An Introduction to
Infinite Dimensional Dynamical Systems

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.

Dynamical systems generated by ODEs:

$$\dot{x}(t) = f(x(t)) \quad ; \quad x(0) = x_0$$

$$f \in C^r(\mathbb{R}^n), r \geq 1 \quad \text{(existence, uniqueness, cont. dependence)}$$

$$x(t) = T(t)x_0 \quad , \quad t \in I \subset \mathbb{R}$$

Determinism: $x(t+s) = T(t+s)x_0 = T(t)x(s) = T(t)T(s)x_0$

Group property: $T(0) = I \quad ; \quad T(t+s) = T(t)T(s)$

$T(t)$ is a nonlinear flow generated by the ODE.

Equilibrium solutions: $\mathcal{E} = \{x_j : f(x_j) = 0\}$

Let $x = x_j + z$ $(x_j \equiv 0)$: $\dot{z} = Az + g(z)$

$$z(t) = e^{At} z_0 + \int_0^t e^{A(t-\tau)} g(z(\tau))\, d\tau$$

Fixed point: $z(t) = T(t) z_0$

$$T(t) = T_g(t) \quad ; \quad T_0(t) = e^{At}$$

Spectral mapping theorem: $\sigma(T_0(t)) = e^{\sigma(A)t}$

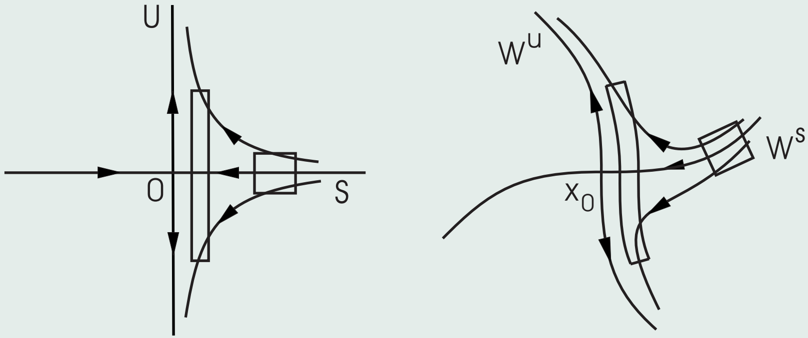
$$\sigma(A) = \{\lambda_j : (A - \lambda_j I) \text{ is singular}\}$$

A is hyperbolic if: $\operatorname{Re} \sigma(A) \neq 0$

Hartman–Grobman theorem: If A is hyperbolic

$\exists$ homeomorphism $h = h_g \in C^0(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$

$$h(T_g(t)x_0) = T_0(t)h(x_0)\ ,\ x_0 \in \Omega \qquad h \circ T_g = T_0 \circ h$$



Stable and unstable manifolds: For A hyperbolic, $\mathbb{R}^n = S \oplus U$,

$$W^s = \{x_0 \in \Omega : T(t)x_0 \to 0 \text{ as } t \to +\infty\} = \operatorname{graph}\Phi^s\ ,\ \Phi^s \in C^r(S \cap \Omega, U)$$

$$W^u = \{x_0 \in \Omega : T(t)x_0 \to 0 \text{ as } t \to -\infty\} = \operatorname{graph}\Phi^u\ ,\ \Phi^u \in C^r(U \cap \Omega, S)$$

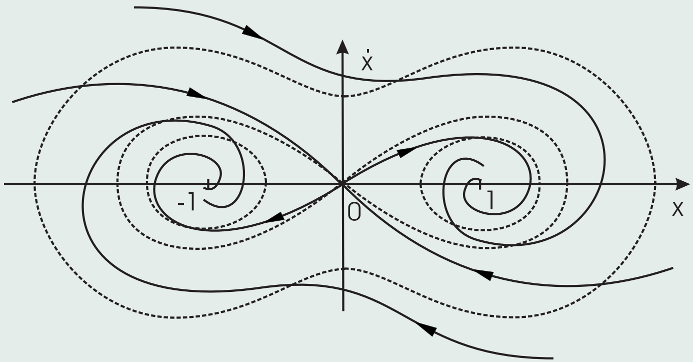
Orbits: $\gamma(x_0) = \{T(t)x_0 : t \in I \subset \mathbb{R}\}$

α and ω-limits:

$$\omega(x_0) = \{y : \exists t_k \to +\infty\ ,\ T(t_k)x_0 \to y\}$$

$$\alpha(x_0) = \{y : \exists t_k \to -\infty\ ,\ T(t_k)x_0 \to y\}$$

Phase portrait: Set of all orbits with the time orientation



$$\ddot{x} + \beta\dot{x} - x + x^3 = 0 \quad , \quad \beta > 0$$

$$\mathcal{E} = \{-1, 0, 1\} \qquad x_j = 0\ ,\ A = \begin{bmatrix} 0 & 1 \\ 1 & -\beta \end{bmatrix}$$

Dynamical systems theory concerns the study of the global orbit structure for most systems

If $\operatorname{Re}\sigma(A) < 0$, there is $B = B'$ so that $A'B + BA = -I$.

Liapunov function: $V(x) = \langle x, Bx \rangle$

$$\frac{d}{dt}V(x) = \langle x, (A'B + BA)x \rangle + 2\langle x, Bg(x) \rangle$$

$$= -\|x\|^2 + 2\langle x, Bg(x) \rangle \quad \leq 0 \ , \text{ for } x \in \Omega$$

If $\omega(x_0) \subset \Omega$ for all $x_0 \in \mathbb{R}^n$ the system is dissipative.

ω-limit sets: equilibria, periodic orbits, ... — invariant sets

Global attractor $\mathcal{A}$: maximal compact invariant set

$$\mathcal{A} = \{ \text{ globally defined bounded solutions } \}$$

Gradient systems: $\dot{x} = -\operatorname{grad} V(x)$

$$\frac{d}{dt} V(x) = -\| \operatorname{grad} V(x) \|^2 \leq 0$$

$$\mathcal{A} = \bigcup_j W^u(x_j)$$

Parameter dependence: $f = f_\lambda \quad \mathcal{A} = \mathcal{A}_\lambda$

Nonlinear control problems: $\lambda = u(t)$ (state feedback)

$$\dot{x}(t) = f(x(t), u(t)) \quad , \quad u(t) = v(t) - Cx(t)$$

Infinite dimensional dynamical systems

- Equations involving heredity (Volterra)

- Retarded functional differential equations:

$$\dot{x}(t) = \int_{-r}^{0} g(\theta, x(t+\theta))\, d\theta$$

Let $\quad x_t(\theta) = x(t+\theta) \quad$ with $\quad x_t \in C([-r,0],\mathbb{R}^n)$

$$\dot{x}(t) = f(x_t) \qquad , \qquad X = C([-r,0],\mathbb{R}^n)$$

- Partial differential equations:

$$(S) \qquad u_t = u_{xx} + f(u) \quad , \quad 0 < x < \pi \quad , \quad u_x(t,0) = u_x(t,\pi) = 0$$

Let $\quad u(t) = u(t,x) \quad$ with $\quad u(t) \in L^2(0,\pi)$

$$u_t = Au + g(u) \quad , \quad Au = u_{xx} - au$$

$$D(A) = \{u \in L^2 : Au \in L^2\} = H^2 \quad , \quad X = H^1$$

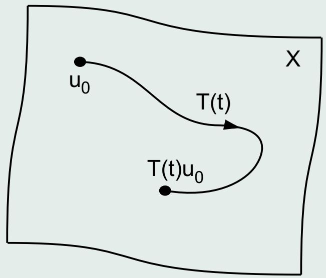
$$u_0(x) = \sum_{k=0}^{+\infty} a_k \cos kx \quad , \quad T_0(t)u_0 = e^{-at} \sum_{k=0}^{+\infty} a_k e^{-k^2 t} \cos kx = e^{At} u_0$$

Dynamical systems $(X, T(t))$

X complete metric space

$T(t) : X \to X$ continuous flow

$T(0) = I$; $T(t+s) = T(t)T(s)$



C^r-semiflows: $t \geq 0$, $u_0 \mapsto T(t)u_0 \in C^r$

(S) generates a dynamical system in $X = H^1(0, \pi)$:

$$T(\cdot) : \mathbb{R} \times X \to X \ , \quad T(t)u_0(x) = u(t,x)$$

$$u(t) = T(t)u_0 = e^{At}u_0 + \int_0^t e^{A(t-s)} f(u(s)) \ ds$$

A is the infinitesimal generator of an analytical semigroup

$$e^{At} = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} e^{\lambda t} \, d\lambda$$

and $(\lambda I - A)^{-1}$ is compact for $\lambda \notin \sigma(A)$

$$T(t) : X \to D(A) \subset X \quad , \quad t > 0$$

$T(t) : X \to X, t \geq 0$, is a gradient system:

$$V(u) = \int_0^{\pi} (\frac{1}{2} u_x^2 - F(u)) dx \quad , \quad F' = f$$

$$\frac{d}{dt} V(u) = - \int_0^{\pi} u_t^2 \, dx \leq 0$$

$T(t)$ has a global attractor $\mathcal{A}$, the maximal compact invariant subset of X,

$$\mathcal{A} = \ \{ \text{ globally defined bounded solutions } \}$$

Equilibria of $T(t)$: $\mathcal{E} = \{u_j : \text{ solutions of (E)}\}$

$$(E) \qquad v_{xx} + f(v) = 0 \quad , \quad v_x(0) = v_x(\pi) = 0$$

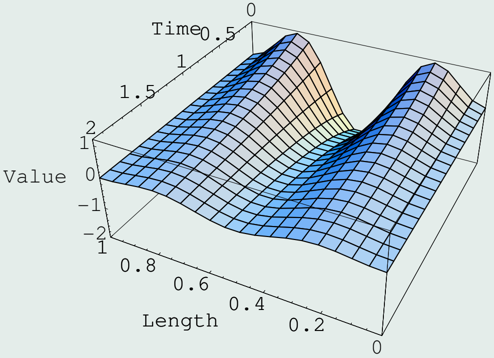
$$\mathcal{A} = \bigcup_j W^u(u_j)$$

Heteroclinic orbits: For all $u_0 \in \mathcal{A}$,

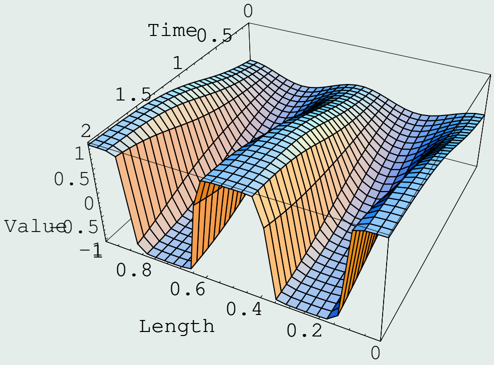
$$\gamma(u_0) \subset \mathcal{A} \text{ connects } \alpha(u_0) \in \mathcal{E} \text{ to } \omega(u_0) \in \mathcal{E}.$$

The asymptotic behavior of (S) is embodied in $\mathcal{A} = \mathcal{A}_f$.

$u_t = u_{xx}$



$u' = f(u)$



The balance $\{$ diffusion $u_{xx} \leftrightarrow$ reaction $f(u)$ $\}$ creates interesting dynamics.

Characterize all the possible dynamics – Classify all the possible attractors

• In general:

$$u_t = u_{xx} + f(x, u, u_x) \quad , \quad u_x(\cdot, 0) = u_x(\cdot, \pi) = 0$$

has a gradient structure and $\mathcal{A} = \bigcup_{v \in \mathcal{E}} W^u(v)$

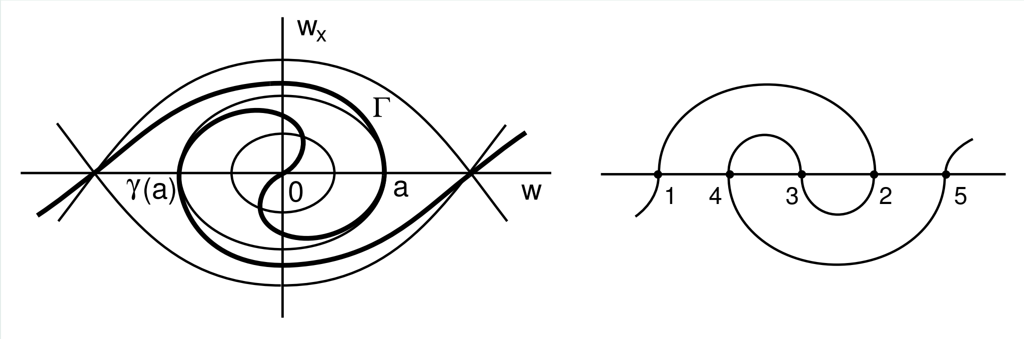
$$\mathcal{E} = \{v_j : \ \textit{solutions of}\ (P)\}$$

$$(P) \qquad v_{xx} + f(x, v, v_x) = 0 \quad , \quad v_x(0) = v_x(\pi) = 0 \ .$$

To find the heteroclinic orbits, solve (P) by shooting:

$$(M) \quad w_{xx} + f(x, w, w_x) = 0 \ , \quad w(0, a) = a \ , \ w_x(0, a) = 0 \ .$$

Meander curve: $\Gamma = \{\gamma(a), a \in \mathbb{R}\} \subset \mathbb{R}^2 \quad , \quad \gamma : a \mapsto (w(\pi, a), w_x(\pi, a))$



$\Gamma = \Gamma_f$ intersects the axis $\{w_x = 0\}$ at n points corresponding to the solutions v_j of (M), i.e., the equilibria.

Meander Permutation: $\sigma = \{1, 4, 3, 2, 5\} = (2\ 4)$

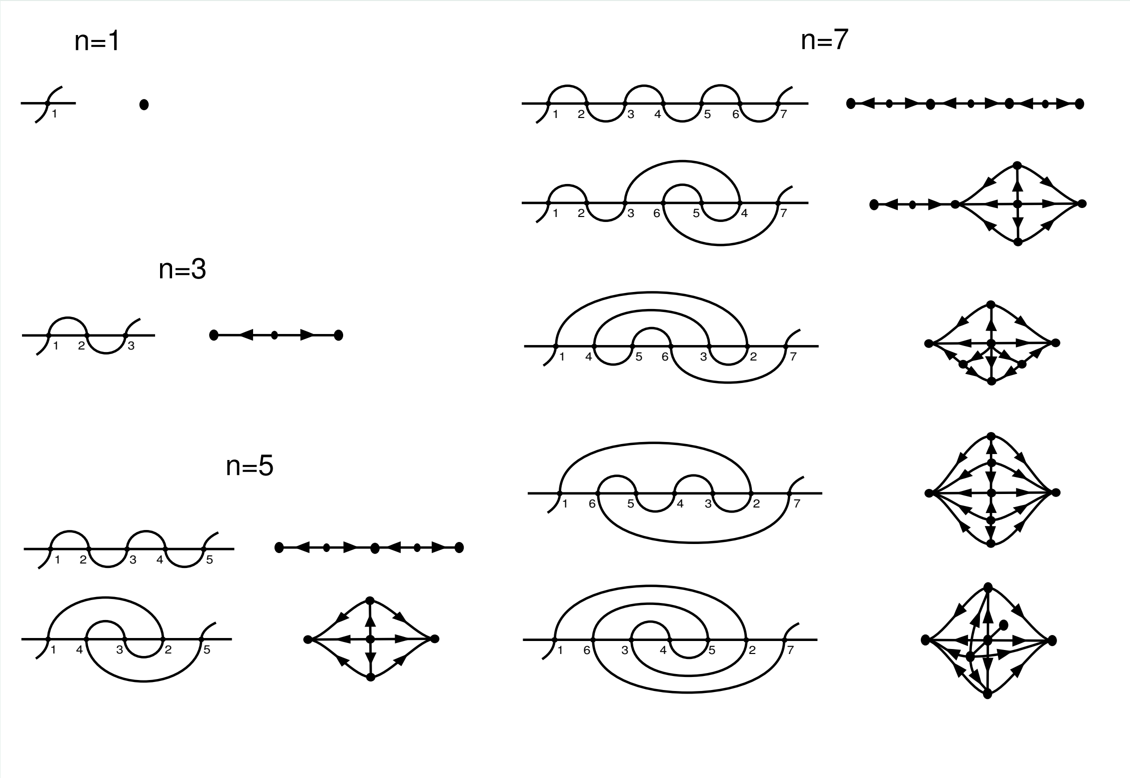
$\sigma = \sigma_f$ determines the Morse indices of equilibria: $\#\{\lambda_j : \lambda_j > 0\}$

$$i(v_j) := \dim W^u(v_j) = i_j(\sigma)$$

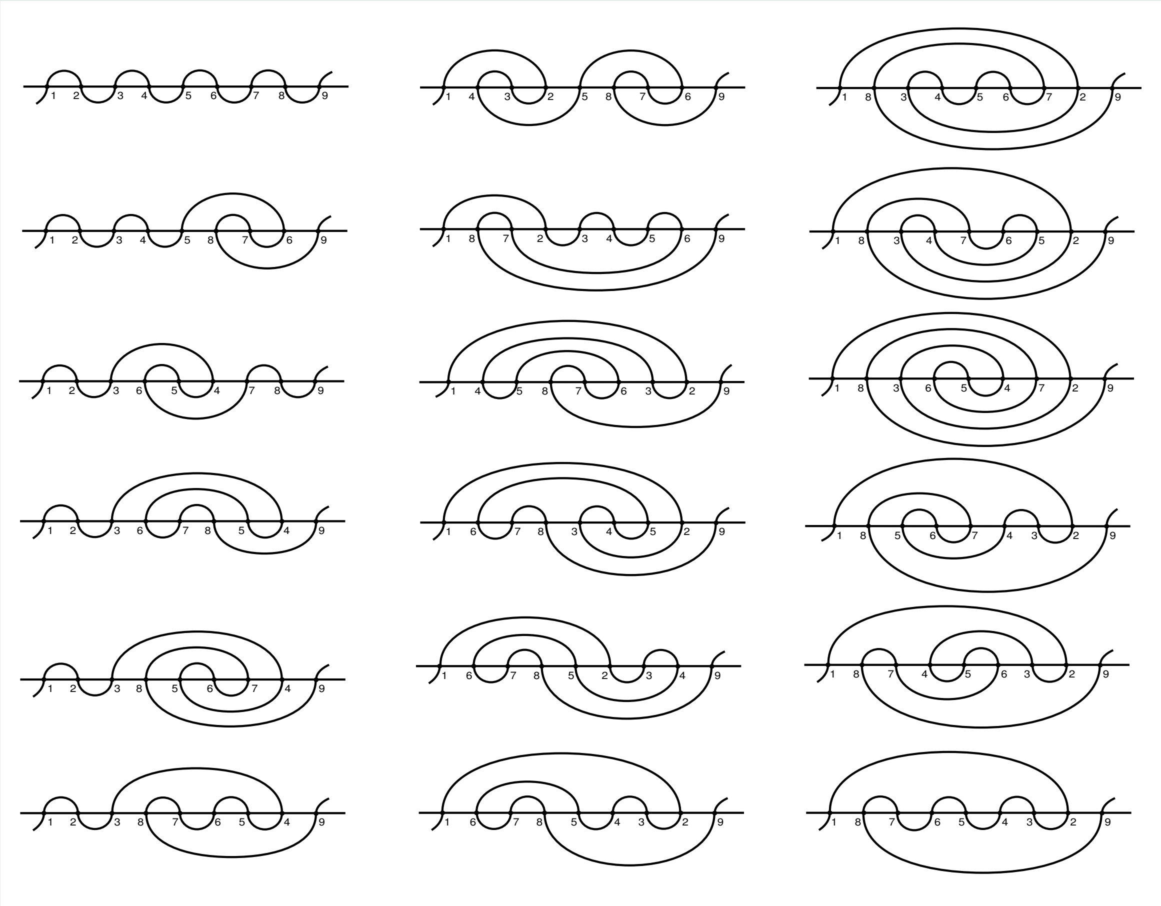
$$i_j(\sigma) = \sum_{k=1}^{j-1} (-1)^{k+1} \operatorname{sign}(\sigma^{-1}(k+1) - \sigma^{-1}(k))$$

$\sigma = \sigma_f$ determines $\mathcal{A}_f$ up to homeomorphism: $\quad \sigma_f = \sigma_g \;\Rightarrow\; \mathcal{A}_f \cong \mathcal{A}_g$.

List of attractors with $n \leq 7$ equilibria:



Dissipative, Morse, meander permutations in $S(9)$:



Infinite dimensional control problem: stabilize the appropriate patterns.