Sample Exam - Lie Algebras and Lie Groups

DMIST - 2 Feb. 2011

1. Let L be the Lie algebra defined as $\mathbb{C}^3 = \mathbb{C}\langle x, y, z \rangle$ with bracket given by

$$[x, y] = z, \qquad [x, z] = y, \qquad [y, z] = 0.$$

   (a) Prove that L is solvable. Is L nilpotent?

   (b) Show that there are no surjective homomorphisms from $\mathsf{sl}(2, \mathbb{C})$ to L.

2. A Lie algebra L is called *reductive* if its adjoint representation is completely reducible. Show that if L is reductive, then L is isomorphic to the direct sum $Z(L) \oplus [L, L]$.

3. Let $\Phi \subset E$ be a root system and $\Phi' \subset \Phi$ a non-empty subset such that $\Phi' = -\Phi'$ and satisfying the property: If $\alpha, \beta \in \Phi'$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi'$. Let E' be the euclidean space generated by Φ'.

   (a) Show that Φ' is a root system in E'.

   (b) Give an example where Φ' does not coincide with $\Phi \cap E'$.

4. Show that any homomorphism (one-parameter subgroup) $\phi : \mathbb{C}^* \to \mathsf{SL}(n, \mathbb{C})$ is conjugated to
$$\tilde{\phi}(\lambda) := \mathrm{diag}(\lambda^{k_1}, ..., \lambda^{k_n}),$$
   for certain integers $k_1, ..., k_n$ satisfying $k_1 + ... + k_n = 0$.

5. Show that $\mathbb{R}^2$ equipped with the multiplication

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 + e^{b_1} a_2, b_1 + b_2)$$

   is a Lie group, and there there is a left invariant integral which is not right invariant and is not conjugation invariant.

6. Let $G \subset GL(n, \mathbb{C})$ be a compact Lie group with identity e, and χ be the character of a representation of G. Prove that, for all $g \in G$, $|\chi(g)| \leq \chi(e)$, with equality if and only if g is a scalar.