Figure 3.5 the sets of trajectories of the stable and unstable manifolds of both equations $\dot{z} = A(0)z$ and $\dot{z} = A(t)z$.

In the following we denote by $BC(\mathbb{R})$ the Banach space of bounded continuous functions $h : \mathbb{R} \to \mathbb{R}^n$, endowed with the uniform norm $||h|| = \sup\{|h(t)| : t \in \mathbb{R}\}$. Similarly, we define the spaces $BC(\mathbb{R}^+)$, $BC(\mathbb{R}^-)$ with $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$, respectively, and denote the corresponding norms by $||\cdot||_+$ and $||\cdot||_-$.

The question of existence and uniqueness of bounded solutions of a nonautonomous linear variational equation with an exponential dichotomy can now be settled.

(3.1) **Theorem**: *If the equation $\dot{z} = A(t)z$ has an exponential dichotomy with projection P and constants $K, \alpha > 0$, then for each function $h \in BC(\mathbb{R}^+)$ there is a unique solution $B^+h \in BC(\mathbb{R}^+)$ of the perturbed equation $\dot{x} = A(t)x + h(t)$ satisfying $PB^+h(0) = 0$. This solution is given by the linear operator $B^+ : BC(\mathbb{R}^+) \to BC(\mathbb{R}^+)$ such that*

$$B^+h(t) = \int_0^t \Psi(t)P\Psi^{-1}(s)h(s)ds - \int_t^{+\infty} \Psi(t)(I-P)\Psi^{-1}(s)h(s)ds$$

and satisfies $||B^+h||_+ \leq \frac{2K}{\alpha}||h||_+$. *Furthermore, if* $x = x(t,\xi)$ *denotes the solution with initial condition* $x(0,\xi) = \xi$, *then* $x(\cdot,\xi) \in BC(\mathbb{R}^+)$ *if and only if it has the form*

$$x(t,\xi) = \Psi(t)P\xi + B^+h(t),$$

or $(I-P)\xi = B^+h(0)$.

Similarly, for each function $h \in BC(\mathbb{R}^-)$ *there is a unique solution* $B^-h \in BC(\mathbb{R}^-)$ *satisfying* $(I-P)B^-h(0) = 0$. *This solution is given by the linear operator* $B^- : BC(\mathbb{R}^-) \to BC(\mathbb{R}^-)$ *such that*

$$B^-h(t) = \int_0^t \Psi(t)(I-P)\Psi^{-1}(s)h(s)ds + \int_{-\infty}^t \Psi(t)P\Psi^{-1}(s)h(s)ds$$

and satisfies $||B^-h||_- \leq \frac{2K}{\alpha}||h||_-$. *Furthermore, the solution* $x = x(t,\xi)$ *satisfies* $x(\cdot,\xi) \in BC(\mathbb{R}^-)$ *if and only if it has the form*

$$x(t,\xi) = \Psi(t)(I-P)\xi + B^-h(t),$$

or $P\xi = B^-h(0)$.

Also, for $h \in BC(\mathbb{R})$ *there is a unique solution* $x(t,\xi)$ *in* $BC(\mathbb{R})$ *and its initial value is* $\xi = B^-h(0) + B^+h(0)$.

Proof: We prove this result only for the case $h \in BC(\mathbb{R}^+)$, since the case $h \in BC(\mathbb{R}^-)$ can be reduced to the first by reversing the time direction $t \to -t$. By a direct computation it can be checked that B^+h verifies the differential equation $\dot{x} = A(t)x + h(t)$ and that $PB^+h(0) = 0$. Since the equation is linear the solution is unique. Furthermore, we have the following estimate

$$\begin{aligned}
||B^+h||_+ &= \sup_{t\geq 0} \left| \int_0^t \Psi(t)P\Psi^{-1}(s)h(s)ds - \int_t^{+\infty} \Psi(t)(I-P)\Psi^{-1}(s)h(s)ds \right| \\
&\leq \sup_{t\geq 0} \left[\int_0^t |\Psi(t)P\Psi^{-1}(s)h(s)|ds + \int_t^{+\infty} |\Psi(t)(I-P)\Psi^{-1}(s)h(s)|ds \right] \\
&\leq \sup_{t\geq 0} \left[K\int_0^t e^{-\alpha(t-s)}|h(s)|ds + K\int_t^{+\infty} e^{+\alpha(t-s)}|h(s)|ds \right] \\
&\leq K \; ||h||_+ \; \sup_{t\geq 0} \; \left[e^{-\alpha t}\int_0^t e^{\alpha s}ds + e^{\alpha t}\int_t^{+\infty} e^{-\alpha s}ds \right] \leq \frac{2K}{\alpha}||h||_+.
\end{aligned}$$

Hence, $B^+h \in BC(\mathbb{R}^+)$. Finally, the general solution of the perturbed equation is of the form

$$w(t) = \Psi(t)\eta + B^+h(t) = \Psi(t)P\eta + \Psi(t)(I-P)\eta + B^+h(t),$$

and since $\sup_{t\geq 0} |\Psi(t)P\eta| \leq K|\eta|$ we have $\Psi(\cdot)P\eta \in BC(\mathbb{R}^+)$. On the other hand, we have

$$|\Psi(t)(I-P)\eta| = |\Psi(t)(I-P)(I-P)\eta| \leq Ke^{+\alpha(t-s)}|\Psi(s)(I-P)\eta|, \qquad s \geq t,$$

and taking $t = 0$ we obtain

$$|\Psi(s)(I-P)\eta| \geq \frac{1}{K}e^{\alpha s}|(I-P)\eta|, \quad s \geq 0.$$

Hence, $\Psi(s)(I-P)\eta$ is in $BC(\mathbb{R}^+)$ if and only if $(I-P)\eta = 0$. Since $PB^+h(0) = 0$, we conclude that a solution $x = x(t,\xi)$ of the perturbed equation is in $BC(\mathbb{R}^+)$ if and only if it has the form

$$x(t,\xi) = \Psi(t)P\xi + B^+h(t).$$

Now, suppose $h \in BC(\mathbb{R})$. In order to have a solution $x(t,\xi)$ in $BC(\mathbb{R})$, it follows from the above that we must have $P\xi = B^-h(0)$ and $(I-P)\xi = B^+h(0)$. Since $B^-h(0) \in P\mathbb{R}^n$ and $B^+h(0) \in (I-P)\mathbb{R}^n$, this happens only for $\xi = B^-h(0) + B^+h(0)$.

QED

In Figure 3.6 we sketch, for an example, the sets of trajectories of the stable and unstable manifolds of a linear equation of the form $\dot{z} = A(t)z$ with an exponential dichotomy and also the sets of trajectories of the perturbed equation $\dot{x} = A(t)x + h(t)$ which are bounded forwards or backwards in time.

3.4 Bounded solutions for weakly nonlinear equations with exponential dichotomy

We consider now perturbations of the linear variational equation of the form

$$\dot{x} = A(t)x + g(t,x),$$

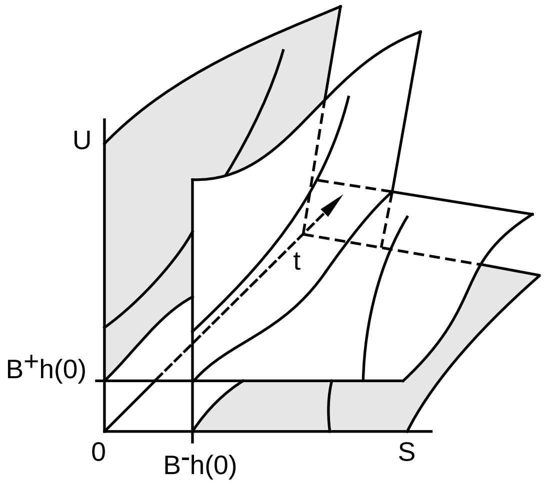

Figure 3.6:

where $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is in the space L_b of continuous functions wich satisfy $g(t, 0) = 0$ and are globally lipschitzian in the second variable with lipschitz constant b, that is, for every $t \in \mathbb{R}$ and x, $y \in \mathbb{R}^n$ we have

$$|g(t, x) - g(t, y)| \leq b|x - y|.$$

We are interested in characterizing the sets of solutions of the perturbed equation which are bounded for positive or negative time. We call these sets respectively stable and unstable and, as in the autonomous case, denote by W^s and W^u:

$$\begin{aligned} W^s &= \{\xi \in \mathbb{R}^n : x(t, \xi) \text{ is bounded for } t \geq 0\}, \\ W^u &= \{\xi \in \mathbb{R}^n) : x(t, \xi) \text{ is bounded for } t \leq 0\}. \end{aligned}$$

It will be shown that, under the appropriate conditions, the solutions with initial conditions on these sets exhibit the expected exponential behaviour. From the previous theorem, we expect to obtain an analytic characterization of the solutions which are bounded for, respectively, positive and negative time by considering the solutions of the integral equations

$$\begin{aligned} z(t, \zeta) &= \Psi(t)\zeta + B^+[g(\cdot, z(\cdot, \zeta))](t), \quad \zeta \in S, \\ z(t, \zeta) &= \Psi(t)\zeta + B^-[g(\cdot, z(\cdot, \zeta))](t), \quad \zeta \in U, \end{aligned}$$

where $g(\cdot, z(\cdot, \zeta))$ denotes the function h such that $h(t) = g(t, z(t, \zeta))$. The solutions of these equations are fixed points of the mappings $T^{\pm} z(t, \zeta) = \Psi(t)\zeta + B^{\pm}[g(\cdot, z(\cdot, \zeta))](t)$ in adequate function spaces. In order to establish the existence and uniqueness of these solutions one can use the following well known fixed point theorem.

(3.2) **Theorem** (Contraction mapping theorem): *Let Z be a complete metric space with metric $d : Z^2 \to \mathbb{R}$ and $T : Z \to Z$ a function which contracts distances, in the sense that there exists a constant $k \in [0, 1)$ such that $d(Tz_1, Tz_2) \leq kd(z_1, z_2)$ for $z_1, z_2 \in Z$. Then T has a unique fixed point $z \in Z$.*

To establish the exponential behaviour of the solutions with initial conditions on the stable or unstable sets we also need the following result.

(3.3)**Lemma** (Gronwall's inequality): *If c, β are nonnegative constants and u is a continuous nonnegative function $u : [0, r] \to \mathbb{R}^+$ satisfying*

$$u(t) \leq c + \int_0^t \beta u(s) ds$$

then, for $t \in [0, r]$, we have

$$u(t) \leq ce^{\beta t}.$$

Proof : If $c > 0$, let $\phi(t) = c + \int_0^t \beta u(s) ds$. For $t \in [0, r]$ we have $\phi(t) > 0$ and $u(t) \leq \phi(t)$. Since $d\phi(t)/dt = \beta u(t)$ we also have

$$\frac{d}{dt} \log \phi(t) = \frac{d\phi(t)/dt}{\phi(t)} = \frac{\beta u(t)}{\phi(t)} \leq \beta,$$

and, after an integration we obtain $\log \phi(t) \leq \log \phi(0) + \beta t$. Taking the exponential of both sides we get

$$u(t) \leq \phi(t) \leq \phi(0)e^{\beta t} = ce^{\beta t}.$$

The case $c = 0$ is obtained by taking the limit as $c \to 0$.

QED

If H is a linear subspace of $\mathbb{R}^n$ and $\Omega \geq 0$ is a real number, we denote by $C(H, \Omega)$ the cone around H with aperture Ω ,

$$C(H, \Omega) = \{\zeta + \eta \in \mathbb{R}^n : \zeta \in H, |\eta| \leq \Omega|\zeta|\}.$$

(3.4) **Theorem**: *If the linear equation $\dot{z} = A(t)z$ has an exponential dichotomy with constants $K, \alpha > 0$ as in the definition of exponential dichotomy and stable and unstable linear manifolds S and U, $g \in L_b$ with $b > 0$ sufficiently small, then the stable and unstable sets W^s and W^u for the perturbed equation $\dot{x} = A(t)x + g(t, x)$ are graphs of globally lipschitzian functions defined on S and U, respectively, which intersect exactly at the origin and are contained in cones*

$$W^s \in C(S, \Omega), \qquad W^u \in C(U, \Omega)$$

with apertures $\Omega = \Omega(b)$ satisfying $\Omega(b) \to 0$ as $b \to 0$. Moreover, there are positive constants c , β such that

$$|x(t, \xi)| \leq ce^{-\beta t}|\xi| \text{ for } t \geq 0 \text{ and } \xi \in W^s,$$

$$|x(t, \xi)| \leq ce^{+\beta t}|\xi| \text{ for } t \leq 0 \text{ and } \xi \in W^u,$$

and $\beta = \beta(b) \to \alpha$ as $b \to 0$.

Proof: We prove this result only for the stable set, the proof for the unstable being entirely similar.

Let Z denote the space of continuous functions $z : \mathbb{R} \times S \to \mathbb{R}^n$, such that:

(i) for each fixed $\zeta \in S$, $z(t, \zeta)$ is bounded for $t \geq 0$;

(ii) $z(t, \zeta)$ is globally lipschitzian in ζ with constant $K+1$, that is, for every $t \geq 0$ and every $\zeta_1, \zeta_2 \in S$

$$|z(t, \zeta_1) - z(t, \zeta_2)| \leq (K + 1)|\zeta_1 - \zeta_2|.$$

For each $z \in Z$ and each positive integer μ we define

$$||z||_m = \sup\{|z(t, \zeta)| : t \geq 0, |\zeta| \leq m\}$$

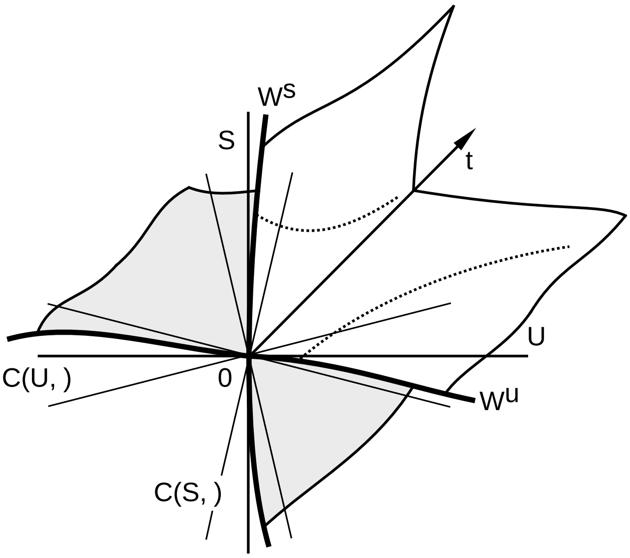

Figure 3.7:

and

$$d(z_1, z_2) = \sum_{m=1}^{+\infty} 2^{-m} ||z_1 - z_2||_m, \qquad z_1, z_2 \in Z.$$

It is easy to see that the function $d : Z^2 \to \mathbb{R}$ defined by this formula is a metric and that Z with this metric is a complete metric space.

We first show that the mapping T defined for $z \in Z$ by

$$Tz(t, \zeta) = \Psi(t)\zeta + B^+[g(\cdot, z(\cdot, \zeta))](t), \quad \zeta \in S$$

takes Z into Z and is a contraction mapping. Given $z \in Z$, for each $\zeta \in S$ the function $Tz(t, \zeta)$ is continuous and bounded for $t \geq 0$. In fact, using the exponential dichotomy for the linear equation and the inequality $||B^+h||_+ \leq 2K\alpha^{-1}||h||_+$, we have

$$\begin{aligned} |Tz(t,\zeta)| &\leq |\Psi(t)\zeta| + \sup_{t\geq 0} |B^+[g(\cdot, z(\cdot,\zeta))](t)| \\ &\leq K|\zeta| + 2K\alpha^{-1} \sup_{t\geq 0} |g(t, z(t,\zeta))| \leq K|\zeta| + 2K\alpha^{-1} b \sup_{t\geq 0} |z(t,\zeta)|, \end{aligned}$$

since for $g \in L_b$ we obtain $|g(t, z(t, \zeta))| = |g(t, z(t, \zeta)) - g(t, 0)| \leq b|z(t, \zeta))|$. Moreover, $Tz(t, \zeta)$ is globally lipschitzian, since for ζ_1, $z_2 \in S$ we have

$$
\begin{aligned}
& |Tz(t,\zeta_1) - Tz(t,\zeta_2)| \\
\leq\; & |\Psi(t)(\zeta_1-\zeta_2)| + \sup_{t\geq 0} |B^+[g(\cdot, z(\cdot,\zeta_1)) - g(\cdot, z(\cdot,\zeta_2))](t)| \\
\leq\; & K|\zeta_1-\zeta_2| + 2K\alpha^{-1} \sup_{t\geq 0} |g(t, z(t,\zeta_1)) - g(t, z(t,\zeta_2))| \\
\leq\; & K|\zeta_1-\zeta_2| + 2K\alpha^{-1} b \sup_{t\geq 0} |z(t,\zeta_1) - z(t,\zeta_2)| \\
\leq\; & [K + 2K(K+1)\alpha^{-1}b]|\zeta_1-\zeta_2)|
\end{aligned}
$$

Also, from this expression we conclude that if $b > 0$ is such that $2K(K+1)\alpha^{-1}b < 1$ then $Tz(t,\zeta)$ is globally lipschitzian with constant $K+1$, that is, T takes Z into Z. For z_1, $z_2 \in Z$ and any positive integer m, we have:

$$
\begin{aligned}
& ||Tz_1 - Tz_2||_m = \sup_{|\zeta|\leq m} \sup_{t\geq 0} |Tz_1(t,\zeta) - Tz_2(t,\zeta)| \\
=\; & \sup_{|\zeta|\leq m} \sup_{t\geq 0} |B^+[g(\cdot, z_1(\cdot,\zeta)) - g(\cdot, z_2(.,\zeta))](t)| \\
\leq\; & \sup_{|\zeta|\leq m} 2K\alpha^{-1} \sup_{t\geq 0} |g(t, z_1(t,\zeta)) - g(t, z_2(t,\zeta))| \\
\leq\; & 2K\alpha^{-1}b \sup_{|\zeta|\leq m} \sup_{t\geq 0} |z_1(t,\zeta) - z_2(t,\zeta)| \leq 2K\alpha^{-1}b||z_1 - z_2||_m.
\end{aligned}
$$

We define $k = 2K\alpha^{-1}b$ and note that $k < 2K(K+1)\alpha^{-1}b < 1$. Then $||Tz_1 - Tz_2||_m \leq k||z_1 - z_2||_m$ and

$$
d(Tz_1, Tz_2) = \sum_{m=1}^{+\infty} 2^{-m}||Tz_1 - Tz_2||_m \leq k \sum_{m=1}^{+\infty} 2^{-m}||z_1 - z_2||_m \leq kd(z_1, z_2).
$$

Consequently, $T : Z \to Z$ is a contraction.

Let $z(t,\zeta)$ denote the unique fixed point of T, $Tz = z$. Since $z(\cdot,\zeta) \in BC(R^+)$ for each $z \in S$ and $PB^+h(0) = 0$ for every $h \in BC(R^+)$, it follows that

$$
Pz(0,\zeta) = PTz(0,\zeta) = P\zeta = \zeta.
$$

Since for each $\zeta \in S$ there is only one solution x of the perturbed equation which is bounded for $t \geq 0$ and satisfies $Px(0) = \zeta$, this solution must be $x(t) = z(t, \zeta)$ and, consequently,

$$W^s = \{z(0, \zeta) : \zeta \in S\}.$$

Therefore, W^s is the graph of the globally lipschitzian function $\zeta \to z(0, \zeta)$ defined on S and satisfying $z(0, \zeta) - \zeta \in U$.

Next, we establish the exponential behaviour of the solutions in W^s for $t \geq 0$. Let us show first that $|z(t, \zeta)| \to 0$ as $t \to +\infty$. If $\rho = \limsup_{t \to +\infty} |z(t, \zeta)|$ satisfies $\rho > 0$ we derive a contradiction in the following way. For $r > 1$ we can find $\tau > 0$ such that $|z(t, \zeta)| \leq r\rho$ for every $\tau \geq \tau$. Also for $t \geq \tau$ we have

$$\begin{aligned}
&|z(t, \zeta)| = |Tz(t, \zeta)| \leq |\Psi(t)\zeta| + |B^+[g(\cdot, z(\cdot, \zeta))](t)| \\
&\leq Ke^{-\alpha t}|\zeta| + \int_0^\tau Ke^{-\alpha(t-s)} b|z(s, \zeta)|ds + \int_\tau^t Ke^{-\alpha(t-s)} br\rho ds \\
&+ \int_t^{+\infty} Ke^{+\alpha(t-s)} br\rho ds \leq Ke^{-\alpha t}\left[|\zeta| + e^{\alpha t}\alpha^{-1} b \sup_{t \geq 0} |z(t, \zeta)|\right] + 2K\alpha^{-1} br\rho.
\end{aligned}$$

From the relation $2K(K+1)\alpha^{-1}b < 1$ taking $r < K + 1$ we have that $2K\alpha^{-1}br < 1$ and letting $t \to +\infty$ we obtain $\rho \leq 2K\alpha^{-1}br\rho < \rho$, a contradiction. We conclude that $\lim_{t \to +\infty} |z(t, \zeta)| = 0$.

Defining $u(t, \zeta) = \sup_{s \geq t} |z(s, \zeta)|$, we have that for each $t \geq 0$ there is $\tau \geq t$ such that $u(s, \zeta) = u(\tau, \zeta) = |z(\tau, \zeta)|$ for $t \leq s \leq \tau$, and accordingly, we obtain in a similar way as for the previous estimate

$$\begin{aligned}
& u(t, \zeta) = u(\tau, \zeta) = |z(\tau, \zeta)| = |Tz(\tau, \zeta)| \\
\leq\ & Ke^{-\alpha\tau}|\zeta| + \int_0^\tau Ke^{-\alpha(\tau-s)} b|z(s, \zeta)|ds + \int_\tau^{+\infty} Ke^{+\alpha(\tau-s)} b|z(s, \zeta)|ds \leq Ke^{-\alpha t}|\zeta| \\
+\ & \int_0^t Ke^{-\alpha(t-s)} bu(s, \zeta)ds + \int_t^\tau Ke^{-\alpha(\tau-s)} bu(s, \zeta)ds + \int_t^{+\infty} Ke^{+\alpha(t-s)} bu(s, \zeta)ds \\
\leq\ & Ke^{-\alpha t}|\zeta| + \int_0^t Ke^{-\alpha(\tau-s)} bu(s, \zeta)ds + 2K\alpha^{-1} bu(t, \zeta).
\end{aligned}$$

Since $2K\alpha^{-1}b < 1$, this relation is equivalent to

$$e^{\alpha t} u(t, \zeta) \leq K(1 - 2K\alpha^{-1}b)^{-1}\left[|\zeta| + \int_0^t be^{\alpha s} u(s, \zeta)ds\right]$$

and using Gronwall's inequality we obtain

$$e^{\alpha t} u(t,\zeta) \le K_0 |\zeta| e^{bK_0 t}, \quad \text{with} \quad K_0 = K(1 - 2K\alpha^{-1} b)^{-1}.$$

Denoting by $x(t,\xi)$ the solution of the perturbed equation with initial condition $\xi = z(0,\zeta) \in W^s$ we have that $\zeta = P\xi$ and $x(t,\xi) = z(t,\zeta)$. We conclude that

$$|x(t,\xi)| = |z(t,\zeta)| \le u(t,\zeta) \le K_0 |P\xi| e^{-(\alpha - bK_0)t}$$

and letting $c = K_0 \sup_{|\xi|=1} |P\xi|/|\xi|$ and $\beta = \alpha - bK_0$ we obtain

$$|x(t,\xi)| \le c e^{-\beta t} |\xi|.$$

Clearly $\beta = \beta(b) \to \alpha$ as $b \to 0$. Since $K \ge 1$, the inequality $2K(K+1)\alpha^{-1} b < 1$ implies that $\beta > 0$.

To show that $W^s \in C(S,\Omega)$, let $\xi = z(0,\zeta) = \zeta + \eta$ where $\zeta \in S$, $\eta \in U$. Since $Pz(0,\zeta) = \zeta$, we have

$$\eta = (I-P)z(0,\zeta) = (I-P)Tz(0,\zeta) = -\int_0^{+\infty} (I-P)\Psi^{-1}(s) g(s, z(s,\zeta)) ds.$$

From the exponential estimate for $x(t,\xi) = z(t,\zeta)$ we also obtain $|z(t,\zeta)| \le K_0|\zeta|$ for $t \ge 0$. The above integral can, then, be estimated as follows

$$\begin{aligned} |\eta| &\le \int_0^{+\infty} K e^{-\alpha s} |g(s, z(s,\zeta))| ds \le \int_0^{+\infty} K e^{-\alpha s} b |z(s,\zeta)| ds \\ &\le \int_0^{+\infty} K e^{-\alpha s} b K_0 |\zeta| ds = \frac{K^2 \alpha^{-1} b}{1 - 2K\alpha^{-1} b} |\zeta| = \Omega(b)|\zeta|. \end{aligned}$$

We conclude that $\Omega(b) \to 0$ as $b \to 0$.

As $S \oplus U = \mathbb{R}^n$, $C(S,\Omega(b)) \cap C(U,\Omega(b)) = 0$ for $b > 0$ sufficiently small. Therefore, $W^s \cap W^u = \{0\}$ and $x(t) = 0$ is the only solution which is bounded for $t \in \mathbb{R}$.

QED

It is a straightforward exercise to adapt the preceeding proof to the case where $g(t,0)$ is not required to vanish, but only to be bounded for $t \in \mathbb{R}$. In this case, it can be shown that W^s and W^u intersect at exactly one point ξ_0, with $x(t,\xi_0)$ being the unique solution which is bounded for $t \in \mathbb{R}$, W^s, W^u belong to cones with apertures $\Omega(b) \to 0$ as $b \to 0$, around the sets

$S+\{\xi_0\}, U+\{\xi_0\}$, respectively, and the exponential estimates are satisfied by $x(t,\xi)-x(t,\xi_0)$ in the form $|x(t,\xi)-x(t,\xi_0)| \leq ce^{-\beta t}|\xi-\xi_0|$ for $t \geq 0$, $\xi \in W^s$ and $|x(t,\xi)-x(t,\xi_0)| \leq ce^{+\beta t}|\xi-\xi_0|$ for $t \geq 0$, $\xi \in W^u$, with $\beta = \beta(b) \to \alpha$ as $b \to 0$.

The last theorem establishes that W^s and W^u are graphs of globally lipschitzian functions defined on S and U, respectively, whenever $g(t,0) = 0$ for all $t \in \mathbb{R}$ and $g(t,x)$ is continuous and globally lipschitzian in x with sufficiently small lipschitz constant $b > 0$. It is frequently necessary to have more regularity.

(3.5) **Theorem:** *If, besides the hypothesis of the preceding theorem, $g(t,x)$ has bounded continuous derivatives relative to x up to order $r \geq 1$, then for $b > 0$ sufficiently small and $g \in L_b$, W^s and W^u are graphs of globally lipschitzian functions of class C^r defined on S and U, respectively.*

Proof: Again we prove this result only for the case of the stable manifold W^s. We have shown that $W^s = \{z(0,\zeta) : \zeta \in S\}$ where $z = Tz$ is the unique fixed point of the mapping T in Z . Now we will prove that the function $\zeta \to z(0,\zeta)$ is of class C^r by induction in r. Let us consider the case C^1. The first idea would be to prove that $z = Tz$ is a fixed point of the mapping T in a subspace Z^1 of globally lipschitz functions of class C^1. To ensure that Z^1 is complete we need to choose a norm controlling both $|z|$ and $|z'|$, for example $||z||_1 = ||z|| + ||D_\zeta z||$, otherwise there are sequences like $|\zeta|^{1+1/n} \to |\zeta|$ converging to C^0 functions that are not C^1. Since

$$Tz(t,\zeta) = \Psi(t)\zeta + \int_0^t \Psi(t)P\Psi^{-1}(s)g(s,z(s,\zeta))ds$$
$$- \int_t^{+\infty} \Psi(t)(I-P)\Psi^{-1}(s)g(s,z(s,\zeta))ds$$

it is clear that $T(Z^1) \subset Z^1$, but to show that T is a contraction we need to estimate $||D_\zeta Tz(t,\zeta_1) - D_\zeta Tz(t,\zeta_2))||$ in terms of the distance $||\zeta_1 - \zeta_2||$, where

$$D_\zeta Tz(t,\zeta) = \Psi(t) + \int_0^t \Psi(t)P\Psi^{-1}(s)\frac{\partial g}{\partial z}(s,z(s,\zeta))D_\zeta z(s,\zeta)ds$$
$$- \int_t^{+\infty} \Psi(t)(I-P)\Psi^{-1}(s)\frac{\partial g}{\partial z}(s,z(s,\zeta))D_\zeta z(s,\zeta)ds.$$

This can be done assuming that $\frac{\partial g}{\partial z}$ is globally lipschitz, that is, if g besides being globally lipschitz and C^1 in z, has globally lipschitz derivative. Nevertheless, these lipschitz conditions on the derivatives of g are unnecessary for the result which we can establish in the following way. Since $z = Tz$ we must have $D_\zeta z = D_\zeta Tz$. Hence, we look for the derivative of z as a fixed point of the function F defined in the Banach space V of continuous bounded functions $v : \mathbb{R}^+ \times S \to L(S, R^n)$ with the norm $||v|| = \sup_{t\geq 0} \sup_{\zeta \in S} |v(t, \zeta)|$, where $|v(t, \zeta)| = \sup_{|\xi|\leq 1} |v(t, \zeta)\xi|$ is the uniform norm in $L(S, \mathbb{R}^n)$,

$$\begin{aligned} Fv(t,\zeta) &= \Psi(t) + \textstyle\int_0^t \Psi(t)P\Psi^{-1}(s)\frac{\partial g}{\partial z}(s, z(s,\zeta))v(s,\zeta)ds \\ &- \textstyle\int_t^{+\infty} \Psi(t)(I-P)\Psi^{-1}(s)\frac{\partial g}{\partial z}(s, z(s,\zeta))v(s,\zeta)ds. \end{aligned}$$

We readily obtain the following estimate

$$\begin{aligned} |Fv(t,\zeta)| &= K + \int_0^t Ke^{-\alpha(t-s)}|\frac{\partial g}{\partial z}(s, z(s,\zeta)||v(s,\zeta)|ds \\ &+ \int_t^{+\infty} Ke^{+\alpha(t-s)}|\frac{\partial g}{\partial z}(s, z(s,\zeta))||v(s,\zeta)|ds \\ &\leq K + \frac{2K}{\alpha} \sup_{t\geq 0} |\frac{\partial g}{\partial z}(t, z(t,\zeta))| \sup_{t\geq 0} |v(t,\zeta)|, \end{aligned}$$

and since $g \in L_b$, $|\partial g/\partial z| \leq b$ and v is bounded we conclude that Fv is bounded. Moreover, for v_1, $v_2 \in V$ we have

$$\begin{aligned} ||Fv_1 - Fv_2|| &= \sup_{t\geq 0, \zeta \in S} |Fv_1(t,\zeta) - Fv_2(t,\zeta)| \\ &\leq \frac{2K}{\alpha} \sup_{t\geq 0, \zeta\in S} |\frac{\partial g}{\partial z}(t, z(t,\zeta))v_1(t,\zeta) - \frac{\partial g}{\partial z}(t, z(t,\zeta))v_2(t,\zeta)| \\ &\leq \frac{2K}{\alpha} b\, ||v_1 - v_2||, \end{aligned}$$

and for sufficiently small b so that $\frac{2K}{\alpha} b < 1$ we have that F is a contraction. Let $v = Fv$ be its unique fixed point in V. We need to show that $v = \frac{\partial z}{\partial \zeta}$. For this we define

$$\gamma(\epsilon) = \sup_{t\geq 0, |\Delta|\leq \epsilon} \frac{|z(t,\zeta+\Delta) - z(t,\zeta) - v(t,\zeta)\Delta|}{|\Delta|}$$

for every $\epsilon > 0$ sufficiently small, and prove that $\gamma(\epsilon) \to 0$ as $\epsilon \to 0$. Let $d(t,\zeta,\Delta) = z(t,\zeta+\Delta) - z(t,\zeta) - v(t,\zeta)\Delta$. From $v = Fv$, $z = Tz$ and using the first order Taylor approximation for g we obtain

$$\begin{aligned} d(t,\zeta,\Delta) &= \int_0^t \Psi(t)P\Psi^{-1}(s)\, G(s,\zeta,\Delta)\, ds \\ &- \int_t^{+\infty} \Psi(t)(I-P)\Psi^{-1}(s)\, G(s,\zeta,\Delta)\, ds \\ &= \int_0^t \Psi(t)P\Psi^{-1}(s)\, \frac{\partial g}{\partial z}(s,z(s,\zeta))\, \bar{d}(s,\zeta,\Delta)\, ds \\ &- \int_t^{+\infty} \Psi(t)(I-P)\Psi^{-1}(s)\, \frac{\partial g}{\partial z}(s,z(s,\zeta))\, \bar{d}(s,\zeta,\Delta)\, ds. \end{aligned}$$

where for convenience we wrote $G(s,\zeta,\Delta) = g(s,z(s,\zeta+\Delta)) - g(s,z(s,\zeta)) - \frac{\partial g}{\partial z}(s,z(s,\zeta))v(s,\zeta)\Delta$ and $\bar{d}(s,\zeta,\Delta) = d(s,\zeta,\Delta) + o(|z(s,\zeta+\Delta) - z(s,\zeta)|)$. From this expression and for ϵ sufficiently small we obtain the estimate

$$\gamma(\epsilon) \le \frac{2K}{\alpha} b[\gamma(\epsilon) + (K+1)\, o(\epsilon)]$$

and, since $\frac{2K}{\alpha}b < 1$ we have

$$\gamma(\epsilon) \le \frac{K+1}{1 - \frac{2K}{\alpha}b}\, o(\epsilon)$$

and we conclude that $\gamma(\epsilon) \to 0$ as $\epsilon \to 0$. This completes the proof for the case of $r = 1$. To prove the general case by induction on r we assume that the result holds for $r \ge 1$ and that the hypothesis is verified for $r+1$. Letting

$$w = \frac{\partial^r z}{\partial \zeta^r}$$

and differentiating $(r-1)$ times $v = Fv$ we obtain

$$w(t,\zeta) = \int_0^t \Psi(t)P\Psi^{-1}(s)\frac{\partial g}{\partial z}(s,z(s,\zeta))w(s,\zeta)ds$$

$$- \textstyle\int_t^{+\infty} \quad \Psi(t)(I-P)\Psi^{-1}(s)\dfrac{\partial g}{\partial z}(s,z(s,\zeta))w(s,\zeta)ds + R(t,\zeta),$$

where $R(t,\zeta)$ contains the terms of lower order derivatives, hence not involving w. Then, we need to show that $\frac{\partial w}{\partial \zeta}$ exists and is continuous. We use the same procedure we used before. The derivative , if it exists, must be a fixed point of the mapping H defined by

$$Hu(t,\zeta) = \int_0^t \Psi(t)P\Psi^{-1}(s)\frac{\partial g}{\partial z}(s,z(s,\zeta))w(s,\zeta)ds$$
$$- \int_t^{+\infty} \Psi(t)(I-P)\Psi^{-1}(s)\frac{\partial g}{\partial z}(s,z(s,\zeta))w(s,\zeta)ds + R(t,\zeta)$$

in the space U of continuous bounded functions $u : \mathbb{R}^+ \times S \to L(S, L((\mathbb{R}^n)^r, \mathbb{R}))$ with the uniform norm. One easily verifies that $H : U \to U$ and

$$||Hu_1 - Hu_2|| \leq \frac{2K}{\alpha} b\, ||u_1 - u_2||,$$

hence H is a contraction in U for $\frac{2K}{\alpha}b < 1$. Denoting by u the unique fixed point $u = Hu$ and defining

$$\gamma(\epsilon) = \sup_{t\geq 0, |\Delta|\leq\epsilon} \frac{|w(t,\zeta+\Delta) - w(t,\zeta) - u(t,\zeta)\Delta|}{|\Delta|}$$

as it was done before one proves that $\gamma(\epsilon) \to 0$ as $\epsilon \to 0$ completing the proof that $\frac{\partial^{r+1}z}{\partial\zeta^{r+1}}$ exists and is equal to u.

QED

3.5 Solutions with bounded exponential rates for weakly nonlinear equations

The separation between the different exponential rates can be obtained in a similar way as before. For $\mu \in \mathbb{R}$, performing the change of variables $y = e^{-\mu t}x$ transforms the linear equation

$$\dot{x} = A(t)x$$

to

$$\dot{y} = A_\mu(t)y.$$

where $A_\mu(t) = A(t) - \mu I$. If this equation has an exponential dichotomy, then there exist a projection P_μ and constants K, α such that

$$|\Psi_\mu(t)P_\mu\xi| \leq Ke^{-\alpha(t-s)}|\Psi_\mu(s)\xi|, \quad t \geq s$$

$$|\Psi_\mu(t)(I-P_\mu)\xi| \leq Ke^{+\alpha(t-s)}|\Psi_\mu(s)\xi|, \quad t \leq s,$$

and since $\Psi_\mu(t) = e^{-\mu t}\Psi(t)$, taking $P = P_\mu$ we obtain

$$|\Psi(t)P\xi| \leq Ke^{(\mu-\alpha)(t-s)}|\Psi(s)\xi|, \quad t \geq s$$

$$|\Psi(t)(I-P)\xi| \leq Ke^{(\mu+\alpha)(t-s)}|\Psi(s)\xi|, \quad t \leq s.$$

This implies that the set of initial conditions is a direct sum of two spaces; solutions with initial conditions in one of them exhibit exponential rates lower than $e^{(\mu-\alpha)t}$ for $t \geq 0$, and those with initial conditions in the other exhibit exponential rates greater than $e^{(\mu+\alpha)t}$ for $t \leq 0$. The only solution with an exponential rate above $\mu - \alpha$ for $t \geq 0$ and below $\mu + \alpha$ for $t \leq 0$ is the trivial solution. Hence, the exponential rate corresponding to $e^{\mu t}$ does not occur and can be separated from the exponential rates that can occur. Naturally, we denote by S_μ and U_μ the subspaces corresponding to this separation of exponential rates, $S_\mu \oplus U_\mu = \mathbb{R}^n$, and again we have that this type of decomposition is persistent under small nonlinear perturbations. In fact, if we consider the equation $\dot{x} = A(t)x + g(t,x)$ with $g \in L_b$ and $b > 0$ sufficiently small we have that the sets

$$W^s_\mu = \{\xi \in \mathbb{R}^n : e^{-\mu t}x(t,\xi) \to 0 \text{ as } t \to +\infty\}$$

$$W^s_\mu = \{\xi \in \mathbb{R}^n : e^{-\mu t}x(t,\xi) \to 0 \text{ as } t \to -\infty\}$$

are graphs of globally lipschitz functions defined on S_μ and U_μ and contained in cones $C(S_\mu, \Omega_u)$ and $C(U_\mu, \Omega_s)$, respectively, with apertures approaching zero as $b \to 0$. Moreover, if we assume g to be of class C^r then these functions will be of class C^r. If $\gamma < \mu < \lambda$ are such that the linear equation has an exponential dichotomy for γ and λ but not for μ, then we define as before the sets $C = S_\lambda \cap U_\gamma, S = S_\gamma$ and $U = U_\lambda$ obtaining the decomposition $\mathbb{R}^n = S \oplus C \oplus U$. Similarly, we define $W^c = W^s_\lambda \cap W^u_\gamma$ and since $S_\lambda \oplus U_\lambda = \mathbb{R}^n$ with $U_\lambda \subset U_\gamma$ we conclude in the C^1 case that W^c is the graph of a C^1 function defined on C and contained in a cone around C with aperture $\Omega \to 0$ as $b \to 0$.

As it was shown above, it is important to distinguish the values of μ for which the linear equation $\dot{y} = [A(t) - \mu I]y$ has an exponential dichotomy from the others. In the autonomous case, $A(t) = A$, these values of μ correspond to values different from the real part of any eigenvalue of A. Generalizing this idea, we call **spectrum** $\Sigma(A)$ of the equation $\dot{x} = A(t)x$ the set of real numbers μ for which the equation $\dot{y} = A_\mu(t)y$ has no exponential dichotomy, thus corresponding to exponential rates $e^{-\mu t}$ that cannot be separated by gaps from the exponential rates occuring in the solutions of the given equation. When $A(t) = A$ we have that $\Sigma(A) = \{Re\lambda : \lambda \text{ is an eigenvalue of } A\}$ is a finite union of points. In general this is not the case when $A(t)$ depends on t. As a simple example consider the scalar equation $\dot{x} = a(t)x$ where a is a continous function defined on $\mathbb{R}$ and such that $a(t) = 0$ for $t \leq 0$ and $a(t) = 1$ for $t \geq 1$. For this equation we have $a_\mu(t) = a(t) - \mu$ and the solutions of the equation $\dot{y} = a_\mu(t)y$ are given by

$$y(t,\xi) = \begin{cases} e^{-\mu t}\xi, & t \leq 0 \\ e^{(1-\mu)t}y(1,\xi) & t \geq 1 \end{cases}$$

We immediately conclude that $S_\mu = \mathbb{R}$ if $\mu > 1$, $S_\mu = 0$ if $\mu \leq 1$, $U_\mu = 0$ if $\mu \geq 0$ and $U_\mu = \mathbb{R}$ if $\mu < 0$. Hence, we have $S_\mu \oplus U_\mu = \mathbb{R}$ if $\mu > 1$ or $\mu < 0$ but $S_\mu \cap U_\mu = \{0\}$ if $0 \leq \mu \leq 1$, and consequently the spectrum of the equation is the closed interval $\Sigma(a) = [0, 1]$.

We have seen that the spectrum of a linear equation can contain intervals. The general situation is not whorse as stated in the following theorem.

(3.6) **Theorem:** *If $A(t)$ is continuous and bounded then the spectrum $\Sigma(A)$ is the finite union of disjoint compact intervals that may degenerate to points*

$$\Sigma(A) = \bigcup_{i=1}^{k} [a_i, b_i]$$

with $1 \leq k \leq n$ and $a_i \leq b_i$. To each **spectral interval** $[a_i, b_i]$ *there is associated a linear subspace $V_i \subset \mathbb{R}^n$, called the* **spectral fiber** *with* $\dim V_i = n_i$ *and satisfying*

(i) $n_i \geq 1$ and $n_1 + \ldots + n_k = n$

(ii) $V_i \cap V_j = \{0\}$ if $i \neq j$ and $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$

(iii) $V_i = S_\lambda \cap U_\mu$ if $(\mu, \lambda) \cap \Sigma(A) = [a_i, b_i]$.

We remark that the spectral fibers V_i correspond to the set of initial conditions of solutions exhibiting exponential rates greater than $e^{\lambda t}$ for $t \geq 0$ and smaller than $e^{\mu t}$ for $t \leq 0$.