Geometric Quantization, Complex Structures and the Coherent State Transform

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Abstract

It is shown that the heat operator in the Hall coherent state transform for a compact Lie group $K$ [Ha1] is related with a Hermitian connection associated to a natural one-parameter family of complex structures on $T^*K$. The unitary parallel transport of this connection establishes the equivalence of (geometric) quantizations of $T^*K$ for different choices of complex structures within the given family. In particular, these results establish a link between coherent state transforms for Lie groups and results of Hitchin [Hi] and Axelrod, Della Pietra and Witten [AdPW].
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1 Introduction

In the present paper we relate the appearance of the heat operator in the Hall coherent state transform (CST) for a compact connected Lie group $K$ [Ha1] with a one-parameter family of complex structures on the cotangent bundle $T^*K$, in the framework of geometric quantization. The heat equation appears also in the quantization of $\mathbb{R}^{2n}$ and of Chern-Simons theories [AdPW, Hi] and in the related theory of theta functions, where it is associated with the so-called Knizhnik-Zamolodchikov-Bernard-Hitchin (KZBH) connection [Fa, Las, Ra, FMN]. The general case was studied from a cohomological point of view in [Hi].

Our main motivation is to give a differential geometric interpretation to the appearance of the heat equation in the Kähler quantization of $T^*K$, thus answering a question raised in [Ha3], §1.3. This interpretation is based in the projection of the prequantization connection to the quantum sub-bundle in geometric quantization, as proposed in [AdPW]. The main advantage of this approach consists in the fact that it ensures the that the quantum connection is Hermitian. Our method is also complementary to the one of Thiemann [Th1, Th2] where he considers generalized canonical transformations generated by complex valued functions on the phase space. The heat equation appears then naturally as the Schrödinger equation for these complex Hamiltonians or complexifiers.

For $\mathbb{R}^{2n}$ the heat equation is associated with independence of the quantization with respect to the choice of a complex structure within the family of complex structures which are invariant under translations.

We consider on $T^*K$ a one-parameter family of complex structures $\{J_s\}_{s \in \mathbb{R}_+}$ induced by the diffeomorphisms

$$T^*K \simeq K \times \mathfrak{k}^* \simeq K \times \mathfrak{k} \xrightarrow{\psi_s} K_{\mathbb{C}} \quad \psi_s(x, Y) \mapsto xe^{isY}, \quad (1.1)$$

where $K_{\mathbb{C}}$ is the complexification of $K$. Here we identify $T^*K$ with $K \times \mathfrak{k}^*$ by means of left-translation and then with $K \times \mathfrak{k}$ by means of an $Ad$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{k} = \text{Lie}(K)$.

Together with the canonical symplectic structure $\omega$, the pair $(\omega, J_s)$ defines on $T^*K$ a Kähler structure for every $s \in \mathbb{R}_+$. Hall has shown [Ha3] that, when one considers geometric quantization of $T^*K$, the CST gives a unitary map between the vertically polarized Hilbert space and the Kähler polar-
ized Hilbert space and provided that one takes into account the half-form correction.

We collect the prequantum and quantum Hilbert spaces for all $s \in \mathbb{R}_+$ in a prequantum and a quantum Hilbert bundles over $\mathbb{R}_+$,

$$\mathcal{H}^{sQ} \to \mathbb{R}_+$$

and

$$\mathcal{H}^Q \to \mathbb{R}_+$$

and show that the natural Hermitian connection on $\mathcal{H}^{sQ}$ induces on $\mathcal{H}^Q$ a connection given by a heat operator (Theorem 1). This connection turns out to be naturally equivalent to the connection obtained by varying $\hbar$ in the CST of Hall (Theorem 4).

Contrary to the flat $\mathbb{R}^{2n}$ case, and its infinite dimensional generalization considered in [AdPW], our family of complex structures is not generated by acting on a fixed one with a family of canonical transformations. It is generated by the flux of a vector field which is not symplectic, but rescales $\omega$. The symplectic structure on $T^*K$ however will be kept fixed throughout the paper.

Notice that, as could be expected from [Ha3], the use of the half-form correction to define the Hermitian structure on $\mathcal{H}^{sQ}$ plays a decisive role in the appearance of the heat equation for the connection induced on the quantum sub-bundle (see also remark 1).

2 The quantum connection and the heat equation

Let $K$ be a compact, connected Lie group. We will consider first the case when $K$ is semisimple and will comment briefly on the case of compact tori, $K = U(1)^n$, at the end of section 2.4.

We start by recalling from [Ha3, Wo] aspects of the geometric prequantization of $T^*K$ but with a natural one-parameter family of complex structures generalizing the fixed complex structure considered by Hall.

The prequantum Hilbert bundle $\mathcal{H}^{sQ}$ over this family is endowed with a natural Hermitian connection, $\delta^{sQ}$. The quantum connection $\delta^Q$ induced from $\delta^{sQ}$ by orthogonal projection on the quantum Hilbert sub-bundle is then automatically Hermitian. Our main result in the present section is Theorem
1 in which we show that $\delta^q$ corresponds in a precise sense to a family of Laplace operators on $T^*K$.

2.1 Complex structures and the prequantum Hilbert bundle

Consider an $Ad$-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g} = \text{Lie}(K)$ and $\{X_i\}_{i=1}^n$, $n = \text{dim } K$, a corresponding orthonormal basis for $\mathfrak{g}$ viewed as the space of left-invariant vector fields on $K$. The canonical 1-form on $T^*K$ is given by

$$\theta = \sum_{i=1}^n y^i w^i,$$

where $(y^1, \ldots, y^n)$ are the global coordinates on $K$ corresponding to the basis $\{X_i\}_{i=1}^n$, and $\{w^i\}_{i=1}^n$ is the basis of left-invariant 1-forms on $K$ dual to $\{X_i\}_{i=1}^n$, pulled-back to $T^*K$ by the canonical projection. The canonical symplectic 2-form is defined as $\omega = -d\theta$. We let $\epsilon$ denote the Liouville volume form on $T^*K$, given by

$$\epsilon = \frac{1}{n!} \omega^n. \quad (2.1)$$

Following the geometric quantization program we consider the trivial complex line bundle $L$ over $T^*K$, $L = T^*K \times \mathbb{C}$, with the trivial Hermitian structure. Sections of this bundle are thus just functions on $T^*K$.

Using the diffeomorphisms $\psi_s$ between $T^*K$ and $K_C$ introduced in (1.1) we produce a family, parameterized by $s \in \mathbb{R}^+$, of complex structures $J_s$ on $T^*K$ by pulling back the canonical complex structure $J$ from $K_C$. Explicitly,

$$J_s = \psi_{s}^{-1} \circ J \circ \psi_{ss},$$

where $\psi_{ss}$ denotes the push-forward of the map $\psi$.

**Proposition 1.** The pair $(\omega, J_s)$ defines a Kähler structure on $T^*K$ for every $s \in \mathbb{R}^+$, whose Kähler potential is $\kappa_s(x, Y) = s|Y|^2$.

**Proof.** The family of complex structures $J_s$ can also be generated by pulling back a fixed complex structure on $T^*K$ with the family of diffeomorphisms $\varphi_s(x, Y) = (x, sY)$ since $J_s = (\psi_1 \circ \varphi_s)^{-1} \circ J \circ (\psi_1 \circ \varphi_s)_s = \varphi_{ss}^{-1} \circ J \circ \varphi_{ss}$. Therefore $(\omega, J_s)$ defines a Kähler structure on $T^*K$ for every $s \in \mathbb{R}^+$ if and only if $(\varphi_{s}^{-1})^* \omega = \omega / s, J_1)$ defines a Kähler structure on $T^*K$. This follows from the fact that $(T^*K, \omega, J_1)$ is a Kähler manifold as shown in [Ha3]. In this reference, the Kähler potential of $(T^*K, \omega, J_1)$ is computed to be $\kappa(x, Y) = |Y|^2$. Then, the Kähler potential $\kappa_s$ for $(\omega, J_s)$ is $\kappa_s = (\varphi_s^* \kappa) / s = s|Y|^2$. \qed
Let $\tilde{X}_j, j = 1, \ldots, n$, be the vector fields on $T^*K$ generating the right action of $K$ lifted to $T^*K$ and given by
\[
\tilde{X}_j(x, Y) = (X_j, [Y, X_j]).
\]
(2.2)

Therefore,
\[
\psi_{s*}\tilde{X}_j(x, Y) = X_j(x) + iJ_s \tilde{w}_j(x, Y),
\]
where $\tilde{w}_j$ denotes the natural extension of $X_j$ from a left-invariant vector field on $K \subset K_C$ to the corresponding left-invariant vector field on $K_C$. Let $\{\tilde{w}_j\}$ be the basis of left invariant 1-forms dual to $\{\tilde{X}_j\}$. For every $s \in \mathbb{R}_+$, a basis of left invariant $J_s$-holomorphic 1-forms is then
\[
\{\tilde{\eta}_j = \tilde{w}_j - iJ_s \tilde{w}_j\}_{j=1}^n,
\]
where $(J_s w)(X) = w(J_s X)$, for a vector field $X$ and a 1-form $w$ on $T^*K$. Consider also the $J_s$-canonical bundle on $T^*K$ whose sections are $J_s$-holomorphic $n$-forms with natural Hermitian structure defined as follows. For a $J_s$-holomorphic $n$-form $\alpha_s$, let $|\alpha_s| = \alpha_s \wedge \alpha_s$ be the unique non-negative $C^\infty$ function on $T^*K$ such that $|\alpha_s| = b \epsilon |\alpha_s|^2 b \epsilon$, where $b = (2i)^n (-1)^{n(n-1)/2}$. Following [Ha3] we write
\[
|\alpha_s|^2 = \frac{\alpha_s \wedge \alpha_s}{b \epsilon}.
\]

A global nowhere vanishing (trivializing) $J_s$-holomorphic section of the $J_s$-canonical bundle is given by
\[
\Omega_s \equiv \tilde{\eta}_1 \wedge \cdots \wedge \tilde{\eta}_n.
\]

Let us introduce the bundle of half-forms. Let $\delta$ denote a square root of the $J_s$-canonical bundle with a fixed trivializing section whose square is $\Omega_s$. As in [Ha3] we denote this section by $\sqrt{\Omega_s}$. Smooth sections of $L \otimes \delta$ are of the form
\[
\sigma_s = f \sqrt{\Omega_s}, \quad f \in C^\infty(T^*K).
\]

The Hermitian structure on the line bundle $L \otimes \delta$ is given by
\[
\langle \sigma_s, \bar{\sigma}_s \rangle = \bar{f} f \left( \frac{\Omega_s \wedge \Omega_s}{b \epsilon} \right)^{1/2} = \bar{f} f |\Omega_s|.
\]
(2.3)
Definition 1. The prequantum bundle $\mathcal{H}^{prQ} \to \mathbb{R}_+$ is the Hilbert vector bundle with fiber over $s \in \mathbb{R}_+$ given by

$$\mathcal{H}_s^{prQ} = \overline{V}_s^{prQ}$$

where the bar denotes norm completion, $V_s^{prQ}$ is

$$V_s^{prQ} = \{ \sigma_s \in \Gamma^{\infty}(L \otimes \delta_s) : ||\sigma_s||^{prQ}_s < \infty \},$$

$\Gamma^{\infty}(L \otimes \delta_s)$ denotes the space of $C^\infty$ sections of the bundle $L \otimes \delta_s$, and

$$\langle \sigma_s, \sigma_s \rangle^{prQ}_s = \int_{T^*K} |\sigma_s|^2 c.$$

From (2.3) it is easy to see that sections of $\mathcal{H}^{prQ}$ of the form

$$f \sqrt{|\Omega_s|} / \sqrt{\Omega_s} \in L^2(T^*K, \epsilon)$$

have $s$-independent norm.

We choose the smooth Hilbert bundle structure on $\mathcal{H}^{prQ}$ as the one compatible with the global trivializing map

$$\mathbb{R}_+ \times L^2(T^*K, \epsilon) \to \mathcal{H}^{prQ},$$

$$(s, f) \mapsto f \sqrt{|\Omega_s|} / \sqrt{\Omega_s}$$

2.2 The prequantum connection $\delta^{prQ}$

We now introduce a natural Hermitian connection on $\mathcal{H}^{prQ}$. Before giving its precise definition we state the following proposition, which is a straightforward consequence of [Ha2, Ha3]. Let $\eta(Y)$ be the $Ad_K$-invariant function defined for $Y$ in a Cartan subalgebra by the following product over the set $R^+$ of positive roots of $\mathfrak{k}$,

$$\eta(Y) = \prod_{\alpha \in R^+} \frac{\sinh \alpha(Y)}{\alpha(Y)}.$$  

Let $dg$ be the Haar measure on $K_C$. We then have,

Proposition 2. The following identities hold:
1. $|\Omega_s| \equiv \sqrt{\frac{\Omega_s \wedge \Omega_s}{b \epsilon}} = s^2 \eta(sY)$;

2. $dg_s := (\psi_s)^*(dg) = s^n \eta^2(sY)\epsilon = |\Omega_s|^2 \epsilon$,

where $\eta$ is the function on $T^*K$ defined by equation (2.5).

Proof. In [Ha2, Ha3] it is shown that $b \eta_s^2(Y) = \Omega_1 \wedge \Omega_1$. This is exactly the first identity with $s = 1$. Recall from Proposition 1 that $J_s = \varphi_{ss}^{-1} \circ J_1 \circ \varphi_{ss}$. This implies the equality $\varphi_s^*(J_1(\varphi_s^* \beta))$ for all 1-forms $\beta$ on $T^*K$. Therefore,

$$\varphi_s^*(\tilde{\eta}_i) = \varphi_s^*(\tilde{w}^i) = \tilde{w}^i - iJ \tilde{w}^i = \tilde{\eta}_i.$$  

Moreover $\varphi_s^* \epsilon = s^n \epsilon$ and this proves the first equation. For the second identity, let

$$\{\eta_C^i = \tilde{w}^i_C - iJ \tilde{w}^i_C\}_{i=1}^n$$

be a basis of $J$-holomorphic 1-forms on $K_C$, where $\tilde{w}^i_C$ is the natural extension of $w^i$ from $K$ to $K_C$, obtained by left translations. Then, the Haar measure on $K_C$ is given by

$$dg = \frac{1}{b} \Omega_C \wedge \Omega_C,$$

where $\Omega_C = \eta_C^1 \wedge \cdots \wedge \eta_C^n$. Since $\psi_s^* \circ J = J_s \circ \psi_s^*$ for all 1-forms on $K_C$, and $\tilde{w}^i = \psi_s^*(\tilde{w}^i_C)$, we have

$$\psi_s^*(\eta_C^i) = \psi_s^*(\tilde{w}^i_C) - i\psi_s^*(J \tilde{w}^i_C) = \tilde{w}^i - iJ \tilde{w}^i = \tilde{\eta}_s^i.$$  

Using the previous result we get the desired identity.  

**Definition 2.** The prequantum connection $\delta^{\text{prQ}}$ on $\mathcal{H}^{\text{prQ}}$ is the connection for which sections of the form (2.4) are horizontal

$$\delta^{\text{prQ}} \left( \frac{f}{\sqrt{|\Omega_s|}} \right) = 0,$$

for all $f \in L^2(T^*K, \epsilon)$.

Note that the prequantum connection is Hermitian, that is, it is compatible with the Hermitian structure on $\mathcal{H}^{\text{prQ}}$ in the following sense

$$\frac{d}{ds} \langle \sigma, \zeta \rangle^{\text{prQ}} = \langle \delta^{\text{prQ}} \sigma, \zeta \rangle^{\text{prQ}} + \langle \sigma, \delta^{\text{prQ}} \zeta \rangle^{\text{prQ}}$$

for all smooth sections $\sigma, \zeta$ of $\mathcal{H}^{\text{prQ}}$.  

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2.3 The induced quantum connection – $\delta^Q$

The Kähler polarizations $(\omega, J_s)$ enter already the definition of the prequantum Hilbert spaces $\mathcal{H}_s^{\omega Q}$ through the half-form bundles $\delta_s$ and the Hermitian structures (2.3). To define the fibers $\mathcal{H}_s^Q$ of the quantum Hilbert sub-bundle $\mathcal{H}_s^Q \subset \mathcal{H}_s^{\omega Q}$, one considers polarized, or $J_s$-holomorphic, sections of $L \otimes \delta_s$.

Explicitly, for every $s \in \mathbb{R}_+$, consider the frame of left $K$-invariant vector fields on $T^*K$

$$\left\{ Z_{j,s} = \frac{1}{2}(X_j - iJ_sX_j) \right\}_{j=1}^n.$$

Let the polarizations be given, for every $s \in \mathbb{R}_+$, by

$$\mathcal{P}_{(x, Y)}^s = \text{span}_\mathbb{C} \left\{ \bar{Z}_{j,s}(x, Y) \right\}_{j=1}^n,$$

where $\bar{Z}_{j,s} = \frac{1}{2}(X_j + iJ_sX_j)$. We use the notation $\bar{Z} \in \mathcal{P}^s$ for $\bar{Z}_{(x,Y)} \in \mathcal{P}_{(x, Y)}^s$ for all $(x, Y) \in T^*K$.

**Definition 3.** The quantum bundle $\mathcal{H}^Q \to \mathbb{R}_+$ is the Hilbert sub-bundle of $\mathcal{H}^{\omega Q}$ with fiber over $s > 0$ given by

$$\mathcal{H}_s^Q = \mathcal{V}_s^Q$$

where

$$\mathcal{V}_s^Q = \left\{ f \sqrt{\Omega_s} \in \mathcal{V}_s^{\omega Q} : \nabla_{\bar{Z}} f = 0, \ \forall \ Z \in \mathcal{P}^s \right\},$$

and $\nabla_{\bar{Z}} = \bar{Z} - \frac{1}{\hbar_0} \theta(\bar{Z})$ is the geometric quantization connection defined on the trivial bundle $L$ and $\hbar_0$ is Planck's constant. We call the solutions of $\nabla_{\bar{Z}} f = 0$ the polarized sections of $L$.

**Proposition 3.** For every $s > 0$, the $\mathcal{P}^s$-polarized (or $J_s$-holomorphic) sections of $L$ are the $C^\infty$ functions $f$ on $T^*K$ of the form

$$f = F e^{-2|\bar{Z}|^2/\hbar_0}$$

where $F$ is an arbitrary $J_s$-holomorphic function on $T^*K$.

**Proof.** From [Ha3, Proposition 2.3] the solutions of $\nabla_{\bar{Z}_{j,s}} f = 0$ are

$$f = F e^{-\kappa_s/2\hbar_0},$$

where $F$ is a $J_s$-holomorphic function on $T^*K$, so that the result follows from proposition 1.

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Let us denote by \((\cdot, \cdot)_Q\) the Hermitian structure on \(\mathcal{H}^Q\) inherited from \(\mathcal{H}^{\text{pr}Q}\).

We conclude that the fibers \(\mathcal{H}_s^Q\) of \(\mathcal{H}^Q\) are given by

\[
\mathcal{H}_s^Q = \left\{ \sigma_s = Fe^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s}, \text{\(F\) is } J_s\text{-holomorphic and } ||\sigma_s||_Q^s < \infty \right\}.
\]

The quantum Hilbert bundle inherits from \((\mathcal{H}^{\text{pr}Q}, \delta^{\text{pr}Q})\) a Hermitian connection \(\delta^Q\) which we call the *quantum connection*. The parallel transport with respect to this connection is automatically unitary and it establishes the invariance of the quantization of \(T^*K\) with respect to the choice of polarization within the family \(\{P^s\}_{s \in \mathbb{R}_+}\).

**Definition 4.** The *quantum connection* \(\delta^Q\) is the Hermitian connection induced on \(\mathcal{H}^Q\) by the natural connection \(\delta^{\text{pr}Q}\) on \(\mathcal{H}^{\text{pr}Q}\)

\[
\delta^Q = P \circ \delta^{\text{pr}Q},
\]

where \(P\) denotes the orthogonal projection \(\mathcal{H}^{\text{pr}Q} \to \mathcal{H}^Q\).

Below in theorem 1 we will obtain an explicit expression for \(\delta^Q\). Consider the second order differential operator \(\Delta^s_C\) on \(T^*K\) given by the pull-back, with respect to \(\psi_s\), of the second order Casimir operator on \(K_C\),

\[
\Delta^s_C = \sum_{i=1}^n (\tilde{X}_i)^2 - (J_s \tilde{X}_i)^2.
\]

Note that \(\Delta^s_C\) takes \(J_s\)-holomorphic functions to \(J_s\)-holomorphic functions.

Let \(\hat{\Delta}^s_C\) be the (unbounded) operator on \(\mathcal{H}^Q_s\) defined on its dense domain by

\[
\hat{\Delta}^s_C \left(Fe^{-s|Y|^2/\hbar_0} \sqrt{\Omega_s} \right) = \Delta^s_C \left[F \right] e^{-s|Y|^2/\hbar_0} \sqrt{\Omega_s}.
\]

**Theorem 1.** Let \(F\) be the function on \(\mathbb{R}_+ \times T^*K\) obtained, for every \(s \in \mathbb{R}_+\), as the pull-back of a given \(J\)-holomorphic function \(\hat{F}\) on \(K_C\),

\[
F(s, x, Y) = \hat{F}(xe^{isY}),
\]

such that \(F(s, \cdot)e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s} \in \mathcal{H}_s^Q\) is in the domain of the operator \(\hat{\Delta}^s_C\). The quantum connection \(\delta^Q\) acts on sections of \(\mathcal{H}^Q\) of the form

\[
Fe^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s} \quad \text{(2.10)}
\]
as
\[
\delta^\mathcal{Q}_g \left[ F e^{-\frac{s|\gamma|^2}{2\hbar_0}} \sqrt{\Omega_s} \right] = \frac{\hbar_0}{2} \left( -\frac{1}{2} \Delta_s + |\rho|^2 \right) \left[ F \right] e^{-\frac{s|\gamma|^2}{2\hbar_0}} \sqrt{\Omega_s},
\]
(2.11)
where \( \rho \) is half the sum of the positive roots of \( \mathbb{R} \).

Notice that choosing the sections of \( \mathcal{H}^\mathcal{Q} \) in the form (2.9) and (2.10) corresponds to choosing moving frames, more precisely a class of global moving frames related by \( s \)-independent transformations. Having made such a choice the covariant derivative in the direction of \( \partial/\partial s \) is defined by a linear operator acting on the fibers as in (2.11).

We will divide the proof of this theorem in several lemmata. Let us denote by \( W \) the following vector field on \( T^*K \),
\[
W = i \sum_{j=1}^{n} y^j Z_{j,s}.
\]

Note that, from (2.2),
\[
W = \frac{i}{2} \sum_{j=1}^{n} y^j (X_j - iJ_s X_j) = \frac{i}{2} \sum_{j=1}^{n} y^j (\tilde{X}_j - iJ_s \tilde{X}_j).
\]

**Lemma 1.** Let \( \hat{F} \) be a fixed \( C^\infty \) function on \( K_C \), \( F \) be the function on \( \mathbb{R}_+ \times T^*K \), given by \( F(s,\cdot) = \psi^*_s \hat{F} \), and let \( f \in C^\infty(T^*K) \) be a function only of \( Y \). We have,

i) If \( \hat{F} \) is holomorphic then
\[
\frac{\partial F}{\partial s} = WF
\]

ii) If \( \hat{F} \) is right \( K \)-invariant then
\[
\frac{\partial F}{\partial s} = 2WF = 2\tilde{W}F
\]

iii) \[
Wf = \frac{1}{2s} \sum_{j=1}^{n} y^j \frac{\partial}{\partial y^j} f.
\]
Proof. A direct computation gives,

\[ \frac{\partial}{\partial s}(\psi_s^* \hat{F})(x, Y) = \frac{\partial}{\partial s} \hat{F}(xe^{isY}) = \sum_{j=1}^n y_j J_s \tilde{X}_j F. \]

The special cases i) and ii) above follow from this equation. The identity in iii) follows from

\[ \sum_{j=1}^n y_j J_s \tilde{X}_j F = \frac{1}{s} \sum_{j=1}^n y_j \frac{\partial}{\partial y_j}. \]

Lemma 2. Let \( X \) be a smooth vector field on \( T^* K \), with \( |X(y^i)| < c \exp(\alpha |Y|) \) for some positive constants \( c \) and \( \alpha \), and let \( \phi \in C^\infty(T^* K) \) be such that

\[ |\phi(x, Y)| < e^{-\delta |Y|^2}, \]

for \( |Y| > R \) and fixed positive constants \( R, \delta \). Then,

\[ \int_{T^* K} L_X(\phi \epsilon) = 0. \]

Proof. Using Cartan’s formula, \( L_X = d \circ \iota_X + \iota_X \circ d \), we have \( L_X(\phi \epsilon) = d(\phi \iota_X \epsilon) \), where the symplectic volume form is \( \epsilon = \omega^1 \wedge dy^1 \wedge \cdots \wedge \omega^n \wedge dy^n \). Since \( T^* K \cong K \times \mathfrak{g} \), using Stokes formula we only need to show that

\[ \lim_{R \to \infty} \int_{K \times S^{n-1}_R} \phi \iota_X \epsilon = 0, \]

where \( S^{n-1}_R \) denotes the sphere of radius \( R \) in \( \mathfrak{g} \) centered at the origin. We have

\[ \left| \int_{K \times S^{n-1}_R} \phi \iota_X \epsilon \right| < c' e^{-\frac{1}{2} R^2}, \]

for some positive constant \( c' \), which proves the lemma.

Let \( \mathcal{B} \) be the subspace of the space \( \mathcal{H}(K_{\mathbb{C}}) \) of holomorphic functions on \( K_{\mathbb{C}} \), spanned by the holomorphic functions

\[ \text{tr} (\pi(g)A), \quad (2.12) \]
where $\pi$ is an irreducible finite-dimensional representation of $K$ extended to an holomorphic representation of $K_{C}$ and $A \in \text{End} \ V_\pi$. Let $\mathcal{F}_s$ denote the subspace of $\mathcal{H}_s^0$ given by sections of the form (2.9) and (2.10) with $\bar{F} \in \mathcal{B}$. It follows from the Lemma 10 of [Ha1] that $\mathcal{F}_s$ is dense in $\mathcal{H}_s^0$.

**Lemma 3.** Let $\sigma, \zeta \in \Gamma(\mathcal{H}_s^0)$ with $\sigma_s, \zeta_s \in \mathcal{F}_s$, $\forall s \in \mathbb{R}^+$, and

$$\sigma = F e^{-\frac{s|Y|^2}{\hbar_0}} \sqrt{\Omega_s}, \quad \zeta = G e^{-\frac{s|Y|^2}{\hbar_0}} \sqrt{\Omega_s}.$$ 

Then we have,

$$\int_{T^*K} (\bar{W} - \frac{|Y|^2}{\hbar_0} + \frac{1}{2} \frac{\partial \ln |\Omega_s|}{\partial s} + \frac{n}{4} s) \left[ \bar{F} \right] G e^{-\frac{s|Y|^2}{\hbar_0}} |\Omega_s| \epsilon = 0. \quad (2.13)$$

**Proof.** The vector field $W$ satisfies $\bar{W}(y^j) = \frac{1}{2} s y^j$ and we can apply lemma 2 with $0 < \delta < \frac{s}{\hbar_0}$ to the first term above. Integrating by parts we obtain

$$- \int_{T^*K} \bar{W} \left[ \bar{F} \right] G e^{-\frac{s|Y|^2}{\hbar_0}} |\Omega_s| \epsilon = \int_{T^*K} \bar{F} G \bar{W} \left[ e^{-s|Y|^2/\hbar_0} \right] |\Omega_s| \epsilon$$

$$+ \int_{T^*K} \bar{F} G e^{-s|Y|^2/\hbar_0} \bar{W} \ln |\Omega_s| |\Omega_s| \epsilon$$

$$+ \int_{T^*K} \bar{F} G e^{-s|Y|^2/\hbar_0} |\Omega_s| \mathcal{L}_\bar{W}(\epsilon). \quad (2.13)$$

For the first term on the r.h.s. of (2.13) we have from $iii)$ in lemma 1

$$\bar{W}[e^{-s|Y|^2/\hbar_0}] = \frac{1}{2s} \sum_{j=1}^{n} y^j \frac{\partial}{\partial y^j} e^{-s|Y|^2/\hbar_0} = \frac{|Y|^2}{\hbar_0} e^{-s|Y|^2/\hbar_0}.$$

From $ii)$ in lemma 1, we see that

$$\bar{W} \ln |\Omega_s| = \frac{1}{2} \frac{\partial \ln |\Omega_s|}{\partial s} - \frac{n}{4} s. \quad (2.15)$$

For the third term in (2.13), we have

$$\mathcal{L}_\bar{W}(\epsilon) = \frac{n}{n!} \mathcal{L}_\bar{W}(\omega) \wedge \omega^{n-1}.$$ 

From $\omega = -d\theta$, Cartan’s formula $\mathcal{L}_\bar{W} = d \circ \iota_W + \iota_W \circ d$ and

$$\iota_W d\theta = \frac{1}{2s} \theta + \frac{i}{4} d|Y|^2,$$

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we obtain $L_W(\omega) = (1/2s)\omega$, which implies that

$$L_W(\epsilon) = \frac{n}{2s} \epsilon. \quad (2.16)$$

Substituting (2.14), (2.15) and (2.16) into (2.13) we obtain the desired result.

Recall that the prequantum connection is Hermitian, so it satisfies equation (2.7). Consider sections of $\mathcal{H}^2$,

$$\sigma = F e^{-\frac{|Y|^2}{2\hbar_0}} \sqrt{\Omega_s}, \quad \zeta = G e^{-\frac{|Y|^2}{2\hbar_0}} \sqrt{\Omega_s}, \quad (2.17)$$

where $F, G$ are as in (2.9). These sections also satisfy (2.7) and we have

$$\langle \delta^{Q^2}_{\hat{\pi}} \sigma, \zeta \rangle^Q = \langle \delta^{m^2}_{\hat{\pi}} \sigma, \zeta \rangle^{m^2}. \quad (2.18)$$

**Lemma 4.** Let $\sigma, \zeta$ be as above. Then, the following identity holds

$$\langle \delta^{Q^2}_{\hat{\pi}} \sigma, \zeta \rangle^Q = s^{-n/2} \int_{K_C} \tilde{F} \tilde{G} \ h_0 \left( \frac{|Y|^2}{2\hbar^2} - \frac{n}{4\hbar} \right) e^{-|Y|^2/\hbar} \eta(Y) \ |\Omega_s| \ \epsilon. \quad (2.19)$$

where $dg$ is the Haar measure on $K_C$, $g = xe^{iY}$ and $\hbar = s\hbar_0$.

**Proof.** From the definition of horizontal sections (2.6) and from (2.18), we obtain

$$\langle \delta^{Q^2}_{\hat{\pi}} \sigma, \zeta \rangle^Q = \int_{T^*K} \left( \frac{\partial F}{\partial s} - \frac{|Y|^2 F}{2\hbar_0} + \frac{F \partial \ln |\Omega_s|}{2} \right) \tilde{G} e^{-\frac{|Y|^2}{\hbar_0}} |\Omega_s| \ |\epsilon|.$$

We now use lemmata 1 and 3 to simplify the equation above for the quantum connection to get,

$$\langle \delta^{Q^2}_{\hat{\pi}} \sigma, \zeta \rangle^Q = \int_{T^*K} \tilde{F} \tilde{G} \left( \frac{|Y|^2}{2\hbar_0} - \frac{n}{4s} \right) e^{-|Y|^2/\hbar} |\Omega_s| \ |\epsilon|,$$

which with the help of $\psi_s$ and proposition 2 gives (2.19).

Let us introduce the $K$-averaged heat kernel measure $d\nu_h$ on $K_C$ given by [Ha3]

$$d\nu_h(g) = \nu_h(g) \ dg = c_h \left( \frac{e^{-|Y|^2/\hbar}}{\eta(Y)} \right) \ dg, \quad (2.20)$$
and \( c_h = (\pi \hbar)^{-n/2} e^{-|\rho|^2 \hbar} \), \( \rho \) being half the sum of the positive roots. Recall from [Ha1] that \( \nu_h \) satisfies the equation

\[
\frac{\partial \nu_h}{\partial \hbar} = -\frac{1}{4} \Delta_C \nu_h
\]

(2.21)
on \( K_C \), where

\[
\Delta_C = \sum_{i=1}^n (X_{i,C})^2 - (JX_{i,C})^2,
\]
is the Casimir operator for \( K_C \). The equation (2.21) is equivalent to the following equality:

**Lemma 5.**

\[
\left( \left( \frac{|Y|^2}{2\hbar^2} - \frac{n}{4\hbar} \right) e^{-|Y|^2/\hbar} \frac{\eta(Y)}{\eta(Y)} \right) = \left( -\frac{1}{8} \Delta_C + \frac{|\rho|^2}{2} \right) e^{-|Y|^2/\hbar} \eta(Y).
\]

\[
\square
\]

We are now ready to prove Theorem 1.

**Proof.** Consider \( \sigma \) and \( \zeta \) as in (2.17) and let

\[
Z_j = \frac{1}{2} (X_{j,C} - iJX_{j,C}), \quad j = 1, \cdots, n,
\]

so that

\[
\Delta_C = 2 \sum_{j=1}^n Z_j^2 + \bar{Z}_j^2.
\]

From (2.19) and Lemma 5, since

\[
X_{j,C}(e^{-|Y|^2/\hbar} \eta(Y)) = 0, \quad \text{for all } j,
\]

we obtain

\[
\langle \delta^Q_0 \sigma, \zeta \rangle^Q = s^{-n/2} \int_{K_C} \tilde{F} \tilde{G} \, \hbar_0 \left( -\frac{1}{8} \Delta_C + \frac{|\rho|^2}{2} \right) \left[ \frac{e^{-|Y|^2/\hbar}}{\eta(Y)} \right] \, dg = \]

\[
= s^{-n/2} \int_{K_C} \tilde{F} \tilde{G} \, \hbar_0 \left( -\sum_{j=1}^n \bar{Z}_j^2 + |\rho|^2 \right) \left[ \frac{e^{-|Y|^2/\hbar}}{\eta(Y)} \right] \, dg.
\]

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Lemma 2 can be applied since $\bar{Z}_j(y^j)$ grows linearly with $|Y|$ (see section 6 in [Ha3]). Using it to integrate twice by parts with respect to $\bar{Z}_j$ and noticing that, from the bi-invariance of $dg$, $L\bar{Z}_j dg = 0$ and also $\bar{Z}_j(\hat{G}) = 0$ we obtain the statement of the theorem

$$\langle \frac{\delta Q}{\delta \sigma} \partial_s \sigma, \zeta \rangle_Q = \int_{T^*K} \frac{h_0}{2} \left( -\frac{1}{2} \hat{\Delta}_s^c + |\rho|^2 \right) G e^{-s|\gamma|^2/h_0} |\Omega_s| \epsilon,$$  \hspace{1cm} (2.22)

for sections $\sigma, \zeta$ with values in $\mathcal{F}_s \ni \sigma_s, \zeta_s$, for all $s \in \mathbb{R}_+$. The operator $\hat{\Delta}_s^c$ in (2.8) is essentially self-adjoint (it has a basis of eigenvectors, with $\hat{F}$’s given by matrix elements of finite dimensional holomorphic representations as in (2.12), with real, and nonpositive, eigenvalues) and the space $\mathcal{F}_s$ is dense in $\mathcal{H}_s^Q$. Therefore, the expression (2.22) implies that $\delta Q/Q$ is given by (2.11) for sections of $\mathcal{H}^Q$ in the form (2.9) and (2.10) and with values in the domain of $\hat{\Delta}_s^c$, for all $s \in \mathbb{R}_+$. \hfill \Box

**Remark 1.** Note that the simple expression (2.19) would not be valid without the inclusion of the half-form correction in the definition of the Hermitian structure on $\mathcal{H}^{pr}$ (see (2.3) and proposition 2). The half-form leads to the cancellation of the term proportional to $\partial \ln |\Omega_s|/\partial s$. This is what ultimately leads to the heat operator in the quantum connection.

### 2.4 The heat equation

Let $\Psi$ be the diffeomorphism

$$\begin{align*}
\Psi : \mathbb{R}_+ \times T^*K & \to \mathbb{R}_+ \times K_C \\
(s, (x, Y)) & \mapsto (s, xe^{isY}),
\end{align*}$$

so that $\Psi(s,.) = \psi_s$, for all $s \in \mathbb{R}_+$. Consider sections $\sigma$ of $\mathcal{H}^Q$ of the form

$$\sigma_s = \frac{1}{\sqrt{a_s}} F(s, \cdot) e^{-s|Y|^2/2h_0} \sqrt{\Omega_s},$$  \hspace{1cm} (2.23)

where $a_s = (\pi h_0)^{n/2} e^{2|\rho|^2 h_0 s}$,

$$F = \Psi^* \tilde{F}$$  \hspace{1cm} (2.24)

and $\tilde{F}$ is a $C^\infty$ function on $\mathbb{R}_+ \times K_C$, holomorphic in the second variable. We then have,
**Theorem 2.** A section of $\mathcal{H}^Q$ of the form (2.23), (2.24) with $\sigma_s$ in the domain of $\tilde{\Delta}_s$, for all $s \in \mathbb{R}_+$, is $\delta^Q$-horizontal if and only if $\tilde{F}$ is a solution of the following heat equation on $K_C$,

$$
\frac{\partial}{\partial s} \tilde{F} = \frac{\hbar_0}{4} \Delta_C \tilde{F}.
$$

**Proof.** Substituting (2.23) in (2.11) we obtain

$$
\delta^Q \sigma = \Psi^* \left( \frac{\partial}{\partial s} \tilde{F} - \frac{\hbar_0}{4} \Delta_C \tilde{F} \right) e^{-\frac{s|Y|^2}{2\hbar_0}} \frac{e^{-\frac{|Y|^2}{2\hbar_0}}}{\sqrt{\Omega_s}}.
$$

Besides establishing the relation of the heat equation on complex semisimple Lie groups $K_C$ with the geometric quantization of $T^*K$ this result also provides the link with the coherent state transform introduced by Hall in [Ha1].

Concerning the case $K = U(1)^n$, note that the results in proposition 2 and equations (2.20) and (2.21) are still valid if we replace $\eta(sY)$ by 1 and the Weyl vector $\rho$ by 0. Therefore, all the results above apply also to this case.

### 3 The quantum connection, the heat equation and the CST

The coherent state transform for $K$ defines a parallel transport on a Hilbert bundle over $\mathbb{R}_+$ for an Hermitian connection that we denote by $\delta^H$. In this section we show that this parallel transport is naturally equivalent to the parallel transport of the quantum connection $\delta^Q$. This follows from propositions 2 and 3, and from (2.3) and (2.20).

#### 3.1 The CST connection - $\delta^H$

Let $\rho_\hbar$, $\hbar > 0$, be the heat kernel for the Laplacian $\Delta$ on $K$ associated to the $Ad$-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$, $\Delta = \sum_{i=1}^n X_i^2$. As proved in [Ha1], $\rho_\hbar$ has a unique analytic continuation to $K_C$, also denoted by $\rho_\hbar$. The
$K$-averaged coherent state transform (CST) is defined as the map

$$C_h : L^2(K, dx) \to \mathcal{H}(K_C)$$

$$\quad (C_h f)(g) = \int_K f(x) \rho_h(x^{-1}) \, dx, \quad f \in L^2(K, dx), \; g \in K_C, \quad (3.1)$$

where $dx$ is the normalized Haar measure on $K$. For each $f \in L^2(K, dx)$, $C_h f$ is the analytic continuation to $K_C$ of the solution of the heat equation on $K$,

$$\frac{\partial u}{\partial \hbar} = \frac{1}{2} \Delta u,$$

with initial condition given by $u(0, x) = f(x)$. Therefore, $C_h f$ is given by

$$(C_h f)(g) = (C \circ \rho_h \ast f)(g) = \left( C \circ e^{\frac{\hbar}{2} \Delta} f \right)(g),$$

where $\ast$ denotes the convolution in $K$ and $C$ denotes analytic continuation from $K$ to $K_C$. Recall that the $K$-averaged heat kernel measure $d\nu_h$ on $K_C$ is given by (2.20). Hall proves the following:

**Theorem 3 (Hall).** For each $h > 0$, the mapping $C_h$ defined in (3.1) is an unitary isomorphism from $L^2(K, dx)$ onto the Hilbert space $\mathcal{H}L^2(K_C, d\nu_h) := L^2(K_C, d\nu_h) \cap \mathcal{H}(K_C)$.

This transformation defines a Hilbert vector bundle $\mathcal{H}^h$ over the one-dimensional base $\mathbb{R}_+$ with global coordinate $h$,

$$\mathcal{H}^h \to \mathbb{R}_+, \quad \mathcal{H}^h_h = \mathcal{H}L^2(K_C, d\nu_h).$$

which we call the **CST bundle**. The Hilbert space structure on the fibers of $\mathcal{H}^h$ is defined by

$$\langle F_h, G_h \rangle_h := \int_{K_C} F_h(g) G_h(g) \, d\nu_h, \quad F_h, G_h \in \mathcal{H}^h_h.$$

From Theorem 3 we conclude that the operators

$$U^h_{h_2h_1} = C_{h_2} \circ C^{-1}_{h_1} : \mathcal{H}^h_{h_1} \to \mathcal{H}^h_{h_2} \quad (3.2)$$

define unitary transformations between the fibers which satisfy

$$U^h_{h_3h_2} \circ U^h_{h_2h_1} = U^h_{h_3h_1}.$$
These operators correspond to the parallel transport with respect to an Hermitian connection $\delta^H$ on $\mathcal{H}^H$. Let $\hat{K}$ denote the set of (equivalence class of) irreducible unitary representations of $K$. It follows from Theorem 3 that by choosing an orthonormal basis in the vector spaces $V_R$ of all the representations $R \in \hat{K}$ the matrix entries $R_{ij}(\cdot)$ analytically continued to $K_C$ form an orthogonal basis of $\mathcal{H}_h^H$

$$\{R_{ij}(\cdot)\}_{R \in \hat{K}, i,j=1,\ldots,\dim V_R} \subset \mathcal{H}_h^H$$

for all $h > 0$ and

$$\int_{K_C} \overline{R_{ij}(g)} R_{ij'}(g) \, d\nu_h = e^{i c_R \delta_{ij'} \delta_{jj'}}$$

where $c_R$ is the eigenvalue of $-\Delta$ on the representation $R$. Therefore the sections of $\mathcal{H}^H$

$$R_{ij}(\cdot)$$

form a global orthogonal frame and obviously the sections

$$F_{Rij}^R = e^{-h c_R/2} R_{ij}(\cdot) \quad (3.3)$$

form a global orthonormal frame, that is, their norms do not depend on $h$. Moreover

$$U_{h_2 h_1}^H (R_{ij}) = e^{-(h_2 - h_1)c_R/2} R_{ij} \iff U_{h_2 h_1}^H (F_{Rij}^R) = F_{Rij}^R \quad (3.4)$$

**Definition 5.** The CST connection $\delta^H$ on $\mathcal{H}^H$ is the (unique) connection for which the sections (3.3) are horizontal

$$\delta^H(F_{Rij}) = 0. \quad (3.5)$$

The CST connection is automatically Hermitian. From (3.3), $\delta^H$ can be equivalently defined through

$$\frac{\delta^H_{\Delta c}}{2} R_{ij} = \frac{c_R}{2} R_{ij} = -\frac{\Delta c}{4} R_{ij}.\quad \text{(3.6)}$$

**Proposition 4.** The unitary parallel transport $U$ corresponding to the connection (3.5) is given by the unitary operators (3.2), $U = U^H$. \hfill \text{(3.7)}
Proof. Since the sections \( F_{Rij} \) satisfy (3.5) the parallel transport for them is

\[
U_{h_2 h_1} F_{Rij}^{h_1} = F_{Rij}^{h_2} \quad \forall h_1, h_2 \in \mathbb{R}_+.
\] (3.6)

This map of orthonormal basis extends to a unique unitary isomorphism \( U_{h_2 h_1} : \mathcal{H}^H_{h_1} \to \mathcal{H}^H_{h_2} \).

From (3.4) and (3.6) we see that \( U_{h_2 h_1} \) and \( U_{h_2 h_1} \) coincide on the basis vectors and therefore they coincide as unitary operators. \( \square \)

3.2 Equivalence between \( \delta^H \) and \( \delta^Q \)

Our main result in this section is

**Theorem 4.** The quantum connection \( \delta^Q \) and the CST connection \( \delta^H \) are equivalent in the sense that there exists a natural unitary isomorphism \( S : \mathcal{H}^H \to \mathcal{H}^Q \) such that

\[
\delta^Q_{\frac{\partial}{\partial \hbar}} \circ S = S \circ \delta^H_{\frac{\partial}{\partial \hbar}}. \tag{3.7}
\]

**Proof.** We start by constructing a natural unitary isomorphism between the CST bundle \( \mathcal{H}^H \) and the quantum bundle \( \mathcal{H}^Q \). Let \( \sigma, \zeta \) be two sections of \( \mathcal{H}^Q \) of the form (2.9), (2.10),

\[
\sigma_s = \hat{F}(xe^{isY}) e^{-\frac{|Y|^2}{2h_0}} \sqrt{\Omega_s} a_s
\]

\[
\zeta_s = \hat{G}(xe^{isY}) e^{-\frac{|Y|^2}{2h_0}} \sqrt{\Omega_s} a_s.
\]

From proposition 2 we obtain

\[
\langle \sigma_s, \zeta_s \rangle^Q = a_s \int_{K_C} \overline{F} \hat{G} dv_h,
\]

where \( h = sh_0 \) and \( a_s \) was defined in (2.23). Therefore, the bundle morphism

\[
S_h : \mathcal{H}^H_{h} \to \mathcal{H}^Q_s \quad \hat{F} \mapsto \psi_s^* (\hat{F}) e^{-\frac{|Y|^2}{2h_0}} \sqrt{\frac{\Omega_s}{a_s}},
\]

is a unitary isomorphism. To show (3.7) it is sufficient to see that the frame of horizontal sections

\[
\{ F_{Rij} = e^{-h_{CR}/2}R_{ij} \}
\]

is mapped to an horizontal frame. This follows directly from theorem 2. \( \square \)
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